ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 2, Spring 1971

EMBEDDINGS OF SURFACES IN E^3

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CONTENTS

Received by the editors March 9, 1970.
AMS 1970 *subject classifications.* Primary 55A30, 55A35, 57A05, 57A10, 57A35, 57A40, 57A45, 57A50.
[1]Support from the National Science Foundation under GP-12025.
[2]National Science Foundation Postdoctoral Fellow.

1. **Introduction.** "Antoine's necklace," "Alexander's horned sphere," "the Fox-Artin feeler," "Bing's hooked rug," "crumpled cubes"— some of the most colorful terminology in mathematics has appeared in connection with pathological, or wild, embeddings of the 2-dimensional sphere in Euclidean 3-dimensional space E^3.

The existence of unusual topological embeddings of the 2-sphere in E^3 was implied about fifty years ago with Antoine's example of a wild Cantor set in E^3 [**8**], [**9**]. Alexander [**5**] observed how Antoine's example could be used to construct such a sphere and also constructed a different example [**4**]. (The latter example is now known as Alexander's horned sphere.) This occurred shortly after he had announced the erroneous result that pathological, or wild, embeddings of the 2-sphere in E^3 do not exist. (See [**4**, p. 10].)

We suspect that these results of Antoine and Alexander in the 1920's were at first extremely surprising and perhaps difficult to believe. This was not long after Schoenflies [**154**] had shown that no such phenomenon occurred for embeddings of the 1-sphere, or simple closed curve, in the plane E^2. In particular, Schoenflies showed that every simple closed curve in E^2 is embedded topologically in E^2 like a circle; that is, for any simple closed curve K in E^2 there is a homeomorphism of E^2 onto itself that carries K onto a circle. Thus, in a topological sense, there is only one class of embeddings of a circle in E^2. The examples described by Antoine and Alexander made it apparent that this was not the case for 2-spheres in E^3.

Hampered by a lack of suitable tools at the time of the discoveries by Antoine and Alexander, mathematicians made no real progress for nearly thirty years on the problem of determining and understanding the types of topological embeddings of the 2-sphere in E^3. Only with basic work of Moise and of Bing begun in the early 1950's has the problem become tractable.

Our main purpose in this paper is to present a summary of the developments of knowledge concerning embeddings of 2-spheres, and of subsets of them, in E^3. We hope that this will be useful as a reference for the research mathematician and as an introduction to the subject for the student. For those who have only an introductory knowledge of topology, we hope that this will offer a general description of the types of problems that arise and perhaps, in some cases, arouse a curiosity for a further study of the problems. For those who wish to pursue a deeper study of the problems, we present an outline that we hope will indicate the general structure of the development and, at the same time, suggest what we consider to be the best order in which to begin a study of the papers in the references. As is frequently the case in the initial stages of an extensive development of a topic by many different mathematicians, the chronological order in which some of the papers have appeared is not the natural order in which to study them.

We believe that students who have reached the second-year graduate level in topology should be able to pursue a thorough study of the embedding problems discussed here. It is interesting to note that many of the papers listed as references were developed as doctoral theses or as sequels to them.

Though not necessary for an understanding of many of the references, a knowledge of some plane topology beyond that often encountered in a first-year graduate course in topology is nevertheless extremely useful. We recommend [**170**, Chapters I, II, and III] as a particularly good summary of the results which we have in mind. However, some of the basic results of Bing require a more extensive knowledge of plane topology. In §4.8 we include a brief outline of some essential theorems about the plane and suggest some references for a more thorough study.

We suggest the following outline, along with a survey of this paper, as a method of beginning a study of the literature on surfaces in E^3.

(1) Cut and paste techniques: Understand thoroughly Theorem 4.2.4 and its proof in §4. A study of the techniques outlined for some of the theorems about E^2 in §4.8 offers a good introduction to similar problems in E^3. An introduction to some of these techniques for E^3 can be found in [**84**].

(2) Lister's form of Bing's side approximation theorem (Theorem 4.6.3): Assume Bing's side approximation theorem (Theorem 4.6.2) and Dehn's Lemma (Theorems 4.5.1 and 4.5.4) for the time being; become familiar with the side approximation theorem for 2-spheres by seeing how it can be put in its most conveniently applied form (Theorem 4.6.3). Study some of the special cases in Bing's proof of the side approximation theorem [**35**].

(3) Bing's 1-ULC theorem (Theorem 6.4.1): Assume Theorem 6.2.1 and obtain a basic characterization of tame 2-spheres by studying our alternative proof of Theorem 6.4.1.

(4) Spanning disks for 2-spheres (Theorem 6.5.2): Study the use of Theorem 6.4.1 and a consequence of the Tietze extension theorem (Theorem 4.2.1) to obtain a second useful characterization of tame 2-spheres.

(5) Existence of tame Sierpiński curves in 2-spheres [**33**]: Avoid some of the difficulties by using Theorem 4.6.3 and our outline of an alternative proof of Theorem 5.1.1.

(6) Hosay-Lininger theorem (Daverman's proof [**67**]): It is sometimes convenient to know that the boundary of crumpled cube C in E^3 is tame from $E^3 - C$. The Hosay-Lininger theorem (Theorem 10.2.1) shows that one can often make this convenient assumption. A study of Daverman's proof [**67**] illustrates a convenient use of Theorem 4.6.3.

(7) Pushing a 2-sphere into its complement [**36**]: The Hosay-Lininger theorem (Theorem 10.2.1) can be used to make some of the pushing arguments more transparent. A study of some of the proofs in [**55**] would be helpful at this stage.

(8) Sets which can be missed by side approximation of spheres [**86**], [**122**], [**58**]: Begin with Gillman's proof that if A is a tame arc on a 2-sphere S in E^3, then S can be side approximated missing A [**86**, p. 462]. Then study Cannon's alternative proof [**58**] of Gillman's conjecture, which was stated in [**86**] and first proved by Loveland in [**122**]. (See §7.1.) Finally, study [**122**, Theorems 1 and 6]. The remainder of [**122**] should then be easily accessible; and, if one uses Lister's form of the side approximation theorem, Loveland's arguments can be simplified.

(9) Taming sets for 2-spheres: Study Bing's proof that every Sierpiń-

ski curve is a taming set for 2-spheres [**36**, Theorem 8.2]. Then study Cannon's characterization of taming sets for 2-spheres [**54**].

(10) Geometric properties which imply tameness of 2-spheres: Start with the papers by Eaton [**79**] and Hosay [**111**] discussed in §9.

Anyone who has followed the outline given above should be able to read without serious difficulty the basic work of Bing (Theorems 6.2.1. and 4.6.2) and many other papers in the references.

Some previous summaries of developments about embeddings of 2-spheres in E^3 can be found in papers by Harrold [**97**], Bing [**38**], and Burgess [**50**]. The one by Harrold was a summary of what had been done before 1957 and thus did not include the accomplishments of the last twelve years. In [**38**], Bing discussed only a restricted part of the entire problem, and Burgess [**50**] presented a brief summary with numerous questions, most of which have by now been answered. Developments that have occurred during the last five years give us a much clearer view of the problem.

For convenience, we restrict most of our discussion to 2-spheres in E^3, and we will indicate in §11 how some of the definitions and theorems can be adapted to 2-manifolds in a 3-manifold. In some cases it is convenient to replace E^3 with a 3-sphere S^3, which is the one-point compactification of E^3. Definitions and theorems for E^3 almost always have obvious analogues in S^3. We shall use those analogues without further comment.

We include an extensive list of references for the topics we discuss, and other important references can be found among the papers cited in this list. A preliminary manuscript of this paper was distributed to numerous mathematicians who are interested in these topics. This has enabled us to learn of several other papers that should be included as references, so we have inserted the supplementary list of references for this purpose. Also, with some of the questions, we have inserted parenthetical remarks to indicate recent developments that were not included in the earlier versions. In some cases, these remarks indicate that the questions have been completely answered. We are very grateful to many of our acquaintances who offered some suggestions that were very helpful in improving the manuscript.

2. **Definitions.** Except for §2.1, in which we describe notation, §2 should be referred to only as needed for clarification of terms. The headings of the subsections indicate where one might look for a definition. However, perhaps we should warn the reader of several things.

(1) Some definitions appear at appropriate places in other sections.

(2) We distinguish between an n-manifold and an n-manifold-with-boundary.

(3) We give only a restricted definition of complexes.

(4) We have adopted a notion of tame set which allows us to speak of tame compact sets which are not topological polyhedra.

2.1. *Euclidean spaces.* *Euclidean space* of dimension n will be denoted by $E^n = \{(x_1, \cdots, x_n) \mid x_i \in E^1, i = 1, \cdots, n\}$ where E^1 is the set of real numbers, and *Euclidean half-space* by

$$E_+{}^n = \{(x_1, \cdots, x_n) \in E^n \mid x_n \geqq 0\}.$$

The *Euclidean metric* will be denoted by ρ and is defined by $\rho((x_1, \cdots, x_n), (y_1, \cdots, y_n)) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}$ for each pair of points $(x_1, \cdots, x_n), (y_1, \cdots, y_n)$ in E^n.

If A and A' are subsets of E^n and ϵ is a positive number, then we define:

(1) Diam $A = \sup \{\rho(x, y) \mid x, y \in A\}$.

(2) $\rho(A, A') = \inf \{\rho(a, a') \mid a \in A, a' \in A'\}$.

(3) $N(A, A'; \epsilon) = \{a' \in A' \mid \rho(a', A) < \epsilon\}$. (If $A' = E^n$, we write $N(A, A'; \epsilon) = N(A; \epsilon)$.)

The numbers defined by (1), (2), and (3) are called the *diameter of* A, the *distance between* A *and* A', and the ϵ-*neighborhood of* A in A', respectively.

The set A in E^n is called an ϵ-*set* if Diam $A < \epsilon$. A map (or homeomorphism) f from A into E^n is called an ϵ-*map* (ϵ-*homeomorphism*) if f moves no point as far as ϵ.

The closure of a set A will be denoted by Cl A.

2.2. *Manifolds-with-boundary.* An n-*manifold-with-boundary* M is a separable metric space, each point of which has an open neighborhood which is homeomorphic to E^n or $E_+{}^n$. The set consisting of those points of M which have neighborhoods homeomorphic to E^n is called the *interior of* M and is denoted by Int M. The set $M -$ Int M is called the *boundary of* M and is denoted by Bd M. If $M =$ Int M, then M is called an n-*manifold.* Note that every n-manifold is an n-manifold-with-boundary but not conversely.

Certain 1-, 2-, and 3-manifolds-with-boundary will be of particular interest to us. First of all, two general definitions are in order:

$$S^n = \{x \in E^{n+1} \mid \rho(x, 0) = 1\}, \qquad B^n = \{x \in E^n \mid \rho(x, 0) \leqq 1\}.$$

(Note that $S^{n-1} =$ Bd B^n.) The space S^n is termed the *standard n-sphere*; B^n is termed the *standard n-cell.* A space homeomorphic to S^1 is called a *simple closed curve*; to B^1, an *arc*; to S^2, a 2-*sphere*; to B^2, a *disk* (or 2-*cell*); to B^3, a 3-*cell* (or *ball* or *cube*); to Int B^3, an *open* 3-*cell*. If $f: S^1 \to E^n$ is a continuous function, then f (or sometimes, the image of f) is called a *loop* in E^n. A loop f in E^n is said to be *unknotted* if f has an extension $f_1 : B^2 \to E^n$ which is an embedding of B^2 in E^n. *If* $g : B^2 \to E^n$ is a continuous function, then g (or sometimes, the image of g) is called a *singular disk*. If g is an embedding, then g (or sometimes, the image of g) is often called *nonsingular* for emphasis.

As far as compact connected 2-manifolds are concerned, we shall usually be considering only the *spheres-with-handles* or compact connected *orientable* 2-manifolds, which are the only compact connected 2-manifolds which can be embedded in E^3 and S^3. The *torus* is a sphere with one handle; the *double torus*, a sphere with two handles; etc.

If D is a disk and $D_1, D_2, \cdots, D_n$ are disjoint disks in Int D, then $D - \bigcup_{i=1}^{n}$ Int D_i is called a *disk-with-holes* and each Int D_i is called a *hole* of D.

2.3. *Piecewise linear* (PL) *structures*. We are interested, for the most part, only in triangulations of 2- and 3-manifolds-with-boundary. Thus we give the following restricted definitions of complexes and triangulations. (Some more general definitions, together with additional references, can be found in [**107**, Chapter 5].)

A 2-*complex* K is a locally finite collection of triangles in some E^n such that if two of the triangles intersect, then their intersection is an edge or a vertex of each. The union of the triangles of K is called the *underlying space* of K and is denoted by $|K|$. If $T : |K| \to X$ is a homeomorphism from $|K|$ onto X, then T induces on X the linear structure of $|K|$. T is called a *triangulation* of X, and X is said to be *triangulable* or *triangulated* as a 2-complex. The images of the triangles (2-simplexes) in K are called the 2-*simplexes of* T, the images of the edges (1-simplexes) of triangles on K are called the 1-*simplexes of* T, and the images of vertices (0-simplexes) of triangles in K are called the 0-*simplexes of* T. If $i = 0, 1$, or 2, the i-*skeleton* of T is the union of all i-simplexes of T.

A 3-*complex* K is a locally finite collection of tetrahedra (3-simplexes) in E^n such that if two of the tetrahedra intersect, then their intersection is a face, an edge, or a vertex of each. The terms *underlying space*, *triangulable space*, *triangulation*, *simplexes of a triangulation*, 3-*skeleton*, 2-*skeleton*, 1-*skeleton* and 0-*skeleton* of a triangulation are defined for 3-complexes in direct analogy to the corresponding definitions for 2-complexes.

A triangulation T' of X is said to be a *subdivision* of T if each simplex of T' is contained in some simplex of T and has the linear structure induced by that simplex of T. We say that two triangulations T and T' of X are *compatible* if they have a common subdivision. If U is open in X and $T(U)$ is a triangulation of U, then we say that $T(U)$ *is compatible with* T if there is a subdivision $T(U)'$ of $T(U)$ each simplex of which is contained in a simplex of T and has the linear structure imposed by that simplex of T.

A closed set F in a triangulated space X is said to be a *polyhedron* if there is a subdivision T' of the triangulation of X such that F is a union of simplexes of T'.

A closed set F in a triangulated space X is said to be *locally polyhedral* at a point $x \in X$ if there is a neighborhood N of x such that $\mathrm{Cl}(N) \cap F$ is a polyhedron.

Suppose X and Y are triangulated spaces with triangulations T_1 and T_2 and $f: X \to Y$ is a map. Then f is said to be *piecewise linear* (PL) if there are subdivisions T_1' of T_1 and T_2' of T_2 such that each simplex of T_1' is taken by f linearly into a simplex of T_2'.

2.4. *Tame sets, wild sets, 2-spheres, crumpled cubes, and collared sets.* We shall adopt two definitions of tame sets. The first is applicable for arbitrary topological polyhedra in a 3-manifold. The second is applicable for arbitrary compact subsets of a 2-sphere in E^3. The two definitions are equivalent on the overlap (see §4.4).

Definition 1. The image X of a compact polyhedron in a triangulated 3-manifold M is said to be *tame, or tamely embedded,* (in M) if there is a homeomorphism $h: M \to M$ such that $h(X)$ is a polyhedron in M.

Definition 2. A compact subset X of a 2-sphere in E^3 (S^3) is said to be *tame* if there is a homeomorphism $h: E^3 \to E^3$ ($h: S^3 \to S^3$) such that $h(X) \subset S^2$, where S^2 is the standard 2-sphere in E^3. (Recall §2.2 and note that $S^2 = S^3 \cap E^3$ in E^4.)

The definition which is appropriate in any given situation should be apparent to the reader. If the condition for tameness which is appropriate for a given set X is not satisfied, then X is said to be *wild* or *wildly embedded.*

A compact set X is said to be *locally tame at a point* $p \in X$ if there is an open set U containing p such that $\mathrm{Cl}(U \cap X)$ is tame, where the appropriate definition is used. The set X is said to be *locally tame* if it is locally tame at each of its points. If F is a subset of X and X is locally tame at each point of $X - F$, then X is said to be *tame modulo* F.

If S is a 2-sphere in E^3, then the bounded and unbounded components of $E^3 - \mathrm{S}$ are denoted by Int S and Ext S, respectively. (See Theorem 4.1.1.) The space $C = \mathrm{S} \cup \mathrm{Int\ S}$ is called a *crumpled cube,* as is any space homeomorphic with C. The 2-sphere S is called the boundary of C (denoted Bd C), and Int S is called the interior of C (denoted Int C). If S is a 2-sphere in S^3, then the closure of each component of $S^3 - S$ is a crumpled cube; and the two crumpled cubes thus bounded by S are called *complementary crumpled cubes.*

It follows easily that a 2-sphere S in S^3 is tame if and only if both of the complementary crumpled cubes bounded by S in S^3 are 3-cells. If one of them is a 3-cell, which we denote by C, then S is said to be *tame from* Int C.

Suppose that U is an open subset of a compact 2-manifold M and that M is embedded in a 2-sided manner in a 3-manifold M^3. Let V be a component of $M^3 - M$. We say that U is *collared* in V if there is an embedding $h : U \times [0, 1) \to U \cup V$ which is open in $U \cup V$ such that $h(u, 0) = u$ for $u \in U$. We say that U is bicollared in M^3 if there is an embedding $h : U \times (-1, 1) \to M^3$ such that $h(u, 0) = u$ for $u \in U$.

2.5. *Decomposition spaces and cellular sets.* Let G be a collection of compact subsets of E^n such that each point of E^n is contained in one and only one element of G. The collection G is said to be a *decomposition* of E^n. If in addition, for each open subset O of E^n, $\bigcup\{g \in G \mid g \subset O\}$ is an open set, then G is said to be an *upper semicontinuous decomposition* of E^n. There is a natural map P, called the *projection map,* which assigns to each element x of E^n the unique element g_x of G which contains x. The collection G can be topologized by calling a subset U of G open if and only of $P^{-1}(U)$ is open. The space thus obtained is denoted by E^n/G and is called the *decomposition space associated with* G. (See [**168**, p. 123] for a discussion of upper semicontinuous decompositions.)

A subset g of E^n is said to be *cellular* in E^n if, for each neighborhood U of g in E^n, there is an n-cell (i.e., a space homeomorphic with the standard n-cell) in U which contains g in its interior. A decomposition G of E^n is said to be *cellular* if each of its elements is a cellular set.

3. **Examples of wild spheres.** All of the known examples of wild 2-spheres in E^3 are based on four basic types of spheres. We believe that the two examples described below in §§3.1 and 3.2 should be distinguished in this classification since the former depends upon an entanglement of the horns and the latter upon the existence of a wild Cantor set. (See remark in last paragraph of [4, p. 9].)

3.1. *The Alexander horned sphere.* A sketch of this 2-sphere, which is pictured in Figure 1, was presented by Alexander [4] in 1924. We consider starting with a cylindrical 2-sphere that begins to grow horns in successive stages so that there is a Cantor set C of wild points on the sphere that is constructed. We can see that it is wild by observing that the simple closed curve K (Figure 1) is the boundary of a disk in E^3 but not in the exterior of the sphere. A precise argument for this can be found in [**28**]. Also, in [**178**], [**185**], the fundamental group of the complement of this sphere is shown to be nontrivial. Bing [**38**] has indicated a way to describe this sphere so that the Cantor set of wild points is a subset of a straight line and the sphere can be pierced at each of its points with an interval of a straight line. Fort [**82**] had previously described another wild 2-sphere with this property. Artistic conceptions of the Alexander horned sphere can be found in [**107**, p. 176] and [**37**, p. 117].

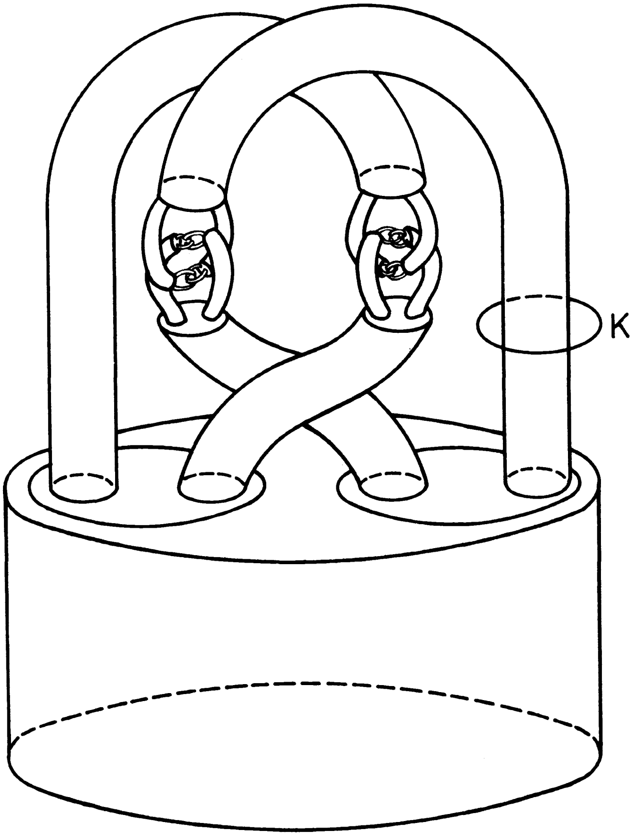

FIGURE 1

3.2. *The Antoine sphere.* In 1920–1921, Antoine [**8**], [**9**, p. 311] described a wild Cantor set in E^3. While he did not include, in these papers, an explicit description of a wild 2-sphere in E^3, Alexander [5] observed that the existence of such a sphere was implicitly included in Antoine's work. More generally, the existence of wild 2-spheres in E^3 is implied by the following theorem and the existence of wild Cantor sets in E^3.

THEOREM 3.2.1. *Each Cantor set in E^3 is a subset of some 2-sphere in E^3.*

Bing [38] has indicated how to prove this theorem with an argument similar to one that can be used to show that, for each Cantor set C in E^2, there is a dendrite having the points of C as its end points. In [30], he characterizes those Cantor sets in E^3 which are subsets of tame spheres.

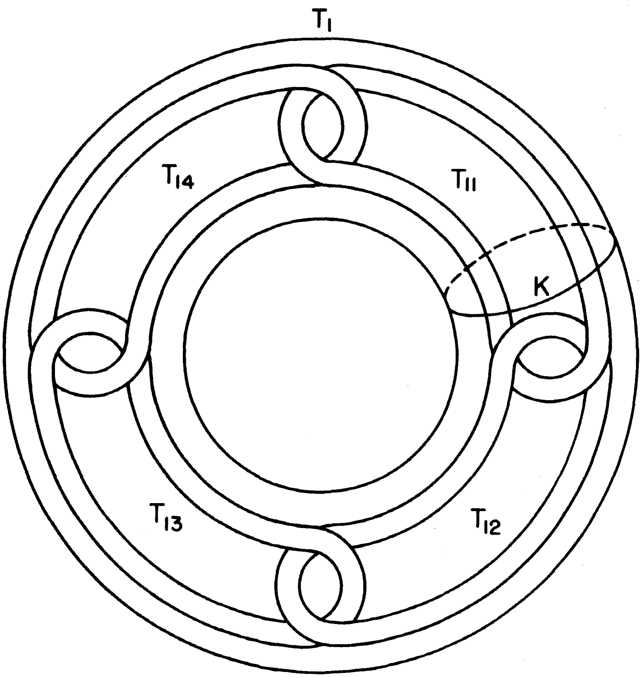

FIGURE 2

Antoine [8], [9] described a wild Cantor set C that is now known as "Antoine's necklace." Let T_1 be a solid torus and let T_{11}, T_{12}, T_{13}, and T_{14} be a chain of four linked solid tori in T_1 as indicated in Figure 2. This process is iterated so that in each T_{ij} there are four solid tori like those in T_1. This process can be continued to obtain a Cantor set

C, where a point belongs to C if and only if it belongs to a solid torus of each stage of the construction. We can see that C is wild by observing that the simple closed curve K indicated in Figure 2 is the boundary of a disk in E^3 but not in $E^3 - C$. A precise argument for this is included in [178], [185]. Sher [157] has discussed variations in this method of constructing a wild Cantor set to show that there are uncountably many different embeddings of a Cantor set in E^3.

Theorem 3.2.1 implies that C, described above, is a subset of a wild 2-sphere in E^3. In [5], such a sphere is constructed by a method that corresponds with the above construction of C, and such a procedure is pictured in [38, Figure 8]. We observe that the wildness of such a 2-sphere depends merely upon the wildness of C and not upon the way the horns are entangled. If C were a tame Cantor set, the process described in [38] might result in a tame 2-sphere.

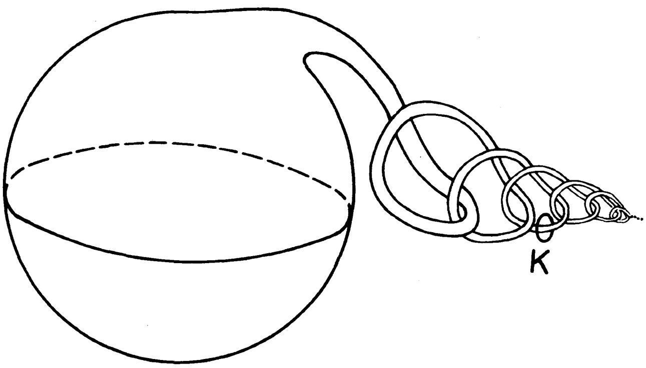

FIGURE 3

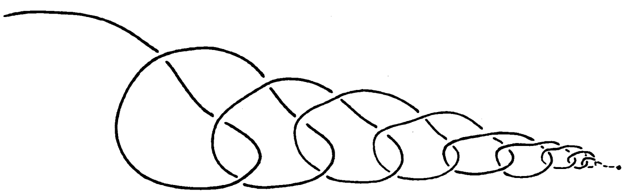

FIGURE 4

3.3. *The Fox-Artin sphere.* While each of the 2-spheres described in §§3.1 and 3.2 has a Cantor set of wild points, Fox and Artin [83] described one in 1948 with only one wild point. This sphere can be described by starting with a thin cone, or feeler, and then entangling it near the vertex as indicated by Figure 3. Another way to picture it is by starting with the Fox-Artin arc indicated in Figure 4 and then making the arc into the entangled cone in Figure 3. A duality between such 2-spheres and arcs is described in [**160**]. We indicate in §6.4 that this 2-sphere S is wild by observing that the simple closed curve K in Figure 3 is not the boundary of a small disk in $E^3 - S$.

It is interesting to observe here that Cantrell [**61**] has shown that for $n \neq 2$ there does not exist an n-sphere in E^{n+1} that fails to be locally flat at only one point.

3.4. *The Bing sphere.* Each of the three spheres described above contains a wild arc. In 1961, Bing [**28**] described a wild 2-sphere in which every arc is tame. The process, involving the use of "eye-bolts," in describing this sphere has sometimes been called the "hooked-rug" procedure. Two steps in the iterative description are shown in Figures 5a and 5b, and further remarks by Bing about the descriptions can be found in [**28**]. His proof that every arc on such a sphere is tame depends upon a characterization of tame arcs in [**100**, Theorem 7]. As with the Alexander horned sphere, it is easy to see that one of the complementary domains of Bing's sphere fails to be simply connected.

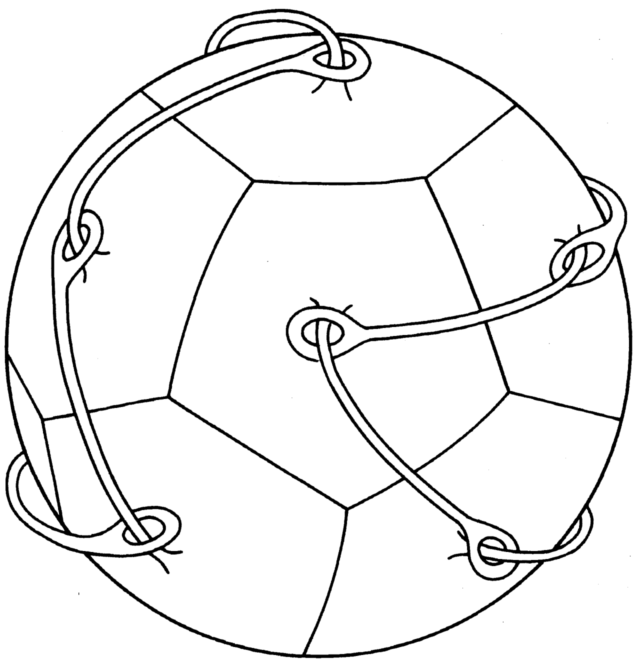

FIGURE 5a

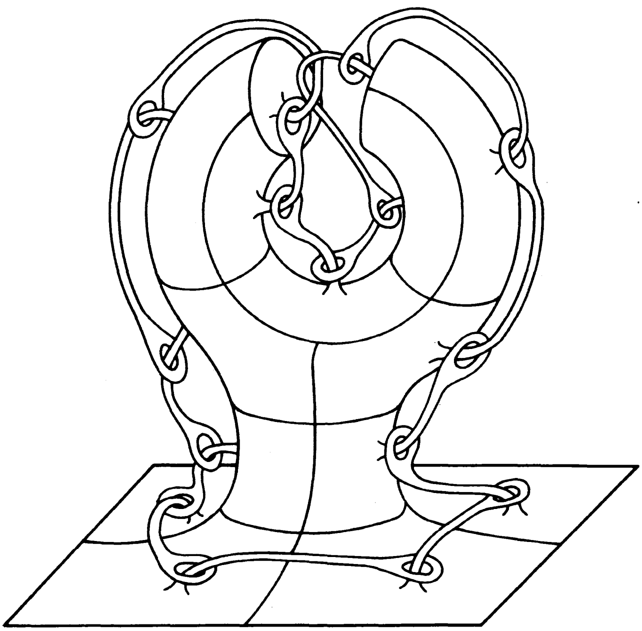

FIGURE 5b

3.5. *Other wild spheres.* Methods similar to those indicated above have been used to describe other wild 2-spheres in E^3. Some different horned spheres have been described by Ball [**15**] and L. O. Cannon [**60**]. Fort [**82**] indicated how the one described by Ball could be changed to furnish the first example of a wild sphere in E^3 that can be pierced at each of its points with a straight interval. These spheres are discussed further in §10 where some additional examples of wild 2-spheres are indicated by sewings of crumpled cubes.

Gillman [**89**] described how to change Bing's sphere so that each of its complementary domains is an open 3-cell and so that it retains the property of being wild at each point, with each of its arcs being tame. Alford [**6**] used a similar procedure to describe a 2-sphere with a simply connected complement such that its set of wild points is an arc.

The procedure described by Fox and Artin [**83**] can be used to describe, for any positive integer n, a 2-sphere with n wild points. Alford and Ball [**7**] have shown how to change the "piercing indices" of the spheres at their wild points so that the embeddings of the spheres are locally topologically different at their wild points. Martin [**135**] used this difference to describe a "rigid" 2-sphere S with the property that any homeomorphism of E^3 onto itself that carries S onto itself is the identity on S. (See the questions in §6.8.) Giffen [**85**], McPherson [**141**], and Sikkema, Kinoshita and Lomonaco [**161**] have announced the existence of uncountably many different embeddings of a 2-sphere in E^3 with only one wild point.

3.6. *Disjoint spheres and disks in* E^3. There exist two disjoint 2-spheres S_1 and S_2 in E^3 such that no 2-sphere separates them in E^3. (Ulam included a question about the existence of two such spheres in [**196**, p. 50].) Following Alexander's procedure for the horned sphere (Figure 1), we construct such an example as indicated in Figure 6. If there were a 2-sphere S which separates S_1 from S_2, then there would be a simple closed curve K in $S \cap \text{Int } S_1 \cap \text{Ext } S_2$, as indicated in Figure 6, which cannot be shrunk to a point in $E^3 - (S_1 \cup S_2)$. However, this is impossible as K would be the boundary of a disk on S that does not intersect $S_1 \cup S_2$. Another such example could be constructed where S_1 and S_2 are spheres of the type described in §3.2.

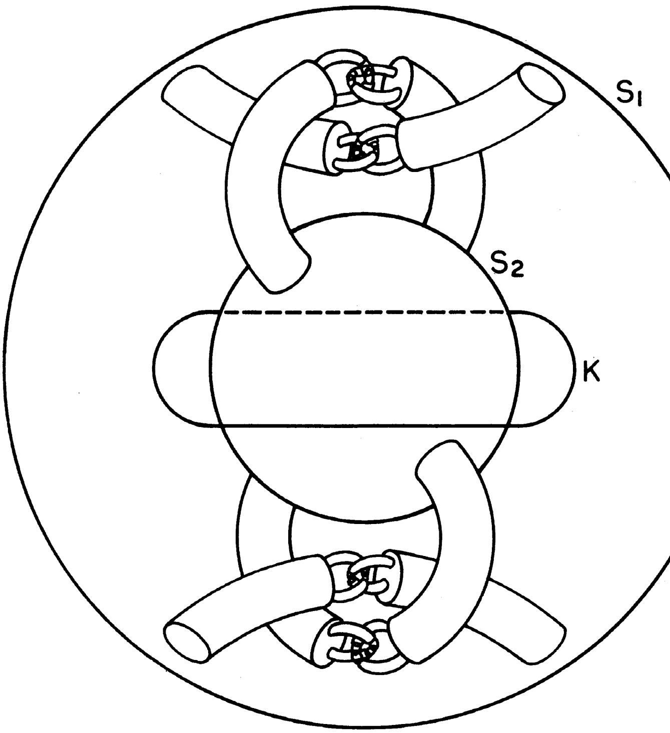

FIGURE 6

Bing [25] has proved, using one of his characterizations of tame 2-spheres, that there do not exist uncountably many disjoint wild 2-spheres in E^3. As his proof has not appeared in print, we outline it here.

THEOREM 3.6.1 (BING [25]). *There do not exist uncountably many disjoint wild 2-spheres in E^3.*

OUTLINE OF PROOF. Suppose that there exists an uncountable collection G of disjoint wild 2-spheres in E^3. By Theorem 4.3.2, some sequence of distinct elements of G converges homeomorphically to an element of G. Furthermore, by using the separation of E^3 by the spheres of G, we can show that there is a sphere S of G such that some sequence $\{S_i\}$ of distinct spheres of G converges homeomorphically to S from Int S and another such sequence $\{S_i'\}$ converges homeomorphically to S from Ext S. (The procedure here is similar to proving that in any uncountable subset H of a line L there is a point of H which is a limit point of H from both directions on L.) It follows from one of Bing's characterizations of tame 2-spheres (Theorem 6.2.1) that S is tame. This contradiction implies that the uncountable collection G cannot exist.

It is interesting, however, that there do exist uncountably many disjoint wild disks in E^3. Such an example has been described by Stallings [162]. He constructed disks with wild boundaries by using variations of the process used by Fox and Artin [83]. As shown by Martin [133], all except countably many disks in such a collection would be locally tame except on their boundaries and, furthermore, would lie on 2-spheres. Sher [156] showed how to construct an uncountable collection of disjoint wild disks in E^3 so that no two of them are equivalently embedded in E^3.

4. **Basic theorems.** Without attempting to include a complete outline of basic theorems that are needed for a study of surfaces in E^3, we discuss some that are essential to such a study. Other important theorems will become apparent in a study of some of the references.

4.1. *Separation and accessibility.* Nearly sixty years ago, Brouwer [44], [173, p. 63] extended the Jordan curve theorem for E^2 to n-spheres in E^{n+1}, and some further extensions were then presented by Alexander [2]. Brouwer [181] also proved that each point of an n-sphere S^n in E^{n+1} is accessible from each component of $E^{n+1} - S^n$. Alexandroff [176] presented a further discussion of accessibility of points of topological polyhedra. We restrict out statement of these theorems to the case where $n = 2$, and we give references to the proofs by Wilder in [173] where the original sources are cited.

THEOREM 4.1.1 ([**173**, p. 63]). *If S is a 2-sphere in E^3, then $E^3 - S$ is the union of the two disjoint connected open sets such that S is the boundary of each of them.*

A set U is defined to be *uniformly locally arcwise connected* (0-ULC) if for each $\epsilon > 0$ there exists a $\delta > 0$ such that any two points of U within δ of each other lie in an ϵ-arc in U.

THEOREM 4.1.2 ([**171**], [**173**, p. 66]). *If S is a 2-sphere in E^3, then each component of $E^3 - S$ is 0-ULC.*

A point p of a set H is *accessible* from a set K if there exists an arc A with p as an end point wich that $A \cap H = p$ and $A - p \subset K$. Theorem 4.1.2 can be used to prove the following theorem about the accessibility of points of a 2-sphere in E^3.

THEOREM 4.1.3 ([**173**, p. 66]). *If S is a 2-sphere in E^3 and U is a component of $E^3 - S$, then each point of S is accessible from U.*

If a closed set K separates two points in E^3 then this separation is preserved under any homotopy of K in the complement of the two points. This property, which is developed in [**112**, p. 97], implies the following:

THEOREM 4.1.4. *If K is a continuum in E^3 which separates two points a and b in E^3, then there exists a $\delta > 0$ such that any image of K in E^3 under a δ-map separates a from b in E^3.*

4.2. *Tietze extension theorem.* For our purposes here, we need only a special case of the more general Tietze extension theorem [**112**, p. 80]. Bing [**39**, Theorem 3] has presented an elementary proof, using calculus, for the following version that we need here.

THEOREM 4.2.1. *Each map of a closed subset X of a metric space Y into a disk D can be extended to map Y into D.*

The following consequence of Theorem 4.2.1 has frequently been used, sometimes without an explicit reference or statement, in various papers about surfaces in E^3.

THEOREM 4.2.2. *If D is a disk in E^3 and f is a map of D into E^3 such that $f(\mathrm{Bd}\ D) \cap D = \emptyset$, then there is a map g of D into E^3 such that*:

(1) *g agrees with f on H, where H is the component of $f^{-1}(f(D) - D)$ that contains* Bd D.

(2) $g(D - H) \subset D$.

PROOF. Let Y denote $D - H$, let X denote $\mathrm{Cl}(H) - H$, and let g

denote $f|X$. The requirements of Theorem 4.2.1 are satisfied by X, Y, D, and g. Thus g can be extended to a map of $D - H$ into D. We extend g to D by letting g agree with f on H, and we observe that g satisfies the requirements of Theorem 4.2.2.

Sometimes the following corollary to Theorem 4.2.2 is useful:

THEOREM 4.2.3. *If D_1 and D_2 are disks in E^3 such that $D_2 \cap \mathrm{Bd}\, D_1 = \emptyset$, then there is a map f of D_1 into E^3 such that*:

(1) *f is the identity map on the component H of $D_1 - D_2$ that contains* $\mathrm{Bd}\, D_1$.

(2) $f(D_1 - H) \subset D_2$.

The following theorem is similar to Theorem 4.2.3, but it can be proved by geometric methods without using the Tietze extension theorem. We include a brief outline of a proof indicated by Bing [29, p. 297].

THEOREM 4.2.4. *If D_1 and D_2 are disks in E^3 such that $D_1 \cap D_2 = \mathrm{Int}\, D_1 \cap \mathrm{Int}\, D_2$ and $\mathrm{Int}\, D_1$ and $\mathrm{Int}\, D_2$ are locally polyhedral, then for each $\epsilon > 0$ there is a disk D_1' such that $\mathrm{Bd}\, D_1' = \mathrm{Bd}\, D_1$, $D_1' \cap D_2 = \emptyset$, $\mathrm{Int}\, D_1'$ is locally polyhedral, and each point of D_1' is in the union of D_1 and an ϵ-neighborhood of D_2.*

BRIEF OUTLINE OF PROOF. By geometric methods, $\mathrm{Int}\, D_1$ can be adjusted slightly so that $D_1 \cap D_2$ is the union of a finite number of disjoint simple closed curves [**96**, Lemma 2]. Let H be the component of $D_1 - D_2$ that contains $\mathrm{Bd}\, D_1$, let $K_1, K_2, \cdots, K_n$ be the components of $D_2 \cap \mathrm{Cl}\, H$, and for each m let H_m denote the subdisk of D_2 that has K_m as its boundary. Now for each i, let K_i' be a polyhedral simple closed curve in H that is homeomorphically very close to K_i. Let H' denote the subset of H that is a disk-with-holes having $\mathrm{Bd}\, D_1 \cup (\bigcup_{i=1}^n K_i')$ as its boundary, and let H_i' be a polyhedral disk near H_i that has K_i' as its boundary and does not intersect D_2. This can be done so that $H' \cup (\bigcup_{i=1}^n H_i')$ is a disk D_1' satsifying the requirements in the conclusion of Theorem 4.2.4.

4.3. *Spaces of functions.* We frequently need some consequences of the following theorem about spaces of continuous functions of compact metric spaces into separable metric spaces. A description of such spaces can be found in [43].

A sequence $K_1, K_2, \cdots$ of sets in a metric space is defined to *converge homeomorphically* to a set K if there exists a sequence $h_1, h_2, \cdots$ such that:

(1) For each i, h_i is a homeomorphism of K_i onto K.

(2) For each $\epsilon > 0$, there exists an integer n such that, for $i > n$, h_i moves no point more than a distance ϵ.

THEOREM 4.3.1 (BORSUK [43, THEOREM 2]). *The space of all maps of a compact metric space into a separable metric space is itself a separable metric space.*

This theorem has the following useful corollary.

THEOREM 4.3.2. *If G is an uncountable collection of homeomorphic compact sets in a separable metric space X, then some sequence of distinct elements of G converges homeomorphically to an element of G.*

PROOF. Let K be some element of G. The space F of all maps of K into X is, by Theorem 4.3.1, a separable metric space. Let H be a subset of F such that (1) each element of H is a homeomorphism of K onto some element of G and (2) each element of G is the image of K under one and only one homeomorphism of H. Since H is an uncountable subset of the separable metric space F, it follows that some sequence $h_1, h_2, \cdots$ of distinct elements of H converges to an element h of H. Now, in the metric space X, the sequence $h_1(K), h_2(K), \cdots$ converges homeomorphically to $h(K)$. To see this, observe that, for each i, $h\, h_i^{-1}$ is a homeomorphism which carries $h_i(K)$ onto $h(K)$ and moves no point more than the distance, in the function space F, from h_i to h.

4.4. *Polyhedral spheres in E^3.* In this section, we define a 2-sphere S in E^3 to be *tame* if there is a homeomorphism of E^3 onto itself that carries S onto a round 2-sphere. Theorem 4.4.1 below implies that this is equivalent to requiring that there be a homeomorphism of E^3 onto itself that carries S onto a polyhedral sphere. Thus the definition we are using is equivalent to the one that has become prevalent in recent years [21, p. 146]. In 1924, Alexander indicated, with an appeal to the logarithmic potential function, how to prove that every polyhedral 2-sphere in E^3 is embedded equivalently with a round sphere. (See remarks in §9.1.) In 1950, Moise and Graeub independently presented more elementary proofs of a more general theorem. (See remarks by Bing [27, p. 63] and Moise [144, p. 407] regarding a step in Moise's proof.)

THEOREM 4.4.1 (ALEXANDER [3]). *Every polyhedral 2-sphere in E^3 is tame.*

THEOREM 4.4.2 (GRAEUB [91] AND MOISE [142]). *If S_1 and S_2 are polyhedral 2-spheres in E^3, then there is a piecewise linear homeomorphism of E^3 onto itself that carries S_1 onto S_2.*

We can see that Theorem 4.4.2 implies Theorem 4.4.1 by observing that the surface of a regular tetrahedron in E^3 is embedded equivalently with a round sphere.

The methods used in proving the above two theorems can be used to prove the following:

THEOREM 4.4.3. *If* S_1 *and* S_2 *are polyhedral 2-spheres in* E^3 *and* h *is a piecewise linear homeomorphism of* S_1 *onto* S_2, *then* h *can be extended to a piecewise linear homeomorphism of* E^3 *onto itself.*

An easy consequence of the fact that S^3 can be realized as the suspension of S^2 (i.e., the double cone over S^2) is the following:

THEOREM 4.4.4. *If* S_1 *and* S_2 *are tame 2-spheres in* E^3 *and* h *is a homeomorphism of* S_1 *onto* S_2, *then* h *can be extended to a homeomorphism of* E^3 *onto itself.*

In working with a 2-sphere S in E^3, it is frequently desirable to assume that S is locally polyhedral except where it is wild. The following theorem, which justifies this assumption, has been proved by Bing [**21**, Theorem 6]. It can also be obtained from some of the work by Moise [**143**, Theorem 2]. We indicate an alternative proof using some more recent results by Bing and Brown.

THEOREM 4.4.5. *If the 2-sphere* S *in* E^3 *is locally tame in the open subset* U *of* S *and* $\epsilon > 0$, *then there is a homeomorphism* h *of* E^3 *onto itself such that:*

(1) *h moves no point more than a distance* ϵ.
(2) *h is the identity homeomorphism on* $S - U$.
(3) *$h(U)$ is locally polyhedral.*

OUTLINE OF PROOF. Let T be a triangulation of E^3. Since S is locally tame at each point of U, it follows from a theorem of Brown [**45**, Theorem 1] that U can be bicollared in $U \cup (E^3 - S)$. This implies that there exist a polyhedral 2-sphere S', an open set V in E^3, and a homeomorphism g of V into E^3 such that

$$g(V \cap S') = g(V) \cap S = U.$$

The open sets V and $g(V)$ have triangulations $T(V)$ and $T(g(V))$, respectively, that are compatible with T. Also, $g(V)$ has another triangulation, which we denote by $g(T(V))$, that is carried by g from the triangulation $T(V)$. Let f be a continuous real function defined on $\mathrm{Cl}\, g(V)$ such that

$$0 < f(x) < \epsilon \quad \text{for} \quad x \in g(V),$$
$$f(x) = 0 \quad \text{for} \quad x \in \mathrm{Bd}\, g(V).$$

It follows from [**27**, Theorem 8] that there is a homeomorphism h of $g(V)$ onto itself such that, for each $x \in g(V)$, $\rho(x, h(x)) < f(x)$ and h

carries the triangulation $g(T(V))$ onto one that is compatible with $T(g(V))$ and thus with T. Extend h to E^3 by letting it be the identity homeomorphism on $E^3 - g(V)$. This extended homeomorphism h satisfies the requirements of the conclusion of Theorem 4.4.5.

4.5. *Dehn's lemma and related theorems.* Papakyriakopoulos [**150**], [**151**] has proved three theorems which are of fundamental importance in the study of 3-manifolds in general and of surfaces in 3-manifolds in particular.

THEOREM 4.5.1 (DEHN'S LEMMA [**150**]). *Let M be a triangulated* 3-*manifold-with-boundary and let D be a* 2-*cell, with self-intersections (singularities), which is piecewise linearly embedded in M and which has as boundary the simple closed polygonal curve C such that there exists a closed neighborhood of C in D which is an annulus (i.e., no point of C is singular). Then there exists a* 2-*cell D_0 with boundary C, semilinearly (i.e., piecewise linearly) embedded in M.*

THEOREM 4.5.2 (SPHERE THEOREM [**150**]). *Let M be an orientable* 3-*manifold-with-boundary such that $\pi_2(M) \neq 0$, and which can be semilinearly embedded in a* 3-*manifold N, having the following property: The commutator quotient group of any nontrivial (but not necessarily proper) finitely generated subgroup of $\pi_1(N)$ has an element of infinite order (n.b. in particular this holds if $\pi_1(N) = 1$). Then there exists a* 2-*sphere S semilinearly embedded in M, such that S is not nullhomotopic in M.*

THEOREM 4.5.3 (LOOP THEOREM [**150**], [**151**]). *Let M be a* 3-*manifold-with-boundary and let the boundary N of M be formed by a number (> 0, $\leqq \infty$) of surfaces closed or not. Let L be a loop belonging to an open set U of an orientable component N' of N such that L is nullhomotopic in M and not on N. Then there exists a simple loop (i.e., a simple closed curve) $L_0 \subset U$ such that L_0 is nullhomotopic in M and not on N.*

Dehn's lemma was first proposed by Dehn in 1910 [**74**], but Kneser found a serious gap in Dehn's proof [**116**]. A satisfactory proof was not forthcoming until many decades later when Papakyriakopoulos presented his proof [**150**].

We have stated the three theorems above essentially as they were originally stated and proved by Papakyriakopoulos. They have since been generalized and the proofs have been in large extent simplified. The interested reader would probably find it desirable to start his reading with [**155**] and [**163**].

Certain improvements which are valuable in a study of surfaces and which have generally not appeared in print in explicit form are immediate consequences of the proofs given by Papakyriakopoulos and others. For example, we have the following:

Addendum to Theorem 4.5.1. *If U is a neighborhood in M of the singularities of D, then D_0 may be chosen to be in $D \cup U$.*

Indication of Proof. The proof given in [**155**] can easily be presented in a constructive form. In that form it contains steps of two types:

(1) Cutting and pasting near singularities in singular 2-cells which are preimages of D in certain covering spaces; (2) replacing preimages of D in certain covering spaces by nonsingular 2-cells which lie in spheres close to those preimages. The cutting and pasting in steps of type (1) can be chosen to alter things only in preimages of U. A linking argument can be used in steps of type (2) to show that the new nonsingular 2-cells can be pushed back, except in preimages of U, into the preimages of D without introducing singularities.

Improvements of still another type can be rendered by use of Bing's polyhedral approximation theorems so that the semilinearity (i.e., piecewise linearity) condition in Theorems 4.5.1, 4.5.2, and 4.5.3 can often be removed. For example, we have the following theorem, which has been used in several places in the literature. We outline a brief proof of it.

Theorem 4.5.4 (Bing's Extension of Dehn's Lemma). *If D is a polyhedral disk, f is a map of D into a triangulated 3-manifold M^3, U is an open set in M^3 containing f (Int D), f is nonsingular in some neighborhood V of* Bd D, *and $f(V) \cap f(D - V) = \emptyset$, then there is a homeomorphism f' of D into $f($*Bd $D) \cup U$ *such that f' is locally piecewise linear except on* Bd D.

Proof. Let D_1 and D_2 be disks such that $D_2 \subset \text{Int } D_1 \subset D_1 \subset \text{Int } D$, $\text{Bd } D_1$ and $\text{Bd } D_2$ are polyhedral, f is nonsingular in the annulus $D - \text{Int } D_2$, and $f(D - D_2) \cap f(D_2) = \emptyset$.

By Bing's polyhedral approximation theorem for open subsets of 2-manifolds (Theorem 4.6.4), there is a map g of D into $U \cup \text{Bd } D$ such that:

(1) g agrees with f on $D_2 \cup \text{Bd } D$.
(2) g is locally piecewise linear in $\text{Int } D - D_2$.
(3) g is nonsingular on $D - \text{Int } D_2$.
(4) $g(D - D_2) \cap g(D_2) = \emptyset$.

Let V be an open subset of U such that $g(D_2) \subset V$ and $(\text{Cl } V) \cap g(D - \text{Int } D_1) = \emptyset$. Now g can be approximated in $D \cup V$ with a map g' such that g' agrees with g except in $g^{-1}(V \cap g(D))$ and is locally piecewise linear except on Bd D. Since $g' \mid D_1$ is a piecewise linear map of D_1 into U and is nonsingular in a neighborhood of Bd D_1, it follows from Dehn's lemma (Theorem 4.5.1 and its addendum) that there is a piecewise linear homeomorphism g'' of D_1 into $g'(D_1) \cup V$ such that g'' agrees with g' in a neighborhood of Bd D_1. Now the homeomorphism that agrees with g'' on D_1 and with g on $D - D_1$ is the required homeomorphism f'.

There seem to be important questions relating certain possible extensions of Dehn's lemma with theorems on surfaces in E^3. We refer the reader to [**50**, p. 81], [**48**, p. 89], and [**32**], and recommend further improvements of Dehn's lemma and the Loop Theorem as possible starting points for an attack on the free sphere problem (see §6.3).

The following geometric forms of the loop theorem and the Sphere Theorem are frequently convenient to use and incorporate some of the improvements that have been made in the original theorems.

THEOREM 4.5.5 (LOOP THEOREM). *If M is a 3-manifold-with-boundary, K is a component of* Bd M, *and L is a loop in K that is nullhomotopic in M but not in K, then there is a disk D such that* Bd $D \subset K$, Int $D \subset$ Int M, *and* Bd D *is not the boundary of a disk in K.* (*See* [**163**].)

THEOREM 4.5.6 (SPHERE THEOREM). *If M is an orientable triangulated 3-manifold and f is a map of a 2-sphere S into an open subset U of M such that f cannot be shrunk to a point in U, then there is a polyhedral 2-sphere S' in U that is not nullhomotopic in U.* (*See* [**197**], [**198**].)

4.6. *Polyhedral approximations of spheres.* By using the local convexity of E^2 and the uniform continuity of a homeomorphism of a circle, it is easy to show that every simple closed curve in E^2 can be homeomorphically approximated with a polygonal simple closed curve. Similarly, it can be shown that any surface in E^3 which is the graph of a continuous function defined on a polyhedral disk in the xy-plane can be approximated with a polyhedral disk. In this case, of course, the surface we are approximating is tame in E^3. Some of the methods developed by Bing [**21**], [**27**] and Moise [**143**] can be used, as in the proof of Theorem 4.4.5, to show that any tame 2-sphere in E^3 can be homeomorphically approximated, in each of its complementary domains, with a polyhedral sphere. The converse of this result is also true and will be discussed in §6.2. (See Theorem 6.2.1.) Thus a 2-sphere S in E^3 that is wild from each component of $E^3 - S$ cannot be homeomorphically approximated with 2-spheres that do not intersect S.

The problem of showing that every 2-sphere in E^3 can be homeomorphically approximated with a polyhedral 2-sphere is very difficult and relies substantially upon a development of the topology of E^2. Bing [**24**] identified this problem, now known as the *approximation theorem for spheres,* and proved it when he was developing a proof, alternative to the one developed by Moise [**143**], that every 3-manifold can be triangulated [**27**]. Some consequences of Theorem 4.6.1 and 4.6.4. were discussed by Harrold [**99**].

THEOREM 4.6.1 (BING'S APPROXIMATION THEOREM FOR SPHERES [**24**, THEOREM 1]). *If S is a 2-sphere in E^3 and $\epsilon > 0$, then there is a homeomorphism h of S into E^3 such that $h(S)$ is polyhedral and h moves no point more than a distance ϵ.*

This approximation theorem and its generalizations have been essential to most of the ensuing work on embeddings of surfaces in E^3. In its original form, stated above, it made no requirement on the way the given 2-sphere intersects the approximating sphere. Bing soon generalized his theorem to a much more useful version known as the *side approximation theorem for spheres* [**35**]. As a result of a delay in the publication of this theorem, some of its consequences appeared in print before the theorem itself appeared. (See [**35**, p. 147].)

THEOREM 4.6.2 (BING'S SIDE APPROXIMATION THEOREM FOR 2-SPHERES [**35**]). *If S is a 2-sphere in E^3 and $\epsilon > 0$, then there is a homeomorphism h of S into E^3 and a finite set of disjoint ϵ-disks, $E_1, E_2, \cdots, E_n$ on $h(S)$ such that:*

(1) *h moves no point more than a distance ϵ.*

(2) *$h(S)$ is polyhedral.*

(3) $h(S) - \bigcup_{i=1}^{n} E_i \subset \operatorname{Int} S$.

As indicated in [**29**], [**33**], and [**86**, Theorem 1], the following requirement can be included in the conclusion of this theorem.

(4) *There exist a finite number of disjoint ϵ-disks $D_1, D_2, \cdots, D_m$ on S such that $S - \bigcup_{i=1}^{m} D_1 \subset \operatorname{Ext} h(S)$.*

Also, a similar theorem can be stated so that the approximating sphere $h(S)$ is in Ext S except for a finite number of ϵ-disks on $h(S)$. In a later paper [**40**], Bing adapted this theorem to surfaces in a 3-manifold and simplified his procedure for approximating open subsets of 2-spheres. Some minor corrections for [**35**] are included at the end of [**40**].

In picturing Theorem 4.6.2, it may be helpful to think of the disks $D_1, D_2, \cdots, D_n$ as painted dots, like polka dots, on S and the disks $E_1, E_2, \cdots, E_m$ as dots, or small painted disks, on $h(S)$. The 2-spheres

S and $h(S)$ do not intersect except in these small disks or dots. However, those on S may be badly entangled with those on $h(S)$ so that one of the D_i's may intersect several of the E_i's, and vice versa. It is easy to see that the wild spheres illustrated in Figures 1, 3, and 5 can be approximated in the manner described here. (Cannon [**182**] has recently announced an alternative proof of Theorem 4.6.2.)

Using Dehn's lemma (Theorem 4.5.1) and Bing's side approximation theorem for spheres (Theorem 4.6.2), Lister [**121**] showed that the small disks on S (the D_i's) can be required to intersect those on $h(S)$ (the E_i's) in a one-to-one manner and that the intersecting disks can be required to correspond under the homeomorphism h. (See Figure 7.) This improved version of the side approximation theorem for spheres (Theorem 4.6.3) has made it possible to simplify proofs of some of the consequences of Theorem 4.6.2 and has led to some additional theorems. A study of Lister's proof reveals that while he changed, or amalgamated, the disks $E_1, E_2, \cdots, E_m$ on $h(S)$, identified above in Theorem 4.6.2, he did not change the disks $D_1, D_2, \cdots, D_n$ on S. It can easily be shown that the same D_i's can be used in approximating S from its two sides, so Lister's improvement of Theorem 4.6.2 can be stated in the following form.

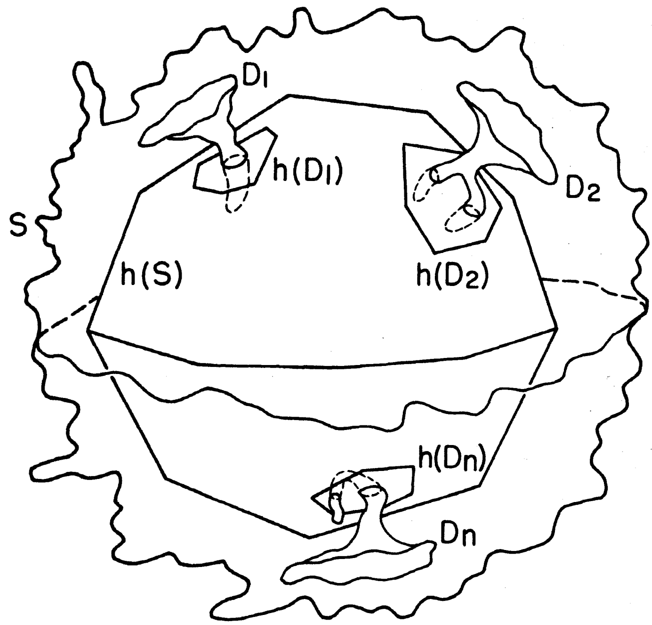

FIGURE 7

THEOREM 4.6.3 (LISTER [121]). *If S is a sphere in E^3 and $\epsilon > 0$, then there exist a finite number of disjoint ϵ-disks $D_1, D_2, \cdots, D_n$ on S and two ϵ-homeomorphisms h and h' of S into E^3 such that:*

(1) *$h(S)$ and $h'(S)$ are disjoint polyhedral spheres.*

(2) $h(S) - \bigcup_{i=1}^n \operatorname{Int} h(D_i) \subset \operatorname{Int} S$.

(3) $h'(S) - \bigcup_{i=1}^n \operatorname{Int} h'(D_i) \subset \operatorname{Ext} S$.

(4) $S - \bigcup_{i=1}^n \operatorname{Int} D_i \subset \operatorname{Ext} h(S) \cap \operatorname{Int} h'(S)$.

(5) *For each i, $D_i \cap h(S) = D_i \cap h(D_i)$ and $D_i \cap h'(S) = D_i \cap h'(D_i)$.*

A theorem proved by Craggs [**65**] can be combined with Theorem 4.6.3 to require, in the conclusion of Theorem 4.6.3, that there be an ϵ-isotopy of E^3 onto itself that moves $h(S)$ to $h'(S)$. An improvement in the way D_i intersects $h(D_i)$ and $h'(D_i)$ can be obtained by applying Bing's theorem on improving the intersection of two surfaces (Theorem 5.5.2). In particular, we could require that $D_i \cap h(D_i)$ be the union of a totally disconnected set and a null sequence of disjoint simple closed curves.

(Eaton [*A two sided side approximation theorem,* Notices Amer. Math. Soc. **17** (1970), 1038. Abstract #679-G19] recently announced an extension of Theorem 4.6.3 where it is concluded that each $D_i \cap (h(D_i) \cup h'(D_i))$ is a subset of the union of a finite number of disjoint disks in D_i such that no one of them intersects both $h(S)$ and $h'(S)$. This has enabled him to obtain a very useful criteria for determining that a sewing of two crumpled cubes yields S^3. See the remark at the end of §10.3.)

Bing [**24**] extended Theorem 4.6.1 to 2-manifolds in a 3-manifold.

THEOREM 4.6.4 (BING [24, THEOREM 7]). *If in a triangulated 3-manifold M^3, M is a 2-manifold and f is a nonnegative continuous function defined on M, then there is a 2-manifold M' and a homeomorphism h of M onto M' such that $\rho(x, h(x)) \leqq f(x)$ and M' is locally polyhedral at $h(x)$ for each $x \in M$ where $f(x) > 0$.*

Bing [**35**], [**40**] extended Theorem 4.6.2 to side approximations of open subsets of 2-spheres in E^3 and to open subsets of 2-manifolds that separate a 3-manifold. Lister [**121**] then showed, as in Theorem 4.6.3, that the small disks can be required to intersect in a one-to-one manner. This can be stated as follows for an open subset of a 2-sphere in E^3.

THEOREM 4.6.5 (BING [40] AND LISTER [121]). *If U is an open subset of a 2-sphere S in E^3 and f is a positive continuous real function on U, then there exist a homeomorphism h of S into E^3 and a locally finite collection of disjoint disks $\{D_i\}$ in U such that:*

(1) *h is the identity on $S - U$.*
(2) *$\rho(x, h(x)) < f(x)$ for each $x \in U$.*
(3) *$h(U)$ is locally polyhedral.*
(4) $h(U - \bigcup \operatorname{Int} D_i) = h(U) - \bigcup \operatorname{Int} h(D_i) \subset \operatorname{Ext} S$.
(5) $U - \bigcup \operatorname{Int} D_i \subset \operatorname{Int} h(S)$.
(6) Diam $D_i <$ *min value of f on D_i.*
(7) $h(D_i) \cap S = h(S) \cap D_i = (\operatorname{Int} h(D_i)) \cap \operatorname{Int} D_i$.

4.7. *Linking.* A very elementary form of homology linking is both adequate and extremely useful in the study of surfaces in E^3. If J and K are disjoint, oriented simple closed curves (i.e., disjoint simple closed curves with a specified direction of traversal) in a simply connected 3-manifold M^3, then an integer $\mathcal{L}(J, K)$ can be assigned to the pair (J, K) which gives a rough indication of how badly J and K are entangled with each other in M^3. Roughly speaking, if D is a (singular) disk in M, bounded by K (such a D exists because M is simply connected), then $\mathcal{L}(J, K)$ is the algebraic number of times J cuts through D. A precise definition requires a more extensive preparation than we develop here. Figure 8 suggests the concept described by $\mathcal{L}(J, K)$.

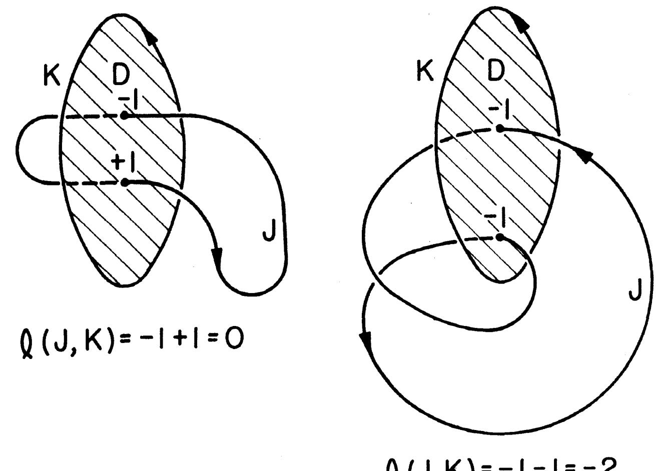

FIGURE 8

The most important facts about the linking of simple closed curves are the following (which are not independent of one another):
(1) $\mathcal{L}(J, K)$ is independent of the choice of D.
(2) $\mathcal{L}(J, K) = \mathcal{L}(K, J)$.
(3) If $\mathcal{L}(J, K) \neq 0$, then J is not nullhomotopic in $M - K$.

(4) If K is homotopic to K_1 in $M - J$, then $\mathfrak{L}(J, K) = \mathfrak{L}(J, K_1)$.

If we count not the algebraic number of times J cuts through D but rather whether J cuts through D an even or an odd number of times, then we obtain an even simpler version of linking. This type of linking is treated nicely and rigorously in [**24**]. General homology linking is treated carefully in [**1**, Chapter 15]. We make no further attempt here to develop a theory of linking except to indicate one of the important linking theorems that is useful in the study of surfaces.

THEOREM 4.7.1. *If* S *is a 2-sphere in* E^3, J *is a simple closed curve in* $S \cup \operatorname{Int} S$, *and* K *is a simple closed curve in* $(S \cup \operatorname{Ext} S) - J$, *then* J *and* K *do not link (i.e.,* $\mathfrak{L}(J, K) = 0$).

PROOF. There is a homotopy $H\colon K \times I \to E^3 - J$ such that $H(k, 0) = k$ and $H(k, 1) \in \operatorname{Ext} S$ for each $k \in K$. If we denote $H \mid K \times 1$ by K_1, then, by property (4) of linking, $\mathfrak{L}(J, K_1) = \mathfrak{L}(J, K)$. Since $S \cup \operatorname{Int} S$ is simply connected (Theorem 4.2.2), J bounds a singular disk in $E^3 - K_1$. By property (3) of linking, $\mathfrak{L}(J, K_1) = 0$. Hence $\mathfrak{L}(J, K) = 0$.

An application of Theorem 4.7.1 appears in [**52**]. Other applications of linking appear, for example, in [**24**], [**48**], [**51**], and [**60**].

4.8. *Brief outline of plane topology.* A thorough treatment of the topology of the plane, E^2, would be much too extensive to include here. We recommend [**118**, Chapter 9] and [**190**, Chapter 9], [**148**, Chapter 4], [**149**, Chapters 5 and 6], [**170**, Chapters 1, 2, and 3], and [**168**, Chapter 6] for a further study of E^2. We outline some of the theorems about E^2 that are needed in a study of surfaces on E^3. These theorems should be helpful in studying Bing's proof of the side approximation theorem for spheres [**35**] and in studying many of the other papers that we list as references. We do not attempt to list the original references for the theorems we discuss, but in most cases these references can be obtained from the books cited above. A *continuum* is defined to be a compact connected metric space.

The following theorem is stated to include both the Jordan curve theorem and the Schoenflies theorem for simple closed curves in E^2.

THEOREM 4.8.1 ([**170**, p. 38], [**154**]). *If* K *is a simple closed curve in* E^2 *and* h *is a homeomorphism of* K *onto a circle in* E^2, *then* h *can be extended to a homeomorphism of* E^2 *onto itself.*

THEOREM 4.8.2. *Every arc in* E^2 *is a subset of a simple closed curve in* E^2.

THEOREM 4.8.3. *If* $\epsilon > 0$ *and* K *is a continuum in* E^2, *then there is a disk-with-holes* F *such that* $K \subset \operatorname{Int} F$, *every point of* F *is within a* *distance ϵ of K, and each component of* Bd F *is polygonal.*

Outline of Proof. There exist a finite number of triangles T_1, T_2, $\cdots$, T_n such that $K \subset \bigcup_{i=1}^n \text{Int } T_i$, each T_i intersects K and has a diameter less than ϵ, and the boundaries of the T_i's are in relative general position. Then $\bigcup_{i=1}^n T_i$ is the desired disk-with-holes F.

Theorem 4.8.4. *If $\epsilon > 0$ and K is a continuum in E^2 which does not separate E^2, then there is a disk D such that $K \subset$ Int D and every point of D is within a distance ϵ of K.*

Proof. Let F be a disk-with-holes satisfying the requirements of Theorem 4.8.3, and suppose that F is chosen so that its number of holes is a minimum. We wish to show that this number is zero so that F is the desired disk. Suppose that F is not a disk. Then there is a polygonal arc A in $E^2 - K$ that intersects two components of $E^2 - F$. Some subarc of A can be thickened and removed from F to obtain a disk-with-holes D such that $D \subset F$, $K \subset$ Int D, and D has fewer holes than F. As F was chosen with a minimal number of holes, we can conclude from this contradiction that F has no holes and is the desired disk.

Theorem 4.8.5. *If H and L are different components of a compact set K in E^2, then there is a polygonal simple closed curve in $E^2 - K$ which separates H from L.*

Outline of Proof. The compact set K is the union of two disjoint closed sets A and B such that $H \subset A$ and $L \subset B$ [**170**, p. 12], [**148**, p. 15], [**149**, p. 82], [**168**, p. 15]. With $0 < \epsilon < \rho(A, B)$, cover A with the interiors of a finite number of ϵ-triangles, as in the proof outlined for Theorem 4.8.3, to obtain a finite number of disjoint disk-with-holes whose interiors cover A. One of these disks-with-holes, which we call D, has H in its interior. Let V be the component of $E^2 - D$ that contains L. Then Bd V is the desired simple closed curve that separates H from L in E^2 and does not intersect K.

Theorem 4.8.6. *If the compact set K in E^2 does not separate E^2, H is a component of K, and $\epsilon > 0$, then there is a disk F such that $H \subset$ Int F, Bd F is polygonal and does not intersect K, and every point of F is within a distance ϵ of H.*

Proof. By Theorem 4.8.4, there is a disk D such that $H \subset$ Int D and every point of D is within a distance ϵ of H. Let L be the component of $K \cup$ Bd D that contains Bd D. By Theorem 4.8.5, there is a polygonal simple closed curve J in $E^2 - (K \cup \text{Bd } D)$ which separates H from L. Thus $J \cup$ Int J is the desired disk F.

THEOREM 4.8.7. *If* $\epsilon > 0$ *and* K *is a compact set in* E^2 *that does not separate* E^2 *such that each component of* K *has a diameter less than* ϵ, *then there exist a finite number of disjoint* ϵ*-disks* $D_1, D_2, \cdots, D_n$ *such that* $K \subset \bigcup_{i=1}^{n} \operatorname{Int} D_i$ *and each* $\operatorname{Bd} D_i$ *is polygonal.*

INDICATION OF PROOF. Theorem 4.8.6 can be used to obtain a finite number of ϵ-disks $E_1, E_2, \cdots, E_m$ such that $K \subset \bigcup_{i=1}^{m} \operatorname{Int} E_i$, each $\operatorname{Bd} E_i$ is a polygonal simple closed curve in $E^2 - K$, and each D_i is essential in this covering of K. The $\operatorname{Bd} E_i$'s can be adjusted so that they are in relative general position, and then the E_i's can be cut apart along their boundaries to obtain the required disks $D_1, D_2, \cdots, D_n$. It should be observed that this process may increase the number of disks; i.e., n may be greater than m.

THEOREM 4.8.8. *If* D *is a disk and* H *and* K *are disjoint continua in* D, *then no two points of* $H \cap \operatorname{Bd} D$ *separate two points of* $K \cap \operatorname{Bd} D$ *in* $\operatorname{Bd} D$.

OUTLINE OF PROOF. Apply Theorem 4.8.3 to obtain a disk-with-holes E such that $H \subset \operatorname{Int} E$ and E does not intersect K. By supposing that some two points of $H \cap \operatorname{Bd} D$ separate two points of $K \cap \operatorname{Bd} D$ on $\operatorname{Bd} D$, we can obtain an arc A in E such that A is a spanning arc of D and the end points of A separate some two points of $K \cap \operatorname{Bd} D$ on $\operatorname{Bd} D$. This is contrary to a property of θ-curves in E^2 [**168**, p. 105].

THEOREM 4.8.9 ([**148**, p. 176], [**168**, p. 105]). *If* K *is a continuum in* E^2, *then each component of* $E^2 - K$ *has a subcontinuum of* K *as its boundary.*

PROOF. Suppose that some component U of $E^2 - K$ has a boundary H that is not connected. By Theorem 4.8.5, there is a simple closed curve J in $E^2 - H$ such that each component of $E^2 - J$ intersects H. This implies that J intersects both U and K. But this is impossible as some point of $J \cap K$ would be a point of the boundary of U.

The following two theorems, first proved by Janiszewski, have been proved in a more general form by Bing [**17**], [**18**]. References to other papers that discuss these and related theorems can be found in [**17**].

THEOREM 4.8.10 ([**148**, p. 173]). *If* H *and* K *are two continua in* E^2 *such that* $H \cap K$ *is connected and neither* H *nor* K *separates* E^2, *then* $H \cup K$ *does not separate* E^2.

THEOREM 4.8.11 ([**118**, p. 354], [**148**, p. 175], [**149**, p. 117]). *If* H *and* K *are two continua in* E^2 *such that* $K \cap H$ *is not connected, then* $K \cup H$ *separates* E^2.

THEOREM 4.8.12. *If the continuum K is not locally connected, then there exist two open sets U and U', with disjoint closures, and a sequence $K_1, K_2, \cdots$ of disjoint continua such that:*

(1) *The sequence $\{K_i\}$ converges to K_1.*

(2) *Each K_i intersects both* Cl U *and* Cl U'.

(3) *Each K_i $(i \geqq 2)$ is a component of $K - (U \cup U')$.*

This theorem can be proved with either [**148**, Theorem 44, p. 111] or [**168**, Theorem 12.1, p. 18]. Where K is in E^2, the sets Cl U and Cl U' can be required to be disks in E^2.

Moore [**146**] presented the first system of axioms that topologically characterize E^2 and, with slight changes, S^2. A discussion of various characterizations of S^2, known in 1935, was given by van Kampen [**115**, p. 75]. One of these, proved by Kuratowski in 1928, is particularly useful. (A continuum H is defined to be *unicoherent* if for any two continua H_1 and H_2 such that $H = H_1 \cup H_2$, the set $H_1 \cap H_2$ is connected.)

THEOREM 4.8.13 (KURATOWSKI [**117**, p. 307], [**118**, p. 374]). *A nondegenerate locally connected continuum K is topologically S^2 if and only if no point of K separates K and each subcontinuum of K that does not separate K is unicoherent.*

In 1945, Bing improved Moore's characterization of S^2 with a proof of the Kline sphere characterization problem.

THEOREM 4.8.14 (KLINE SPHERE CHARACTERIZATION, BING [**19**]). *A nondegenerate locally connected continuum K is topologically S^2 if and only if each simple closed curve in K separates K and no pair of points separates K.*

The following theorem on upper semicontinuous decompositions of E^2 was first proved by Moore [**147**] using his characterizations of E^2 and S^2. The proof we outline is based upon Kuratowski's characterization (Theorem 4.8.13) and is essentially the proof given by Kuratowski [**117**, p. 318]. By using a corresponding characterization of E^2, the same proof is valid for decompositions of E^2. A similar argument can be given using the Kline sphere characterization (Theorem 4.8.14).

THEOREM 4.8.15 (MOORE [**147**]). *If G is an upper semicontinuous decomposition of S^2 such that each element of G is a proper subcontinuum of S^2 that does not separate S^2, then the decomposition space S^2/G is homeomorphic with S^2.*

OUTLINE OF PROOF. The definition of upper semicontinuity and Theorem 4.8.12 can be used to show that S^2/G is a locally connected continuum [**168**, Chapter 7]. We wish to show that S^2/G satisfies the requirements of Theorem 4.8.13. Clearly no element of S^2/G separates S^2/G. Suppose now that some subcontinuum K of S^2/G does not separate S^2/G and is the union of two continua K_1 and K_2 such that $K_1 \cap K_2$ is not connected. Each of $P^{-1}(K_1)$ and $P^{-1}(K_2)$, where P is the projection map associated with G, is a continuum in S^2. Since K does not separate S^2/G, it readily follows that the continuum $P^{-1}(K)$ does not separate S^2. On the other hand, from the supposition that $K_1 \cap K_2$ is not connected and the hypothesis that the elements of G are continua, we can show that $P^{-1}(K_1) \cap P^{-1}(K_2)$ is not connected. This is contrary to Theorem 4.8.11. Thus the requirements of Theorem 4.8.13 are fulfilled, so S^2/G is topologically S^2.

5. **General properties of spheres and crumpled cubes in E^3.** Several of the theorems discussed in §4 indicate some properties which are possessed by every sphere in E^3. In this section we wish to discuss some other general properties of this type.

5.1. *Tame arcs and other tame continua on spheres.* Until about eight years ago, it was not known whether every 2-sphere in E^3 contains a tame arc. Bing [33] then showed the existence of tame arcs on 2-spheres by proving a much stronger and more useful theorem. We can roughly describe his theorem, which is stated below as Theorem 5.1.1, by picturing a null sequence of disjoint small disks on a 2-sphere so that the remaining part of the 2-sphere is tame. This does not imply, however, that the 2-sphere is locally tame at any of the remaining points. Some of the 2-spheres described in §§3.4 and 3.5 are wild at all of their points. However, in view of Theorems 4.6.4 and 8.1.5, we can replace the disks that are removed with disks that are locally polyhedral in their interior to obtain a tame 2-sphere.

Sierpiński [**159**] described a 1-dimensional locally connected plane continuum in which every 1-dimensional plane continuum can be embedded. This continuum, which has been named a *Sierpiński curve,* is what remains in a 2-sphere S after removing from S the interiors of a null sequence of disjoint disks whose union is everywhere dense in S [**169**]. The points of a Sierpiński curve which do not belong to any of these disks are called *inaccessible* points [**34**, p. 594], [**36**, p. 34]. This is an invariant property under different embeddings of the Sierpiński curve in E^2 [**34**, Theorem 3.2].

Theorem 5.1.1 (Bing [**33**, Theorem 1]). *If S is a 2-sphere in E^3 and $\epsilon > 0$, then there is a null sequence $\{E_i\}$ of disjoint ϵ-disks on S such that the continuum $S - \bigcup_{i=1}^{\infty} \operatorname{Int} E_i$ is tame.*

Outline of Proof. Bing's proof [**33**] can be simplified by using

Lister's form of the side approximation theorem (Theorem 4.6.3), which was not available at the time Bing proved Theorem 5.1.1. Following the general procedure in the proofs of [33, Theorem 1] and [122, Theorem 1], we start with two homeomorphisms h_0 and h_0' on S so that $h_0(S)$ and $h_0'(S)$ satisfy requirements (1)—(5) for $h(S)$ and $h'(S)$ in the conclusion of Theorem 4.6.3. Using Theorem 4.6.4, we replace the disks $D_1, D_2, \cdots, D_n$ on S with disks that are locally polyhedral in their interiors to obtain a 2-sphere S_1. Now adjust $h_0(S)$ and $h_0'(S)$ in the $h_0(D_i)$'s and the $h_0'(D_i)$'s so that they miss S_1 and are close homeomorphic approximations of it. We continue to call these adjusted spheres $h_0(S)$ and $h_0'(S)$. Now use Theorem 4.6.3 again to obtain even closer homeomorphic approximations h_1 and h_1' of S_1 so that requirements (1)—(5) in the conclusion of Theorem 4.6.3 are satisfied for $h_1(S_1)$, $h_1'(S_1)$, and S_1. Adjust S_1 to obtain a 2-sphere S_2 that is locally polyhedral in the interiors of the finite number of disjoint small disks that contain the intersection of S_1 with $h_1(S_1)$ and $h_1'(S_1)$. Then adjust $h_1(S_1)$ and $h_1'(S_1)$ so that they miss S_1. As in the proofs of [33, Theorem 1] and [**122**, Theorem 1], we continue this process to define a sequence of spheres $\{S_i\}$ that converges homeomorphically to a 2-sphere S'. Furthermore, this is done so that $\{h_i(S_i)\}$ and $\{h_i'(S_i)\}$ converge homeomorphically to S' from different components of $E^3 - S'$. Thus by Theorem 6.2.1, S' is tame. In the selection of $\{h_i(S_i)\}$ and $\{h_i'(S_i)\}$, we need to be cautious so that $\{S_i\}$ does converge to a 2-sphere, so that there is a Sierpiński curve K in $S \cap S'$, and so that each component of $S - K$ has a diameter less than ϵ. Such a procedure is described in Bing's proof of [33, Theorem 1], with some elaboration in Loveland's proof of [**122**, Theorem 1].

Theorem 5.1.1 can be combined with properties of a Sierpiński curve to obtain the following corollaries.

THEOREM 5.1.2 (BING [**36**, THEOREM 9.2], [**40**, THEOREM 6.3]). *If* S *is a* 2-*sphere in* E^3 and $\epsilon > 0$, *then there is a triangulation* T *of* S *such that* T *has mesh* ϵ *and the* 1-*skeleton of* T *is tame.*

THEOREM 5.1.3 (BING [33, THEOREM 9]). *Every disk on a* 2-*sphere in* E^3 *contains a tame arc.*

This can be combined with Theorem 5.4.2, to weaken the hypothesis of Theorem 5.1.3.

THEOREM 5.1.4 (BING [33, THEOREM 9]). *Every disk* D *in* E^3 *contains a tame arc.*

This theorem has been extended by Martin [**134**] to require that there is such a tame arc which intersects Bd D.

5.2. *Piercing spheres with arcs.* One of the major difficulties encountered by Bing in his proof of the polyhedral approximation theorem for spheres (Theorem 4.6.1) was the fact that the intersection of the 1-skeleton of a triangulation of E^3 with a surface can be extremely bad. It is only within the last few years that Bing [**41**] has given a definitive answer to this difficulty with his proof of Theorem 5.5.1. These earlier difficulties led to the following question.

Can each disk in E^3 be pierced by a tame arc?

If D is a disk and A an arc that intersects D at a single point p in Int D, then A is said to *pierce* D at p if $A - p$ is the union of two half-open arcs that lie, near p, on opposite sides of D. We observe in §5.4 that Theorem 5.4.1 can be used to distinguish the sides of D. Doyle and Hocking [**76**] gave an affirmative answer to a special case of the above question by showing that a disk D can be pierced by a tame arc at each point of Int D provided each arc in D is tame. Since each disk in E^3 contains a subdisk which lies on a 2-sphere in E^3 (Theorem 5.4.1), the following question is equivalent to the previous one:

Can each 2-sphere S in E^3 be pierced, in every open subset of S, by a tame arc?

Bing [**34**] answered this question affirmatively by proving the following theorem and combining it with the theorems discussed in §5.1.

THEOREM 5.2.1 (BING [**34**, THEOREM 5.1]). *A 2-sphere S in E^3 can be pierced by a tame arc at each point of each tame Sierpiński curve on S.*

This theorem was the first of many results on piercing properties of surfaces in E^3. It is of interest to remark that Bing [**22**] has described a simple closed curve in E^3 which pierces no disk. Using Theorem 5.2.1, Gillman [**86**] proved the following:

THEOREM 5.2.2 (GILLMAN [**86**, THEOREM 5]). *A 2-sphere S in E^3 can be pierced by a tame arc at each point of each tame arc on S.*

ALTERNATIVE PROOF OF THEOREM 5.2.2. Let A be a tame arc on S, and let D and E be disks on S such that $D \cap E = A$. It follows from Theorem 4.6.5 that there exist disks D' and E' and null sequences $\{D_i\}$ and $\{E_i\}$ of disjoint disks in Int D' and Int E', respectively, such that:

(1) Int D' and Int E' are locally polyhedral.
(2) Bd D' = Bd D and Bd E' = Bd E.
(3) $\{D_i\}$ and $\{E_i\}$ are locally finite in Int D' and Int E', respectively.
(4) Int $D' - \bigcup_{i=1}^{\infty} D_i \subset$ Int S and Int $E' - \bigcup_{i=1}^{\infty} E_i \subset$ Ext S.
(5) $D' \cap E' = A$.

Now reduce D' and E' slightly to obtain subdisks D'' and E'', respectively, such that $D'' \cap E'' = A$, $D'' - A \subset \text{Int } D'$ and $E'' - A \subset \text{Int } E'$. It follows from [77, Theorem 3] that $D'' \cup E''$ is tame. Let p be a point of A. There exists an arc H in $(D'' - \bigcup_{i=1}^{\infty} D_i) \cup (E'' - \bigcup_{i=1}^{\infty} E_i)$ which intersects both Int D' and Int E'' such that $H \cap A = p$. Thus H is a tame arc which pierces S at p.

Gillman [86] called the points of S at which S can be pierced by a tame arc the *piercing points* of S. He proved the following two theorems:

THEOREM 5.2.3 (GILLMAN [86, THEOREMS 6 AND 10]). *Suppose S is a 2-sphere in E^3 and $p \in S$. Then the following are equivalent:*
(1) *The point p is a piercing point of S.*
(2) *The point p lies on a tame arc in S.*
(3) *The point p can be missed with side approximations of S.*
(*See* §7.1.)

THEOREM 5.2.4 (GILLMAN [86, THEOREM 11]). *If S is a 2-sphere in E^3, then the set of nonpiercing points of S is a 0-dimensional F_σ-set.*

Stallings [162] gave an example of a 2-sphere in E^3 with uncountably many nonpiercing points.

Martin [136] introduced the following notion of piercing points of a crumpled cube.

DEFINITION. Suppose C is a crumpled cube in E^3 and p is a point of Bd C. Then p is a *piercing point* of C if there is a homeomorphism h of C into E^3 such that $h(\text{Bd } C)$ can be pierced by a tame arc at $h(p)$.

Suppose C is a crumpled cube in E^3 and $p \in \text{Bd } C$ is a piercing point of C. Let $h : C \to E^3$ be an embedding such that $h(\text{Bd } C)$ can be pierced by a tame arc at $h(p)$. By Theorem 5.2.3, $h(p)$ can be missed with side approximations of $h(\text{Bd } C)$. By Theorem 7.1.7, $h(C) - h(p)$ is 1-ULC. Since h and h^{-1} are uniformly continuous, it follows that $C - p$ is 1-ULC. Thus we have established one-half of the following theorem. (The other half can be proved similarly.)

THEOREM 5.2.5 (MCMILLAN [139, THEOREM 1]). *If C is a crumpled cube in E^3 and p is a point of* Bd C, *then p is a piercing point of C if and only if $C - p$ is 1-ULC.*

Martin's original application of piercing points of a crumpled cube was the following.

THEOREM 5.2.6 (MARTIN [136, p. 147]). *If S is a 2-sphere in S^3 which bounds crumpled cubes C and C^*, then each point of S is a piercing point of at least one of C and C^*.*

This theorem is similar to the following, which was proved by McMillan [139] and attributed by him to Bing.

THEOREM 5.2.7 (MCMILLAN [**139**, p. 318]). *If S is a 2-sphere in E^3 and $p \in S$, then there is an arc A which pierces S at p such that at least one of the arcs* Cl(A ∩ Int S) *and* Cl(A ∩ Ext S) *is tame.*

Substantially more can be said about the piercing points of a 2-sphere which has open 3-cells as its complementary domains in S^3.

THEOREM 5.2.8 (MCMILLAN [**140**, THEOREM 3] AND LOVELAND [**131**]). *A 2-sphere S in S^3 has at most two nonpiercing points if its complementary domains are open* 3*-cells.*

This theorem is particularly interesting along with the following theorem, which Sher [**158**] proved by assuming the continuum hypothesis.

THEOREM 5.2.9 (SHER [**158**]). *If S is a 2-sphere in E^3 and P is the set of all piercing points of S, then S can be pierced simultaneously at all points of P with a family of disjoint tame arcs.*

Without assuming the continuum hypothesis, Gillman [**86**, Theorem 5] has shown the existence of such a family of piercing arcs for a tame arc in *P*. Such a family of arcs can be obtained by applying [**34**, Theorem 4.2] to the disk $D'' \cup E''$ in the proof outlined above for Theorem 5.2.2.

Some further interesting properties of piercing points of 2-spheres and of crumpled cubes are stated in the following four theorems.

THEOREM 5.2.10 (LOVELAND [**129**, THEOREM 2]). *If S is a 2-sphere in S^3 and p is a point of S at which S can be locally spanned or capped, then p is a piercing point of S.* (*See definitions in* §6.5).

THEOREM 5.2.11 (MCMILLAN [**139**, THEOREMS 2 AND 3]). *If C is a crumpled cube in E^3 and p is a point of* Bd C, *then p is a piercing point of C if and only if there is a tame arc A in E^3 which contains p and lies except for p in $E^3 - C$.*

By Theorem 10.4.1 of §10.4, every crumpled cube *C* can be embedded in S^3 so that Bd *C* is tame from $E^3 - C$. Martin [**136**] has proved the following:

THEOREM 5.2.12 (MARTIN [**136**, LEMMA 1]). *If C is a crumpled cube in E^3 such that* Bd *C is tame from $E^3 - C$, then a point $p \in$* Bd *C is a piercing point of C if and only if* Bd *C can be pierced at p with a tame arc.*

ALTERNATIVE PROOF. By definition, there is some embedding $h : C \to E^3$ such that $h(\mathrm{Bd}\, C)$ can be pierced with a tame arc at $h(p)$.

Thus by Theorem 5.2.3, $h(p)$ is a subset of a tame arc A on $h(\text{Bd } C)$ such that A can be missed with side approximations of $h(\text{Bd } C)$. Since C is tame from $E^3 - C$, it follows from a theorem of Lister [**121**, Theorem 9] that $h^{-1}(A)$ is a tame arc on Bd C. Thus by Theorem 5.2.3, Bd C can be pierced at p by a tame arc.

Daverman [**70**] has announced some conditions which imply that a crumpled cube has only a finite number of nonpiercing points.

5.3. *Neighborhoods of spheres.* A *retraction* r of a space Y onto a subset X of Y is a map of Y onto X that is the identity on X. Where such a map exists, we say that Y can be *retracted* onto X. The following theorem is a special case of the fact that a 2-sphere is an absolute neighborhood retract (ANR) and can be proved by use of the Tietze extension theorem (Theorem 4.2.1). An elementary proof appears in [**38**, Theorem 5].

THEOREM 5.3.1. *If* S *is a* 2-*sphere in an open subset* U *of* E^3, *then there is an open set* V *such that* $S \subset V \subset U$ *and* Cl V *can be retracted onto* S.

Bing [**39**] has described, with an elementary proof, how this theorem leads to the following:

THEOREM 5.3.2 ([**38**, THEOREM 4] AND [**39**, THEOREM 1]). *If* S *is a* 2-*sphere in* E^3, *then* S $\cup$ Ext S *can be retracted onto* S.

Theorem 5.3.2 has the following consequence:

THEOREM 5.3.3. *Every crumpled cube is unicoherent* [**168**, p. 229], 1-*ULC, and contractible.*

McMillan [**138**] has shown that a very specialized type of neighborhood can be obtained for any 2-sphere S in E^3. Roughly, such a neighborhood is formed by considering a 3-dimensional annulus A bounded by side approximations of S, which are supplied by Lister's form of the side approximation theorem (Theorem 4.6.3), and sewing a finite number of disjoint small cubes-with-handles to A along the disks $h(D_i)$ and $h'(D_i)$ of Theorem 4.6.3. McMillan's theorem was stated and proved for a 2-manifold in a 3-manifold, but we specialize it here for a 2-sphere in E^3.

THEOREM 5.3.4 (MCMILLAN [**138**, THEOREM 2]). *If* S *is a* 2-*sphere in* E^3 *and* $\epsilon > 0$, *then there is a polyhedral* 3-*dimensional closed annulus* L *and a finite collection* $H_1, H_2, \cdots, H_n$ *of disjoint polyhedral cubes-with-handles such that:*

(1) *Each component of* Bd L *is homeomorphically within* ϵ *of* S.

(2) *Each* H_i *has diameter less than* ϵ.

(3) *Each* $H_i \cap L$ *is a disk in both* Bd L *and* Bd H_i.
(4) $S \subset \mathrm{Int}(L \cup H_1 \cup H_2 \cup \cdots \cup H_n)$.

McMillan's proof of this theorem depended upon the side approximation theorem for spheres (Theorem 4.6.2), Bing's theorem improving the intersection of a sphere with a line (Theorem 5.5.1), and Armentrout's theorem that 3-manifolds are invariant under cellular decompositions that yield 3-manifolds [**12**], [**13**]. (A longer and more detailed version of [**13**] is forthcoming in the Memoirs of the American Mathematical Society.)

5.4. *Small disks on surfaces in* E^3 *are on small spheres.* By letting the first pair of horns, used in constructing the Alexander horned sphere, grow from opposite sides of a disk, we can construct a disk in E^3 that is not a subset of a 2-sphere [**34**, p. 592], [**16**, p. 207], [**132**, p. 392 and p. 397]. This disk is a subset of a torus in E^3 [**29**, p. 302]. However, with a sequence of such pairs of horns converging to the boundary of a disk, we can construct a disk in E^3 that is not a subset of a 2-manifold in E^3 [**29**, p. 302], [**102**, p. 613]. Hempel [**102**] has shown that a disk in E^3 is a subset of a 2-manifold in E^3 if D is locally tame in a neighborhood of its boundary. Some conditions implying that a disk is a subset of a 2-sphere have been given by Bean [**16**] and by Martin [**132**].

Bing [**29**] has shown that for any disk D in E^3 and any point $p \in \mathrm{Int}\, D$ there is a disk D' such that $p \in \mathrm{Int}\, D' \subset D$ and D' is a subset of a 2-sphere in E^3. This theorem enables some of the theorems about tame arcs on 2-spheres, piercings of 2-spheres with tame arcs, and local tameness of 2-spheres to be extended to disks in E^3.

THEOREM 5.4.1 (BING [**29**, THEOREM 5]). *If D is a disk in E^3, $p \in$ Int D, and $\epsilon > 0$, then there exist a 2-sphere S and a disk D' such that $p \in$ Int $D' \subset D$, $D' \subset S$, and S has diameter less than ϵ.*

This theorem, along with Theorem 4.1.1, can be used to distinguish the sides in a disk. Furthermore, it is valid in a 3-manifold and can be changed so that points of Int S near p are on a given side of D [**48**, Theorem 1].

We will discuss in §11 how the following theorem is useful in extending many of the theorems discussed in this paper to 2-manifolds in a 3-manifold.

THEOREM 5.4.2 (BING [**29**, THEOREM 5]). *If M is a 2-manifold-with-boundary in a 3-manifold M^3, $p \in M -$ Bd M, and $\epsilon > 0$, then there exist a 2-sphere S and a disk D such that:*
(1) $p \in \mathrm{Int}\, D \subset M - \mathrm{Bd}\, M$.
(2) $D \subset S$.

(3) *S has diameter less than ϵ.*
(4) *S is locally tame at each point of $S - D$.*

We wish to observe that in some cases it cannot be concluded that there is such a sphere S so that $S \cap M = D$, for then it would follow that M can be locally spanned at p from some component of $M^3 - M$. The double Alexander horned sphere, which is discussed in §10 as the common boundary in S^3 of two Alexander crumpled cubes, cannot be locally spanned from either of its complementary domains at any point where it is wild. (See §6.5 for a discussion of 2-spheres which can be locally spanned.)

5.5. *Improving intersections of spheres with lines and with other spheres.* In his proof of Theorem 5.3.4, McMillan used the following:

THEOREM 5.5.1 (BING [41, p. 155]). *If S is a 2-sphere in E^3, L is a line intersecting S, and $\epsilon > 0$, then there is an ϵ-homeomorphism h of E^3 onto itself such that $L \cap h(S)$ is a finite set and h is the identity outside of an ϵ-neighborhood of $S \cap L$.*

Bing [41, p. 155] has observed that a proof of this theorem that does not depend upon the side approximation theorem for spheres would enable him to simplify his proof of the side approximation theorem. However, the only known proofs of Theorem 5.5.1 do depend upon that theorem.

Bing [41] has used similar methods, including the existence of tame Sierpiński curves in spheres (Theorem 5.1.1), to prove:

THEOREM 5.5.2 (BING [41, THEOREM 4]). *If S and S' are intersecting 2-spheres in E^3 and $\epsilon > 0$, then there is an ϵ-homeomorphism h of E^3 onto itself such that $S' \cap h(S)$ is the union of a totally disconnected set and the elements of a null sequence of disjoint tame simple closed curves.*

An argument using function spaces (see §4.3) can be used to prove the following:

THEOREM 5.5.3 (BING [26, THEOREM 10.1]). *If S is a 2-sphere in E^3 and $\epsilon > 0$ then there is an ϵ-homeomorphism h of E^3 onto itself such that $h(S)$ contains no interval of a straight line.*

5.6. *Equivalence of complements of crumpled cubes and arcs in E^3.* As indicated in our description of the Fox-Artin sphere S in §3, the exterior of S is homeomorphic with the complement of an arc A in E^3. Furthermore, $E^3/(S \cup \operatorname{Int} S)$ is homeomorphic with E^3/A. (By E^3/A we mean the decomposition space for E^3 where A is the only nondegenerate element. Actually, in the case considered here, E^3/A is E^3 itself.)

By using methods similar to the ones they used in sewings of crumpled cubes, Daverman and Eaton [72] have proved a theorem about 3-cells which, along with Theorem 10.2.1, implies that every crumpled cube in E^3 is equivalent to an arc in the manner indicated above.

THEOREM 5.6.1 (DAVERMAN AND EATON [72]). *If K is a crumpled cube in E^3, then there is an arc A such that $E^3 - K$ is homeomorphic with $E^3 - A$. Furthermore, E^3/K is homeomorphic with E^3/A.*

The following question remains unanswered:

If A is an arc in E^3, is there a crumpled cube K such that $E^3 - K$ is homeomorphic with $E^3 - A$?

5.7. *Pushing a 2-sphere into its complement.* The surface of a crumpled cube in E^3 may be wild both from within the crumpled cube and from without. A consequence of Bing's side approximation theorem (Theorem 4.6.2) is that the wildness from without can be untangled in the large by slight moves of the crumpled cube in E^3. (Such smoothing cannot generally be realized by a homeomorphism of E^3 onto itself, of course.) This fact, if pursued in one direction, leads to the Hosay-Lininger theorem (Theorem 10.2.1). If pursued in another direction, it leads to the result that, for each $\epsilon > 0$, a crumpled cube or 2-sphere in E^3 can be "pushed" by a nice ϵ-map almost into its interior. In this section, we survey results of the latter type. The basic results are three in number and are due to Bing [36].

THEOREM 5.7.1 (BING [36, THEOREM 2.1]). *Suppose S is a 2-sphere in E^3 and U is a component of $E^3 - S$. Then for each $\epsilon > 0$ there exist a Cantor set C on S and an ϵ-map f: $\mathrm{Cl}\ U \to U \cup C$ such that $f = I$ on $U - N(S, \epsilon)$ and f is a homeomorphism on $\mathrm{Cl}\ U - f^{-1}(C)$.*

THEOREM 5.7.2 (BING [36, THEOREM 4.2]). Suppose S *is a 2-sphere in E^3 and U is a component of $E^3 - S$. Then there exists a 0-dimensional F_σ-set F on S such that $U \cup F$ is 1-ULC.*

THEOREM 5.7.3 (BING [36, THEOREM 5.1]). *Suppose S is a 2-sphere in E^3 and U is a component of $E^3 - S$. Then for each $\epsilon > 0$ there exist a Cantor set C on S and an ϵ-map g of S into $U \cup C$ such that g takes $S - C$ homeomorphically into U.*

It is not to be concluded from Theorem 5.7.3 that g keeps C pointwise fixed. It is in general impossible to choose g to be a homeomorphism [78, Corollary 18]. Cannon [55] has shown that g cannot be chosen to be a homeomorphism if S is wild from U and each arc in S is tame. Thus there does not exist such a homeomorphism for the wild 2-sphere described by Bing (Figure 5). Bing's methods can be applied to push open subsets of 2-spheres and other 2-manifolds in E^3 almost to one side. The results are extremely suggestive of Bing's side approximation theorems (Theorems 4.6.2 and 4.6.4) and, with slight alterations, are extremely useful as such. We first indicate a singular side approximation theorem which arises from Theorem 5.7.1 and is like Lister's form of Bing's side approximation theorem. (The word "singular" is used since the map g must in general have singularities.)

THEOREM 5.7.4 (CANNON [55]). *Suppose S is a 2-sphere in E^3 and U is a component of $E^3 - S$. Then for each $\epsilon > 0$ there exist an ϵ-map $g: S \to S \cup U$ and disjoint ϵ-disks $D_1, \cdots, D_n$ in S such that*

(1) $g(S) \cap S$ and $g^{-1}(g(S) \cap S)$ are zero-dimensional subsets of $\bigcup_{i=1}^n \mathrm{Int}\, D_i$,

(2) $g(D_i) \cap g(D_j) = \emptyset$ for $i \neq j$,

(3) $g(S) \cap S = \bigcup_{i=1}^n [\mathrm{Int}\, D_i \cap g(\mathrm{Int}\, D_i)]$, and

(4) $g \mid [S - g^{-1}(g(S) \cap S)]$ is a homeomorphism onto a locally polyhedral set in Int C.

Lister's theorem (Theorem 4.6.3) can be obtained from Theorem 5.7.4 by an application of Dehn's lemma (Theorem 4.5.1) to the singular disks $g(D_1), \cdots, g(D_n)$. The advantage of Theorem 5.7.4 as a side approximation theorem depends in large measure on the following:

ADDENDUM TO THEOREMS 5.7.1, 5.7.2, 5.7.3, AND 5.7.4. Let S and U be as they are identified in these four theorems. Let $X_1, X_2, \cdots$ be a sequence of closed subsets of S. Then $f(C) \cap S$ in Theorem 5.7.1 (the 0-dimensional F_σ-set F in Theorem 5.7.2; the sets $g(S) \cap S$ in Theorems 5.7.3 and 5.7.4) can be chosen to lie in $S - \bigcup_{i=1}^\infty X_i$ for each $\epsilon > 0$ if and only if $(S \cup U) - X_i$ is 1-ULC for each i. (This latter condition on X_i is satisfied, for example, if X_i is tame and has no degenerate component. See Theorems 7.1.5 and 7.1.7.)

The singular side approximation theorem (Theorem 5.7.4) and the Addendum above play important roles in characterizations of tame subsets of 2-spheres (§7), in the results on singular spanning disks (§6.5), in deforming Sierpiński curves into the complement of a 2-sphere (§6.7), and in $*$-taming sets (§8.2).

The fact that a surface in E^3 can almost be pushed free of itself (Theorem 5.7.1) has a converse due to R. L. Wilder.

THEOREM 5.7.5 (WILDER [174, THEOREM 1], [175, p. 105]). *In E^3, let K be a locally connected continuum forming a common boundary of (at least) two domains, and such that $H_1(K) = 0$. If for arbitrary* *given* $\epsilon > 0$ *and* U *either of the given domains complementary to* K, *there exists a closed and totally disconnected subset* T *of* K, *and an* ϵ*-transformation* $f(K) = K'$ *into* $U \cup T$, *then* K *is a 2-sphere.*

In the same references, Wilder gives a corresponding characterization for 2-manifolds, with some generalizations to higher dimensions.

6. **Characterizations of tame spheres.** We discuss in this section some properties which are possessed by every tame 2-sphere in E^3 and which, in turn, imply tameness for any 2-sphere that possesses these properties. For convenience in stating the theorems, we omit the statement, which is quite obvious in all the cases, that every tame 2-sphere possesses the property under consideration. Some other properties, which imply tameness of 2-spheres but are not actual characterizations, are discussed in §9.

6.1. *Locally tame spheres.* The first significant characterization of tameness for 2-spheres was developed independently, about 1954, by Bing [**21**] and Moise [**145**] with their proofs that locally tame sets are tame. Another proof of the same theorem for 2-spheres is included in the subsequent work by Brown [**45**], [**46**], which is valid in higher dimensions as well as in E^3, for surfaces which are locally flat, or locally bicollared. Before the time of the work by Bing and Moise, Harrold [**95**] had given some conditions under which the complement of a 2-sphere in E^3 is 1-ULC. Later work by Bing [**29**] showed that such spheres are tame (Theorem 6.4.1).

THEOREM 6.1.1 (BING [**21**, THEOREM 6] AND MOISE [**145**, THEOREM 8.1]). *A 2-sphere in* E^3 *is tame if it is locally tame.*

This theorem has the following corollary for crumpled cubes:

THEOREM 6.1.2. *A crumpled cube* C *is a 3-cell if* Bd C *is locally tame from* Int C.

Another early characterization of tame 2-spheres, which we do not attempt to describe here, was obtained by Griffith [**92**] as a consequence of Theorem 6.1.1 and his characterizations of tame disks in E^3.

6.2. *Spheres which can be homeomorphically approximated in their complementary domains.* We discussed, with the side approximation theorem in §4, how each 2-sphere S in E^3 can be homeomorphically approximated with a 2-sphere that is "almost" in a given component of $E^3 - S$. In this section we consider a 2-sphere S which can be homeomorphically approximated in each component of $E^3 - S$ with a 2-sphere that does not intersect S. Clearly this is a property of every tame 2-sphere in E^3. Bing [**26**] proved that the converse is true, thus giving one of the most fundamental and useful characterizations of tame 2-spheres in E^3. It has been used as a basis for nearly all of the other characterizations that have subsequently been developed.

THEOREM 6.2.1 (BING [26]). *A 2-sphere S in E^3 is tame if it can be homeomorphically approximated in each component of $E^3 - S$.*

INTRODUCTION TO PROOF. We restrict our discussion to an indication of the general procedure that Bing followed to show that $S \cup \text{Int } S$ is a 3-cell. It follows from the hypothesis that there is a sequence $\{S_i\}$ of 2-spheres in Int S such that:

(1) $\{S_i\}$ converges homeomorphically to S.
(2) For each i, $S_i \subset \text{Int } S_{i+1}$.
(3) $\bigcup_{i=1}^{\infty} \text{Int } S_i = \text{Int } S$.

By applying Theorem 4.6.1, we can further assume that:

(4) Each S_i is polyhedral.

We give only an introduction to the problem of obtaining a homeomorphsim h from Int S to the interior of a round 2-sphere so that h can be extended to S. Let S' be a round 2-sphere in E^3 and let $\{S_i'\}$ be a sequence of concentric round 2-spheres in Int S' so that requirements (1), (2), and (3) above are satisfied for these spheres. Using the homeomorphic convergence of $\{S_i\}$ and $\{S_i'\}$ we obtain a homeomorphism g of $S \cup (\bigcup_{i=1}^{\infty} S_i)$ onto $S' \cup (\bigcup_{i=1}^{\infty} S_i')$. Our problem now is to extend g to a homeomorphism of $S \cup \text{Int } S$ onto $S' \cup \text{Int } S'$.

By Theorem 4.4.4, there is a homeomorphism h_1 of $S_1 \cup \text{Int } S_1$ onto $S_1' \cup \text{Int } S_1'$ that agrees with g on S_1. It follows from Theorem 4.4.4 that there is a homeomorphism h_2 of $S_2 \cup \text{Int } S_2$ onto $S_2' \cup \text{Int } S_2'$ that agrees with h_1 on $S_1 \cup \text{Int } S_1$ and with g on S_2. Continue this process to obtain a sequence $\{h_i\}$ of homeomorphisms, where h_{i+1} is an extension of h_i that carries $S_{i+1} \cup \text{Int } S_{i+1}$ onto $S_1' \cup \text{Int } S_{i+1}'$ and agrees with g on S_{i+1}. Requirement (2) enables us to identify a homeomorphism h of Int S onto Int S', where $h = h_i$ on $S_i \cup \text{Int } S_i$. At this stage we would like to extend h to agree with g on S. However, without further restrictions on the h_i's, the above procedure does not identify a homeomorphism h that can be extended to a homeomorphism of $S \cup \text{Int } S$ onto $S' \cup \text{Int } S'$. Roughly, we can see this difficulty by observing that for each i a radial arc from S_i' to S_{i+1}', might be carried by h_i^{-1} onto an arc of large diameter from S_i to S_{i+1}. (A similar difficulty would be encountered if the same procedure were used in attempting to obtain a homeomorphism between two disks in E^2.) These further restrictions, which we do not attempt to describe here, involve a long and difficult process that accounts for a major portion of Bing's paper [26]. (Cannon [182] has recently used Bing's side approximation theorem (Theorem 4.6.5) to obtain a simple proof of Theorem 6.2.1.)

6.3. *Free 2-spheres in E^3*. We do not know whether a 2-sphere S in E^3 is tame if the homeomorphism required in the hypothesis of Theorem 6.2.1 is replaced with a map of S. A 2-sphere S in E^3 is defined to be *free* in the component U of S if, for each $\epsilon > 0$, there is a map of S into U that moves no point more than a distance ϵ. A 2-sphere in E^3 is defined to be *free* if it is free in each of its complementary domains. Some discussions of free sets in Euclidean spaces can be found in [**90**], [**172**], [**174**], [**175**]. The following question, which remains unanswered, was raised in [**103**, p. 280].

Is a 2-sphere tame in E^3 if it is free?

Affirmative answers for some special cases, including some for spheres that are locally free, have been given by Hempel [**103**], [**104**], Loveland [**126**], Eaton [**78**], and Burgess [**51**]. White [**167**] has recently announced that a free 2-sphere is tame if it is locally tame except on a nowhere dense set. A 2-sphere S is defined to be *locally free* if for each $p \in S$ and each component U of $E^3 - S$ there exists a disk D with $p \in \operatorname{Int} D$ such that for each $\epsilon > 0$ there is a map of D into U that moves no point more than a distance ϵ. Clearly every free 2-sphere is locally free. However, the following question remains unanswered:

Is a 2-sphere in E^3 free if it is locally free?

Perhaps an answer to this will involve answering the following question:

Is a 2-sphere in E^3 tame if it is locally free?

Answers to these questions may require some further improved versions of Dehn's lemma and the Sphere Theorem [**31**], [**32**], [**105**], [**106**].

6.4. *Spheres with complements that are* 1-*ULC*. We indicated in §3 that the Alexander horned sphere S is wild by observing that there is a simple closed curve in Ext S that bounds a disk in E^3 but not in Ext S. This procedure is not possible with the Fox-Artin sphere S' (Figure 3) as every simple closed curve in Ext S' that bounds a disk in E^3 also bounds one in Ext S'. The exterior of S' is simply connected but the exterior of S is not. However, Ext S' fails to be uniformly locally simply connected. We can find a small simple closed curve K in Ext S' and near the wild point of S' such that K bounds a large disk in Ext S' but not a small one. (See Figure 3.) This suggests the following question, which is phrased roughly. Is a 2-sphere S in E^3 tame if every small simple closed curve in $E^3 - S$ bounds a small singular disk in $E^3 - S$? This question was answered affirmatively by Bing [**29**] several years ago, and the resulting theorem, stated below as Theorem 6.4.1, has been a basis of several other characterizations of tame surfaces. To discuss this theorem, we need first to formulate some precise definitions of simple connectedness and local simple connectedness.

Let D denote a disk. A set U is *simply connected* if every map of Bd D into U can be extended to a map of D into U. The set U is *uniformly locally simply connected* (1-ULC) if for each $\epsilon > 0$ there is a $\delta > 0$ such that each map of Bd D into a δ-subset of U can be extended to a map of D into an ϵ-subset of U. We say that a simple closed curve can be *shrunk to a point* in U if some homeomorphism of Bd D onto K can be extended to a map of D into U.

THEOREM 6.4.1 (BING [29, THEOREM 2]). *A 2-sphere S in E^3 is tame if its complement is uniformly locally simply connected* (1-*ULC*).

ROUGH OUTLINE OF PROOF. We restrict our discussion to showing that S is tame from Int S; that is, we indicate why S ∪ Int S is a 3-cell. Use the side approximation theorem (Theorem 4.6.2) to approximate S with a polyhedral sphere S′ that is in Int S except for a finite number of small disks on S′. Remove the interiors of these disks so that what remains of S′ is a "sieve" H in Int S. Now we need to patch the holes of H to obtain a sphere S″ that is homeomorphically close to S, for it would then follow from a one-sided version of Theorem 6.2.1 that S ∪ Int S is a 3-cell. The holes are patched by applying the hypothesis that Int S is 1-ULC to shrink their boundaries to points in small subsets of Int S and then applying Dehn's lemma (Theorem 4.5.1) to replace these singular disks with disks so that an approximating sphere S′ is obtained in Int S. Considerable attention needs to be given here to a procedure that will enable us to show that the resulting disks are disjoint and that the resulting sphere is homeomorphically close to S. Such a procedure is described in Bing's proof of Theorem 6.4.1. As observed by Hempel [**103**, p. 277], Bing's proof is valid where $E^3 -$ S is weakly 1-ULC. (See §7.2.)

We now present an outline of a proof that is based on Lister's improved form of the side approximation theorem (Theorem 4.6.3). This enables us to simplify the proof presented by Bing in [**29**].

ALTERNATIVE PROOF OF THEOREM 6.4.1. Using the hypothesis that $E^3 -$ S is 1-ULC, we shall show that for each $\epsilon > 0$ and each component U of $E^3 -$ S there is an ϵ-homeomorphism h of S into U. This will, of course, imply by Theorem 6.2.1 that S is tame. We give the proof only with $U =$ Int S.

By Theorem 4.6.3 there exist disjoint disks $D_1, D_2, \cdots, D_n$ on S and an ϵ-homeomorphism h from S onto a polyhedral sphere h(S) in E^3 such that:

(1) $h(S) - \bigcup_{i=1}^{n} \operatorname{Int} h(D_i) \subset \operatorname{Int} S$.
(2) $S - \bigcup_{i=1}^{n} \operatorname{Int} D_i \subset \operatorname{Ext} h(S)$.
(3) $S \cap h(S) = \bigcup_{i=1}^{n} [\operatorname{Int} D_i \cap \operatorname{Int} h(D_i)]$.
(4) For each i $(1 \leqq i \leqq n)$, Diam $[D_i \cup h(D_i)] < \epsilon$.

It follows from Theorem 4.2.3 that for each i $(1 \leqq i \leqq n)$, there is a map g_i of D_i into $h(D_i) \cup \operatorname{Int} D_i$ such that g_i agrees with h on that component K_i of $D_i - h^{-1}[h(D_i) \cap D_i]$ which contains Bd D_i and g_i takes $D_i - K_i$ into Int D_i. Let T_i be a triangulation of D_i whose image under g_i has very fine mesh. Since Int S is dense in S $\cup$ Int S and Int S is both 0-ULC (Theorem 4.1.2) and 1-ULC, it follows that the images, under g_i, of the 0-, 1-, and 2-skeletons of T_i can be adjusted successively very slightly, near S only, so as to lie entirely in Int S. Let f_i denote this adjustment of the map g_i. We may require that $f_1(D_1), f_2(D_2), \cdots, f_n(D_n)$ be disjoint singular disks in Int S such that, for each i:

(1) $f_i(D_i)$ has no singularities near $f_i(\text{Bd } D_i) = g_i(\text{Bd } D_i) = h(\text{Bd } D_i)$.
(2) Diam $(f_i(D_i) \cup D_i) < \epsilon$.
(3) $f_i(D_i) \cap h(S - \bigcup_{i=1}^{n} \operatorname{Int} D_i) = f_i(\text{Bd } D_i)$.

By Dehn's lemma (Theorem 4.5.1 and its addendum), each singular disk $f_i(D_i)$ may be replaced by a polyhedral disk E_i in Int S such that $h(S - \bigcup_{i=1}^{n} \operatorname{Int} D_i) \cup (\bigcup_{i=1}^{n} E_i)$ is a polyhedral 2-sphere S' in Int S and Diam $(D_i \cup E_i) < \epsilon$. Let h' be a homeomorphism from S onto S' which agrees with h on $S - \bigcup_{i=1}^{n} \operatorname{Int} D_i$. Then h' is an ϵ-homeomorphism from S into Int S. Similarly, there is an ϵ-homeomorphism from S into Ext S. Thus by Theorem 6.2.1, S is tame.

We now change the above definition of 1-ULC and show, using Theorem 6.4.1, that a 2-sphere S is tame in E^3 if $E^3 - S$ is locally simply connected at each point of S. In particular, Int S is defined to be *locally simply connected* at a point p of S if for every open set U containing p there is an open set V such that $p \in V \subset U$ and every map of Bd D into $V \cap \operatorname{Int} S$ can be extended to a map of D into $U \cap \operatorname{Int} S$. We observe that the exterior of the Fox-Artin sphere (Figure 3) fails to be locally simply connected at the wild point of the sphere.

THEOREM 6.4.2 (BING [29]). *A 2-sphere S in E^3 is tame if each component of $E^3 - S$ is locally simply connected at each point of S.*

PROOF. We indicate how to show that Int S is 1-ULC. Let D be a disk and let ϵ be a positive number. From the hypothesis that Int S is uniformly simply connected at each point of S, it follows that there exist a finite collection G' of open sets covering S such that, for each element g' of G', each map of Bd D into $g' \cap \operatorname{Int} S$ can be extended to a map of D into an ϵ-subset of Int S. Now cover Int $S - \bigcup G'$ with the interiors of a finite number of round 2-spheres of diameter less than ϵ in Int S. Let G denote the collection consisting of these spheres together with the intersections of the elements of G' with $S \cup$ Int S. Let δ be a Lebesgue number for the covering G of $S \cup$ Int S. Thus for any map f of Bd D into a δ-set in Int S, $f(D)$ must be a subset of some element of G. We have constructed G so that f can be extended to map D into an ϵ-subset of Int S. Thus Int S is 1-ULC, so it follows from a one-sided version of Theorem 6.4.1 that $S \cup$ Int S is a 3-cell.

As indicated in [48, pp. 81-85], the above proof can be combined with Theorem 5.4.2 to prove the following:

THEOREM 6.4.3. *If S is a 2-sphere in E^3, U is an open subset of S, and* Int S *is locally simply connected at each point of U, then S is locally tame from* Int S *at each point of U.*

6.5. *Spheres which can be locally spanned.* Any tame 2-sphere S in E^3 has the property that for any $p \in S$ and any $\epsilon > 0$ there is a tame 2-sphere S' of diameter less than ϵ such that $p \in$ Int S' and $S \cap S'$ is a simple closed curve. Harrold [98] first showed that any 2-sphere with this property is tame. Actually, he did not directly require that the simple closed curve $S \cap S'$ be tame, but his requirement of local peripheral unknottedness implies, with a rather lengthy argument [**100**, Theorem 7], that $S \cap S'$ is tame. So long as this is the case, it is not necessary to require that S' be tame, for Theorem 4.4.5 implies that S' can be assumed to be locally polyhedral except on $S \cap S'$. With $S \cap S'$ tame, it then follows from Theorem 8.1.5 that S' is tame. As we will see below in Theorem 6.5.2, we can remove the requirement that $S \cap S'$ be tame.

THEOREM 6.5.1 (HARROLD [98, THEOREM 5]). *A 2-sphere S in E^3 is tame if for each $p \in S$ and each $\epsilon > 0$ there is a 2-sphere S' such that $S \cap S'$ is a tame simple closed curve, $p \in$* Int S', *and* Diam $S' < \epsilon$.

We say that a 2-sphere S in E^3 can be *locally spanned from the component U* of $E^3 - S$ if for each $\epsilon > 0$ there exist ϵ-disks D and D' such that $p \in$ Int $D \subset S$, Int $D' \subset U$, and Bd D' = Bd D. It is convenient to picture this by considering small bubbles or blisters on a surface.

THEOREM 6.5.2 (BURGESS [48, THEOREM 7]). *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if it can be locally spanned from U.*

ROUGH OUTLINE OF PROOF. We prove this theorem by showing that

U is 1-ULC (Theorem 6.4.1). Let R be a disk and let f be a map of Bd R into a small subset of U. Extend f to map R into a small subset of E^3 so that $f(R) \cap S$ is a subset of a small disk on S. (A Lebesgue number for a covering of S is needed here.) Apply Theorem 4.2.2 to change f so that it maps R into a small subset of $S \cup U$. Now S can be locally spanned from U at each point of $f(R) \cap S$. This can be done so that each spanning disk D' separates the interior of the corresponding disk D on S from $f(\text{Bd } R)$ in $S \cup U$. There exists a finite number of such spanning disks $D_1', D_2', \cdots, D_n'$ such that $f(R) \cap S$ is covered by the interiors of the corresponding disks $D_1, D_2, \cdots, D_n$ on S. Theorem 4.2.2 can be applied successively n times to obtain a map g of R into a small subset of U such that $g|\text{Bd } R = f|\text{Bd } R$ and $g(R) \subset f(R) \cup (\bigcup_{i=1}^n D_i')$. A linking argument is needed at each step to conclude that $g(R)$ does not intersect $\bigcup_{i=1}^n D_i$.

Theorem 6.5.2 has the following two-sided version as a corollary.

THEOREM 6.5.3 (BURGESS [48, THEOREM 8]). *A 2-sphere S in E^3 is tame if it can be locally spanned from each component of $E^3 - S$.*

The proof outlined for Theorem 6.5.2 also shows that U is locally simply connected at each point of S. The following theorem thus follows from a combination of this proof and Theorem 6.4.3.

THEOREM 6.5.4 (BURGESS [48, Theorem 10]). *If S is a 2-sphere in E^3 and V is an open subset of S such that S can be locally spanned from* Int S *at each point of V, then S is locally tame from* Int S *at each point of V.*

There are several ways to generalize the above theorem about 2-spheres that can be locally spanned. First we will discuss some generalizations of Theorem 6.5.3 by permitting $S \cap S'$ to be a continuum rather than requiring it to be a simple closed curve. The following general question in this respect remains unanswered, however. (Cannon has recently answered this question affirmatively.)

Is a 2-sphere S in E^3 tame if for each $p \in S$ and each $\epsilon > 0$ there is a 2-sphere S' such that $p \in \text{Int } S'$, Diam $S' < \epsilon$, and $S \cap S'$ is a continuum?

Eaton [80] has obtained an affirmative answer by imposing an additional requirement on $S \cap S'$.

THEOREM 6.5.5 (EATON [80, THEOREM 2]). *A 2-sphere S in E^3 is tame if for each $p \in S$ and each $\epsilon > 0$ there is a 2-sphere S' such that $p \in$* Int S', Diam $S' < \epsilon$ *and $S \cap S'$ is a continuum that irreducibly separates S.*

In his proof, Eaton needs the irreducibility of $S \cap S'$ to obtain a disk in S' that will serve the function of the spanning disk D' in the proof of Theorem 6.5.2.

A theorem of Loveland [**123**, Theorem 12] can be combined with Cannon's characterization of taming sets (Theorem 8.1.6) to enable us to obtain an affirmative answer to the following two special cases of the above question [**192**].

THEOREM 6.5.6. *A 2-sphere S in E^3 is tame if for each $p \in S$ and each $\epsilon > 0$ there is a 2-sphere S' such that $p \in \operatorname{Int} S'$, $\operatorname{Diam} S' < \epsilon$, and $S \cap S'$ is a tame continuum.*

THEOREM 6.5.7. *A 2-sphere S in E^3 is tame if for each $p \in S$ and each $\epsilon > 0$ there is a tame 2-sphere S' such that $p \in \operatorname{Int} S'$, $\operatorname{Diam} S' < \epsilon$ and $S \cap S'$ is a continuum.*

We now consider some generalizations of Theorems 6.5.2, 6.5.3, and 6.5.4 by using some less restrictive definitions of local spanning. Mainly, this involves various ways to permit the spanning disks to have singularities. One way to do this is to permit the boundary of the spanning disk D' to be different from the boundary of the corresponding disk D on the sphere provided Bd D and Bd D' are required to be homotopically close to each other. This can be stated precisely as follows: We say that a 2-sphere S in E^3 can be *locally spanned in the component* U of $E^3 - S$ if for each $\epsilon > 0$ there exists an ϵ-disk D on S with $p \in \operatorname{Int} D$ such that for any $\delta > 0$ there exist an ϵ-disk D' in U and a map f of Bd D onto Bd D' that moves no point more than a distance δ. Under these conditions, we say further that S can be *locally spanned in U on* Bd D.

The following question remains unanswered:

Is a sphere S tame if it can be locally spanned in each component of $E^3 - S$?

A special case was answered affirmatively in [**48**] where it was required that the local spanning be uniform over S. This was then included in Loveland's [**123**] affirmative answer to the following special case.

THEOREM 6.5.8 (LOVELAND [**123**, THEOREM 3]). *A 2-sphere S in E^3 is tame from a component U of $E^3 - S$ if it can be locally spanned in U on tame simple closed curves.*

Loveland [**126**] has proved some similar theorems with other types of local spanning. Another way to permit the spanning disks to have singularities is to require that certain small simple closed curves on S can be shrunk to a point in a small subset of the complement of S.

The following question remains unanswered, and we suspect that an improved version of Dehn's lemma may be needed to answer it.

Is a 2-sphere S in E^3 tame from a component U of $E^3 - S$ if for each $p \in S$ and each $\epsilon > 0$ there exists an ϵ-disk D on S with $p \in \operatorname{Int} D$ such that Bd D can be shrunk to a point in an ϵ-subset of $U \cup \operatorname{Bd} D$?

Cannon [**55**] obtained an affirmative answer for the special case where Bd D is required to be tame. White [**166**] obtained an affirmative answer with a requirement that each tame simple closed curve in some ϵ-neighborhood of p on S can be shrunk to a point in itself together with an ϵ-subset of U. By using his characterization of taming sets (Theorem 8.1.6), Cannon [**55**] has obtained the following affirmative answer for a special case of the above question:

THEOREM 6.5.9 (CANNON [**55**]). *A 2-sphere S in E^3 is tame from a component U of $E^3 - S$ if for each $p \in S$ and each $\epsilon > 0$ there is a map f of a disk D into an ϵ-subset of $S \cup U$ such that $f(D) - S = f(\operatorname{Int} D)$ and $f(\operatorname{Bd} D)$ is tame and links p in some ϵ-neighborhood of p on S.*

Some further generalizations by Cannon [**55**] permit $f(\operatorname{Bd} D)$ to be replaced by a tame continuum K that can be shrunk to a point in $K \cup U$.

6.6. *Spheres which are pierced with disks.* We say that a 2-sphere S in E^3 is *pierced with the disk D on the arc A* of S if D intersects both components of $E^3 - S$, $D \cap S = A$, and A is a spanning arc of D. (A *spanning arc* of a disk D is an arc that has its end points on Bd D and is otherwise in Int D.) Similarly, we say that S is *pierced with the annulus H on the simple closed curve K* of S if $H \cap S = K$ and the two components of Bd H are in different components of $E^3 - S$.

Each of the wild spheres described in §3 can be pierced with a disk on some arc. We observe that the Fox-Artin sphere (Figure 3) can be pierced with a disk on each of its arcs. We can use a sequence of pairs of horns of the type in the Alexander horned sphere to describe a 2-sphere which contains an arc on which it cannot be pierced with a disk. Bing [**32**] has asked the following question:

Is a 2-sphere S in E^3 tame if it can be pierced with a tame disk on each of its arcs?

An affirmative answer was given by Burgess [**51**] for the special case where each component of $E^3 - S$ is an open 3-cell. Shortly afterward, Eaton [**81**] obtained an affirmative answer to Bing's question. His method of proof involved using the fact that S can be pierced with a tame annulus on each simple closed curve [**51**, p. 326] so that he could apply an argument similar to his proof of Theorem 9.1.1.

THEOREM 6.6.1 (EATON [**81**]). *A 2-sphere S in E^3 is tame if it can be pierced with a tame disk on each of its arcs.*

This theorem has the following consequence:

THEOREM 6.6.2. *A 2-sphere S in E^3 is tame if each of its arcs is tame and S can be pierced on each of its arcs with a disk.*

PROOF. Let A be an arc of S and let D be a disk which pierces S on A. Apply Bing's polyhedral approximation theorem for disks (Theorem 4.6.4) to obtain a disk D' which pierces S on A and is locally polyhedral except on the tame arc A. It follows from [**77**, Theorem 3] that such a disk is tame. Thus S can be pierced with a tame disk on each of its arcs. By Theorem 6.6.1, S is tame.

Eaton's proof of Theorem 6.6.1 is valid to show that a 2-sphere is tame from one side where there is only "half piercing" with tame disks.

THEOREM 6.6.3 (EATON [**81**]). *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if for each arc A of S there is a tame disk D such that $S \cap D = S \cap \operatorname{Bd} D = A$ and $D - A \subset U$.*

Daverman [**66**] has recently extended this theorem to permit D to be singular.

6.7. *Pushing sets away from spheres.* The definition that a 2-sphere S in E^3 is tame from the component U of $E^3 - S$ is equivalent to the following requirement: For each $p \in S$ there exist an open subset V of S containing p and a homeomorphism h of $V \times I$ into E^3 such that h is the identity on V and $h(V \times (0, 1])$ is a subset of U. Thus, according to the definition stated in §6.9, V can be *collared* from U. More generally, we say that a subset K of S *can be pushed into U with a homotopy* if there exists a map f of $K \times I$ into $S \cup U$ such that $f \mid K$ is the identity homeomorphism and $f(K \times (0, 1])$ is a subset of U.

The following theorem is a direct consequence of each of several theorems, including Theorem 6.2.1, and Brown's more general theorem [**46**].

THEOREM 6.7.1. *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if S can be collared from U.*

This theorem is also a special case of the following theorem proved by Hempel.

THEOREM 6.7.2 (HEMPEL [**103**, THEOREM 1]). *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if S can be pushed into U with a homotopy.*

The following theorem can be proved by Hempel's methods and is also a direct consequence of Theorems 6.7.5 and 6.7.6.

THEOREM 6.7.3. *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if each point of S is in an open subset of S that can be pushed into U with a homotopy.*

Another way to generalize Theorem 6.7.1 is the following, where the proof involves showing that S can be locally spanned from *U*.

THEOREM 6.7.4 (BURGESS [**48**, THEOREM 15]). *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if each point of S is an inaccessible point of a Sierpiński curve on S that can be collared from U.*

In this theorem, we do not require that the Sierpiński curve be tame and we do not require that every Sierpiński curve on S can be collared from *U*. We observe that each point of the Fox-Artin sphere S (Figure 3) is in a Sierpiński curve on S that can be collared from Ext S. However, the point where S is wild from Ext S is not an inaccessible point of such a Sierpiński curve.

White [**166**, Corollary 6B] generalized Theorem 6.7.2 with the weakened hypothesis that each Sierpiński curve on S can be pushed into *U* with a homotopy, and Cannon [**55**] obtained the more general theorem where the hypothesis of Theorem 6.7.4 is weakened to permit the Sierpiński curves to be pushed into *U* with a homotopy.

THEOREM 6.7.5 (WHITE [**166**]). *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if each Sierpiński curve on S can be pushed into U with a homotopy.*

THEOREM 6.7.6 (CANNON [**55**]). *A 2-sphere S in E^3 is tame from the component U of $E^3 - S$ if each point of S is an inaccessible point of a Sierpiński curve on S that can be pushed into U with a homotopy.*

Some other theorems about 2-spheres on which arcs or simple closed curves can be collared are discussed in §§6.6 and 6.9.

6.8. *Extending homeomorphisms of spheres and of their complements.* By definition, a 2-sphere S in E^3 is tame if some homeomorphism of a tame 2-sphere onto S can be extended to a homeomorphism of E^3 onto itself. By using Theorem 6.7.2, Hempel [**103**] has shown that this extension can be to a map rather than to a homeomorphism. Some related theorems can be found in [**94**], [**195**].

THEOREM 6.8.1 (HEMPEL [**103**, THEOREM 2]). *If S is a tame 2-sphere and f is a map of E^3 onto itself such that $f \mid S$ is a homeomorphism and $f(E^3 - S) = E^3 - f(S)$, then $f(S)$ is tame.*

Hempel observed, with a corollary, that his method of proof permitted the weaker hypothesis where f maps E^3 into itself provided the two components of $E^3 - S$ are mapped into different components of $E^3 - f(S)$.

Armentrout [**10**] has presented a partial converse of Theorem 6.8.1 with a proof of the following theorem. An alternative proof has been given by Price [**152**].

THEOREM 6.8.2 (ARMENTROUT [**10**, THEOREM 1]). *If S is a 2-sphere in S^3 and f is a map of S^3 onto itself such that $f \mid S$ is a homeomorphism, $f(E^3 - S) = E^3 - f(S)$, $f(S)$ is tame, and only countably many points have nondegenerate inverses under f, then S is tame.*

Recent work by Armentrout [**177**] permits the omission of the requirement that only countably many points have nondegenerate inverses, provided it is required that the inverses of points be cellular. It has been recently announced by Boyd [**180**] that for any 2-sphere S in S^3 there is a monotone map f of S^3 onto itself such that $f \mid S$ is a homeomorphism, $f(S)$ is tame, and $f(E^3 - S) = E^3 - f(S)$.

Again by definition, a 2-sphere S in E^3 is tame if there is a homeomorphism of the complement of S onto the complement of a tame sphere which can be extended to a homeomorphism of E^3 onto itself. Loveland [**130**] has shown that tameness is implied without requiring that such a homeomorphism be extended to all of E^3.

THEOREM 6.8.3 (LOVELAND [**130**, COROLLARY 4]). *A 2-sphere S in E^3 is tame if for each point p of S there exists a homeomorphism of $(E^3 - S) \cup p$ onto the complement of a round 2-sphere S' together with a point of S'.*

Loveland [**130**] observed that this hypothesis can be weakened to require that, for some closed 0-dimensional subset K of S, such a homeomorphism should exist only for each point $p \in S - K$.

There is a homeomorphism of the complement of the Fox-Artin sphere S (Figure 3) onto the complement of a round sphere S'. However, Loveland [**130**] made the interesting observation that his methods show that there is no point p of S for which such a homeomorphism can be extended to carry p onto a point of S'.

It follows from the definition of tameness for a 2-sphere in E^3 that every homeomorphism of a tame 2-sphere onto a round 2-sphere can be extended to a homeomorphism of E^3 onto itself. (See Theorem

4.4.4.) The following question remains unanswered.

Is a 2-sphere S in E^3 tame if every homeomorphism of S onto itself can be extended to a homeomorphism of E^3 onto itself?

A related question is the following:

Is a 2-sphere S in E^3 tame if for any two points p and q of S there is a homeomorphism of E^3 onto itself that carries S onto itself and p onto q [32], [88]?

The wild 2-spheres described by Bing [28] and Gillman [89] can be pierced with a tame disk, so it follows from Theorem 6.6.1 that they do not offer negative answers to the first of these two questions. It is not known whether either of these spheres has the property required in the latter question.

6.9. *Spheres which are almost tame.* We consider here some spheres which are locally tame except on a totally disconnected set. This, of course, includes spheres which are tame except on a finite set. It follows from work by Harrold and Moise [**101**] and Cantrell [**62**] that 2-spheres in S^3 with only one wild point have complements like the complement of a tame 2-sphere.

THEOREM 6.9.1. (HARROLD AND MOISE [**101**]). *If the 2-sphere S in S^3 is locally tame except at a point p, then each component of $S^3 - S$ is simply connected and the closure of one of them is a 3-cell. Furthermore, each component of $S^3 - S$ is an open 3-cell (Cantrell* [**62**]).

Following Brown's definitions [**45**], we say that a subset K of a 2-sphere S in E^3 can be *collared* from the component U of $E^3 - S$ if there exists a homeomorphism h of $K \times I$ into $S \cup U$ such that h is the identity on K and $h(K \times (0, 1)) \subset U$. We define S to be *locally peripherally collared* from U if for each $p \in S$ and each $\epsilon > 0$ there is an ϵ-disk D on S with $p \in \text{Int } D$ such that Bd D can be collared from U.

Thus the requirements of this definition are slightly weaker than those in the definition that S can be locally spanned from U. Any 2-sphere S in S^3 that is tame except on a totally disconnected set is locally peripherally collared from each component of $S^3 - S$.

THEOREM 6.9.2 (BURGESS [**51**, COROLLARY 1]). *If the 2-sphere S in S^3 can be locally peripherally collared from each component of $S^3 - S$ and each of these components is an open 3-cell, then S is locally tame except at two points; that is, S has at most two wild points.*

This theorem, together with the observation preceding it, implies the following:

THEOREM 6.9.3 (BURGESS [**51**, COROLLARY 2]). *If the set W of wild points of the 2-sphere S in S^3 is totally disconnected and each component of $S^3 - S$ is an open 3-cell, then W consists of at most two points.*

Daverman [**68**] extended these theorems where Bd D, in the definition of local peripheral collaring, is permitted to be any continuum one of whose complementary domains on S is an ϵ-set containing p. Another extension has been given by Loveland [**126**, Theorem 7].

All of the theorems of this section can be modified for local tameness, except at one point, from a component U of $S^3 - S$. The hypothesis of Theorem 6.9.2 has been weakened by Daverman [**69**] and Taylor [**165**] to permit S to have more than two wild points. Rosen [**153**] and Daverman [**69**] have obtained some related results for 2-manifolds in a 3-manifold. Loveland [**124**], [**125**] has extended Theorems 6.9.2 and 6.9.3 to cellular subsets of 2-spheres.

7. **Tame subsets of spheres in E^3.** Bing's discovery (see Theorem 5.1.4) that each disk in E^3, no matter how wild, contains many tame arcs and his subsequent exploitation of that fact in papers such as [**33**], [**34**], [**36**], and [**40**] has aroused much interest in tame subsets of spheres. This interest has centered especially on problems suggested by the following two general questions:

What properties characterize the tame subsets of a 2-sphere in E^3?

What information about the embedding of a 2-sphere in E^3 can be derived from knowledge concerning its tame subsets?

In this section we consider problems on characterizations and near-characterizations of tame subsets of 2-spheres suggested by the first of these two questions. In §8 we discuss those suggested by the second of these questions. There is, of course, a great deal of overlap between problems in these two areas. Some conditions which imply tameness of finite graphs, which may not be subsets of 2-spheres in E^3, can be found in [**137**]. Discussions of cellular subsets of 2-spheres, where these sets may not be tame, are included in [**140**], [**124**], [**164**].

7.1. *Side approximations missing a set.* The most useful property yet suggested in connection with tame subsets of a 2-sphere S in E^3 is that of missing subsets of S, in a rather strong way, with the approximating spheres in Bing's side approximation theorem (Theorem 4.6.2). More specifically, for a set X on S, there should be approximating spheres so that the small disks D_i (see Theorems 4.6.2 and 4.6.3) do not intersect X. This property handles with complete adequacy only those tame subsets of S which have no degenerate component. But as we shall see in this section and in §8, such sets are precisely those whose tameness is directly related to the embedding of S in E^3.

DEFINITION. We say that a 2-sphere S in E^3 can be *side approximated missing a subset* X of S if for each $\epsilon > 0$ there exist a finite collection of disjoint disks $D_1, D_2, \cdots, D_n$ on S and homeomorphisms h and h' of S onto approximating spheres $h(S)$ and $h'(S)$ which satisfy the requirements in the conclusion of Theorem 4.6.3 such that $X \subset S - \bigcup_{i=1}^{n} D_i$.

In this definition, we have used side approximations of the form developed by Lister [**121**], although Gillman [**86**] used side approximations of the form developed by Bing [**35**] as specified in Theorem 4.6.2. However, Lister [**121**, Theorem 2] showed that X can be missed by side approximations of the type required by Bing in Theorem 4.6.2 if and only if they can be missed by side approximations of the type required by Lister in Theorem 4.6.3. Thus, we will use whichever type of side approximation that we find most convenient. Gillman [**86**] used Property ($*$, X, S) to mean that the 2-sphere S can be side approximated missing X, and we use this notation for convenience in some places. In some instances, we will use a one-sided version of this property.

Gillman characterized tameness of arcs on a 2-sphere as follows:

THEOREM 7.1.1 (GILLMAN [**86**, THEOREM 10]). *An arc A on a 2-sphere in E^3 is tame if and only if S can be side approximated missing A.*

It is apparent that Gillman suspected that Property ($*$, X, S) could be used to characterize a larger class of tame subsets of spheres in E^3. If we know that a 2-sphere S can be side approximated missing a closed set X, we can perform the first step in the outline of the proof of Theorem 5.1.1 so that $X \subset S - \bigcup_{i=1}^{\infty} D_i$. However, in the second step, we adjust S to a 2-sphere S_1 so that $X \subset S \cap S_1$. We could continue with the outline of Theorem 5.1.1 so that X would be a subset of the tame 2-sphere S' provided we knew that S_1 could be side approximated missing X. Gillman [**86**] formulated the problem suggested here as follows:

CONJECTURE (GILLMAN [**86**, p. 467]). *If S and S_1 are 2-spheres in E^3, U is an open subset of both S and S_1, and X is a closed set in U such that S can be side approximated missing X, then S_1 can be side approximated missing X.*

Loveland [**122**] proved this conjecture and several basic theorems about tame subsets of spheres.

THEOREM 7.1.2 (LOVELAND [**122**, THEOREM 3]). *Gillman's conjecture is true.*

As indicated above, this led to the following:

THEOREM 7.1.3 (LOVELAND [**122**, THEOREM 6]). *A closed subset X of a 2-sphere S in E^3 is tame if S can be side approximated missing X.*

He later extended his methods to a one-sided version of the above theorem.

THEOREM 7.1.4 (LOVELAND [**127**, THEOREM 1]). *A closed proper subset X of a 2-sphere S in E^3 is a subset of the boundary of a 3-cell if S can be side approximated, missing X, from one of the components of $E^3 - S$.*

Loveland showed that Property $(*, X, S)$ is a characteristic not only of tame arcs but also of tame Sierpiński curves and tame Sierpiński-like curves [**122**, Theorem 6], and he proved that the property is preserved under finite unions [**122**, Theorem 21].

The next four theorems summarize further work on the relation of tameness of X with Property $(*, X, S)$. In each of these four theorems, let S be a 2-sphere in E^3 and X a closed subset of S.

THEOREM 7.1.5 (CANNON [**54**, THEOREM 1.1 AND COROLLARY 5.4]). *If X has no degenerate component, then X is tame if and only if S can be side approximated missing X.*

THEOREM 7.1.6 (CANNON [**54**, THEOREM 5.3]). *If $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed and satisfies $(*, X_i, S)$, then X satisfies $(*, X, S)$.*

THEOREM 7.1.7 (CANNON [**58**, THEOREMS 2.1 AND 2.4]). *S can be side approximated missing X if and only if the sets $(S \cup \operatorname{Ext} S) - X$ and $(S \cup \operatorname{Int} S) - X$ are 1-ULC.*

THEOREM 7.1.8 (LOVELAND [**122**, THEOREM 14]). *If U is an open subset of S which is locally polyhedral modulo X and S can be side approximated missing X, then U is locally tame.*

Some further theorems about Property $(*, X, S)$ will be discussed in §8.

We wish to observe that Gillman's conjecture, stated above, is an immediate consequence of Theorem 7.1.7. Furthermore, as indicated in [**58**, Theorem 0.3], Theorem 7.1.7 can be proved rather easily with Lister's form (Theorem 4.6.3) of the side approximation theorem.

7.2. *Characterizations of tame subsets of spheres.* The general characterization problem yields to a combination of the results of §7.1, Bing's work on tame 0-dimensional sets [**30**], the results of §8, and the methods of Bing's paper on pushing a 2-sphere into its complement [**36**]. (See Theorem 5.7.1.)

As a basic characterization of tame subsets of spheres, we state the following:

THEOREM 7.2.1 (CANNON [**59**]). *Let X be a closed subset of a 2-sphere S in E^3. Then X is tame if and only if both of the following two conditions are satisfied:*

(1) *If i is a positive integer and X_i is the union of components of X having diameter at least $1/i$, then X_i is tame.*

(2) *If p is a degenerate component of X and $\epsilon > 0$, then there is a 2-sphere R in $E^3 - X$ of diameter less than ϵ such that $p \in \operatorname{Int} R$.*

This theorem can be used to give some other characterizations of tame subsets of 2-spheres. As an outgrowth of Bing's characterization of tame 2-spheres in E^3 as those with complements that are 1-ULC (Theorem 6.4.1) we state the following definition: If X is a closed subset of a 2-sphere S in E^3 and for each $\epsilon > 0$ there exists a $\delta > 0$ such that any δ-loop in $E^3 - S$ bounds a singular ϵ-disk in $E^3 - X$, then we say that $E^3 - S$ is 1-ULC in $E^3 - X$. In some papers [**122**], [**123**], the property described here has been called Property (A, X, S). If for each $\epsilon > 0$ there is a $\delta > 0$ such that every unknotted simple closed curve of diameter less than δ in $E^3 - S$ bounds a singular ϵ-disk in $E^3 - X$, then we say that $E^3 - S$ is *weakly* 1-ULC in $E^3 - X$. In some papers, this property has been called Property (C, X, S) [**127**].

The following definition is suggested by Bing's characterization of tame spheres as those with homeomorphic approximations in their complementary domains (Theorem 6.2.1): If X is a closed subset of a 2-sphere S in E^3 and for each $\epsilon > 0$ there exist ϵ-homeomorphisms h_1 and h_2 of S into E^3 such that $X \subset \operatorname{Int} h_1(S) \cap \operatorname{Ext} h_2(S)$, then we say that S *can be wedged between homeomorphic approximations of* S. In some papers [**122**], this property has been called Property (B, X, S). It should be emphasized that it is not required that $h_1(S)$ and $h_2(S)$ be side approximations satisfying the requirements of Theorem 4.6.3, although this is implied in some special cases. (See Theorem 7.2.2.)

Hosay [**110**] announced that a closed subset X of a 2-sphere S is tame provided that the diameters of the components of X have a positive lower bound and either $E^3 - S$ is 1-ULC in $E^3 - X$ or X can be wedged between homeomorphic approximations of S. Loveland [**122**] proved the following:

THEOREM 7.2.2 (LOVELAND [**122**, THEOREMS 8, 9, AND 10]). *If X is a closed subset of a 2-sphere S in E^3 and the diameters of the components of X have a positive lower bound, then the following three requirements are equivalent*:

(1) *S can be side approximated missing X.*
(2) *$E^3 - S$ is 1-ULC in $E^3 - X$.*
(3) *X can be wedged between homeomorphic approximations of S.*

This theorem, together with Theorem 7.1.3, implies the above results announced by Hosay. Cannon [**59**] recently used Theorem 7.2.1 to prove the following without placing any restrictions on the components of X.

THEOREM 7.2.3 (CANNON [**59**]). *If X is a closed subset of a 2-sphere S in E^3, then the following four requirements are equivalent*:

(1) *X is tame.*
(2) *$E^3 - S$ is 1-ULC in $E^3 - X$.*
(3) *$E^3 - S$ is weakly 1-ULC in $E^3 - X$.*
(4) *X can be wedged between homeomorphic approximations of S.*

Most of the theorems for tame 2-spheres in §§6.6 and 6.7 can be adapted to subsets of 2-spheres.

In extending these results to 2-manifolds embedded in a 3-manifold, the following theorem is useful.

THEOREM 7.2.4 (CANNON [**59**]). *Suppose S is a 2-sphere in E^3, X is a tame subset of S, and $f: S \to [0, \infty)$ is a continuous nonnegative real-valued function on S. Then there is a homeomorphism h from S into E^3 such that:*

(1) *$\rho(x, h(x)) \leqq f(x)$ for each $x \in S$.*
(2) *If $x \in S$ and $f(x) > 0$, then $h(S)$ is locally tame at $h(x)$.*
(3) *$X \subset h(S)$.*

Lister [**121**], [**191**] has proved some theorems about tame sets on the boundary of a crumpled cube C which remain tame under any homeomorphism $h: C \to E^3$ where $h(\mathrm{Bd}\, C)$ is tame from $E^3 - h(C)$.

7.3. *Sequentially 1-ULC subsets of spheres.* Gillman [**87**] showed that a simple closed curve K on a 2-sphere in E^3 is tame if K is the intersection of a decreasing sequence of solid tori that satisfy some uniformity requirements. This can be described with the following definition and theorem.

DEFINITION. A sequence $\{H_i\}$ of tori is *sequentially* 1-ULC if for each $\epsilon > 0$ there exists a $\delta > 0$ and an integer k such that for $n > k$ a simple closed curve J on H_n bounds an ϵ-disk on H_n provided J is of diameter less than δ and bounds a disk on H_n.

THEOREM 7.3.1 (GILLMAN [87, THEOREM 1]). *If A is an arc on a 2-sphere in E^3 and there exist a simple closed curve K in E^3 and a decreasing sequence of solid tori $\{T_i\}$ such that $A \subset K$, $K = \bigcap_{i=1}^{\infty} \operatorname{Int} T_i$, and $\{\operatorname{Bd} T_i\}$ is sequentially 1-ULC, then A is tame.*

Gillman [87] observed that it follows from this theorem that no subarc of the simple closed curve described by Bing in [22] lies on a 2-sphere. (See the comment on this example in §5.2.) The same example shows that it is necessary in Theorem 7.3.2, which is a corollary to Theorem 7.3.1, that K be a subset of a 2-sphere.

THEOREM 7.3.2 (GILLMAN [87, THEOREM 1]). *If K is a simple closed curve on a 2-sphere S in E^3 and $\{T_i\}$ is a decreasing sequence of solid tori such that $K = \bigcap_{i=1}^{\infty} \operatorname{Int} T_i$ and $\{\operatorname{Bd} T_i\}$ is sequentially 1-ULC, then K is tame.*

Burgess and Loveland [53, Theorem 4] proved a similar theorem where K is replaced by a continuum that separates S and the definition of sequentially 1-ULC is weakened to permit small simple closed curves on T_n to be shrunk to points in ϵ-subsets of $E^3 - K$. Also, some further generalizations of Theorem 7.3.2, with weaker definitions of sequentially 1-ULC, are included in [53]. One of these was used by Burgess and Cannon [52] to give the following geometrically appealing characterization of tame tree-like continua on 2-spheres.

THEOREM 7.3.3 (BURGESS AND CANNON [52, p. 400]). *A tree-like continuum X on a 2-sphere S in E^3 is tame if and only if X can be described with trees of 3-cells.*

This means that even though each arc in E^3 can be described with a chain of open sets in E^3, no wild arc on a 2-sphere in E^3 can be described with trees of cubes in E^3. However, there do exist wild arcs in E^3 which can be described with trees, or even chains, of 3-cells in E^3. There is such an arc on a simple closed curve in which there is a sequence of overhand knots converging to a point.

Burgess and Cannon [52] have combined the notions in Theorem 7.3.3 and in the definition of sequentially 1-ULC to obtain the following characterization of those tame subsets of a 2-sphere S that do not separate S and have no degenerate component. (A complete description of the definitions can be found in [52].)

THEOREM 7.3.4 (BURGESS AND CANNON [52, THEOREM 1]). *Suppose K is a closed subset of a 2-sphere S in E^3 such that K does not separate S and the components of S are nondegenerate. Then in order that K should be tame it is necessary and sufficient that there exist a sequence*

$\{M_i\}$ of 3-manifolds with boundary such that:

(1) *K is uniformly described by $\{M_i\}$.*

(2) *Each component of each M_i is a polyhedral 3-cell.*

(3) *$\{M_i\}$ is sequentially 1-ULC.*

Recent work by Cannon (Theorem 7.2.3) shows that it is not necessary to require that the components of K be nondegenerate.

8. **Taming sets for spheres.** Moise [**145**, Theorem 9.3] proved that a 2-sphere in E^3 is tame if it is the union of two tame disks with a common boundary. His methods showed that a 2-sphere S in E^3 is tame if there exists a tame simple closed curve K on S such that S is locally tame at each point of $S - K$. This led to the problem of determining which subsets of 2-spheres share this taming property with tame simple closed curves.

8.1. *Characterization of taming sets.* We define a *taming set for 2-spheres* to be a closed set X in E^3 having the following two properties:

(1) X is a subset of some 2-sphere in E^3.

(2) If S is a 2-sphere in E^3 which contains X and is tame modulo X, then S is tame.

It is an immediate consequence of Theorem 4.6.5 that every such taming set is tame.

A similar definition could, of course, be given for "taming sets for disks." Doyle and Hocking [**77**] proved that a tame arc is a taming set for disks. (In Theorems 8.1.1 and 8.1.2, we use a version of Theorem 4.4.5 for disks in E^3 to replace "locally polyhedral" with "locally tame" in [**77**, Theorems 2 and 3].)

THEOREM 8.1.1 (DOYLE AND HOCKING [**77**, THEOREM 3]). *If D is a disk in E^3, A is a tame arc in D, and D is locally tame at each point of $D - A$, then D is tame.*

This theorem can be combined with the fact, mentioned above, that a tame simple closed curve is a taming set for 2-spheres to prove the following:

THEOREM 8.1.2 (DOYLE AND HOCKING [**77**, THEOREM 2]). *Every tame arc is a taming set for 2-spheres in E^3.*

Combined work by Gillman [**86**] and Loveland [**122**] relates taming sets for 2-spheres to sets which can be missed with side approximations of spheres.

THEOREM 8.1.3 (GILLMAN [**86**, p. 462] AND LOVELAND [**122**, THEOREM 16]). *A closed subset X of a 2-sphere in E^3 is a taming set for 2-*

spheres if and only if for each 2-sphere S in E^3 containing X, the sphere S can be side approximated missing X.

THEOREM 8.1.4 (LOVELAND [**122**, THEOREM 19]). *If $\epsilon > 0$ and X is a closed subset of a 2-sphere S in E^3 such that S can be side approximated missing X, then there exists a null sequence $\{D_i\}$ of ϵ-disks in $S - X$ such that the continuum $S - \bigcup_{i=1}^{\infty}$ Int D_i is a taming set for 2-spheres.*

With the following two theorems, we summarize chronologically recent work toward characterizing taming sets for 2-spheres in E^3.

THEOREM 8.1.5. *A closed subset X of a 2-sphere in E^3 is a taming set for 2-spheres if any one of the following properties is satisfied:*

(1) *X is a tame finite graph (Doyle and Hocking* [**77**, *Theorem* 3]).

(2) *X is a tame Sierpiński curve (Bing* [**36**]).

(3) *X is a nondegenerate tame tree-like continuum (Burgess and Cannon* [**52**]).

(4) *X is a nondegenerate, tame locally connected continuum (Cannon* [**56**]).

(5) *X is the union of countably many taming sets for 2-spheres. (This result was proved implicitly by Bing in* [**40**, *Theorem* 3.1] *and it is a corollary to* [**58**, *Theorem* 0.4].)

Finally, Cannon [**54**] characterized taming sets for 2-spheres as follows:

THEOREM 8.1.6 (CANNON [**54**]). *A closed subset X of a 2-sphere in E^3 is a taming set for 2-spheres if and only if X is tame and has no degenerate component.*

The proof of this theorem first involves a lengthy argument to show that a 2-sphere S in E^3 can be side approximated missing a tame compact subset X that has no degenerate component. (Observe how this is related to Theorem 7.1.8.) Then for any tame compact set X in E^3 with a point p as a component, it is shown that a construction similar to the one used by Fox and Artin (Figure 3) can be used to describe a 2-sphere which contains X and is tame modulo X but which is wild at p.

The proof of Theorem 5.2.2 illustrates how taming sets can be used to study the embeddings of surfaces in E^3. Theorems 8.1.3, 8.1.5 (5), and 8.1.6 are the major connections between tame subsets of a 2-sphere S in E^3 and the embeddings of S in E^3. They enable one to answer, for example, questions of the following type:

For what types of closed tame subsets X and Y of a 2-sphere in E^3 can it be concluded that $X \cup Y$ is tame?

If X and Y have no degenerate component, then they are taming sets by Theorem 8.1.6. It follows from (5) of Theorem 8.1.5 that $X \cup Y$ is a taming set and thus is tame. However, if A is an arc in E^3 which is locally tame at each point except one end point p where it is wild (Figure 3), then A is a subset of a 2-sphere and is the union of two compact tame sets X and Y with $p \in X \cap Y$. Two such sets X and Y can be identified as follows. Let $p_1, p_2, \cdots$ be an increasing sequence of points of A converging to p such that p_1 is the other end point of A. For each i, let $X = p \cup (\bigcup_{i=1}^{\infty} A_{2i-1})$ and $Y = p \cup (\bigcup_{i=1}^{\infty} A_{2i})$, where A_n denotes the subarc of A with end points p_n and p_{n+1}. It is easy to see that X and Y are closed tame sets whose union is the wild arc A.

Some of the theorems discussed in §§6 and 7 are related to the properties of taming sets discussed in this section. Other illustrations of the use of taming sets can be found in [**36**], [**79**], [**113**].

8.2. *$*$-taming sets.* Let C be a crumpled cube in S^3, C^* the complementary crumpled cube $S^3 - \text{Int } C$, and p a point of the 2-sphere Bd C ($=$ Bd C^*). We restate parts of Theorems 5.2.2 and 5.2.11 from §5.2 in new forms for comparison.

THEOREM 8.2.1. *The point p is a piercing point of C^* if p lies on a tame arc in* Bd C. *(See Theorem* 5.2.2.)

THEOREM 8.2.2. *The point p is a piercing point of C^* if p lies on a tame arc which except for p is a subset of* Int C. *(See Theorem* 5.2.11.)

In light of these two theorems, it is natural to ask the following questions.

(1) Is it necessary that the tame arc of Theorem 8.2.1 lie entirely in Bd C and that the tame arc of Theorem 8.2.2 lie except for p in Int C? More specifically, in order that p be a piercing point of C^*, is it sufficient that p lie on a tame arc in C?

We shall see that the latter question can be answered affirmatively and that Theorems 8.2.1 and 8.2.2 are just two of many theorems which can be unified by a generalization of the concept of taming set.

DEFINITION. A *$*$-taming set* X in S^3 is a compact set having the following property: If C is a crumpled cube in S^3, $X \subset C$, and Bd C is locally tame at each point of Bd $C - X$, then C^* is a 3-cell. (We emphasize that X is not required to be a subset of Bd C.) Speaking very loosely, a $*$-taming set in C "tames" C^*. The theory of $*$-taming sets is developed in [**57**], and we survey a number of these results. A very simple one is the following:

THEOREM 8.2.3. *A closed subset X of a 2-sphere in S^3 is a taming set if it is a $*$-taming set.*

PROOF. Let S be a 2-sphere in S^3 which contains X and is tame modulo X, and let C and C^* denote the complementary crumpled cubes bounded by S. Since $X \subset C$, $\operatorname{Bd} C = S$ is tame modulo X, and X is a $*$-taming set, we conclude that C^* is a 3-cell. Since X is also a subset of C^*, a corresponding argument shows that C is a 3-cell. Hence, S is tame, and thus X is a taming set.

The converse statement that a taming set is a $*$-taming set is much more difficult and is included in the following result.

THEOREM 8.2.4 (CANNON [**57**]). *A compact set X in S^3 is a $*$-taming set if it satisfies any of the following conditions:*

(1) *X has no degenerate components and lies on a tame 2-sphere in S^3 (i.e., X is a taming set).*

(2) *X is a crumpled cube and the complementary crumpled cube X^* is a 3-cell. (This is the content of* [**49**, *Theorem* 2].)

(3) *X is a locally peripherally unknotted arc. (See* [**100**].)

(4) *X has no degenerate components and can be uniformly defined by cubes in S^3 which are sequentially 1-ULC. (See* [**52**].)

(5) *X is a countable union of $*$-taming sets.*

All $*$-taming sets have nice properties relative to different embeddings of crumpled cubes.

THEOREM 8.2.5 (CANNON [**57**]). *If C is a crumpled cube in S^3, X is a $*$-taming set contained in C, and $h : C \to S^3$ is an embedding, then the following are equivalent:*

(1) *$h(C)^* - h(X)$ is 1-ULC (where $h(C)^* = S^3 - \operatorname{Int} h(C)$).*

(2) *X and $h(X)$ are equivalently embedded in S^3.*

(3) *$h(X)$ is a $*$-taming set.*

Theorems 8.2.3, 8.2.4, and 8.2.5 together unify and generalize many of the known theorems on tame subsets of spheres. We refer the reader to [**57**] for a further discussion of this fact, and we restrict ourselves here with two examples of it.

(1) Theorems 8.2.1 and 8.2.2 can be unified by the following:

THEOREM 8.2.6 (CANNON [**57**]). *If C and C^* are complementary crumpled cubes in S^3 and $p \in \operatorname{Bd} C$ $(= \operatorname{Bd} C^*)$, then p is a piercing point of C^* if and only if p lies on a $*$-taming set in C.*

(2) Consider the following theorem which appears as Lemma 4 in [**139**].

THEOREM 8.2.7. *Let K be a crumpled cube in S^3, and p a piercing point of the crumpled cube $K^* = S^3 - \operatorname{Int} K$. Suppose A is an arc in K having p as an end point, such that $A \cap S = p$, where $S = \operatorname{Bd} K$. If there exists a homeomorphism $h : K \to S^3$ such that $h(A)$ is tame, then A is tame.*

Reformulated in terms of $*$-taming sets this might become, for example, the following theorem (recall that p is a piercing point of K^* if and only if $K^* - p$ is 1-ULC (Theorem 5.2.5)).

THEOREM 8.2.8 (CANNON [57]). *Let K be a crumpled cube in S^3, and P a closed subset of the sphere $S = \operatorname{Bd} K$ such that $K^* - P$ is 1-ULC, where K^* is the crumpled cube $S^3 - \operatorname{Int} K$. Suppose that A is a compact subset of K having no degenerate component such that $A \cap S = P$. If there exists a homeomorphism $h : K \to S^3$ such that $h(A)$ is a subset of a tame 2-sphere, then A is tame.*

PROOF. By Theorem 8.2.4, $h(A)$ is a $*$-taming set. By Theorem 8.2.5, A and $h(A)$ are equivalently embedded since $K^* - A = K^* - P$ is 1-ULC.

Combined with the characterization of tame subsets of 2-spheres in Theorem 7.2.3, this theorem yields the following further generalization of Theorem 8.2.7.

THEOREM 8.2.9 (CANNON [59]). *Let K be a crumpled cube in S^3, and P a closed subset of the sphere $S = \operatorname{Bd} K$ such that $K^* - P$ is 1-ULC, where K^* is the crumpled cube $S^3 - \operatorname{Int} K$. Suppose that A is a compact subset of K which lies on a 2-sphere in S^3 such that $A \cap S = P$. If there exists a homeomorphism $h : K \to S^3$ such that $h(A)$ is tame, then A is tame.*

The following question remains unanswered.

Can the hypothesis that A lies on a 2-sphere in S^3 be removed from the statement of Theorem 8.2.9? That is, does it follow from the other hypotheses that A lies on a 2-sphere in S^3? The difficulty arises in the case where A has both degenerate and nondegenerate components.

The first theorems that were unquestionably of the $*$-taming set type appeared in two papers by Burgess [47], [49]. He proved, for example, as indicated in the statement of Theorem 8.2.4, that a crumpled cube K is a $*$-taming set if its complementary crumpled cube K^* is a 3-cell. Theorems related to those of Burgess were announced by Hosay [108]. Some consequences of the theory of $*$-taming sets are included in [57], [183], [184].

9. **Geometric properties which imply tameness of spheres.** In this section we discuss several geometric properties which imply that a 2-sphere in E^3 is tame but which are not possessed by every tame 2-sphere in E^3. The theorem by Alexander, discussed in §4 as Theorem 4.4.1, that every polyhedral 2-sphere in E^3 is tame should be included in this class of theorems. We could formulate characterizations by observing, in each case, that a 2-sphere in E^3 is tame if and only if there is a homeomorphism h of E^3 onto itself such that $h(S)$ has the geometric property in question.

9.1. *Spheres with connected horizontal sections.* When Alexander [3] indicated, in 1924, a way to show that polyhedral 2-spheres in E^3 are tame, he suggested that a 2-sphere S in E^3 should be tame if each of its horizontal sections is either a simple closed curve or a point. His method, which involved using the logarithmic potential function, implied the existence of a homeomorphism h of E^3 onto itself that carries S onto a round sphere and each horizontal plane onto itself. While there is such a homeomorphism for the case where S is polyhedral, Bing [38] described a nonpolyhedral sphere, in 1962, for which there is no such homeomorphism. In 1968, both Eaton [79] and Hosay [111] presented proofs of the following theorem, thus solving the problem suggested by Alexander. Their proofs depended upon Theorem 6.4.1, so indirectly upon Dehn's lemma (Theorem 4.5.1), the side approximation theorem for spheres (Theorem 4.6.2), and Bing's characterization of tame spheres (Theorem 6.2.1).

THEOREM 9.1.1 (EATON [79] AND HOSAY [111]). *A 2-sphere in E^3 is tame if each of its horizontal sections is a simple closed curve or a point.*

The methods in [79] and [111] lead to a proof of the following:

THEOREM 9.1.2. *A 2-sphere S in E^3 is tame if for some two points p_1 and p_2 of S every plane containing p_1 and p_2 has a simple closed curve as its intersection with S.*

PROOF. Eaton's and Hosay's methods of proving Theorem 9.1.1 can be used to show that S is locally tame except at p_1 and p_2. But since $p_1 \cup p_2$ is a subset of a tame simple closed curve on S, it follows from Theorem 8.1.5 that S is tame.

Loveland [128] noticed that, in Eaton's proof, a countable number of horizontal sections could be permitted to be continua other than simple closed curves. This required knowing that every nondegenerate tame continuum is a taming set for 2-spheres in E^3 (Theorem 8.1.6). Only a countable number of horizontal sections are different from simple closed curves provided each horizontal section is a locally connected continuum, for there do not exist uncountably many disjoint locally connected continua on S that are neither points, arcs, nor simple closed curves [**148**, Theorem 85, p. 223]. Separation properties require that no more than two horizontal sections, the highest one and the lowest one, can be points or arcs.

THEOREM 9.1.3 (LOVELAND [**128**]). *A 2-sphere in E^3 is tame if each of its horizontal sections is a locally connected continuum.*

Jensen [**113**] has recently proved a further generalization of Theorem 9.1.1. He has adapted Hosay's methods and used Cannon's characterization of taming sets (Theorem 8.1.6) to remove Loveland's requirement that the horizontal sections be locally connected.

THEOREM 9.1.4 (JENSEN [**113**]). *A 2-sphere in E^3 is tame if each of its horizontal sections is connected.*

Perhaps the methods of proving this theorem will lead to an answer to the following question.

What is the greatest integer n such that a 2-sphere in E^3 is tame if each of its horizontal sections has not more than n components?

We observe that $n < 5$, as the Fox-Artin sphere (Figure 3) can be described so that each horizontal section has at most five components. (By using the development of $*$-taming sets discussed in §8.2, Cannon [**184**] has recently answered the above question by showing that $n = 4$. He has further shown [**183**] that a closed subset X of a 2-sphere in E^3 is tame if no horizontal section of X has a degenerate component.)

9.2. *Spheres with tangent 3-cells at each point.* It is a fact, though difficult to prove, that a 2-sphere S can be pierced with a tame arc at each of its points if and only if for each point p of S there exist 3-cells K_1 and K_2 such that (1) $K_1 \cap K_2 = p$, (2) Bd K_1 and Bd K_2 are tame, and (3) $K_1 - p$ and $K_2 - p$ are in different components of $E^3 - S$ [**139**]. (The difficulty arises from the fact that every arc which lies in $K_1 \cup K_2$ and intersects both $K_1 - p$ and $K_2 - p$ may be wild.) As there exist wild 2-spheres which can be pierced with tame arcs at every point (see §3), we need to impose some further requirements on K_1 and K_2 in order to conclude that S is tame. The following question [**38**, p. 358], [**82**, p. 994] remains unanswered.

Is a 2-sphere S tame in E^3 if for each point p of S there exist two round balls B_1 and B_2 such that $B_1 \cap B_2 = p$ and $B_1 - p$ and $B_2 - p$ are in different components of $E^3 - S$?

By using Theorem 6.5.3, Griffith [**93**] has obtained an affirmative answer to the special case of this question where the tangent balls are required to be uniformly large over S; that is, for some $\delta > 0$, each pair of tangent balls B_1 and B_2 are required to be of diameter δ. (Loveland [192] recently answered the above question affirmatively. Bothe [179] answered a more general question where B_1 and B_2 are cones.)

The above question can be generalized as follows:

Is a 2-sphere S in E^3 tame if for each point p of S there exist polyhedral 3-cells K_1 and K_2 such that $K_1 \cap K_2 = p$ and $B_1 - p$ and $B_2 - p$ are in different components of $E^3 - S$?

(Cannon [183] has recently answered a more general question by showing that a crumpled cube C in E^3 is a 3-cell provided there exist a countable number of directions in E^3 such that each point of Bd C is a subset of a straight line interval which has one of these directions and does not intersect Int C.)

9.3. *Intersections of spheres with straight lines.* Fort [82] described Ball's horned sphere [15] so that it can be pierced at each of its points with a straight arc, and Bing [38] then presented a similar description of the Alexander horned sphere. Thus there do exist wild 2-spheres that can be pierced at each point with a straight arc, but no family of such piercing arcs can be continuous [38, Theorem 1].

Both the Alexander horned sphere (Figure 1) and the Fox-Artin sphere (Figure 3) can be described so that no vertical line intersects them in more than four points. Jensen and Loveland [114] have recently shown that four is the smallest number in this respect for wild 2-spheres in E^3.

THEOREM 9.3.1 (JENSEN AND LOVELAND [114]). *A 2-sphere* S *in* E^3 *is tame if no vertical line intersects* S *in more than three points.*

(Loveland has recently shown, using properties of $*$-taming sets [57], [183], [184], that S $\cup$ Int S is a 3-cell if no vertical line intersects S in more than five points.)

The following question remains unanswered:

Is a 2-sphere S in E^3 tame if the intersection of each vertical line with S $\cup$ Int S is connected?

(Loveland [193] has recently used properties of $*$-taming sets [57], [183], [184] to answer this question affirmatively.)

A closed subset X of a 2-sphere S in E^3 is defined to be *visible* from the point $p \in$ Int S if each straight ray which starts at p and intersects X has only one point of intersection with S. Cobb [64] has recently proved the following theorem, with a suggestion of the alternative proof we outline here.

THEOREM 9.3.2 (COBB [64]). *If X is a closed subset of a 2-sphere* S *in* E^3 *such that X is visible from some point* $p \in$ Int S *and* S *is locally tame modulo X, then* S *is tame.*

OUTLINE OF PROOF. For convenience, we assume that $\rho(p, S) > 1$. Let K denote the round 2-sphere with radius 1 and center p. We define as follows two nonnegative real functions f_1 and f_2 on K which are upper semicontinuous and lower semicontinuous, respectively. For each $x \in K$, let R_x denote the straight ray that starts at p and contains x. Let $X' = \{x \mid x \in K \text{ and } X \cap R_x \neq \emptyset\}$. If R_x intersects X, define $f_1(x)$ and $f_2(x)$ such that $f_1(x) = f_2(x) = \rho(x, X \cap R_x)$. If R_x does not intersect X, let $f_1(x) = 0$ and $f_2(x) = \rho(x, S \cap R_x)$. Thus for each $x \in X'$, $f_1(x) = f_2(x)$, and for each $x \in K - X'$, $f_1(x) < f_2(x)$. It follows from [**14**, Theorem 5.4.8, p. 156] that there is a continuous real function f defined on K such that:

(1) $f_1(x) < f(x) < f_2(x)$ for each $x \in K - X'$.

(2) $f_1(x) = f(x) = f_2(x)$ for each $x \in X'$.

For each $x \in K$, let x' be a point of R_x such that $\rho(p, x') > 1$ and $\rho(x, x') = f(x)$. With such a point for each $x \in K$, we have identified a tame 2-sphere S' such that $S' \subset S \cup \text{Int } S$ and $S \cap S' = X$. It follows from [**49**, Theorem 2] that S is tame from Ext S. A similar argument can be used to show that S is tame from Int S.

Theorem 9.3.2 has the following corollary:

THEOREM 9.3.3 (COBB [**64**]). *A closed subset X of a 2-sphere S in E^3 is tame if X is visible from some point of* Int S.

10. **Sewings of crumpled cubes.** We picture sewing two crumpled cubes C_1 and C_2 together by considering a homeomorphism h of Bd C_1 onto Bd C_2 and identifying each point x of Bd C_1 with the point $h(x)$ on Bd C_2. Clearly S^3 is obtained by sewing two round balls together and, similarly, by sewing any two 3-cells together. On the other hand, we can describe a sewing of two crumpled cubes, each bounded by a Fox-Artin sphere (Figure 3), that does not yield S^3. Here we consider the Fox-Artin sphere S of Figure 3 to be in S^3 and use the crumpled cube $S \cup U$, where S is wild at p from U and is locally polyhedral in $S - p$. We call $S \cup U$ a *Fox-Artin crumpled cube* and denote it by K. Let K be in E^3 and consider another copy K' of K obtained under a translation of E^3 so that K and K' do not intersect. Let h be the homeomorphism of Bd K onto Bd K' that agrees with this translation on Bd K. Now sew K and K' together with the homeomorphism h, and suppose that this sewing of K and K' yields S^3. Denote the common boundary of K and K', under the sewing h, by S'. Then S' is a 2-sphere in S^3 and is locally tame except for one point p' where it is wild from each component of $S^3 - S'$. By Theorem 4.4.5, we can assume that $S' - p'$ is locally polyhedral. Harrold and Moise [**101**] have shown that this is impossible. Thus the sewing of

K and K' with the homeomorphism h does not yield S^3. We observe, however, that a sewing of K to K' will yield S^3 provided that the point where Bd K is wild from Int K is not sewn to the point where Bd K' is wild from Int K'.

It is our purpose in this section to discuss the following question and some answers to it.

Under what conditions does a sewing of two crumpled cubes yield S^3?

Precisely, with a *sewing* of two crumpled cubes C_1 and C_2 under a homeomorphism h of Bd C_1 onto Bd C_2, we are considering the space obtained under an upper semicontinuous decomposition G of the space $C_1 \cup C_2$, where C_1 and C_2 are disjoint and the elements of G are the points of Int $C_1 \cup$ Int C_2 and the pairs of points $(x, h(x))$. Sometimes we call the homeomorphism h a *sewing* of C_1 and C_2. Where C_2 is the image of C_1 under a translation of E^3, as was the case for the Fox-Artin crumpled cubes discussed above, and where h agrees with the translation on Bd C_1, then we consider the sewing h to be a *sewing of C_1 to itself under the identity homeomorphism.*

10.1. *Sewings of solid horned spheres.* We consider the Alexander horned sphere S (Figure 1) to be embedded in S^3 and call $S \cup U$ an *Alexander crumpled cube,* where U is the component of $S^3 - S$ that is not simply connected. Bing sketched a picture of such a crumpled cube in [**20**] and called it a solid horned sphere. As observed in his paper [**20**], Bing solved a problem about the fixed point set under a period two homeomorphism of E^3 by showing that S^3 is obtained by sewing an Alexander crumpled cube to itself with the identity homeomorphism.

THEOREM 10.1.1 (BING [**20**, p. 356]). *The sewing of the Alexander crumpled cube to itself with the identity homeomorphism yields* S^3.

Thus, contrary to what is the case with the Fox-Artin crumpled cube, there are sewings of crumpled cubes that yield S^3 and match points where the boundaries of the crumpled cubes are wild from their interiors. Bing's methods involved showing that the space obtained with the sewing is homeomorphic with the space obtained from a certain upper semicontinuous decomposition of E^3 into points and tame arcs and by then showing that the latter space is homeomorphic with E^3. This initiated an extensive study of decompositions of E^3, which has been summarized by Armentrout in [**11**].

Similar methods to those used by Bing can be used to show that S^3 results from sewing an Antoine crumpled cube, described in §3, to itself with the identity homeomorphism. It is interesting to observe that while the set of wild points of the Alexander horned sphere is a tame Cantor set, it becomes a wild Cantor set under the sewing described in Theorem 10.1.1 [**20**, p. 362].

We described above how S^3 is obtained from some sewings of Fox-Artin crumpled cubes and not from other sewings. Casler [**63**] showed, however, that this is not the case with the Alexander crumpled cube by proving the following extension of Theorem 10.1.1. (Eaton [**189**] has recently presented an alternative proof of this theorem.)

THEOREM 10.1.2 (CASLER [**63**]). *Any sewing of the Alexander crumpled cube to itself yields* S^3.

Ball [**15**] modified Alexander's description of the horned sphere, using four instead of two pairs of horns at each stage, to obtain a crumpled cube that can be sewn to itself so as not to yield S^3. He showed that the space obtained under the sewing is homeomorphic with the "dog-bone space," which Bing [**23**] described and proved was different from S^3. The sewing used by Ball was different from the identity, and indeed S^3 does result from sewing this crumpled cube to itself with the identity homeomorphism [**60**, Theorem 1]. This result by Ball can be combined with the above result by Casler (Theorem 10.1.2) to show that the Alexander crumpled cube and the one described by Ball are not homeomorphic.

L. O. Cannon [**60**] has given a general definition of horned spheres and has described a crumpled cube bounded by a horned sphere such that S^3 is not obtained by sewing this crumpled cube to itself with the identity homeomorphism.

10.2. *Universal crumpled cubes.* Casler's result that S^3 is obtained from any sewing of the Alexander crumpled cube to itself suggests the following definition. A crumpled cube K is *self-universal* if any sewing of K to itself yields S^3. Thus the Alexander crumpled cube is self-universal while the one described by Ball is not. The following question has been only partially answered:

What is the class of all crumpled cubes that are self-universal?

We define a crumpled cube K to be *universal* if, for any crumpled cube K', any sewing of K to K' yields S^3. Thus any universal crumpled cube is self-universal.

While it is trivial to notice that every 3-cell is self-universal, it is a significant problem to show that every 3-cell is universal. This was shown independently by Hosay [**109**] and Lininger [**119**], using Bing's side approximation theorem for spheres (Theorem 4.6.2). More recently, Daverman [**67**] has presented a much simpler proof using Lister's form of the side approximation theorem for spheres (Theorem 4.6.3).

THEOREM 10.2.1 (HOSAY [**109**], LININGER [**119**], AND DAVERMAN [**67**]). *Every 3-cell is universal; that is, any sewing of a 3-cell to any crumpled cube yields* S^3.

Daverman and Eaton [**73**] have recently shown the existence of universal crumpled cubes other than 3-cells. Their general theorem on universality of crumpled cubes [**73**, Theorem 5.1], which we do not describe here, has the following corollary:

THEOREM 10.2.2 (DAVERMAN AND EATON [**73**]). *If K is a crumpled cube such that* Int C *is an open 3-cell and each point of* Bd C *is a piercing point of C, then C is universal.*

Using modifications of Bing's description of a wild 2-sphere (Figure 5), Gillman [**89**] and Alford [**6**] each described a wild 2-sphere S in S^3 so that each component of $S^3 - S$ is an open 3-cell and S can be pierced at each of its points with a tame arc. Each such 2-sphere, then, is the boundary of a crumpled cube which is different from a 3-cell and satisfies the hypothesis of Theorem 10.2.2.

THEOREM 10.2.3 (DAVERMAN AND EATON [**73**]). *The crumpled cubes described by Gillman* [**89**] *and Alford* [**6**] *are universal and different from a 3-cell.*

Thus each of these crumpled cubes is also self-universal. It had previously been shown by Lininger [**119**] that the Bing crumpled cube yields S^3 when sewn to itself under the identity homeomorphism. Modifications of the methods used by Gillman [**89**] and Alford [**6**] lead to the following consequence of Theorem 10.2.2.

THEOREM 10.2.4 (DAVERMAN AND EATON [**73**]). *There exists an uncountable family of topologically different universal crumpled cubes.*

The sufficiency condition for universality of crumpled cubes presented by Daverman and Eaton [**73**, Theorem 5.1] does not furnish answers to the following two questions.

Is the crumpled cube described by Bing (Figure 5) universal?

Is the Alexander crumpled cube universal?

(Eaton [**189**] has recently answered the first of these two questions affirmatively by showing that a crumpled cube C is universal if each arc in Bd C is tame. Bass and Daverman [**187**] recently answered the second question negatively, and Daverman presented some further work on universal crumpled cubes in [**188**].)

We wonder whether there is a crumpled cube K which can be considered a "test kit" for determining whether a crumpled cube is universal. This can be phrased as the following question:

Does there exist a crumpled cube K such that a crumpled cube C is universal if and only if every sewing of C to K yields S^3?

A candidate for such an example might be the closure of the complement, in S^3, of a 3-cell, described by Stallings [**162**, Figure 1]. (Bass and Daverman [**187**] recently used this crumpled cube to show that the Alexander crumpled cube is not universal.)

10.3. *Sewings that yield* S^3. Until recently it was not known whether for any two crumpled cubes there is some sewing that yields S^3. This was answered affirmatively when Daverman and Eaton [**71**] showed that there is a dense set of such sewings. Roughly speaking, this means that any sewing of two crumpled cubes is close to a sewing that yields S^3. Precisely, this can be stated as follows, where ρ denotes the metric discussed in §4.3.

THEOREM 10.3.1 (DAVERMAN AND EATON [**71**, THEOREM 1]). *If K and K' are two crumpled cubes, h is a homeomorphism of* Bd K *onto* Bd K', *and* $\epsilon > 0$, *then there is a homeomorphism h' of* Bd K *onto* Bd K' *such that, for* $x \in \text{Bd } K, \rho(h(x), h'(x)) < \epsilon$ *and* S^3 *results from sewing K and K' together under h'.*

The proof of this theorem, which is tedious and difficult, relies strongly on Bing's theorem on the existence of tame Sierpiński curves on 2-spheres in E^3 (Theorem 5.1.1).

There is one point where the space resulting from sewing the Fox-Artin crumpled cube to itself under the identity homeomorphism fails to yield a 3-manifold. The following theorem has been proved by Martin in [**136**], where he attributes an earlier proof to Lininger.

THEOREM 10.3.2 (MARTIN [**136**, p. 150]). *If a sewing of two crumpled cubes yields a* 3-*manifold then it yields* S^3.

Martin's proof of this theorem relies on a substantial theorem, proved by Armentrout [**12**], [**13**] about cellular upper semicontinuous decomposition of E^3 yielding E^3 if they yield 3-manifolds.

(Eaton [**189**] has recently shown that a sewing of two crumpled cubes K_1 and K_2 yields S^3 if and only if there exist disjoint 0-dimensional F_σ-sets F_1 and F_2 in Bd K_1 and Bd K_2, respectively, such that no point of F_1 is sewn to a point of F_2 and each of $F_1 \cup \text{Int } K_1$ and $F_2 \cup \text{Int } K_2$ is 1-ULC.)

10.4. *Consequences of the universality of* 3-*cells*. The first such consequence, which is simply an equivalent form of Theorem 10.2.1, is an important theorem on embedding crumpled cubes in E^3.

THEOREM 10.4.1 (HOSAY [**109**], LININGER [**119**], DAVERMAN [**67**]). *If K is a crumpled cube, then there is a homeomorphism h of K into E^3 such that* Bd $h(K)$ *is tame from $E^3 - h(K)$.*

In his paper on the sewing of two Alexander crumpled cubes [**20**], Bing described an upper semicontinuous decomposition G of a round ball B so that G yields the Alexander crumpled cube such that each nondegenerate element of G is a tame arc that has only an end point on Bd B. The Fox-Artin crumpled cube can be obtained with an upper semicontinuous decomposition G of B where the elements of G are all points except for one element which is a Fox-Artin arc (Figure 4) intersecting Bd B at its tame end point.

Using Theorem 10.4.1, Lininger [**119**] showed, in 1965, that any crumpled cube can be obtained from an upper semicontinuous decomposition of a round ball B into points and cellular continua that intersect Bd B in only one point. We indicate, with the following theorem, how a slightly stronger result can be obtained from Theorem 10.4.1. As indicated below, this has been further strengthened by Lininger and Bing.

THEOREM 10.4.2 (LININGER [**119**], [**120**]). *If K is a crumpled cube, then there is an upper semicontinuous decomposition G of a round ball B into arcs and points such that:*

(1) *For each arc $g \in G$, $g \cap$* Bd B *is an end point of g and g is locally tame except at its other end point.*

(2) *The decomposition space of B under G is K.*

OUTLINE OF PROOF. By Theorem 10.4.1, we consider K embedded in E^3 so that Bd K is tame from $E^3 - K$. Let B be a round ball such that $K \subset$ Int B. Let B' be a round ball in Int B such that $K \subset$ Int B' and Bd B' is concentric with Bd B. Since K is tame from $E^3 - K$, it follows that there is a homeomorphism h of $B -$ Int B' onto $B -$ Int K that is the identity on Bd B. Let G denote the collection consisting of the points of Int K and the arcs which are images under h of the radial arcs from Bd B' to Bd B. Then G is an upper semicontinuous decomposition of B that yields K. The characterization of tame arcs by Harrold, Griffith, and Posey [**100**, Theorem 6], or the one by Bing and Kirkor [**42**], can be used to show that each arc of G is locally tame except at its end point on Bd K. Thus G satisfies the requirements of the conclusion of Theorem 10.4.2.

A slight modification of the above procedure, enables us to require that each arc of G be locally polygonal except at its end point on Bd K. We have described G so that every point of Bd B is an end point of some arc of G. In a recent paper, Lininger [**120**] has obtained a further strengthening of Theorem 10.4.2 where it is required that only the points of a zero-dimensional set on Bd B are end points of arcs of G. Bing announced the same result at the International Congress of Mathematicians in Moscow, 1966. They retain the requirement that each arc of G be locally polyhedral except at its end point in Int B.

11. **2-manifolds in a 3-manifold.** Most of what we have discussed in previous sections about embeddings of 2-spheres, or of subsets of them, in E^3 or S^3 can be extended to 2-manifolds in a 3-manifold. We define a 2-manifold M in a 3-manifold M^3 to be *tame* relative to some triangulation T of M^3 [**143**], [**27**] if there is a homeomorphism of M^3 onto itself that carries M onto a polyhedron relative to T. It follows from work by Moise [**143**, Theorem 2] and Bing [**27**, Theorem 8] that if M is tame relative to some triangulation of M^3, then M is tame relative to every triangulation of M^3. Thus it is appropriate to say merely that M is *tame* in M^3. We observe that, in view of the remarks in the first paragraph of §4.4, this definition is consistent with the one we have used in previous sections for a tame 2-sphere in E^3 or S^3.

We define a compact subset X of a 2-manifold in M^3 to be tame if X is a subset of some tame 2-manifold in M^3. In [**137**], McMillan discusses tameness of finite graphs in a 3-manifold without requiring that they be subsets of a 2-manifold.

The two main tools in extending theorems about 2-spheres in E^3 to 2-manifolds in a 3-manifold M^3 are Theorem 5.4.2, which states roughly that small disks on a 2-manifold in M^3 are subsets of small 2-spheres in M^3, and Theorem 6.1.1, which can be extended to imply that locally tame 2-manifolds in M^3 are tame. In applying Theorem 5.4.2, the 2-sphere is required to be so small that it lies in a neighborhood, relative to M^3, that is homeomorphic with E^3.

The basic theorems discussed in §4 can be adapted to a compact 2-manifold M in a 3-manifold M^3. In cases where M does not separate M^3, some of the theorems on separation and accessibility need to be adapted, with the use of Theorem 5.4.2, for local separation. Similarly, Bing's side approximation theorem needs to be applied locally in case M does not separate M^3. Lister [**121**, Theorem 4] showed that Theorem 4.6.3 is valid for 2-manifolds that separate a 3-manifold. Bing [**40**] showed that Theorem 5.1.1, about tame Sierpiński curves in a 2-sphere, can be suitably extended to manifolds. With the use of Theorem 5.4.2, the theorems in §5.2 on piercing of spheres can be changed to apply to manifolds.

McMillan proved that 2-manifolds in a 3-manifold have canonical neighborhoods similar to what is stated for 2-spheres in Theorem

5.3.4. The intersection of a polygonal arc and a 2-manifold, or the intersection of two 2-manifolds, can be improved, with the aid of Theorem 5.4.2, in the manner discussed in §5.5 for 2-spheres and lines. Craggs [**186**] has done some similar work for complexes.

The theorems in §10 on sewings of crumpled cubes can be adapted to sewings of the closures of complementary domains of 2-manifolds in a 3-manifold. In such cases, the conditions imply that the sewings yield 3-manifolds.

11.1. *Tame subsets of 2-manifolds in a 3-manifold.* The theorems and methods in [**48**] show, with the aid of Theorems 5.4.2 and 11.1.1, that the theorems in §§6.2 through 6.8, inclusive, can be extended to a 2-manifold. It is interesting to remark here that Daverman [**69**] has shown that the conclusion of Theorem 6.9.2 can be strengthened where S is replaced with a torus K in S^3 and a suitable replacement is made for the requirement that each component of $S^3 - S$ be an open 3-cell. In such a case, K cannot have any wild points. Further interesting related theorems are included in [**153**], [**165**].

The following theorem extends Theorem 6.1.1 to closed subsets of a 2-manifold in a 3-manifold.

THEOREM 11.1.1 (CANNON [**59**]). *If X is a locally tame closed subset of a compact 2-manifold-with-boundary M in a 3-manifold M^3 and $\epsilon > 0$, then there is an ϵ-homeomorphism h from M into M^3 such that $X \subset h(M)$ and $h(M)$ is tame in M^3.*

PROOF. There is a triangulation T of Int M whose mesh approaches zero near Bd M such that if D is any topological 2-simplex (or disk) in the 2-skeleton of T, then the following conditions are satisfied:

(1) D lies in an open subset U_D of M^3 which is homeomorphic to E^3.
(2) $X \cap D$ lies on a tame 2-sphere in U_D.
(3) Bd D lies on a tame 2-sphere in U_D.
(4) D lies on a 2-sphere S_D in U_D.

Condition (1) can be satisfied because M^3 is a 3-manifold, condition (2) because X is locally tame, condition (3) because Theorem 5.1.1, about tame Sierpiński curves in a 2-sphere, can be suitably extended to manifolds [**40**], and condition (4) because of Theorem 5.4.2.

For each disk D in the 2-skeleton of T, let $f_D : S_D \to [0, \infty)$ be a continuous function such that:

(5) $f_D(x) < \frac{1}{3} \min [\rho(x, M - D), \epsilon, \text{Diam } D]$ for $x \in$ Int D.
(6) Bd $D = f_D^{-1}(0)$.

By Theorem 7.2.4, there is a homeomorphism h_D from S_D into U_D such that:

(7) $\rho(x, h_D(x)) \leqq f_D(x)$ for each $x \in S_D$.

(8) $X \cap D \subset h_D(D)$.

(9) $h_D(S_D)$ is locally tame modulo Bd D.

Define $h : M \to M^3$ by $h|D = h_D|D$ for each element D of the 2-skeleton of T and define $h \mid \text{Bd } M$ to be the identity homeomorphism. Then h is an embedding because of (5) and (7) and because the mesh of T approaches zero near Bd M. The set $h(D)$ is tame for each D because $h(D)$ is a disk that is tame modulo a tame boundary (Theorem 8.1.5). Each interior point p of $h(M)$ lies in the interior of a disk E in Int $h(M)$ which is the union of tame subdisks of the $h(D)$'s and which lies on a 2-sphere in some coordinate neighborhood of M^3 (Theorem 5.4.2). Since the union of finitely many tame disks on a 2-sphere in E^3 is tame (Theorem 8.1.5), E is a tame disk. Hence $h(M)$ is locally tame at p. That is, Int $h(M)$ is locally tame at each point.

There is a collar C for Bd $h(M)$ in $h(M)$. Since Int $h(M)$ is locally tame, C can be shoved slightly to one side of $h(M)$, leaving Bd $h(M)$ fixed, so as to form a collar V for Bd $h(M)$ in $M^3 - \text{Int } h(M)$ that is locally tame modulo Bd $h(M)$. Hence there is an $\epsilon/3$-homeomorphism h_1 from $h(M)$ into $h(M) \cup V$ such that $h(M) \subset \text{Int } h_1h(M)$. The map h_1 simply stretches $h(M)$ slightly near Bd $h(M)$ to cover part of the collar V.

Finally, by the procedure indicated in the first half of this proof, there is an $\epsilon/3$-homeomorphism h_2 from $h_1h(M)$ into M^3 such that $h_2h_1h(M)$ is locally tame everywhere and yet contains X. But $h_2h_1h(M)$ is tame because it is locally tame (Theorem 6.1.1, extended to 2-manifolds in a 3-manifold).

11.2. *Taming sets for 2-manifolds in a 3-manifold.* We extend the definition of taming sets for 2-spheres in E^3, stated in §8, to the following: A *taming set for 2-manifolds* in a 3-manifold M^3 is a closed set X in M^3 having the following two properties:

(1) X is a subset of some 2-manifold in M^3.

(2) If M is a 2-manifold in M^3 which contains X and is tame modulo X, then M is tame.

Cannon's characterization of taming sets for 2-spheres in E^3 (Theorem 8.1.6) can be stated as follows for manifolds:

THEOREM 11.2.1. *A closed subset X of a tame 2-manifold in a 3-manifold M^3 is a taming set for 2-manifolds if and only if X has no degenerate component.*

PROOF. Let X be a tame subset of a 2-manifold in M^3 such that X has no degenerate component and let M be a 2-manifold in M^3 that contains X and is locally tame modulo X. We show that M is tame by showing that it is locally tame [**21**]. Let p be a point of X

and let N be a neighborhood of p that is homeomorphic with E^3. By Theorem 5.4.2, there exist a disk D and a 2-sphere S in N such that:
(1) $p \in \operatorname{Int} D$.
(2) $D \subset S \cap M$.
From the requirement that X is a subset of a tame 2-manifold in M^3, we can require that $D \cap X$ is a tame subset of N. Let D' be a disk in $\operatorname{Int} D$ such that $p \in \operatorname{Int} D'$, and let X' denote the union of all components of $X \cap D$ that intersect D'. Thus X' is a tame subset of S that has no degenerate component. By Theorem 5.1.1, we can require that $\operatorname{Bd} D'$ be tame. By Theorem 4.6.4 there is a 2-sphere S' in N that contains $D' \cup X'$ and is locally tame modulo $D' \cup X'$. It follows from Theorem 8.1.2, applied locally, that S' is locally tame modulo X'. Thus by Theorem 8.1.6, S' is tame. Since $D' \subset S' \cap M$ and $p \in \operatorname{Int} D'$, it follows that M is locally tame at p. Thus M is locally tame and hence is tame.

Now let X be a tame subset of a 2-manifold in M^3 such that some point is a component of X. In the same manner that we indicated for tame subsets of 2-spheres in E^3 (Theorem 8.1.6), we can show that there is a wild 2-manifold that contains X and is locally tame modulo X.

We can further extend the definition of taming sets to *taming sets for 2-manifolds-with-boundary* in a 3-manifold M^3 by replacing "2-manifold" by "2-manifold-with-boundary" in the above definition. Then we can extend Theorem 11.2.1 as follows:

THEOREM 11.2.2. *A closed subset X of a tame 2-manifold-with-boundary in a 3-manifold M^3 is a taming set for 2-manifolds-with-boundary if and only if X has no degenerate component.*

PROOF. Let X be a tame subset of a 2-manifold-with-boundary in M^3 such that X has no degenerate component, and let M be a 2-manifold-with-boundary in M^3 that contains X and is locally tame modulo X. The procedure in the first part of the proof of Theorem 11.2.1 can be followed to show that M is locally tame modulo $\operatorname{Bd} M$. Thus it remains for us to show that M is locally tame at each point of $X \cap \operatorname{Bd} M$. Let p be such a point, and let N be a neighborhood of p that is homeomorphic with E^3. There exist two disks D and D' in $M \cap N$ such that $D' \subset M - \operatorname{Cl}(M - D)$ and $\operatorname{Bd} D \cap \operatorname{Bd} M$ and $\operatorname{Bd} D' \cap \operatorname{Bd} M$ are arcs having p as an interior point. As in the proof of Theorem 11.2.1, we can require that $D \cap X$ is a tame subset of N. Let X' denote the union of all components of $X \cap D$ that intersect D'. Since $\operatorname{Int} M$ is locally tame, it follows that $\operatorname{Int} D$ is locally tame and thus, by a thickening process similar to the one indicated for

Theorem 4.4.5, that D is a subset of a 2-sphere S in N that is locally tame modulo X' [77, Theorem 0]. Since X' is a tame subset of S having no degenerate component, it follows from Theorem 8.1.6 that S is tame in N. Thus M is locally tame at p and hence is tame [21].

For the other part of the proof, we can follow the procedure indicated in the proof of Theorem 8.1.6 to show that X is not a taming set if it has a degenerate component.

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