ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 4, Fall 1971

EXAMPLES OF MODULAR ANNIHILATOR ALGEBRAS

BRUCE A. BARNES

1. **Introduction.** The theory of topological algebras with minimal ideals has developed to the point where now there is considerable literature on the subject. The earliest paper to appear is probably that of W. Ambrose [4] in which he introduced H^*-algebras, certain algebras that generalize the L^2 algebra of a compact topological group. Other early papers dealt with dual algebras (I. Kaplansky [26]), completely continuous algebras (I. Kaplansky [26]), and annihilator algebras (F. Bonsall and A. Goldie [18]). The most recent development has been the introduction of compact Banach algebras (K. Vala [50] and J. C. Alexander [1]). The definition of each of these algebras involves the topology of the algebra. In [54] B. Yood defined modular annihilator rings. We abbreviate modular annihilator to m.a. A ring A is a m.a. ring if every modular maximal left (right) ideal of A has a nonzero right (left) annihilator (note that this definition is purely algebraic). An equivalent formulation when A is semisimple is: A is a m.a. ring if and only if A/S_A is a radical ring, where S_A is the socle of A. There are many topological algebras with this property including (assuming semisimplicity) H^*-algebras, dual and annihilator Banach algebras, Banach algebras with dense socle, completely continuous normed algebras, and compact Banach algebras. The concept of m.a. algebras unifies the study of these various algebras. For example it is true for a general m.a. ring A that the structure space of primitive ideals of A is discrete in the hull-kernel topology; see [8, Theorem 4.2, p. 569]. This is proved for completely continuous Banach algebras in [27, Theorem 5.1], for annihilator B^*-algebras in [34, Corollary (4.10.15)] and for compact Banach algebras in [1, Theorem 6.1]. The basic properties of m.a. algebras can be found in [8] and [57].

The purpose of this paper is to provide examples of m.a. algebras which occur in analysis and in the theory of topological algebras with minimal ideals. Some of the results are taken from unpublished portions of [7]. A bibliography of papers concerning algebras with minimal ideals is included.

Received by the editors March 2, 1970 and, in revised form, June 22, 1970.
AMS 1970 *subject classifications.* Primary 46H05.

2. **Preliminaries.** We assume throughout that A is a complex semisimple algebra. S_A will denote the socle of A. $\mathrm{Sp}_A(a)$ ($\mathrm{Sp}_A'(a)$) is the (nonzero) spectrum of an element $a \in A$. Given $a \in A$, L_a and R_a are the linear operators defined on A by $L_a(b) = ab$ and $R_a(b) = ba$, $b \in A$.

Let X be a locally convex topological linear space. We denote by $\mathcal{C}(X)$ the algebra of all continuous linear operators on X, by $\mathcal{F}(X)$ the set of all $T \in \mathcal{C}(X)$ which have finite dimensional range, and by $\mathcal{K}(X)$ the algebra of all compact operators on X. I is the identity operator on X. When T is a linear operator on X, $\mathcal{N}(T)$ is the null space of T. The ascent of T is the smallest nonnegative integer such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ or $+\infty$ if no such integer exists.

Now we prove a useful lemma.

LEMMA 2.1. *Assume that A is a topological algebra. Then $a \in A$ is quasi-regular in A if and only if L_a is quasi-regular in $\mathcal{C}(A)$.*

PROOF. First assume that a is quasi-regular in A. Then by definition there exists $b \in A$ such that $a + b - ab = a + b - ba = 0$. Then $L_a + L_b - L_aL_b = L_a + L_b - L_bL_a = 0$, so that L_a is quasi-regular in $\mathcal{C}(A)$. Conversely assume that $S \in \mathcal{C}(A)$ and $S + L_a - SL_a = S + L_a - L_aS = 0$. Then $I - L_a$ maps A onto A, and therefore $(1 - a)A = A$. Then there exists $b \in A$ such that $(1 - a)b = -a$. Therefore $L_b + L_a - L_aL_b = 0$. This implies that $L_b = S$. Therefore $L_b + L_a - L_bL_a = 0$. Then $(b + a - ba)A = 0$, and semisimplicity of A implies that $b + a - ba = 0$. Then b is the quasi-inverse of a.

3. **Criteria for an algebra to be a modular annihilator algebra.** In this section we present a number of sufficient conditions (C1 through C11) for an algebra to be a m.a. algebra. Most of the conditions listed are known, but C4, C5, C8, and C10 have not been previously published, and the proof of C7 is new. Recall that by our standing assumption A is always a complex semisimple algebra.

C1. A is a m.a. algebra if and only if every modular maximal left ideal of A has a nonzero right annihilator.

C2. A is a m.a. algebra if and only if A/S_A is a radical algebra.

Both C1 and C2 follow from [**57**, Theorem 3.4, p. 38].

C3. A is a m.a. algebra if and only if S_A is not contained in any primitive ideal of A.

A simple algebraic argument shows that C2 and C3 are equivalent.

C4. Let I be the intersection of all the primitive ideals of A which contain S_A. If J is any two-sided ideal of A such that $S_A \subset J \subset I$, then J is a m.a. algebra.

PROOF. Note that $S_J = S_A$. By C3 it is sufficient to show that there is no primitive ideal Q of J such that $S_J \subset Q$. Suppose such an ideal Q does exist. Let $R = \{x \in A \mid xJ \subset Q\}$. Then R is a primitive ideal of A and $Q = R \cap J$ by [24, Proposition 3, p. 206]. Then $S_A \subset R$ so that $I \subset R$, and therefore $J \subset R$. But then $J = Q$, a contradiction.

C5. Assume that A is a topological algebra such that every primitive ideal of A is closed. If J is a two-sided ideal of A such that $S_A \subset J$ and $J^n \subset \overline{S_A}$ for some positive integer n, then J is a m.a. algebra.

PROOF. By C4 it is enough to show that whenever P is a primitive ideal of A such that $S_A \subset P$, then $J \subset P$. P is a closed ideal of A so that $\overline{S_A} \subset P$. Then $J^n \subset P$. By [34, Theorem (2.2.9)(iv), p. 54] $J \subset P$.

C6. A semisimple Banach algebra with dense socle is a m.a. algebra.

C6 follows immediately from C5 with $n = 1$. C6 was first proved by B. Yood [57, Lemma 3.11, p. 41].

The next proposition is proved in [57, Theorem 3.7, p. 40]. We give a different proof.

C7. Let A be a m.a. algebra and assume that J is a two-sided ideal of A. Then J is a m.a. algebra.

PROOF. By C3 it is enough to show that there are no primitive ideals Q of J such that $S_J \subset Q$. Suppose Q were such an ideal. Then $R = \{x \in A \mid xJ \subset Q\}$ is a primitive ideal of A and $Q = R \cap J$ by [24, Proposition 3, p. 206]. But then $S_A \cdot J \subset S_J \subset Q$ so that $S_A \subset R$. This contradicts C3.

When E is a subset of A, $R[E]$ denotes the right annihilator of E.

C8. Assume that A is an algebra of operators on a linear space X with the properties:

(1) When $T \in A$ is not left quasi-regular, then there exists a nonzero idempotent $E \in S_A$ such that $R[A(I - T)] = EA$.

(2) The ascent of $I - T$ is finite for all $T \in A$. Then A is a m.a. algebra.

PROOF. Let M be a modular maximal left ideal of A and assume that $A(I - U) \subset M$. We shall show that $R[M] \neq 0$. If not, by [57, Lemma 3.3, p. 38] $S_A \subset M$. By (2) we may assume that the ascent of $I - U$ is a finite nonnegative integer n. $A(I - U)^n \subset M$, and therefore the element $I - (I - U)^n$ is not left quasi-regular in A. By (1) there exists a nonzero idempotent $E \in S_A$ such that $R[A(I - U)^n] = EA$. Consider $I - W = (I - U)^n - E$. Since $E \in S_A \subset M$, then $A(I - W) \subset M$. Therefore W is not left quasi-regular. By (1) there exists $F \in A$ such that $F \neq 0$ and $(I - W)F = 0$. Then $(I - U)^n F = EF$. Therefore $(I - U)^{2n} F = (I - U)^n EF = 0$. By the choice of

n, $\mathcal{N}((I - U)^{2n}) = \mathcal{N}((I - U)^n)$. Then $F(X) \subset \mathcal{N}((I - U)^n)$, and thus $(I - U)^n F = 0$. Therefore $EF = 0$, and since $F \in R[A(I - U)^n] = EA$, $F = EF = 0$. This is a contradiction. We have proved that whenever M is a modular maximal left ideal of A, then $R[M] \neq 0$. It follows that A is a m.a. algebra by C1.

C9. Assume that A is a Banach algebra with the property that for every $a \in A$ $\mathrm{Sp}_A(a)$ has no nonzero limit points. Then A is a m.a. algebra.

C9 is proved in [**12**, Theorem 4.2, p. 516].

We call a linear operator T on a normed linear space X asymptotically quasi-compact if

$$\inf_{S \in \mathcal{K}(X)} \|T^n - S\|^{1/n} \to 0 \quad \text{as } n \to +\infty.$$

This definition is that of A. F. Ruston in [37].

C10. Assume that A is a Banach algebra with the property that the operator $S_a = L_a R_a$ is asymptotically quasi-compact for each $a \in A$. Then A is a m.a. algebra.

Proof. By [37, Lemma 3.5, p. 324] $\mathrm{Sp}'_{\mathcal{C}(A)}(S_a)$ is a finite set or an infinite sequence converging to zero for all $a \in A$. Given $a \in A$, let $C(a)$ be the commutator of a. Then $\mathrm{Sp}'_{\mathcal{C}(a)}(a^2) = \mathrm{Sp}_A{}'(a^2)$. Therefore if $1 \in \mathrm{bndry}(\mathrm{Sp}_A{}'(a^2))$, then there exists $\{u_n\} \subset C(a)$ such that $\|u_n\| = 1$ for all n and $\|(I - S_a)u_n\| \to 0$, so that $1 \in \mathrm{Sp}'_{\mathcal{C}(A)}(S_a)$. It follows that for every $a \in A$, $\mathrm{bndry}(\mathrm{Sp}_A{}'(a^2)) \subset \mathrm{Sp}'_{\mathcal{C}(A)}(S_a)$. Therefore $\mathrm{Sp}_A{}'(a^2)$ is a finite set or an infinite sequence converging to zero. By [34, Theorem (1.6.10), p. 32] the same must be true for $\mathrm{Sp}_A{}'(a)$. It follows from C9 that A is a m.a. algebra.

We note here that the converse of C10 holds. For assume that A is a normed m.a. algebra. $A/\overline{S_A}$ is a normed radical algebra. Given $a \in A$ it follows from [34, Theorem (2.3.4), p. 56] that there exists $\{t_n\} \subset S_A$ such that

$$\|a^n - t_n\|^{1/n} = \epsilon_n \to 0 \quad \text{as } n \to \infty.$$

$S_{t_n} \in \mathcal{G}(A)$ for all n by [1, Theorem 7.2, p. 14]. Then

$$\begin{aligned} \|S_a{}^n(u) - S_{t_n}(u)\| &= \|a^n u a^n - t_n u t_n\| \\ &\leqq \|a^n u a^n - t_n u a^n\| + \|t_n u a^n - t_n u t_n\| \\ &\leqq \|a^n - t_n\| \, \|u\| \, \|a^n\| + \|a^n - t_n\| \, \|t_n\| \, \|u\|. \end{aligned}$$

Therefore

$$\begin{aligned} (\|S_a{}^n - S_{t_n}\|)^{1/n} &\leqq \epsilon_n(\|a^n\| + \|t_n\|)^{1/n} \\ &\leqq \epsilon_n((2\|a^n\|)^{1/n} + \epsilon_n) \to 0 \end{aligned}$$

as $n \to \infty$. Therefore S_a is asymptotically quasi-compact.

C11. Assume that A is a Banach algebra and a m.a. algebra. Then every semisimple closed subalgebra of A is a m.a. algebra.

C11 follows easily from C9; see [**12**, p. 517].

4. **Examples.** This section is devoted to examples of m.a. algebras. When A is a m.a. algebra, then any two-sided ideal in A is also an example of a m.a. algebra by C7. Furthermore when A is a Banach algebra which is a m.a. algebra, then every semisimple closed subalgebra of A is also a m.a. algebra by C11.

4.1. *Algebras with dense socle.* Any semisimple Banach algebra with dense socle is a m.a. algebra by C6. In particular annihilator or dual Banach algebras are m.a. algebras since these algebras have dense socle by [**34**, Corollary (2.8.16), p. 100]. By [**26**, Corollary to Theorem 12, p. 697] any H^*-algebra is dual. When G is a compact topological group, then $L^p(G)$ is dual for $1 \leqq p < \infty$ by [**26**, Theorem 15, p. 699]. Also $C(G)$, the algebra of all continuous functions on G with convolution multiplication and uniform norm, is dual by the same theorem. More general examples involving algebras of vector valued functions on topological groups can be found in [**47**]. When G is any locally compact topological group, then $AP(G)$, the algebra of almost periodic functions on G with convolution multiplication and uniform norm, is dual; see [**56**]. The algebra of compact operators on a Hilbert space H has dense socle $\mathcal{F}(H)$ by the spectral theorem for selfadjoint compact operators. More generally if X is any Banach space then $\overline{\mathcal{F}(X)}$ has dense socle by definition. We note here that there are normed algebras with dense socle that are not m.a. algebras. The algebra B constructed in [**57**, Example 2.6, p. 31] is an example.

4.2. *Algebras of linear operators.* Let X be a locally convex topological linear space. We use C8 to prove that $\mathcal{K}(X)$ is a m.a. algebra. Assume that $T \in \mathcal{K}(X)$ is not left quasi-regular, i.e., $I - T$ has no left inverse in $\mathcal{C}(X)$. Then $\mathcal{N}(I - T)$ is a nonzero finite dimensional subspace of X by [**36**, Proposition 1, p. 144] and [**36**, Corollary 2, p. 147]. Choose E any projection of X onto $\mathcal{N}(I - T)$ such that $E \in \mathcal{C}(X)$. Then $E \in \mathcal{F}(X)$ which is the socle of $\mathcal{K}(X)$, and $R[\mathcal{K}(X)(I - T)] = E\mathcal{K}(X)$. This verifies (1) of C8. For any $T \in \mathcal{K}(X)$, $I - T$ has finite ascent by [**36**, Proposition 2, p. 146]. Thus (2) of C8 holds, and $\mathcal{K}(X)$ is a m.a. algebra.

Assume that X is a Banach space. Let I be the intersection of all primitive ideals P of $\mathcal{C}(X)$ such that $\mathcal{F}(X) \subset P$. I is also the intersection of all primitive ideals of $\mathcal{C}(X)$ such that $\mathcal{K}(X) \subset P$. This ideal I is called the ideal of inessential operators by D. Kleinecke

[**28**]. I is a m.a. algebra by C4. Kleinecke proves that if J is any two-sided ideal of $\mathcal{C}(X)$ such that whenever $T \in J$, $\mathrm{Sp}'_{\mathcal{C}(X)}(T)$ consists of isolated eigenvalues of finite multiplicity, then $J \subset I$ [**28**, Theorem 1, p. 864]. By C4 any such ideal is also a m.a. algebra.

Now let X be the Banach space of all continuous complex valued functions on a compact Hausdorff space. Let Y be $L^1(G)$ where G is n-dimensional Euclidean space or the circle group. Let $\mathcal{W}(X)$ ($\mathcal{W}(Y)$) be the ideal of all weakly compact operators on $X(Y)$. $\mathcal{W}(X)^2 \subset \mathcal{K}(X)$ by [**21**, Corollary 5, p. 494]. $\mathcal{W}(Y)^2 \subset \mathcal{K}(Y)$ by [**20**, Corollary to Theorem 3.1.9, p. 370]. Then $\mathcal{W}(X)$ and $\mathcal{W}(Y)$ are m.a. algebras by C5.

We note here that when X is an incomplete normed linear space, the ideal of all precompact operators on X need not be a m.a. algebra. For in the example on p. 590 of [**35**], J. Ringrose constructs a precompact operator with spectrum the whole plane. However any element u in a normed m.a. algebra A has the property that $\mathrm{Sp}_A'(u)$ is finite or an infinite sequence converging to zero by [**11**, Theorem 3.4, p. 502].

4.3. *Completely continuous algebras.* Let A be a normed algebra. A is a left (right) completely continuous algebra if $L_a(R_a)$ is compact for all $a \in A$. A is a completely continuous algebra if A is both left and right completely continuous. We abbreviate these designations to LCC, RCC, and CC. CC algebras were defined and discussed by I. Kaplansky in [**26**] and [**27**]. Using C8 we prove that a normed LCC algebra is a m.a. algebra. The proof presented here is a simplification of the proof given in [**8**, Theorem 7.2, p. 576]. Assume that A is a semisimple normed LCC algebra. A is isomorphic to $B = \{L_u \mid u \in A\}$ which is a subalgebra of $\mathcal{C}(A)$. For any $u \in A$, L_u is quasi-regular in B if and only if L_u is quasi-regular in $\mathcal{C}(A)$ by Lemma 2.1. Thus if L_u is not left quasi-regular in B, then L_u is not quasi-regular in $\mathcal{C}(A)$. Then $\mathcal{N}(I - L_u)$ is a nonzero finite dimensional right ideal of A by [**36**, Proposition 1, p. 144] and [**36**, Corollary 2, p. 147]. By [**8**, Lemma 6.1, p. 572] there exists an idempotent $e \in S_A$ such that $\mathcal{N}(I - L_u) = eA$. Let $E = L_e$. Then $E \in S_B$ and $R[B(I - L_u)] = EB$. Also $I - L_u$ has finite ascent for any $u \in A$ by [**36**, Proposition 2, p. 146]. Therefore B, and hence A, is a m.a. algebra by C8.

Any commutative semisimple Banach algebra with dense socle is a CC algebra. Also $L^p(G)$ and $C(G)$ are CC algebras where G is a compact topological group, $1 \leqq p < \infty$. We note here that CC algebras need not have dense socle (Theorem 1, p. 368, of [**32**] states that any LCC algebra is a left annihilator algebra, but the proof is incorrect). For let A be the commutative Banach algebra with dense socle constructed by B. E. Johnson in [25]. The ideal $\tilde{J}$ constructed by Johnson (p. 408) is closed and therefore is a CC algebra. But $\tilde{J}$ does not have dense socle (the closure of the socle of $\tilde{J}$ is the ideal J (p. 408), and as Johnson shows $J \neq \tilde{J}$).

4.4. *Compact Banach algebras.* A Banach algebra A is a compact algebra if the operator $L_a R_a$ is compact for all $a \in A$. Compact Banach algebras were introduced and discussed by J. C. Alexander in [1] and K. Vala in [50]. It is immediate from C10 that every compact Banach algebra is a m.a. algebra. The converse is not true. When X is a Banach space then $\mathcal{K}(X)$ is a compact algebra by [1, Theorem 5.4, p. 9]. This theorem also implies that $\mathcal{K}(X)$ is the largest ideal of $\mathcal{C}(X)$ which is a compact Banach algebra. Therefore $\mathcal{W}(X)$ for certain X is an example of a m.a. algebra which is not a compact algebra (see paragraph 3 of 4.2).

Every Banach algebra with dense socle is a compact algebra by [1, Theorem 7.3, p. 15]. Also every LCC algebra is a compact algebra. Since there are LCC Banach algebras which do not have dense socle, then a compact Banach algebra need not have dense socle; see the example in 4.2. We give a less subtle example of a compact Banach algebra which does not have dense socle. Let $L^\infty(T)$ be the algebra of all essentially bounded functions on the circle group T with convolution multiplication and uniform norm. The socle of this algebra is the set of trigonometric polynomials. The closure of the socle is $C(T)$, the continuous functions on T, by Fejer's Theorem. $L^\infty(T) * L^\infty(T) \subset C(T)$. In fact, if $f = g * h$ where g, $h \in L^\infty(T)$ then the sequence of Fourier coefficients of f is absolutely summable. For any $f \in L^\infty(T)$ the operator $L_f R_f(g) = f * f * g$ and $f * f \in C(T)$ which has dense socle. Therefore $L^\infty(T)$ is a compact Banach algebra which does not have dense socle.

References

1. J. C. Alexander, *Compact Banach algebras*, Proc. London Math. Soc. (3) **18** (1968), 1-18. MR **37** #4618.

2. F. E. Alexander, *On complemented and annihilator algebras,* Glasgow Math. J. **10** (1969), 38-45. MR **39** #6068.

3. F. E. Alexander and B. J. Tomiuk, *Complemented B*-algebras,* Trans. Amer. Math. Soc. **137** (1969), 459-480. MR **38** #5009.

4. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364-386. MR **7**, 126.

5. G. F. Bachelis, *Homomorphisms of annihilator Banach algebras*, Pacific J. Math. **25** (1968), 229-247. MR **39** #6076.

6. ———, *Homomorphisms of annihilator Banach algebras.* II, Pacific J. Math. **30** (1969), 283-291. MR **40** #7798.

7. B. A. Barnes, *Modular annihilator algebras*, Thesis, Cornell University, Ithaca, N.Y., 1964.

8. ———, *Modular annihilator algebras*, Canad. J. Math. **18** (1966), 566-578. MR 33 #2681.
9. ———, *Algebras with the spectral expansion property*, Illinois J. Math. **11** (1967), 284-290. MR 35 #747.
10. ———, *Some theorems concerning the continuity of algebra homomorphisms*, Proc. Amer. Math. Soc. **18** (1967), 1035-1038. MR 36 #3125.
11. ———, *A generalized Fredholm theory for certain maps in the regular representations of an algebra*, Canad. J. Math. **20** (1968), 495-504. MR 38 #534.
12. ———, *On the existence of minimal ideals in a Banach algebra*, Trans. Amer. Math. Soc. **133** (1968), 511-517. MR 37 #2008.
13. ———, *The Fredholm elements of a ring*, Canad. J. Math. **21** (1969), 84-95. MR 38 #5823.
14. ———, *Subalgebras of modular annihilator algebras*, Proc. Cambridge Philos. Soc. **66** (1969), 5-12.
15. ———, *Algebras with minimal left ideals which are Hilbert spaces*, Pacific J. Math. **35** (1970), 537-548.
16. F. F. Bonsall, *Compact linear operators*, Yale University Notes, New Haven, Conn., 1967.
17. ———, *Compact linear operators from an algebraic standpoint*, Glasgow Math. J. **8** (1967), 41-49. MR 34 #4902.
18. F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc. (3) **4** (1954), 154-167. MR 15, 881.
19. J. Duncan, *$B^\#$ modular annihilator algebras*, Proc. Edinburgh Math. Soc. **15** (1966/67), 89-102. MR 36 #6945.
20. N. Dunford and B. J. Pettis, *Linear operators on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323-392. MR 1, 338.
21. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.
22. M. Freundlich, *Completely continuous elements of a normed ring*, Duke Math. J. **16** (1949), 273-283. MR 10, 612.
23. L. C. Grove, *A generalized group algebra for compact groups*, Studia Math. **26** (1965), 73-90. MR 33 #586.
24. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R.I., 1956. MR 18, 373.
25. B. E. Johnson, *A commutative semisimple annihilator Banach algebra which is not dual*, Bull. Amer. Math. Soc. **73** (1967), 407-409. MR 34 #6556.
26. I. Kaplansky, *Dual rings*, Ann. of Math. (2) **49** (1948), 689-701. MR 10, 7.
27. ———, *Normed algebras*, Duke Math. J. **16** (1949), 399-418. MR 11, 115.
28. D. C. Kleinecke, *Almost-finite, compact, and inessential operators*, Proc. Amer. Math. Soc. **14** (1963), 863-868. MR 27 #5136.
29. M. A. Naĭmark, *Normed rings*, GITTL, Moscow, 1956; English transl., Noordhoff, Groningen, 1959. MR 19, 870; MR 22 #1824.
30. T. Ogasawara and K. Yoshinaga, *A characterization of dual B^*-algebras*, J. Sci. Hiroshima Univ. Ser. A. **18** (1954), 179-182. MR 16, 1126.
31. A. Olubummo, *Left completely continuous $B^\#$-algebras*, J. London Math. Soc. **32** (1957), 270-276. MR 19, 665.
32. ———, *$B^\#$-algebras with a certain set of left completely continuous elements*, J. London Math. Soc. **34** (1959), 367-369. MR 21 #5913.
33. ———, *Weakly compact $B^\#$-algebras*, Proc. Amer. Math. Soc. **14** (1963), 905-908. MR 27 #5137.
34. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, N.J., 1960. MR 22 #5903.

35. J. R. Ringrose, *Precompact linear operators in locally convex spaces*, Proc. Cambridge Philos. Soc. **53** (1957), 581-591. MR **19**, 869.

36. A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge Tracts in Math. and Math. Phys., no. 53, Cambridge Univ. Press, New York, 1964. MR **28** #5318.

37. A. F. Ruston, *Operators with a Fredholm theory*, J. London Math. Soc. **29** (1954), 318-326. MR **15**, 965.

38. P. P. Saworotnow, *On a generalization of the notion of H*-algebra*, Proc. Amer. Math. Soc. **8** (1957), 49-55. MR **19**, 47.

39. ———, *On the embedding of a right complemented algebra into Ambrose's H*-algebra*, Proc. Amer. Math. Soc. **8** (1957), 56-62. MR **19**, 47.

40. ———, *On a realization of a complemented algebra*, Proc. Amer. Math. Soc. **15** (1964), 964-966. MR **29** #6331.

41. ———, *On continuity of multiplication in a complemented algebra*, Pacific J. Math. **14** (1964), 1399-1403. MR **30** #2361.

42. ———, *Discrete complemented algebra*, Proc. Amer. Math. Soc. **17** (1966), 499-501. MR **33** #3124.

43. ———, *On two-sided H-algebras*, Pacific J. Math. **16** (1966), 365-370. MR **32** #4562.

44. J. F. Smith and P. P. Saworotnow, *On some classes of scalar-product algebras*, Pacific J. Math. **11** (1961), 739-750. MR **25** #5405.

45. M. F. Smiley, *Right H*-algebras*, Proc. Amer. Math. Soc. **4** (1953), 1-4. MR **14**, 660.

46. ———, *Right annihilator algebras*, Proc. Amer. Math. Soc. **6** (1955), 698-701. MR **17**, 386.

47. D. Z. Spicer, *Group algebras of vector-valued functions*, Pacific J. Math. **24** (1968), 379-399. MR **37** #2001.

48. ———, *Semisimplicity of group algebras of vector-valued functions*, Proc. Amer. Math. Soc. **19** (1968), 573-577. MR **37** #2002.

49. B. J. Tomiuk, *Structure theory of complemented Banach algebras*, Canad. J. Math. **14** (1962), 651-659. MR **26** #626.

50. K. Vala, *Sur les éléments compacts d'une algèbre normée*, Ann. Acad. Sci. Fenn. Ser. A I No. 407 (1967). MR **36** #5692.

51. K. G. Wolfson, *Annihilator rings*, J. London Math. Soc. **31** (1956), 94-101. MR **17**, 647.

52. G. V. Wood, *A generalization of the Peter-Weyl theorem*, Proc. Cambridge Philos. Soc. **63** (1967), 937-945. MR **36** #704.

53. K. Ylinen, *Compact and finite-dimensional elements of normed algebras*, Ann. Acad. Sci. Fenn. Ser. A I No. 428 (1968). MR **38** #6365.

54. B. Yood, *Homomorphisms on normed algebras*, Pacific J. Math. **8** (1958), 373-381. MR **21** #2924.

55. ———, *Faithful *-representations of normed algebras*, Pacific J. Math. **10** (1960), 345-363. MR **22** #1826.

56. ———, *Noncommutative Banach algebras and almost periodic functions*, Illinois J. Math. **7** (1963), 305-321. MR **27** #5140.

57. ———, *Ideals in topological rings*, Canad. J. Math. **16** (1964), 28-45. MR **28** #1505.

58. ———, *On algebras which are pre-Hilbert spaces*, Duke Math. J. **36** (1969), 261-272.

UNIVERSITY OF OREGON, EUGENE, OREGON 97403