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EIGENFUNCTION EXPANSIONS AND SCATTERING THEORY FOR PERTURBATIONS OF $-\Delta$

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1. **Survey of results.** Let Ω be the unbounded domain exterior to a compact C^2 hypersurface Γ in R^n ($n \geqq 2$) and $q(x)$ a real-valued function such that $e^{2a|x|}q(x)$ is bounded and uniformly α-Hölder continuous in $\Omega \cup \Gamma$ for certain constants $a > 0$ and $0 < \alpha < 1$.

We let H denote the selfadjoint operator $-\Delta + q$ in $L^2(\Omega)$ acting on functions which are zero on Γ. Specifically, we define

$$(1.1)\qquad \begin{aligned} D(H) &= \{g : (\partial/\partial x)^\alpha g \in L^2(\Omega) \text{ for } |\alpha| \leqq 2 \text{ and } g|_\Gamma = 0\}, \\ Hg &= -\Delta g + qg \text{ for } g \in D(H). \end{aligned}$$

Here differentiation is interpreted in the space $\mathfrak{D}'(\Omega)$ of distributions on Ω and $g|_\Gamma$ is interpreted in an L^2 sense (see §4).

We treat H as a perturbation of the selfadjoint operator $H_0 = -\Delta$ in $L^2(R^n)$,

$$(1.2)\qquad \begin{aligned} D(H_0) &= \{f : (\partial/\partial x)^\alpha f \in L^2(R^n) \text{ for } |\alpha| \leqq 2\}, \\ H_0 f &= -\Delta f \quad \text{for } f \in D(H_0). \end{aligned}$$

The Fourier transform

$$(1.3)\qquad \hat{f}(\xi) = \text{l.i.m.}\ (2\pi)^{-n/2} \int_{R^n} f(x)\, e^{-ix\cdot\xi} dx \quad (\xi \in R^n)$$

is a unitary map

$$L^2(R^n) \ni f \rightarrow \hat{f} \in L^2(R^n)$$

which "diagonalizes" H_0, i.e., which transforms H_0 into multiplication by $|\xi|^2$,

$$(1.4)\qquad (H_0 f)\hat{\ }(\xi) = |\xi|^2 \hat{f}(\xi).$$

The plane wave $e^{ix\cdot\xi}$ is an eigenfunction of the differential operator $-\Delta$,

$$-\Delta e^{ix\cdot\xi} = |\xi|^2 e^{ix\cdot\xi},$$

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but is not in $L^2(R^n)$ and is not an eigenfunction of H_0. Nevertheless, $\hat{f}(\xi)$, the formal inner product of f with $e^{ix\cdot\xi}$, may be thought of as the corresponding Fourier coefficient of f. Then the inversion formula

$$f(x) = \text{l.i.m.}\, (2\pi)^{-n/2} \int_{R^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi \tag{1.5}$$

expresses f as the "sum" (integral) of the Fourier coefficients times the corresponding eigenfunctions. The Fourier transform is called an improper eigenfunction expansion for H_0, and $\{e^{ix\cdot\xi} : \xi \in R^n\}$ is called a complete set of improper eigenfunctions of H_0.

Write $H = \int_{-\infty}^{\infty} \lambda\, dE_\lambda$ and set $P = \int_{0+}^{\infty} dE_\lambda$.

We will see that the spectrum of H in $(-\infty, 0]$ consists of eigenvalues $k_j^2 \leqq 0$ with corresponding eigenfunctions ϕ_j,

$$H\phi_j = k_j^2\, \phi_j, \quad \int_\Omega |\phi_j|^2 dx = 1,$$

while HP is absolutely continuous and has two complete sets of improper eigenfunctions $\{\phi_+(x, \xi) : 0 \neq \xi \in R^n\}$ and $\{\phi_-(x, \xi) : 0 \neq \xi \in R^n\}$. The so-called distorted plane waves $\phi_\pm$ are determined by the equations

$$\begin{aligned} (-\Delta + q(x) - k^2)\phi_\pm(x, \xi) &= 0 \quad \text{for } x \text{ in } \Omega\,, \\ \phi_\pm(x, \xi) &= 0 \quad \text{for } x \text{ on } \Gamma \end{aligned} \tag{1.6}$$

and the requirement that

$$v_+(x, \xi) = \phi_+(x, \xi) - e^{ix\cdot\xi}\ [v_-(x, \xi) = \phi_-(x, \xi) - e^{ix\cdot\xi}]$$

satisfy the outgoing [incoming] Sommerfeld radiation conditions

$$\begin{aligned} v_\pm(x, \xi) &= O(|x|^{(1-n)/2}), \\ \left(\frac{\partial}{\partial |x|} \mp i|\xi|\right) v_\pm(x, \xi) &= o(|x|^{(1-n)/2}) \end{aligned} \tag{1.7}$$

as $|x| \to \infty$.

For g in $L^2(\Omega)$, we set

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)^* dx \tag{1.8}$$

and

$$\hat{g}^\pm(\xi) = \text{l.i.m.}\, (2\pi)^{-n/2} \int_\Omega \phi_\pm(x, \xi)^* g(x) dx \quad (0 \neq \xi \in R^n). \tag{1.9}$$

We will see that for $g \in L^2(\Omega)$

$$\int_\Omega |g(x)|^2 dx = \sum |\hat{g}_j| + \int_{R^n} |\hat{g}^\pm(\xi)|^2 d\xi, \tag{1.10}$$

and

$$g(x) = \text{l.i.m.} \sum \hat{g}_j \phi_j + \text{l.i.m.}(2\pi)^{-n/2} \int \phi_\pm(x, \xi)\, \hat{g}^\pm(\xi) d\xi \ ; \tag{1.11}$$

that

$$(Hg)\hat{}_j = k_j^2\, \hat{g}_j, \qquad (Hg)\hat{}^{\,\pm}(\xi) = |\xi|^2 \hat{g}^\pm(\xi), \tag{1.12}$$

for $g \in D(H)$; and that

$$\hat{}^{\,\pm} \ : \ PL^2(\Omega) \rightarrow L^2(R^n)$$

are unitary.

For $f \in D(H_0)$ and $g \in D(H)$, the solutions $u_0(t)$ and $u(t)$ of

$$\frac{1}{i}\frac{d}{dt} u_0(t) = H_0 u(t), \qquad u_0(0) = f \tag{1.13}$$

and

$$\frac{1}{i}\frac{d}{dt} u(t) = Hu(t), \qquad u(0) = g \tag{1.14}$$

are given by $u_0(t) = e^{-itH_0} f$ and $u(t) = e^{-itH} g$.

To compare the unitary groups e^{-itH_0} and e^{-itH}, we introduce a cut-off function $\mu(x) \in C^\infty(R^n)$ with $\mu(x) = 0$ inside and near Γ and $\mu(x) = 1$ for large x. Then for $f \in L^2(R^n)$, the limits

$$W_\pm f = \lim_{t \rightarrow \pm\infty} e^{itH} \mu e^{-itH_0} f \tag{1.15}$$

exist and define the *wave operators*

$$W_\pm \ : \ L^2(R^n) \rightarrow L^2(\Omega).$$

$W^\pm$ are independent of the choice of μ, are isometric, and satisfy

$$e^{-itH}\, W_\pm = W_\pm\, e^{-itH_0} \tag{1.16}$$

for all t and

$$e^{-itH}\, W_\pm f \sim e^{-itH_0} f \ \text{ as } \ t \rightarrow \pm\infty \tag{1.17}$$

We will see that

$$W_\pm \ : \ L^2(R^n) \rightarrow PL^2(\Omega)$$

are unitary and that the wave operators are related to the eigenfunctions by the equations

$$(W_+f)^{\wedge +} = \hat{f} = (W_-f)^{\wedge -} \quad \text{for } f \in L^2(R^n). \tag{1.18}$$

Thus we have the commuting diagram of unitary operators

$$\begin{array}{ccc} L^2(R^n) & \xrightarrow{\ \wedge\ } & L^2(R^n) \\ & {\scriptstyle W_+}\searrow & \uparrow {\scriptstyle \wedge -} \\ & & PL^2(\Omega) \end{array}$$

and the corresponding diagram for W_-.

Now consider $g \in L^2(\Omega)$ and set $u(t) = e^{-itH}g$. *If* g is an eigenfunction ϕ_j of H, then $u(t) = \phi_j \exp(-itk_j^2)$ does not tend to zero over bounded sets as $t \to \pm\infty$ and therefore cannot behave asymptotically like a free-space solution $e^{-itH_0}f$.

If $Pg = g$, however, then we set $f_\pm = W_\pm^* g$ to obtain the unique functions in $L^2(R^n)$ satisfying $W_\pm f_\pm = g$ and we have

$$e^{-itH_0}f_- \underset{t \to -\infty}{\sim} e^{-itH}g \underset{t \to +\infty}{\sim} e^{-itH_0}f_+. \tag{1.19}$$

The *scattering operator* $\mathcal{S}$ maps the free-space solution on the left side of (1.19) into the free-space solution on the right side. We label these solutions by their values at $t = 0$. Thus we set $\mathcal{S}f_- = f_+$ or $\mathcal{S} = W_+^* W_-$ and $\mathcal{S}$ is a unitary operator on $L^2(R^n)$.

The Fourier transform of $\mathcal{S}$ has the form

$$(\mathcal{S}f)^\wedge(k,\omega) = S(k)\hat{f}(k, \cdot\,)(\omega). \tag{1.20}$$

Here we have introduced polar coordinates $\xi = (k, \omega)$ with $k > 0$ and $\omega \in S^{n-1}$, the $(n-1)$-sphere, and $S(k)$, the *scattering matrix*, is for each $k > 0$ a unitary operator on $L^2(S^{n-1})$. We will derive a specific formula for $S(k)$ which will show that it has a meromorphic extension to $|\mathrm{Im}\, k| < a$ (and $k \neq 0$, $-\pi < \arg k \leqq \pi$ if n is even).

Example. Consider $n = 2$, $q = 0$, and $\Gamma =$ unit circle parametrized by θ, $0 \leqq \theta \leqq 2\pi$. Then in this case

$$S(k)\left(\sum_{-\infty}^{\infty} a_m e^{im\theta}\right) = -\sum_{-\infty}^{\infty} \frac{H_m^{(2)}(k)}{H_m^{(1)}(k)} a_m e^{im\theta}$$

where $H_m^{(1)}$ and $H_m^{(2)}$ are the Hankel functions (multiple-valued analytic function of $k \neq 0$ with a logarithmic branch point at $k = 0$).

The poles of $S(k)$ for $-\pi < \arg k \leqq \pi$ are at the zeros of $H_m^{(1)}(k)$ and lie in the lower half k-plane. Suppose k_0 is such a pole. Then

$$u(r, \theta) = e^{im\theta} H_m^{(1)}(k_0 r)$$

is a nonzero solution of

$$(-\Delta - k_0^2)u = 0, \quad r > 1,$$
$$u = 0, \quad r = 1,$$

satisfying the generalized outgoing radiation condition to be described in the next section. We call such a function a *resonant state*. In §5 we will show, with specific formulas, that this example is typical, i.e., that k ($\operatorname{Im} k < 0$) is a pole of $S(k)$ if and only if there exist resonant states at k.

2. **Generalized outgoing radiation conditions.** As in the last section, let a and α be constants with $a > 0$ and $0 < \alpha < 1$. Let $\mathcal{K}$ be the subset $\{k\colon |\operatorname{Im} k| > -a\}$ of the complex plane, if n is odd, and $\{k \neq 0\colon |\operatorname{Im} k| > -a$ and $-\pi < \arg k \leqq \pi\}$ if n is even. Consider k in $\mathcal{K}$.

The outgoing fundamental solution $F_k^+(x)$ for $-\Delta - k^2$ in R^n is given by

$$(2.1) \qquad F_k^+(x) = \frac{i}{4}\left(\frac{k}{2\pi|x|}\right)^p H_p^{(1)}(k|x|), \quad p = (n-2)/2,$$

and has the asymptotic form

$$(2.2) \qquad \left(\frac{\partial}{\partial |x|}\right)^m F_k^+(x) = (ik)^m \left(\frac{k}{2\pi i |x|}\right)^{(n-1)/2} \frac{i}{2k} e^{ik|x|}\left(1 + O\left(\frac{1}{|x|}\right)\right)$$

$(m = 0, 1, 2, \cdots)$ as $|x| \to \infty$.

We say that a function $u \in C^1(\Omega \cup \Gamma) \cap C^{2+\alpha}(\Omega)$ is *outgoing at* k if

$$(2.3) \qquad (-\Delta - k^2)u(x) = O(e^{-a|x|}) \quad \text{as } x \to \infty$$

and if for each x in Ω

$$(2.4) \qquad \int_{|x|=N}\left[u(y)\frac{\partial}{\partial \nu_y} F_k^+(x-y) - F_k^+(x-y)\frac{\partial}{\partial \nu_y} u(y)\right] dS_y \to 0$$

as $N \to \infty$. Green's formula shows that for functions u satisfying (2.3), condition (2.4) is equivalent to the equation

$$u(x)= \int_\Gamma \left[u(y)\frac{\partial}{\partial \nu_y} F_k^+(x-y) - F_k^+(x-y)\frac{\partial}{\partial \nu_y} u(y) \right] dS_y \tag{2.5}$$
$$+ \int_\Omega F_k^+(x-y)(-\Delta - k^2)u(y)dy.$$

(ν_y is the unit exterior normal at y.)

Thus the outgoing radiation condition requires that $u(x)$ be built up from the outgoing fundamental solution.

Because of the asymptotic form (2.2) of F_k^+, condition (2.4) is equivalent for $\operatorname{Im} k \geqq 0$, to the Sommerfeld outgoing radiation condition

$$\begin{aligned} u(x) &= o(|x|^{(1-n)/2}) \\ \left(\frac{\partial}{\partial x} - ik\right) u(x) &= o(|x|^{(1-n)/2}) \end{aligned} \qquad \text{as } x \to \infty \tag{2.6}$$

For $\operatorname{Im} k < 0$, however, nonzero outgoing functions grow exponentially as $|x| \to \infty$.

(A thorough discussion of the material in §§2 and 3 can be found in [1].)

3. **The Dirichlet problem with the outgoing radiation condition.** Let Λ denote the region interior to the surface Γ and for $\beta \geqq 0$ let $\bar{C}^\beta(\Omega \cup \Lambda)$ denote those functions $f \in C^\beta(\Omega \cup \Lambda)$ which have C^β extensions from Ω to $\Omega \cup \Gamma$ and from Λ to $\Lambda \cup \Gamma$. $\bar{C}^\beta(\Omega)$ is defined analogously. Set

$$B_2 = \{\tau \in \bar{C}(\Omega \cup \Lambda): \quad e^{-a|x|}\tau(x) \text{ is bounded}\},$$
$$B_2' = \{\tau \in \bar{C}^\alpha(\Omega \cup \Lambda): \quad e^{-a|x|}\tau(x) \text{ is bounded and uniformly } \alpha\text{-Hölder continuous in } \Omega \cup \Lambda\}.$$

B_2 and B_2' are Banach spaces with the norms

$$\|\tau\|_2 = \sup_{x \in \Omega \cup \Lambda} |\tau(x)|,$$

and

$$\|\tau\|_2' = \|\tau\|_2 + \left[\sup_{x,y \in \Omega} + \sup_{x,y \in \Lambda}\right] \frac{|e^{-a|x|}\tau(x) - e^{-a|y|}\tau(y)|}{|x-y|^\alpha},$$

respectively. Set $B = C^\alpha(\Gamma) \times B_2$ and $B' = C^{1+\alpha}(\Gamma) \times B_2'$. For $[\eta, \tau]$ in B and x in R^n, we define the double-layer and volume potentials

$$(D(k)\eta)(x) = 2\int_\Gamma \eta(y)\frac{\partial}{\partial \nu_y} F_k^+(x-y)\, dS_y,$$

and

$$(V(k)\tau)(x) = \int_{R^n} \tau(y)\, F_k^+(x-y)\, e^{-2a|y|} dy.$$

Let f_1 be a function in $C^{1+\alpha}(\Gamma)$ and f_2 a function such that $e^{a|x|}f_2(x)$ is bounded and uniformly α-Hölder continuous in $\Omega\cup\Gamma$. We will look for outgoing solutions $u \in \bar{C}^1(\Omega)\cap C^{2+\alpha}(\Omega)$ of

$$\begin{aligned} &\lim_{\epsilon\downarrow 0} u(x+\epsilon\nu_x) = f_1(x) \quad \text{for } x \text{ on } \Gamma, \\ &(-\Delta + q(x) - k^2)u(x) = f_2(x) \quad \text{for } x \text{ in } \Omega \end{aligned} \tag{3.1}$$

in the form

$$u(x) = \mathcal{U}(k)[\eta, \tau](x) = D(k)\eta(x) + V(k)\tau(x), \qquad x \in \Omega, \tag{3.2}$$

with $[\eta, \tau] \in B'$.

To describe this construction, we need to summarize some results from potential theory. The operator $\mathcal{U}(k)$ is an analytic function of $k \in \mathcal{K}$ with values in the countably seminormed space

$$L(B', \bar{C}^1(\Omega\cup\Lambda)\cap C^{2+\alpha}(\Omega\cup\Lambda))$$

of continuous linear operators from B' to $\bar{C}^1(\Omega\cup\Lambda)\cap C^{2+\alpha}(\Omega\cup\Lambda)$.

Set

$$\begin{aligned} \lambda(x,k) &= q(x)\, e^{2a|x|} \quad \text{for } x \text{ in } \Omega, \\ &= (k^2+i)\, e^{2a|x|} \quad \text{for } x \text{ in } \Lambda \end{aligned} \tag{3.3}$$

and

$$M(k)\begin{bmatrix} \eta \\ \tau \end{bmatrix} = \begin{bmatrix} D(k) & V(k) \\ \lambda D(k) & \lambda V(k) \end{bmatrix}\begin{bmatrix} \eta \\ \tau \end{bmatrix} \tag{3.4}$$

Then $M(k)$ is an analytic function of $k \in \mathcal{K}$ with values in the Banach space $L(B,B')$ of bounded linear operators from B to B'.

Consider $[\eta, \tau]$ in B' and define $u(x)$ by (3.2) for x in $\Omega\cup\Lambda$. Then u is in $\bar{C}^1(\Omega\cup\Lambda)\cap C^{2+\alpha}(\Omega\cup\Lambda)$ and satisfies the equations

$$\lim_{\epsilon\downarrow 0} u(x\pm\epsilon\nu_x) = \pm\eta(x) + D\eta(x) + V\tau(x) \quad \text{for } x \text{ on } \Gamma, \tag{3.5}$$

$$(-\Delta - k^2)u(x) = \tau(x)\, e^{-2a|x|} \quad \text{for } x \text{ in } \Omega\cup\Lambda. \tag{3.6}$$

We rewrite equation (3.6) in the form

$$(-\Delta + q(x) - k^2)u(x) = e^{-2a|x|}[\tau(x) + q(x)\, e^{2a|x|}u(x)]$$

for x in Ω and

$$(-\Delta+i)u(x) = e^{-2a|x|}[\tau(x)+(k^2+i)\,e^{2a|x|}u(x)]$$

for x in Λ. Noting definition (3.3) of λ we obtain

$$
\begin{aligned}
e^{-2a|x|}[\tau + \lambda D\eta + \lambda V\tau](x) &= (-\Delta+q(x)-k^2)u(x) \quad \text{for } x \text{ in } \Omega, \\
&= (-\Delta+i)u(x) \quad \text{for } x \text{ in } \Lambda\ .
\end{aligned}
\tag{3.7}
$$

With equations (3.5) and (3.7) we obtain the following lemma.

LEMMA 3.1. *Define $f_2|\Lambda$ to be any function in $\bar{C}^{\alpha}(\Lambda)$. Suppose that $[\eta,\tau] \in B$ is a solution of*

$$(I+M(k))[\eta,\tau] = [f_1, e^{2a|x|}f_2]\,. \tag{3.8}$$

Then $[\eta,\tau]$ is in B'. Furthermore $u = \mathcal{U}(k)[\eta,\tau]$ is in $\bar{C}^1(\Omega\cup\Lambda) \cap C^{2+\alpha}(\Omega\cup\Lambda)$ and is a solution of (3.1) *and of the auxiliary differential equation*

$$(-\Delta+i)u(x) = f_2(x) \quad \text{for } x \text{ in } \Lambda. \tag{3.9}$$

PROOF. $[f_1, e^{2a|x|}f_2]$ is in B' and $M(k)$ maps B into B'. Hence (3.8) implies that $[\eta,\tau]$ is in B' and the lemma is a consequence of the above comments. Q.E.D.

We now turn to the proof that the function $u(x)$ of Lemma 3.1 is outgoing.

LEMMA 3.2. *Let each of the functions $u(x)$ and $\tilde{u}(x)$ be in one of the forms*

(i) $F_k^+(x-z)$,
(ii) $\int_\Gamma a(y)F_k^+(x-y)dS_y$,
(iii) $\int_\Gamma a(y)(\partial/\partial\nu_y)F_k^+(x-y)dS_y$, *or*
(iv) $\int_{R^n} b(y)F_k^+(x-y)dy$

with k in $\mathcal{K}$, z a point in Ω, $a(y)$ a continuous function on Γ, and $b(y)$ a Hölder continuous function on $\Omega\cup\Lambda$ such that $|b(y)| \leqq C\,e^{-a|y|}$ for all $y \in \Omega\cup\Lambda$. Then

$$
\begin{aligned}
&\int_{|x|=N}\left(\frac{\partial\tilde{u}}{\partial\nu} - \tilde{u}\frac{\partial u}{\partial\nu}\right)dS_x \\
&\qquad = \int_{|x|\geqq N}[u(-\Delta-k^2)\tilde{u}-\tilde{u}(-\Delta-k^2)u]\,dx \to 0
\end{aligned}
\tag{3.10}
$$

as $N\to\infty$.

PROOF. For $0 < \operatorname{Im} k < a$, the functions u, $\tilde{u}$ and their first and second order derivatives tend to zero exponentially as $|x|\to\infty$ so that

(3.10) is immediate. Both integrals in (3.10) are analytic functions of $k \in \mathcal{K}$. Consequently, the equation in (3.10) holds for all $k \in \mathcal{K}$. The volume integral tends to zero for any $k \in \mathcal{K}$ by a direct estimate.

COROLLARY 3.1. *For any* $[\eta, \tau] \in B'$, $u = \mathcal{U}(k)[\eta, \tau]$ *is outgoing for* $k \in \mathcal{K}$.

PROOF. Set $\tilde{u}(x) = F_k^+(x-z)$ $(z \in \Omega)$ in (3.10) to obtain (2.4) for $u = D\eta$ and $u = V\tau$. Q.E.D.

COROLLARY 3.2. *If* u *and* $\tilde{u}$ *are both in* $\bar{C}^1(\Omega) \cap C^{2+\alpha}(\Omega)$ *and are both outgoing for the same value of* $k \in \mathcal{K}$, *then Green's formula*

$$\int_\Gamma \left[u\frac{\partial \tilde{u}}{\partial \nu} - \tilde{u}\frac{\partial u}{\partial \nu} \right] dS_x \tag{3.11}$$
$$= \int_\Omega [u(-\Delta - k^2)\tilde{u} - \tilde{u}(-\Delta - k^2)u]\, dx$$

holds.

PROOF. Equation (2.5) shows that each of u and $\tilde{u}$ is a sum of functions in the forms studied in Lemma 3.2. Q.E.D.

Let $O(k)$ denote the vector space of outgoing solutions $v \in \bar{C}^1(\Omega) \cap C^{2+\alpha}(\Omega)$ of

$$\begin{aligned} v(x+0\nu_x) &= 0 \quad \text{for } x \text{ on } \Gamma, \\ (-\Delta + q(x) - k^2)v(x) &= 0 \quad \text{for } x \text{ in } \Omega. \end{aligned} \tag{3.12}$$

Corollary 3.2 yields the following necessary condition on f_1, f_2 for there to exist outgoing solutions of (3.1).

COROLLARY 3.3. *If there exists an outgoing solution* u *of* (3.1), *then*

$$\int_\Gamma f_1 \frac{\partial v}{\partial \nu} dS_x + \int_\Omega f_2 v\, dx = 0 \tag{3.13}$$

for all v *in* $O(k)$.

We will see shortly that (3.13) is also a sufficient condition for there to be outgoing solutions of (3.1).

Let $\mathcal{G}(k)$ denote the space of those $[f_1, e^{2a|x|}f_2]$ in B which satisfy (3.13) for all v in $O(k)$.

LEMMA 3.3. dim $O(k) =$ codim $\mathcal{G}(k)$.

LEMMA 3.4. Image $(I + M(k)) \subset \mathcal{G}(k)$.

Proof. The expression on the left side of (3.13) is of the form $T(v)[f_1, e^{2a|x|}f_2]$ with $T(v)$ a continuous linear functional on B. Hence $\mathcal{G}(k)$ is a closed subspace of B. Lemma 3.3 is immediate because linearly independent functions v in $O(k)$ yield linearly independent linear functionals $T(v)$ on B.

For $[\eta, \tau]$ in B' with τ of bounded support, we define $[f_1, e^{2a|x}f_2] = (I + M(k))[\eta, \tau]$.

Then $u = \mathcal{U}(k)[\eta, \tau]$ is an outgoing solution of (3.1) so that, by Corollary 3.3, $[f_1, e^{2a|x|}f_2]$ is in $\mathcal{G}(k)$. This establishes Lemma 3.4 because such $[\eta, \tau]$ are dense in B.

This procedure of constructing outgoing solutions of (3.1) is an adaptation and extension of a procedure used by Peter Werner [3] to study (3.1) for Im $k \geqq 0$. The next lemma generalizes the key calculation in [3].

Lemma 3.5. *$\mathcal{U}(k)$: Null space $(I + M(k)) \to O(k)$ is one-to-one for each $k \in \mathcal{K}$.*

Proof. Lemma 3.1 and Corollary 3.1 show that $\mathcal{U}(k)$ maps the null space of $I + M(k)$ into $O(k)$. Suppose that $[\eta_1, \tau_1]$ and $[\eta_2, \tau_2]$ are in the null space of $I + M(k)$ and yield the same function in $O(k)$. Set $[\eta, \tau] = [\eta_1, \tau_1] - [\eta_2, \tau_2]$ and $v = \mathcal{U}(k)[\eta, \tau]$. v is in $\bar{C}^1(\Lambda) \cap C^{2+\alpha}(\Lambda)$ and is identically zero in Ω. The normal derivatives of $D\eta$ and of $V\tau$ are continuous across Γ, so

$$\lim_{\epsilon \downarrow 0} \frac{\partial}{\partial \nu_x} v(x - \epsilon \nu_x) = 0 \quad \text{for } x \text{ in } \Gamma.$$

Also

$$(-\Delta + i)v(x) = 0 \quad \text{for } x \text{ in } \Lambda$$

so by Green's formula

$$0 = \int_\Gamma v \frac{\partial \bar{v}}{\partial \nu} dS_x = \int_\Lambda (|\nabla v|^2 + i|v|^2)dx.$$

The imaginary part of the last equation implies that $v(x)$ is also identically zero in Λ. Hence

$$\tau(x)e^{-2a|x|} = (-\Delta - k^2)v(x) = 0 \quad \text{for } x \text{ in } \Omega \cup \Lambda$$

and

$$2\eta(x) = \lim_{\epsilon \downarrow 0} [v(x + \epsilon \nu_x) - v(x - \epsilon \nu_x)] = 0 \quad \text{for } x \text{ in } \Gamma.$$

This shows that η and τ are zero and completes the proof of the lemma.

From Lemmas 3.3, 3.4, and 3.5 we obtain the inequalities

$$\begin{aligned} &\dim O(k) = \operatorname{codim} \mathcal{G}(k) \leqq \operatorname{codim} \text{Image}\,(I + M(k)) \\ &\dim O(k) \geqq \dim \text{null space}\,(I + M(k)). \end{aligned} \tag{3.14}$$

Since $M(k)$ is compact, the numbers on the right sides of inequalities (3.14) are equal and finite, and all the numbers in (3.14) are equal. Referring again to Lemmas 3.4 and 3.5, we see that

$$\mathcal{G}(k) = \text{Image}\,(I + M(k)) \tag{3.15}$$

and that

$$\mathcal{U}(k) : \text{null space}\,(I + M(k)) \rightarrow O(k) \tag{3.16}$$

is an isomorphism. Thus we have established the following theorem.

THEOREM 3.1. *Consider a fixed $k \in \mathcal{K}$ and functions f_1 and f_2 satisfying the conditions stated above problem* (3.1). *Problem* (3.1) *has an outgoing solution $u \in \bar{C}^1(\Omega) \cap C^{2+\alpha}(\Omega)$ if and only if condition* (3.13) *is satisfied for all v in $O(k)$.*

Set $f_2 = 0$ in Λ. Then $\mathcal{U}(k)$ is an isomorphism between the finite dimensional space of solutions $[\eta, \tau]$ of (3.8) *and the space of outgoing solutions of* (3.1).

We next state without proof the following well-known result.

LEMMA 3.6. *Outgoing solutions of* (3.1) *are unique for* $0 \leqq \operatorname{Im} k$.

Lemmas 3.5 and 3.6 combine to show that $I + M(k)$ is one-to-one for $0 \leqq \operatorname{Im} k$. $M(k) : B \rightarrow B$ is compact for each k because $M(k) : B \rightarrow B'$ is bounded. Hence $(I + M(k))^{-1} : B \rightarrow B$ is analytic for $0 \leqq \operatorname{Im} k$, and, by a general result of Steinberg [2], $(I + M(k))^{-1} : B \rightarrow B$ is meromorphic for $-a < \operatorname{Im} k < 0$. Since $M(k) : B \rightarrow B'$ is analytic, $M(k)(I + M(k))^{-1}$ is analytic [meromorphic] as an operator from B to B' and hence from B' to B' for $0 \leqq \operatorname{Im} k$ $[-a < \operatorname{Im} k < 0]$. The equation

$$(I + M(k))^{-1} = I - M(k)(I + M(k))^{-1}$$

thus shows that $(I + M(k))^{-1}$ is analytic [meromorphic] with values in $L(B', B')$ for $0 \leqq \operatorname{Im} k$ $[-a < \operatorname{Im} k < 0]$. Since $\mathcal{U}(k) : B' \rightarrow \bar{C}^1(\Omega \cup \Lambda) \cap C^{2+\alpha}(\Omega \cup \Lambda)$ is analytic, $\mathcal{U}(k)(I + M(k))^{-1} : B' \rightarrow C^{-1}(\Omega \cup \Lambda) \cap C^{2+\alpha}(\Omega \cup \Lambda)$ is analytic for $0 \leqq \operatorname{Im} k$ and meromorphic for $-a < \operatorname{Im} k < 0$. By Theorem 3.1, k is a pole of $(I + M(k))^{-1}$ if and only if outgoing solutions of (3.1) are not unique. Set

$$\mathcal{K}_1 = \mathcal{K} \setminus \{k : k \text{ is a pole of } (I + M(k))^{-1}\}.$$

We have established the following result.

THEOREM 3.2. *Consider f_1 in $C^{1+\alpha}(\Gamma)$ and $e^{a|x|}f_2(x)$ bounded and uniformly α-Hölder continuous on $\Omega \cup \Gamma$. Set $f_2(x) = 0$ for x in Λ. Then*

$$u(x, k) = \mathcal{U}(k)(I + M(k))^{-1}[f_1, e^{2a|x|}f_2] \tag{3.17}$$

is an analytic function of $k \in \mathcal{K}_1$ and a meromorphic function of $k \in \mathcal{K}$, and is, for each k in $\mathcal{K}_1$ the unique outgoing solution of (3.1).

If $[f_1, e^{2a|x|}f_2] \in B'$ depends continuously or analytically on an auxiliary parameter, then the corresponding dependence of $u \in \bar{C}^1(\Omega) \cap C^{2+\alpha}(\Omega)$ can be read off from (3.17). We give two such applications of Theorem 3.2 which will be needed below.

For $\omega \in S^{n-1}$, the unit sphere in R^n, and for $k \in \mathcal{K}_1$, let $v_+(x, k, \omega)$ be the outgoing solution of

$$\begin{aligned} v_+(x, k, \omega) &= -e^{ikx\cdot\omega} \quad \text{for } x \text{ on } \Gamma, \\ (-\Delta + q(x) - k^2)v_+(x, k, \omega) &= -q(x)e^{ikx\cdot\omega} \quad \text{for } x \text{ in } \Omega. \end{aligned} \tag{3.18}$$

(For $0 \neq \xi \in R^n$, we write $v_+(x, \xi)$ for $v_+(x, |\xi|, \xi/|\xi|)$.) Then the diffracted plane wave v_+ is an analytic function of $k \in \mathcal{K}_1$ and a meromorphic function of $k \in \mathcal{K}$.

For $\xi \in R^n$ and $k \neq 0$ with $\operatorname{Im} k \geqq 0$ let $v_1(x, k; \xi)$ be the outgoing solution of

$$\begin{aligned} v_1(x, k; \xi) &= -e^{ix\cdot\xi} \quad \text{for } x \text{ on } \Gamma, \\ (-\Delta + q(x) - k^2)v_1(x, k; \xi) &= -q(x)e^{ix\cdot\xi} \quad \text{for } x \text{ in } \Omega. \end{aligned} \tag{3.19}$$

v_1 is only used to prove the eigenfunction expansion theorem in the next section. We have the following estimates which are needed in that discussion.

Let K be a compact subset of $\{k \neq 0 : 0 \leqq \arg k \leqq \pi$ and k is not a pole of $(I + M(k))^{-1}\}$. There is a constant C depending only on K such that

$$\begin{aligned} &|v_1(x, k; \xi)| + \sum_{j=1}^{n} \left| \frac{\partial}{\partial x_j} v_1(x, k, \xi) \right| \\ &\qquad \leqq C(1 + |\xi|^2) \exp\{-\min(a, \operatorname{Im} k)|x|\} \end{aligned} \tag{3.20}$$

for $x \in \Omega \cup \Gamma$, $\xi \in R^n$ and $k \in K$.

4. **Eigenfunction expansions for H.** We now consider the exterior boundary value problem

$$\begin{aligned} (-\Delta + q(x) - k^2)u(x) &= f(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \Gamma, \end{aligned} \tag{4.1}$$

in the L^2-sense, i.e. we consider the problem of solving (4.1) in $L^2(\Omega)$ for the case $f \in L^2(\Omega)$. This leads to the study of the selfadjoint operator H, and in particular to the development of an eigenfunction expansion for H.

For functions f in

$$D(H') = \{f | f \in C^1(\Omega \cup \Gamma) \cap C^2(\Omega), f|_\Gamma = 0, \text{and } f, -\Delta f + qf \in L^2(\Omega)\}$$

define the operator H' by setting

$$H'f(x) = -\Delta f(x) + q(x) f(x), \qquad x \in \Omega.$$

H' is a real, symmetric, densely-defined operator in $L^2(\Omega)$, and it will be shown to be esentially selfadjoint with closure H. The spectral theorem allows us to express H in the form

$$H = \int_{-\infty}^{\infty} \lambda \, dE_\lambda.$$

We will calculate the spectral projections E_λ from the well-known formula

$$\text{(4.2)} \quad \begin{aligned} &\frac{1}{2}((E_{b+} + E_{b-})f, f) - \frac{1}{2}((E_{a+} + E_{a-})f, f) \\ &= \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \int_a^b \| [H - (\mu - i\epsilon)]^{-1} f \|^2 d\mu. \end{aligned}$$

For functions f in $C_0^\infty(R^n)$, $\xi \in R^n$ and k a regular point of $[I + M(k)]^{-1}$ with $\operatorname{Im} k > 0$, set

$$\text{(4.3)} \quad W_k f(x) = (2\pi)^{-n/2} \int_{R^n} \hat{f}(\xi) [|\xi|^2 - k^2]^{-1} w_+(x, k, \xi) d\xi, \; x \in \Omega,$$

where

$$w_+(x, k, \xi) = e^{ix\cdot\xi} + v_1(x, k, \xi),$$

and v_1 is given by (3.19). The definition of v_1 and the Fourier inversion formula show that $W_k f$ can be written in the form

$$\begin{aligned} W_k f(x) = &(H_0 - k^2)^{-1} f(x) \\ &+ (2\pi)^{-n/2} \int_{R^n} \hat{f}(\xi) [|\xi|^2 - k^2]^{-1} \\ &\cdot \mathcal{U}(k) [I + M(k)]^{-1} [-e^{ix\cdot\xi}, -q(x) e^{ix\cdot\xi}] \, d\xi, \end{aligned}$$

which is simply

$$\text{(4.4)} \qquad W_k f(x) = h_k(x) + \mathcal{U}(k) [I + M(k)]^{-1} [-h_k, -q h_k],$$

with

$$h_k(x) = (H_0 - k^2)^{-1} f.$$

Equation (4.4) shows that $W_k f$ is the outgoing solution of

$$(4.5) \qquad \begin{aligned} (-\Delta + q(x) - k^2)u(x) &= f(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \Gamma. \end{aligned}$$

If $w = (H - k^2)^{-1}(f|_\Omega)$, then w is also an outgoing solution of (4.5). This is a consequence of the following lemma, which is proved in [4].

LEMMA 4.1. *If $u \in D(H)$ and $(H - k^2)u = f|_\Omega$, with $f \in C_0^\infty(R^n)$, then*
(i) *u is bounded on Ω,*
(ii) *u is outgoing.*

The uniqueness of outgoing solutions therefore implies that

$$(4.6) \qquad W_k f(x) = (H - k^2)^{-1}(f|_\Omega)(x), \qquad x \in \Omega.$$

Thus $W_k f \in D(H)$ and $(H - k^2)W_k f(x) = f(x)$, $x \in \Omega$. But (4.4) shows that $W_k f \in C^1(\Omega \cup \Gamma) \cap C^2(\Omega)$, so that $W_k f \in D(H')$ and $(H' - k^2)W_k f = f$. H' therefore has deficiency indices (0, 0), and is essentially self-adjoint, with closure H.

Equation (4.6) gives a representation of $(H - k^2)^{-1}(f|_\Omega)$ in terms of the distorted plane waves w_+. We will apply (4.2) by determining a representation of $\|[H - (\mu - i\epsilon)]^{-1} f\|^2$ in terms of the functions w_+, as follows.

For $f \in C_0^\infty(\Omega)$, and $g \in C_0^\infty(R^n)$ we have

$$\begin{aligned} \int_\Omega (H - \bar{k}^2)^{-1} f(x) g(x)^* dx &= \int_\Omega f(x)\, (H - k^2)^{-1}(g|_\Omega)^*(x) dx \\ &= \int_\Omega f(x)\, W_k g(x)^* dx. \end{aligned}$$

Substitute the defining expression for $W_k g$ into the last integral and reverse the order of integration to obtain the equation

$$(4.7) \qquad \int_\Omega (H - \bar{k}^2)^{-1} f(x) g(x)^* dx = \int_{R^n} [|\xi|^2 - \bar{k}^2]^{-1} \hat{f}_k(\xi) \hat{g}(\xi)^* d\xi,$$

where

$$(4.8) \qquad \hat{f}_k(\xi) = (2\pi)^{-n/2} \int_\Omega f(x) w_+(x, k, \xi)^* dx, \qquad \xi \in R^n.$$

Estimates (3.20) and the construction of $v_1(x, k, \xi)$ show that $\hat{f}_k \in C(R^n)$. Concerning the integrability of $\hat{f}_k$ we have the following

LEMMA 4.2. *For each $f \in C_0^\infty(\Omega)$ and each compact subset K of $\{k \mid \operatorname{Im} k \geqq 0,\ [I + M(k)]^{-1}$ is regular$\}$, there exists a constant $C(f, K)$ such that*

$$\int_{R^n} \frac{|\hat{f}_k(\xi)|^2}{(1+|\xi|^2)^2} d\xi \leqq C(f, K) \tag{4.9}$$

for all $k \in K$.

We will give proof of Lemma 4.2 following the eigenfunction expansion Theorem 4.1.

If we extend the function $(H-\bar{k}^2)^{-1}f$ appearing on the left side of (4.7) by setting $(H-\bar{k}^2)^{-1}f(x) = 0$ for x in Λ, the Plancherel Theorem and (4.7) show that

$$[(H - \bar{k}^2)^{-1}f]^\wedge(\xi) = (|\xi|^2-\bar{k}^2)^{-1}\hat{f}_k(\xi), \qquad \xi \in R^n, \tag{4.10}$$

and

$$\|(H-\bar{k}^2)^{-1}f\|^2_{L^2(\Omega)} = \|(|\xi|^2-\bar{k}^2)^{-1}\hat{f}_k(\xi)\|^2_{L^2(R^n)}, \tag{4.11}$$

for $f \in C_0^\infty(\Omega)$ and k a regular point of $[I + M(k)]^{-1}$ with $\operatorname{Im} k > 0$.

Define transforms $\hat{f}^+$ for functions $f \in C_0^\infty(\Omega)$ by setting

$$\hat{f}^+(\xi) = (2\pi)^{-n/2} \int_\Omega f(x)\phi_+(x, \xi)^* dx, \qquad 0 \neq \xi \in R^n, \tag{4.12}$$

with

$$\phi_+(x, \xi) = e^{ix\cdot\xi} + v_+(x, \xi), \qquad v_+(x, \xi) = v_+(x, |\xi|, \xi)$$

given by (3.18). The transform $\hat{f}^+$ belongs to $C(R^n - \{0\})$. Set $P = \int_{0+}^\infty dE_\lambda$. We consider first the part of H in $PL^2(\Omega)$.

THEOREM 4.1. (i) *For $f \in C_0^\infty(\Omega)$, $\hat{f}^+$ is in $L^2(R^n)$ and*

$$\int_{R^n} |\hat{f}^+(\xi)|^2 d\xi = \int_\Omega |Pf(x)|^2 dx. \tag{4.13}$$

The map $f \to \hat{f}^+$ can be extended to a unitary mapping of $PL^2(\Omega)$ onto a closed subspace M of $L^2(R^n)$.[1]

(ii) *For $f \in L^2(\Omega)$, $0 < a < b < \infty$,*

$$(E_b - E_a)f(x) = (2\pi)^{-n/2} \int_{a<|\xi|^2<b} \hat{f}^+(\xi)\phi_+(x, \xi) d\xi \tag{4.14}$$

in $L^2(\Omega)$, the integral converging absolutely for each x in $\Omega \cup \Gamma$ to a

[1] It will be shown subsequently that $M = L^2(R^n)$. See §5.

function in $C^1(\Omega\cup\Gamma)\cap C^{2+\alpha}(\Omega)$ *which satisfies the boundary condition* $u = 0$ *on* Γ. *In* $L^2(\Omega)$,

$$Pf(x) = \lim_{a\to 0;\, b\to\infty} \int_{a<|\xi|^2<b} \hat{f}^+(\xi)\phi_+(x,\xi)d\xi.$$

(iii) $f \in D(HP) \Longleftrightarrow \int_{R^n} |\xi|^4|\hat{f}^+(\xi)|^2 d\xi < \infty$ *and*

$$(HPf)^{\wedge +}(\xi) = |\xi|^2\hat{f}^+(\xi) \quad \textit{for } f \in D(HP).$$

PROOF. Let a,b be fixed with $0 < a < b < \infty$, and choose $\delta > 0$ so that $\delta < a < b < \delta^{-1}$. Write $k^2 = \mu + i\epsilon$ ($\operatorname{Im} k \geqq 0$) for $\delta \leqq \mu \leqq \delta^{-1}$, $0 \leqq \epsilon \leqq \epsilon_0$ where $\epsilon_0 > 0$ is chosen so as to have

$$K = \{k \mid k^2 = \mu + i\epsilon,\ \delta \leqq \mu \leqq \delta^{-1},\ 0 \leqq \epsilon \leqq \epsilon_0\}$$

belong to the domain of regularity of $[I + M(k)]^{-1}$. For $f \in C_0^\infty(\Omega)$, (4.2) and (4.11) give

$$\tag{4.15} \begin{aligned} &\tfrac{1}{2}((E_{b+} + E_{b-})f,f) - \tfrac{1}{2}((E_{a+} + E_{a-})f,f) \\ &= \lim_{\epsilon\to 0} \frac{\epsilon}{\pi} \int_a^b d\mu \int_{R^n} \frac{|\hat{f}_k(\xi)|^2 d\xi}{(|\xi|^2-\mu)^2 + \epsilon^2}. \end{aligned}$$

The integral over R^n we split into two parts:

$$\int_{R^n} = \int_{|\xi|^2<\delta^{-1}} + \int_{|\xi|^2\geqq\delta^{-1}},$$

and treat each of the corresponding terms in (4.15) separately. For the second term, $|\xi|^2 \geqq \delta^{-1}$ and so for μ with $a \leqq \mu \leqq b$,

$$\frac{\epsilon}{(|\xi|^2 - \mu)^2 + \epsilon^2} \leqq \frac{\epsilon}{(|\xi|^2 + C)^2}$$

where $C = \min(a-\delta, \delta^{-1}-b)$. Lemma 4.2 therefore implies that

$$\tag{4.16} \lim_{\epsilon\to 0} \frac{\epsilon}{\pi}\int_a^b d\mu \int_{|\xi|^2\geqq\delta^{-1}} \frac{|\hat{f}_k(\xi)|^2 d\xi}{(|\xi|^2-\mu)^2 + \epsilon^2} = 0,$$

For $\xi^2 < \delta^{-1}$ we observe that $\hat{f}_k(\xi)$ is bounded uniformly;

$$|\hat{f}_k(\xi)| \leqq M \text{ for } k \in K,\quad |\xi|^2 \leqq \delta^{-1},$$

so that

$$\tag{4.17} \frac{\epsilon}{\pi}\int_a^b \frac{|\hat{f}_k(\xi)|^2 d\mu}{(|\xi|^2-\mu)^2+\epsilon^2} \leqq M^2 \left| \left[\operatorname{Tan}^{-1}\frac{|\xi|^2-\mu}{\epsilon}\right]_a^b \right| \leqq C.$$

Fubini's Theorem therefore applies, and

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\int_a^b d\mu\int_{|\xi|^2<\delta^{-1}}\frac{|\hat{f}_k(\xi)|^2d\xi}{(|\xi|^2-\mu)^2+\epsilon^2}$$

(4.18)

$$=\lim_{\epsilon\to 0}\int_{|\xi|^2<\delta^{-1}}d\xi\,\frac{\epsilon}{\pi}\int_a^b\frac{|\hat{f}_k(\xi)|^2d\mu}{(|\xi|^2-\mu)^2+\epsilon^2}\,.$$

For each $\xi\in R^n$, $\hat{f}_k(\xi)$ is continuous in $k\in K$ and so is continuous in ϵ and μ for $0\leqq\epsilon\leqq\epsilon_0$, $\delta\leqq\mu\leqq\delta^{-1}$, with

$$\lim_{k\to|\xi|;\ \mathrm{Im}\ k>0}\hat{f}_k(\xi)=\hat{f}^+(\xi).$$

Therefore (see Titchmarsh, *Introduction to the Theory of Fourier Integrals*, p. 31)

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\int_a^b\frac{|\hat{f}_k(\xi)|^2d\mu}{(|\xi|^2-\mu)^2+\epsilon^2}&=|\hat{f}^+(\xi)|^2\quad\text{if } a<|\xi|^2<b,\\&=0\quad\text{if } |\xi|^2<a\quad\text{or } |\xi|^2>b.\end{aligned}\tag{4.19}$$

Formulas (4.17), (4.18) and (4.19) together with the Lebesgue Bounded Convergence Theorem now show that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\int_a^b d\mu\int_{|\xi|^2<\delta^{-1}}\frac{|\hat{f}_k(\xi)|^2d\xi}{(|\xi|^2-\mu)^2+\epsilon^2}=\int_{a<|\xi|^2<b}|\hat{f}^+(\xi)|^2d\xi.\tag{4.20}$$

(4.15), (4.16) and (4.20) then give

$$\frac{1}{2}((E_{b+}+E_{b-})f,f)-\frac{1}{2}((E_{a+}+E_{a-})f,f)=\int_{a<|\xi|^2<b}|\hat{f}^+(\xi)|^2d\xi.\tag{4.21}$$

Let a approach b in (4.21) to find

$$((E_{b+}-E_{b-})f,f)=0,$$

so that $E_{b+}=E_{b-}=E_b$ for all $b>0$, and consequently (4.21) becomes simply

$$((E_b-E_a)f,f)=\int_{a<|\xi|^2<b}|\hat{f}^+(\xi)|^2d\xi\,.\tag{4.22}$$

Let $a\to 0$ and $b\to\infty$ in (4.22) to obtain formula (4.13) of part (i) of the theorem. The extension to arbitrary functions $f\in L^2(\Omega)$ is obvious.

By polarization, (4.22) gives

$$(4.23)\qquad ((E_b - E_a)f, g) = \int_{a<|\xi|^2<b} \hat{f}^+(\xi)\hat{g}^+(\xi)^* d\xi,$$

which shows that $\{(E_b - E_a)f\}^{\hat{}\,+}$ is the projection of $\chi_{[a,b]}(|\xi|^2)\hat{f}^+(\xi)$ onto the range of "$\hat{}\,+$". Here $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$. Since $(E_b - E_a)f \in PL^2(\Omega)$ for $0 < a < b < \infty$, and because "$\hat{}\,+$" is an isometry from $PL^2(\Omega)$ to $L^2(R^n)$, it follows that

$$(4.24)\qquad \{(E_b - E_a)f\}^{\hat{}\,+}(\xi) = \chi_{[a,b]}(|\xi|^2)\hat{f}^+(\xi).$$

Part (iii) of the thoerem is a consequence of (4.24).

To prove (ii), let $f \in L^2(\Omega)$ and $g \in C_0^\infty(\Omega)$, substitute the defining expression for $\hat{g}^+$ into (4.23) and invert the order of integration to find

$$(4.25)\qquad ((E_b - E_a)f, g) = (2\pi)^{-n/2}\int_\Omega g(x)^* dx \int_{a<|\xi|^2<b} \hat{f}^+(\xi)\phi_+(x, \xi)d\xi.$$

The interchange of order of integration is valid because $\hat{f}^+ \in L^1$ $(a \leqq |\xi|^2 \leqq b)$ and $\phi_+(x, \xi)$ is bounded for $a \leqq |\xi|^2 \leqq b$, $x \in \operatorname{supp}(g)$. Part (ii) of the theorem now follows easily from (4.25).

The above theorem gives an expansion for functions in $PL^2(\Omega)$ in terms of the generalized eigenfunctions $\phi_+(x, \xi)$, i.e. solutions of

$$(-\Delta + q(x) - |\xi|^2)\phi_+(x, \xi) = 0, \qquad x \text{ in } \Omega,$$
$$\phi_+(x, \xi) = 0, \qquad x \text{ in } \Gamma,$$

of the form $\phi_+(x, \xi) = e^{ix\cdot\xi} + v_+(x, \xi)$, $v_+(x, \xi)$ outgoing. A corresponding result holds with $\phi_+(x, \xi)$ replaced by $\phi_-(x, \xi) = \phi_+(x, -\xi)^*$.

The part of H in $(I-P)L^2(\Omega)$ has pure point spectrum. This is a consequence of the

LEMMA 4.3. *For* $\operatorname{Im} k > 0$, $O(k) =$ Null space $(H - k^2)$.

The proof of this lemma follows easily from Theorem 3.1 and Lemma 4.1.

Lemma 3.5 states the $O(k)$ is nonempty if and only if the null space of $I + M(k)$ is nonempty, i.e. if and only if k is a pole of $[I + M(k)]^{-1}$. Thus Lemma 4.3 shows that if $H - k^2$ is one-one, then k must be a regular point of $[I + M(k)]^{-1}$, and so k is a pole of $[I + M(k)]^{-1}$ if and only if k^2 is an eigenvalue of H. Let $k_1 = i\alpha_1$, $k_2 = i\alpha_2$, $0 < \alpha_1 < \alpha_2$, be two consecutive poles of $[I + M(k)]^{-1}$ (necessarily on the positive imaginary axis). If $a, b < 0$ are chosen so that $-\alpha_2^2 < a < b < -\alpha_1^2$, then the segment $k = it$, $\sqrt{(-b)} \leqq t \leqq \sqrt{(-a)}$ is free of poles of $[I + M(k)]^{-1}$. Set

$$K = \{k | k^2 = \mu + i\epsilon,\ a \leqq \mu \leqq b,\ 0 \leqq \epsilon \leqq \epsilon_0\}$$

where $\epsilon_0 > 0$ is chosen so small that K belongs to the domain of regularity of $[I + M(k)]^{-1}$. A glance at formula (4.15) with this choice of a, b and K shows that the factor $[(|\xi|^2 - \mu)^2 + \epsilon^2]^{-1}$ is dominated by $[|\xi|^2 + b]^{-2}$, so that Lemma 4.2 and equation (4.15) yield

$$(E_{b+} + E_{b-}) - (E_{a+} + E_{a-}) = 0,$$

i.e. there is no other spectrum of H in $(-\infty, 0)$ other than the point spectrum.

An immediate consequence of Lemma 4.3 is

COROLLARY 4.1. *The multiplicity of each eigenvalue k^2 of H is finite, and equal to* dim $O(k)$.

In case of odd space dimensions n, $[I + M(k)]^{-1}$ is meromorphic in $\operatorname{Im} k > -a$, so that the lower semiboundedness of H and Lemma 4.3 gives

COROLLARY 4.2. *The point spectrum of H is finite if n is odd.*

PROOF OF LEMMA 4.2. The formula (4.9) is true with $\hat{f}_k$ replaced by $\hat{f}$, so it will suffice to prove that

$$\left| \int_{R^n} \hat{g}(\xi)[\hat{f}_k(\xi) - \hat{f}(\xi)^*]\, d\xi \right| \leqq C(f, K) \int_{R^n} [1 + |\xi|^2]^2 |\hat{g}(\xi)|^2 d\xi \tag{4.26}$$

holds for all $g \in C_0^\infty(R^n)$, $k \in K$.

The defining equation (4.8) for $\hat{f}_k$ gives

$$\hat{f}_k(\xi) - \hat{f}(\xi) = (2\pi)^{-n/2} \int_\Omega f(x) v_1(x, k, \xi)^* dx,$$

so by reversing the order of integration,

$$\int_{R^n} \hat{g}(\xi)[\hat{f}_k(\xi) - \hat{f}(\xi)]^* d\xi = \int_\Omega Q_k g(x) f(x)^* dx, \tag{4.27}$$

with

$$\begin{aligned} Q_k g(x) &= (2\pi)^{-n/2} \int_{R^n} \hat{g}(\xi) v_1(x, k, \xi) d\xi \\ &= \mathcal{U}(k)[I + M(k)]^{-1}[-g, -qg] \end{aligned} \tag{4.28}$$

(The last equation follows exactly as in (4.4).) The above equation identifies $Q_k g$ as the outgoing solution of the inhomogeneous boundary value problem

$$(4.29)\qquad \begin{aligned}(-\Delta+q(x)-k^2)u(x) &= -q(x)g(x), && x\in\Omega,\\ u(x) &= -g(x), && x\in\Gamma.\end{aligned}$$

In view of equation (4.27), it will suffice to prove the estimate

$$(4.30)\qquad \|Q_k g\|^2_{L^2(D)} \leqq C(f)\int_{R^n}[1+|\xi|^2]^2|\hat{g}(\xi)|^2 d\xi$$

with $D = \operatorname{supp}(f)$, for the Schwartz inequality applied to (4.27), together with (4.30), implies (4.26). Estimate (4.30) is an immediate consequence of special estimates for the Green's function $G(x, y, k)$ for the boundary value problem (4.1). We state below the properties of G which imply (4.30). The construction of G and the proof of the estimates depend on a careful application of the potential theory developed in the initial sections of this paper, and can be found in §5 of [1].

THEOREM 4.2. *Let $y\in\Omega$, and let K be as in Lemma 4.2.*

(i) *The Green's function $G(x, y, k)$ exists for $k\in K$, and is the outgoing solution of*

$$\begin{aligned}(-\Delta_x+q(x)-k^2)G(x,y,k) &= 0, && x\in\Omega-\{y\},\\ G(x,y,k) &= 0, && x\in\Gamma,\end{aligned}$$

which behaves like $F_k^+(x-y)$ near $x-y=0$.

(ii) *If u is outgoing for k and $u\in H^2(\bar{\Omega})\cap C^{2+\alpha}(\Omega)$, then*

$$(4.31)\qquad \begin{aligned}u(x) = &\int_\Gamma\left[u(y)\frac{\partial}{\partial\nu_y}G(y,x,k)-G(y,x,k)\frac{\partial u}{\partial\nu_y}(y)\right]dS_y\\ &+\int_\Omega G(y,x,k)(-\Delta+q(y)-k^2)u(y)dy\end{aligned}$$

for all $x\in\Omega$.

(iii) *For each compact subset D of Ω there is a constant $C(D, K)$ such that*

$$(a)\qquad |(\partial/\partial y)^m G(y,x,k)|\leqq C(D,K)$$

for $|m|\leqq 1$, $y\in\Gamma$, $k\in K$ and $x\in D$;

$$(b)\qquad \int_\Omega |q(y)G(y,x,k)|dy \leqq C(K,D)$$

for $x\in D$, $k\in K$;

(c) $$\int_D |G(y, x, k)|dx \leqq C(K, D)$$

for $y \in \Omega$, $k \in K$.

In order to prove the estimate (4.31), apply (4.30) to represent $Q_k f$ in terms of G. The estimates in (iii) of Theorem 4.2 can then be used to show that (4.30) is true. We only remark that it is necessary to use the estimate (see [5])

$$\sum_{|m|\leqq 1} \int_\Gamma |(\partial/\partial x)^m g(x)|^2 \, dx \leqq M \sum_{|m|\leqq 2} \int_\Omega |(\partial/\partial x)^m g(x)|^2 \, dx \tag{4.32}$$

in obtaining (4.30) from the representation of $Q_k f$.

5. **The operators** $W_\pm$ **and S.** Let $\mu \in C_0^\infty(R^n)$ be a function which vanishes inside of and near Γ, and equals 1 for large x. Define J: $L^2(R^n) \to L^2(\Omega)$ by $(Jf)(x) = \mu(x)f(x)$. Clearly $\|J\| \leqq 1$, and J maps $D(H_0)$ into $D(H)$. Set

$$W(t) = e^{itH}Je^{-itH_0}, \qquad -\infty < t < \infty, \tag{5.1}$$

and define the wave operators $W_\pm$ by

$$W_\pm = \operatorname*{s-lim}_{t \to \pm\infty} W(t). \tag{5.2}$$

It is clear that

$$\|W(t)\| \leqq 1, \quad -\infty < t < \infty. \tag{5.3}$$

THEOREM 5.1. (i) $W_\pm$ *exist, are independent of the choice of* μ, *and are unitary maps of* $L^2(R^n)$ *onto* $PL^2(\Omega)$.

(5.4) (ii) $$(W_+f)^{\wedge -} = \hat{f} = (W_-f)^{\wedge +} \quad \textit{for } f \in L^2(R^n).$$

PROOF. We shall consider the case $n \geqq 3$, as the case $n = 2$ requires an additional argument. Let $f \in L^2(R^n)$ be such that $\hat{f} \in C_0(R^n)$, $\hat{f}(\xi) = 0$ near $\xi = 0$. Choose a, b, $0 < a < b < \infty$, so that $\hat{f}(\xi) = 0$ if $|\xi|^2 \notin \Delta = [a, b]$. An application of Theorem 4.1 gives

$$\begin{aligned} &[E(\Delta)W(t)f]^{\wedge -}(\xi) \\ &= \chi_\Delta(|\xi|^2)(2\pi)^{-n/2} \int_\Omega \phi_-(x, \xi)^* \mu(x)(e^{it[|\xi|^2 - H_0]}f)(x)dx. \end{aligned} \tag{5.5}$$

Since f belongs to the Schwartz class $\mathcal{S}$, $e^{it[|\xi|^2 - H_0]}f \in \mathcal{S}$, and for ξ with $|\xi|^2 \in \Delta$,

$$|(e^{it[|\xi|^2 - H_0]}f)(x)| + \left| \frac{d}{dt} (e^{it[|\xi|^2 - H_0]}f)(x) \right| \leqq C(f)t^{-n/2} \tag{5.6}$$

for $|t| \geqq 1$, $x \in R^n$. (See the Kato-Kuroda lectures for estimate (5.6).) Differentiate (5.5) with respect to t, and then integrate by parts:

$$\begin{aligned} &\frac{d}{dt}[E(\Delta)W(t)f]^{\wedge -}(\xi) \\ (5.7)\quad &= \chi_\Delta(|\xi|^2)i(2\pi)^{-n/2}\int_\Omega \phi_-(x,\xi)^*\mu(x)(\Delta+|\xi|^2)(e^{it[|\xi|^2-H_0]}f)(x)dx \\ &= \chi_\Delta(|\xi|^2)i(2\pi)^{-n/2}\int_\Omega (e^{it[|\xi|^2-H_0]}f)(x)(\Delta+|\xi|^2)[\mu(x)\phi_-(x,\xi)^*]\,dx. \end{aligned}$$

The last line of (5.7) shows that

$$\frac{d}{dt}[E(\Delta)W(t)f]^{\wedge -}(\xi) = O(t^{-n/2}) \quad \text{as } |t| \to \infty, \tag{5.8}$$

uniformly in ξ, since

$$\begin{aligned} (\Delta+|\xi|^2)\mu(x)\phi_-(x,\xi) &= \mu(x)q(x)\phi_-(x,\xi) + 2\nabla\mu(x)\cdot\nabla\phi_-(x,\xi) \\ &\quad + (\Delta\mu(x))\phi_-(x,\xi) \end{aligned}$$

is bounded in $L^1(R^n)$ for ξ with $|\xi|^2 \in \Delta$, and estimate (5.6) holds for such ξ. Therefore $[E(\Delta)W(T)f]^{\wedge -}(\xi)$ converges uniformly in ξ as $T \to \infty$, as integration of (5.8) readily shows. We now calculate $\lim_{T\to\infty}[E(\Delta)W(T)f]^{\wedge -}(\xi)$ by taking the Abelian limit,

$$\begin{aligned} &\lim_{T\to\infty}[E(\Delta)W(T)f]^{\wedge -}(\xi) - [E(\Delta)Jf]^{\wedge -}(\xi) \\ (5.9)\quad &= \lim_{\epsilon\to 0^+}\int_0^\infty e^{-\epsilon t}\frac{d}{dt}[E(\Delta)W(t)f]^{\wedge -}(\xi)dt. \end{aligned}$$

Multiply both sides of (5.7) by $e^{-\epsilon t}$, $\epsilon > 0$, integrate over $[0, \infty]$ with respect to t and use the equation

$$\int_0^\infty \epsilon^{-\epsilon t}(e^{it[|\xi|^2-H_0]}f)(x)dt = i([\Delta+|\xi|^2+i\epsilon]^{-1}f)(x)$$

to obtain

$$\begin{aligned} &\int_0^\infty e^{-\epsilon t}\frac{d}{dt}[E(\Delta)W(t)f]^{\wedge -}(\xi)dt \\ (5.10)\quad &= -(2\pi)^{-n/2}\chi_\Delta(|\xi|^2)\int_\Omega u_+(x,k)(\Delta+|\xi|^2)(\mu(x)\phi_-(x,\xi)^*)dx \end{aligned}$$

with

$$u_+(x,k) = -\int_{R^n} F_k^+(x,y)f(y)dy, \qquad k^2 = |\xi|^2+i\epsilon.$$

As $\epsilon \to 0^+$, $u_+(x, k)$ converges to

$$u_+(x) = -\int_{R^n} F^+_{|\xi|}(x, y)f(y)dy \tag{5.11}$$

uniformly in $x \in R^n$. u_+ is the outgoing solution of

$$(\Delta + |\xi|^2)u_+(x) = f(x), \quad x \in R^n. \tag{5.12}$$

Consequently, (5.9), (5.10) and (5.11) yield

$$\begin{aligned} &\lim_{T\to\infty} [E(\Delta)W(T)f]^{\wedge -}(\xi) - [E(\Delta)Jf]^{\wedge -}(\xi) \\ &\quad = -(2\pi)^{-n/2}\chi_\Delta(|\xi|^2) \int_\Omega u_+(x)(\Delta + |\xi|^2)[\mu(x)\phi_-(x, \xi)^*]\, dx. \end{aligned} \tag{5.13}$$

We now make two trivial modifications in the integral on the right side of (5.13). Since $(\Delta + |\xi|^2)\mu\phi_-$ has its support in Ω, we can replace the region of integration Ω by R^n. We can replace the term $(\Delta + |\xi|^2)\mu\phi_-$ by $(\Delta + |\xi|^2)(\mu(x)\phi_-(x, \xi) - e^{-ix\cdot\xi})$ since $(\Delta + |\xi|^2)e^{-ix\cdot\xi} = 0$. The result of these modifications in (5.13) is

$$\begin{aligned} &\lim_{T\to\infty} [E(\Delta)W(T)f]^{\wedge -}(\xi) - [E(\Delta)Jf]^{\wedge -}(\xi) \\ &\quad = -(2\pi)^{-n/2}\chi_\Delta(|\xi|^2)\int_{R^n} u_+(x)(\Delta + |\xi|^2)(\mu(x)\phi_-(x, \xi) - e^{ix\cdot\xi})^* dx. \end{aligned} \tag{5.14}$$

For large values of x, $\mu(x)\phi_-(x, \xi) - e^{ix\cdot\xi\,*}$ equals $v_-(x, \xi)^*$, which is outgoing. Since $u_+(x)$ is also outgoing we can integrate by parts in (5.14) and apply (5.12), with the result

$$\begin{aligned} &\lim_{T\to\infty} [E(\Delta)W(T)f]^{\wedge -}(\xi) - [E(\Delta)Jf]^{\wedge -}(\xi) \\ &\quad = -(2\pi)^{-n/2}\chi_\Delta(|\xi|^2) \int_{R^n} (\mu(x)\phi_-(x, \xi)^* - e^{-ix\cdot\xi})f(x)dx \\ &\quad = \chi_\Delta(|\xi|^2)[(Jf)^{\wedge -}(\xi) - \hat{f}(\xi)]. \end{aligned} \tag{5.15}$$

This shows that

$$\lim_{T\to\infty} [E(\Delta)W(T)f]^{\wedge -}(\xi) = \hat{f}(\xi) \tag{5.16}$$

uniformly in $\xi \in R^n$, since both sides of (5.15) are zero if $|\xi|^2 \notin \Delta$. Consequently,

$$[E(\Delta)W(T)f]^{\wedge -} \underset{T\to\infty}{\to} \hat{f} \text{ in } L^2(R^n). \tag{5.17}$$

Since $E(\Delta)W(T)f \in PL^2(\Omega)$, (4.13) and (5.17) yield

$$\lim_{T\to\infty} \|E(\Delta)W(T)f\|_{L^2(\Omega)} = \|\hat{f}\|_{L^2(R^n)} = \|f\|_{L^2(R^n)}. \tag{5.18}$$

Because of (5.3) we have

$$\|(I - E(\Delta))W(T)f\|^2 = \|W(T)f\|^2 - \|E(\Delta)W(T)f\|^2$$
$$\leqq \|f\|^2 - \|E(\Delta)W(T)f\|^2,$$

so that (5.18) implies

$$\lim_{T\to\infty} \|(I-E(\Delta))W(T)f\|_{L^2(\Omega)} = 0. \tag{5.19}$$

Therefore

$$[(I-E(\Delta))W(T)f]^{\wedge -} \underset{T\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(R^n),$$

which together with (5.17) shows that

$$\lim_{T\to\infty} [W(T)f]^{\wedge -} = \hat{f} \quad \text{in } L^2(R^n). \tag{5.20}$$

Finally, (4.13) and (5.20) imply that $PW(T)f$ converges in $L^2(\Omega)$ as $T \to \infty$. However, from (5.19) it follows that

$$(I-P)W(T)f \underset{T\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(\Omega)$$

so we obtain the existence of

$$\lim_{T\to\infty} [PW(T)f + (I-P)W(T)f] = \lim_{T\to\infty} W(T)f = W_+f$$

in $L^2(\Omega)$. Equation (5.20) now implies that

$$[W_+f]^{\wedge -} = \hat{f}, \tag{5.21}$$

which proves half of (5.4), at least for the class of functions f considered. The extension of (5.21) to arbitrary $f \in L^2(R^n)$ is immediate.

(5.21) shows that the transformation "$\wedge -$" is a unitary transformation from $PL^2(\Omega)$ onto $L^2(R^n)$. Let $F_- : PL^2(\Omega) \to L^2(R^n)$ denote the transformation $f \to \hat{f}^-$, and let $F : L^2(R^n) \to L^2(R^n)$ denote the Fourier transformation $f \to \hat{f}$. Then (5.21) can be written as

$$F_-W_+ = F, \tag{5.22}$$

which implies that

$$W_+ = F_-^*F \tag{5.23}$$

is unitary from $L^2(R^n)$ onto $L^2(R^n)$. In addition, (5.23) yields the formula

$$W_+f(x) = (2\pi)^{-n/2} \int_{R^n} \hat{f}(\xi)\phi_-(x, \xi)d\xi \quad \text{in } L^2(R^n). \tag{5.24}$$

If we define $F_+ : PL^2(\Omega) \to L^2(R^n)$ to be the transformation $f \to \hat{f}^+$, a proof analogous to the above shows the existence and unitarity of W_-; $L^2(R^n) \to PL^2(\Omega)$, and

$$W_- = F_+^* F. \tag{5.25}$$

This completes the proof of Theorem 5.1 and establishes the unitarity of F_+ and F_-, and so completes the proof of Theorem 4.1 also (see the footnote following Theorem 4.1).

The scattering operator $\mathcal{S} = L^2(R^n) \to L^2(R^n)$ is defined by

$$\mathcal{S} = W_+^* W_-, \tag{5.26}$$

and is clearly unitary. $\mathcal{S}$ can also be written in the form

$$\mathcal{S} = F^* F_- F_+^* F. \tag{5.27}$$

The "scattering matrix" S is defined by setting

$$S = F_- F_+^*, \tag{5.28}$$

and is unitary since $\mathcal{S} = F^* SF$. We shall now obtain a representation of the scattering matrix S.

Let $f \in L^2(R^n)$ be such that $\hat{f} \in C_0^\infty(R^n)$, with $\hat{f}(\xi) = 0$ near $\xi = 0$. It follows from (5.25) that

$$S\hat{f} = F_- W_- f = [W_- f]^{\wedge -}. \tag{5.29}$$

Proceeding as in the proof of Theorem 5.1, we calculate $S\hat{f}$ as an Abelian limit:

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{0} e^{\epsilon t} \frac{d}{dt} [W(t)f]^{\wedge -}(\xi)dt = (Jf)^{\wedge -}(\xi) - (S\hat{f})(\xi). \tag{5.30}$$

In this situation, however, we obtain

$$\begin{aligned} &\int_{-\infty}^{0} e^{\epsilon t} \frac{d}{dt} [W(t)f]^{\wedge -}(\xi)dt \\ &\quad = (2\pi)^{-n/2} \int_{R^n} u_-(x, k)(\Delta + |\xi|^2)(\mu(x)\phi_-(x, \xi)^*)dx, \end{aligned} \tag{5.31}$$

where

$$u_-(x, k) = -\int_{R^n} F_k^-(x, y)f(y)dy, \qquad k^2 = |\xi|^2 - i\epsilon.$$

As $\epsilon \to 0^+$, $u_-(x, k)$ converges to

$$u_-(x) = -\int_{R^n} F_{|\xi|}^-(x, y)f(y)dy$$

uniformly in $x \in R^n$. u_- is the incoming solution of

$$(\Delta + |\xi|^2)u_-(x) = f(x), \qquad x \in R^n.$$

As before, we modify the integrand on the right side of (5.31) by inserting the term $0 = (\Delta + |\xi|^2)e^{-ix\cdot\xi}$, and let $\epsilon \to 0^+$ to obtain from (5.30) and (5.31) the equation

$$(Jf)\hat{}^{-}(\xi) - (S\hat{f})(\xi) = (2\pi)^{-n/2}\int_{R^n} u_-(x)(\Delta + |\xi|^2)(\mu(x)\phi_-(x,\xi) - e^{ix\cdot\xi})^* dx. \tag{5.32}$$

Here the factor $\mu(x)\phi_-(x, \xi) - e^{ix\cdot\xi\,*}$ equals $v_-(x, \xi)^*$ for large values of x, and so is outgoing for large x, while the factor $u_-(x)$ is incoming. Integration by parts in (5.32) will now introduce a boundary term:

$$\begin{aligned}(Jf)\hat{}^{-}(\xi) - (S\hat{f})(\xi) &= (2\pi)^{-n/2}\int_{R^n} \phi_-(x,\xi)^*\mu(x)f(x)dx - (2\pi)^{-n/2}\int_{R^n} \hat{f}(\xi)e^{ix\cdot\xi} \\ &\quad - \lim_{R\to\infty}(2\pi)^{-n/2}\int_{|x|=R}\left[u_-(x)\frac{\partial \bar{v}_-}{\partial|x|}(x,\xi) - \bar{v}_-(x,\xi)\frac{\partial u_-}{\partial |x|}(x)\right] dS_x.\end{aligned} \tag{5.33}$$

The boundary integral in (5.33) may be evaluated by utilizing the asymptotic expressions for $\bar{v}_-$ and u_-. Setting $x = R\theta$, where $R = |x|$, $\theta = |x|^{-1}x$, and $\xi = |\xi|\omega$, with $\omega = |\xi|^{-1}\xi$, we have

$$u_-(R\theta) \sim \left[\frac{-i}{2|\xi|}\left(\frac{i|\xi|}{2\pi}\right)^{(n-1)/2}(2\pi)^{n/2}\hat{f}(|\xi|, -\theta)\right]\frac{e^{-i|\xi|R}}{R^{(n-1)/2}},$$

$$\frac{\partial u_-}{\partial |x|}(R\theta) \sim -i|\xi|u_-(R\theta),$$

$$v_-(R\theta, |\xi|, \omega)^* \sim s_-(\theta, |\xi|, \omega)^*\frac{e^{i|\xi|R}}{R^{(n-1)/2}},$$

$$\frac{\partial v_-}{\partial |x|}(R\theta, |\xi|, \omega)^* \sim i|\xi|v_-(R\theta, |\xi|, \omega)^*$$

as $R \to \infty$. Therefore, we obtain

$$\begin{aligned}\lim_{R\to\infty}(2\pi)^{-n/2}\int_{|x|=R}\left[u_-(x)\frac{\partial v_-^*}{\partial |x|} - v_-^*(x,\xi)\frac{\partial u_-}{\partial |x|}(x)\right] dS_x &= \left(\frac{i|\xi|}{2\pi}\right)^{(n-1)/2}\int_{S^{n-1}} s_-(\theta, |\xi|, \omega)^*\hat{f}(|\xi|, -\theta)d\theta \\ &= \left(\frac{i|\xi|}{2\pi}\right)^{(n-1)/2}\int_{S^{n-1}} s_-(-\theta, |\xi|, \omega)^*\hat{f}(|\xi|, \theta)d\theta.\end{aligned} \tag{5.34}$$

(5.33) and (5.34) now establish the

THEOREM 5.2. *The scattering matrix S has the following representation:*

$$S\hat{f}(\xi) = \hat{f}(\xi) + \left(\frac{i|\xi|}{2\pi}\right)^{(n-1)/2} \int_{S^{n-1}} s_-(-\theta, |\xi|, \omega)^* \hat{f}(|\xi|, \theta) d\theta,$$

where $s_-(\theta, |\xi|, \omega)$ *is given by*

$$\phi_-(x, \xi) - e^{ix\cdot\xi} \sim s_-(\theta, |\xi|, \omega) \frac{e^{-i|\xi|(x)}}{|x|^{(n-1)/2}}$$

as $|x| \to \infty$, $x = |x|\theta$, *and* $\xi = |\xi|\omega$.

For $k > 0$ we define the scattering matrix $S(k) : L^2(S^{n-1}) \to L^2(S^{n-1})$ by setting

$$(S(k)h)(\theta) = h(\theta) + \left(\frac{ik}{2\pi}\right)^{(n-1)/2} \int_{S^{n-1}} s_-(-\omega, k, \theta)^* h(\omega) d\omega.$$

The next section will be devoted to studying the meromorphic continuation of $S(k)$ to the strip $|\mathrm{Im}\, k| < a$.

6. **Resonant states and poles of the scattering matrix.** For simplicity we restrict our attention in this section to the case of $n = 3$ and $\Omega = R^3$. Thus we assume that $e^{2a|x|}q(x)$ is bounded and uniformly Hölder continuous in R^3. The general case is discussed in [6].

A nonzero outgoing solution of

$$(6.1) \qquad (-\Delta + q - k^2)u(x) = 0, \qquad x \in R^3,$$

is an eigenfunction of H if $\mathrm{Im}\, k > 0$, but grows exponentially as $|x| \to \infty$ if $\mathrm{Im}\, k < 0$. We call a nonzero outgoing solution of (6.1) with $\mathrm{Im}\, k < 0$ a *resonant state at k*. We show in this section that resonant states occur at k $(-a < \mathrm{Im}\, k < 0)$ if and only if k is a pole of the scattering matrix $S(k)$.

The discussion is given in two parts. We first consider k with $\mathrm{Im}\, k < 0$ and k^2 not an eigenvalue of H and then the more difficult case of k on the negative imaginary axis such that k^2 is an eigenvalue of H.

It was shown in the last section that the scattering matrix $S(k)$: $L^2(S^2) \to L^2(S^2)$ is for each $k > 0$ a unitary operator.

$$(6.2) \qquad S(k)h(\omega) = h(\omega) + \frac{ki}{2\pi} \int_{|\theta|=1} h(\theta) s_-(-\theta, k, \omega)^* d\theta$$

where s_- is the transmission coefficient, the radiation pattern of the incoming diffracted plane wave

$$v_-(x, k, \omega) \sim s_-(\theta, k, \omega)\ e^{-ikr}/r \quad \text{as } r \to \infty \tag{6.3}$$

$(x = r\theta)$. Here v_- satisfies the integral equation

$$v_-(x, k, \omega) = -\frac{1}{4\pi} \int \frac{e^{-ik|x-y|}}{|x-y|} q(y)\phi_-(y, k, \omega)dy \tag{6.4}$$

with

$$\phi_-(x, k, \omega) = e^{ikx\cdot\omega} + v_-(x, k, \omega) \tag{6.5}$$

the generalized eigenfunction of H used in the eigenfunction expansion $\hat{\ }^-$.

Equations (6.3) and (6.4) show that

$$\begin{aligned} s_-(\theta, \bar{k}, \omega)^* &= \frac{-1}{4\pi} \int e^{-iky\cdot\theta} q(y)\phi_-(y, \bar{k}, -\omega)^* dy \\ &= \frac{-1}{4\pi} \int e^{-iky\cdot\theta} q(y)\phi_+(y, k, -\omega)dy. \end{aligned} \tag{6.6}$$

We have used the equation $\phi_-(y, \bar{k}, \omega)^* = \phi_+(y, k, -\omega)$ which follows from the radiation conditions. Since, by the results of §3, $\phi_+(y, k, -\omega)$ is a meromorphic function of k with $|\text{Im } k| < a$, $S(k)$ is also meromorphic in that region and a pole of $S(k)$ must be a pole of $\phi_+(x, k, \omega)$ and hence of $(I + M(k))^{-1}$. Therefore Theorem 3.1 gives the following result.

LEMMA 6.1. *If k_0 with $-a < \text{Im } k_0 < 0$ is a pole of $S(k)$, then $O(k_0) \neq \{0\}$, i.e., then there exist resonant states at k_0.*

Similarly, Lemma 4.3 shows that a nonzero pole of S in the closed upper half plane must lie on the positive imaginary axis and be the square root of an eigenvalue of H.

Case I. Consider k_0 with $\text{Im } k_0 < 0$ and k_0^2 not an eigenvalue of H. Since $S(k)$ is unitary for $k > 0$, its inverse is $S(k)^*$. Formula (6.2) yields

$$S(\bar{k})^*(\theta) = h(\theta) + \frac{k}{2\pi i} \int_{|\omega|=1} h(\omega)s_-(-\theta, k, \omega)d\omega. \tag{6.7}$$

Hence $S(\bar{k})^*$ is also meromorphic for $|\text{Im } k| < a$ and k_0 is not a pole of $S(\bar{k})^*$ since it is not a pole of $\phi_+(y, \bar{k}, -\omega)^*$ (see equation (6.6)). Now, by analytic continuation

$$S(k) = [S(\bar{k})^*]^{-1} \tag{6.8}$$

in a deleted neighborhood of k_0.

LEMMA 6.2. *k_0 is a pole of $S(k)$ if and only if* Null space $S(\bar{k})^* \neq \{0\}$.

PROOF. If $S(\bar{k}_0)^*h = 0$ and $h \neq 0$, then equation (6.8) gives $h = S(k)S(\bar{k})^*h$ for k near k_0. Since $S(\bar{k})^*h \to 0$ as $k \to k_0$, S must have a pole at k_0.

If the null space of $S(\bar{k}_0)^*$ is trivial then $[S(\bar{k})^*]^{-1}$ exists and is analytic in a neighborhood of k_0, so that by (6.8) S is analytic at k_0. Q.E.D.

We will introduce some notation and make some preliminary calculations before stating our results.

For $f(y)$ with $|f(y)| \leqq ce^{a|y|}$ for all y in R^3 and for x in R^3 and θ in S^2, set

$$T_k^{\pm}f(x) = \frac{-1}{4\pi}\int_{R^3} \frac{e^{\pm ik|x-y|}}{|x-y|}\, q(y)f(y)dy \tag{6.9}$$

and

$$B_k^{\pm}f(\theta) = \frac{-1}{4\pi}\int_{R^3} e^{-(\pm iky\cdot\theta)}q(y)f(y)dy. \tag{6.10}$$

Then

$$T_k^{\pm}f(r\theta) \sim B_k^{\pm}f(\theta)\, e^{\pm ikr}/r \quad \text{as } r \to \infty \tag{6.11}$$

and

$$B_k^{+}f(\theta) = B_k^{-}f(-\theta). \tag{6.12}$$

$f = T_k^{\pm}f$ if and only if f is an {outgoing (incoming)} solution of $(-\Delta + q - k^2)f = 0$.

Also, noting equation (6.4), we obtain $v_- = T_k^-\phi_-$ which implies

$$(I - T_k^-)\phi_-(\cdot, k, \omega)(x) = e^{ikx\cdot\omega} \tag{6.13}$$

and

$$s_-(\theta, k, \omega) = B_k^-\phi_-(\cdot, k, \omega)(\theta). \tag{6.14}$$

LEMMA 6.3. *For any f such that $|f(y)| \leqq ce^{a|y|}$ for all y,*

$$(T_k^+ - T_k^-)f(x) = \frac{ki}{2\pi}\int e^{ikx\cdot\omega}B_k^+f(\omega)d\omega. \tag{6.15}$$

PROOF. Since

$$\frac{e^{ik|x|}}{|x|} - \frac{e^{-ik|x|}}{|x|} = \frac{ki}{2\pi}\int_{|\omega|=1} e^{ikx\cdot\omega}d\omega,$$

we have

$$(T_k^+ - T_k^-)f(x) = \frac{-1}{4\pi} \int_{R^3} q(y)u(y) \left(\frac{ki}{2\pi} \int_{|\omega|=1} e^{ik(x-y)\cdot\omega} d\omega \right) dy$$

$$= \frac{ki}{2\pi} \int e^{ikx\cdot\omega} \left(\frac{-1}{4\pi} \int_{R^3} e^{-ikx\cdot\omega} q(y) dy \right) d\omega$$

which is the desired result. Q.E.D.

THEOREM 6.1. *Consider* k_0 *with* $\operatorname{Im} k_0 < 0$ *and* k_0^2 *not an eigenvalue of* H.

(i) *Suppose* $0 \neq h \in L^2(S^2)$ *is in the null space of* $S(\bar{k}_0)^*$. *Then*

$$U(x) = \int_{|\omega|=1} h(\omega)\phi_-(x, k_0, \omega) d\omega \tag{6.16}$$

is a resonant state at k_0 *and*

$$U(r\theta) \sim \frac{2\pi}{ik_0 r} h(\theta)\, e^{ik_0 r} \quad \text{as } r \to \infty. \tag{6.17}$$

(ii) *Suppose that* $u(x)$ *is a resonant state at* k_0. *Define* $h \in L^2(S^2)$ *by*

$$u(r\theta) \sim \frac{2\pi}{ik_0 r} h(\theta)\, e^{ik_0 r} \quad \text{as } r \to \infty$$

and define $U(x)$ *by* (6.16). *Then* $u = U$ *and* $S(\bar{k}_0)^* h = 0$, $h \neq 0$.

(*Thus equations* (6.16) *and* (6.17) *provide an isomorphism between resonant states at* k_0 *and the null space of* $S(\bar{k})^*$.)

PROOF. (i) Equation (6.13) yields

$$(I - T_{k_0}^-) U(x) = \int_{|\omega|=1} h(\omega)\, e^{ik_0 x\cdot\omega} d\omega, \tag{6.18}$$

while (6.12), (6.14), and (6.7) give

$$\begin{aligned} B_{k_0}^+ U(\theta) = \ B_{k_0}^- U(-\theta) &= \int_{|\omega|=1} h(\omega) s_-(-\theta, k_0, \omega) d\omega \\ &= \frac{2\pi}{ik_0} [h(\theta) - S(\bar{k}_0)^* h(\theta)]. \end{aligned} \tag{6.19}$$

From the last equation and (6.15) we have

$$(T_{k_0}^+ - T_{k_0}^-)\, U(x) = \int_{|\omega|=1} e^{ik_0 x\cdot\omega}\, [h(\omega) - S(\bar{k}_0)^*\, h(\omega)]\, d\omega. \tag{6.20}$$

Subtract (6.20) from (6.18) to obtain

$$(6.21)\qquad (I - T^+_{k_0})U(x) = \int_{|\omega|=1} e^{ik_0 x\cdot\omega}\, S(\bar{k}_0)^* h(\omega)d\omega.$$

Since $S(\bar{k}_0)^*h = 0$, we have $U = T^+_{k_0}U$ and therefore U is a resonant state and

$$U(r\theta) \sim B^+_{k_0}\, U(\theta)\frac{e^{ik_0 r}}{r} = \frac{2\pi}{ik_0 r} h(\theta)\, e^{ik_0 r}$$

by (6.19). U is not zero because h is not zero.

(ii) Since u is a resonant state, $u = T^+_{k_0}u$ and by (6.15)

$$(I - T^-_{k_0})u(x) = (T^+_{k_0} - T^-_{k_0})u(x) = \frac{k_0 i}{2\pi}\int_{|\omega|=1.} e^{ik_0 x\cdot\omega}h(\omega)d\omega$$

which equals $(I - T^-_{k_0})U(x)$. Thus $(U - u) = T^-_{k_0}(U - u)$ is an incoming (L^2) solution of $(-\Delta + q - k_0^2)(U - u) = 0$. Since all such solutions are zero, we have $u = U$. From (6.21) we obtain

$$0 = \int e^{ik_0 x\cdot\omega}\, S(\bar{k}_0)^* h(\omega)d\omega$$

for all x in R^3. For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ any triple of nonnegative integers we have

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial x}\right)^{\alpha} \int e^{ik_0 x\cdot\omega}\, S(\bar{k}_0)^* h(\omega)d\omega|_{x=0} \\ &= \int (ik_0\omega)^{\alpha}\, S(\bar{k}_0)^* h(\omega)d\omega \end{aligned}$$

and hence $S(\bar{k}_0)^*h \neq 0$. h is not zero because u, given by (6.16), is not zero.

Case II. Consider k_0 with $\operatorname{Im} k_0 < 0$ and k_0^2 an eigenvalue of H. In this case, $\phi_-(x, k, \omega)$ and $S(\bar{k})^*$ do not exist at $k = \bar{k}_0$, and Lemma 6.2 no longer can be used to assert the existence of a pole of $S(k)$ at k_0. We now obtain new conditions which guarantee a pole of $S(k)$ at k_0, and characterize the resonant states at k_0.

For k near k_0, the resolvent operator $(k^2 - H)^{-1}$ can be written in the form

$$(6.22)\qquad (k^2 - H)^{-1} = \frac{R_{-1}}{k^2 - k_0^2} + R_0 + R_1(k), \quad R_1(k_0) = 0,$$

with R_{-1}, R_0, $R_1(k)$ bounded linear operators in $L^2(R^3)$ mapping $L^2(R^3)$

into $D(H)$. R_{-1} is the projection onto the eigenspace of H at k_0^2, i.e. if $\{\phi_j\}_{j=1}^N$ are an orthonormal basis for the eigenspace of H at k_0^2,

$$H\phi_j = k_0^2\phi_j, \quad j = 1, 2, \cdots, N,$$

then

$$(R_{-1}f)(x) = \sum_{j=1}^{N} (f, \phi_j)\phi_j(x) .$$

The ϕ_j can be taken to be real, since $k_0^2 - H$ is real. The reduced resolvent R_0 satisfies

$$(k_0^2 - H)R_0 = I - R_{-1}. \tag{6.23}$$

An alternate expression for $\phi_-(x, k, \omega)$ for k near k_0 is given by

$$\phi_-(x, k, \omega) = e^{ikx\cdot\omega} + (k^2 - H)^{-1}(E(\cdot, k, \omega))(x) \tag{6.24}$$

with

$$E(x, k, \omega) = q(x)\, e^{ik\omega\cdot x}.$$

$E(\cdot, k, \omega) : \{k|-a < \operatorname{Im} k < 0\} \to L^2(R^3)$ is analytic, so (6.24) and (6.6) allow us to obtain a Laurent expansion for $s_-(\theta, k, \omega)$ in a neighborhood of k_0:

$$s_-(\theta, k, \omega) = s_{-1}(\theta, \omega)/(k-k_0) + s_0(\theta, \omega) + s_1(\theta, k, \omega). \tag{6.25}$$

If we define $s_j(\theta)$ and $\sigma_j(\theta)$ by

$$\phi_j(r\theta) \sim s_j(\theta) \frac{e^{-ik_0r}}{r} = \frac{-e^{ik_0r}}{4\pi r} \int_{R^3} e^{ik_0y\cdot\theta} q(y)\phi_j(y)dy,$$

$$\sigma_j(\theta) = \frac{d}{dk_0} s_j(\theta),$$

then straightforward but tedious calculations show that

$$s_{-1}(\theta, \omega) = \frac{-2\pi}{k_0} \sum_{j=1}^{N} s_j(\theta)s_j(\omega), \tag{6.26}$$

$$\begin{aligned} s_0(\theta, \omega) = &-\frac{\pi}{2k_0^2} \sum_{j=1}^{N} s_j(\theta)s_j(\omega) \\ &+ \frac{1}{2k_0} \sum_j [\sigma_j(\theta)s_j(\omega) + \sigma_j(\omega)s_j(\theta)] \\ &+ [B_k\bar{\phi}_0(\cdot, k_0, \omega)](\theta), \end{aligned} \tag{6.27}$$

with

$$\phi_0(x, k_0, \omega) = e^{ik_0 x\cdot\omega} + R_0(E(\cdot, k_0, \omega))(x). \tag{6.28}$$

Equation (6.25) and the definition (6.7) of $S(\bar{k})^*$ lead to the following Laurent expansion for $S(\bar{k})^*$ near $k = k_0$:

$$S(\bar{k})^* = S_{-1}/(k-k_0) + S_0 + S_1(k), \tag{6.29}$$

where

$$\begin{aligned}(S_{-1}k)(\theta) &= \frac{k_0}{2\pi i}\int_{|\omega|=1} s_{-1}(-\theta, \omega)h(\omega)d\omega \\ &= \frac{k_0}{2\pi i}\sum_{j=1}^{N}\left(\int_{|\omega|=1} h(\omega)s_j(\omega)d\omega\right)s_j(-\theta),\end{aligned} \tag{6.30}$$

$$(S_0h)(\theta) = h(\theta) + \frac{k_0}{2\pi i}\int_{|\omega|=1} s_0(-\theta, \omega)h(\omega)d\omega, \tag{6.31}$$

$$S_1(k) \text{ is analytic at } k_0, \qquad S_1(k_0) = 0. \tag{6.32}$$

(6.30) shows that Range S_{-1} is the space spanned by the functions $\{s_j(-\theta)\}$.

We shall need to use the following integral equation which is satisfied by ϕ_0:

$$\begin{aligned}(I - T_{k_0}^-)\phi_0(\cdot, k_0, \omega)(x) &= e^{ik_0 x\cdot\omega} \\ &\quad - 4\pi\sum_{j=1}^{N} s_j(\omega)\int_{R^3} F_{k_0}^-(x-y)\phi_j(y)dy.\end{aligned} \tag{6.33}$$

Equation (6.33) follows from (6.23), which shows that

$$\begin{aligned}(k_0^2 - q(x) + \Delta)R_0(E(\cdot, k_0, \omega))(x) &= (I-R_{-1})E(\cdot, k_0, \omega)(x) \\ &= q(x)\, e^{ik_0\omega. x} - 4\pi\sum_{j=1}^{N} s_j(\omega)\phi_j(x),\end{aligned}$$

which in turn implies that

$$\begin{aligned}(I - T_{k_0}^-)&R_0(E(\cdot, k_0, \omega))(x) \\ &= T_{k_0}^-(e^{ik_0\omega\cdot})(x) - 4\pi\sum s_j(\omega)\int_{R^3} F_{k_0}^-(x-y)\phi_j(y)dy.\end{aligned} \tag{6.34}$$

Equation (6.33) now follows readily from (6.34).

Lemma 6.3 and (6.27) yield

$$
\begin{aligned}
&(T_{k_0}^{+} - T_{k_0}^{-})\phi_0(\cdot, k_0, \omega)(x) \\
(6.35)\quad &= \frac{k_0 i}{2\pi}\int_{|\theta|=1} e^{ik_0 x\cdot\theta}\Big\{s_0(-\theta,\omega) + \frac{\pi}{2k_0^2}\sum s_j(\omega)s_j(-\theta) \\
&\qquad - \frac{1}{2k_0}\sum[\sigma_j(-\theta)s_j(\omega) + \sigma_j(\omega)s_j(-\theta)]\Big\}d\theta .
\end{aligned}
$$

LEMMA 6.4. *Suppose $h \neq 0$ satisfies*

$$(6.36)\qquad S_{-1}h = 0, \qquad S_0 h \in \text{Range } S_{-1}.$$

Then k_0 is a pole of $S(k)$.

PROOF. If $S(k)$ does not have a pole at k_0, then in a neighborhood of k_0 we can write

$$(6.37)\qquad S(k) = G_0 + G_1(k-k_0) + G_2(k)(k-k_0)^2$$

with $G_2(k)$ analytic at k_0. Consequently

$$(6.38)\quad I = S(k)S(\bar{k})^* = G_0 S_{-1}/(k-k_0) + G_0 S_0 + G_1 S_{-1} + G_3(k)$$

with $G_3(k)$ analytic at k_0. Apply (6.38) to h and let $k \to k_0$ to obtain $h = G_0 S_0 h$. But $S_0 h = S_{-1}\ell$ for some ℓ, so

$$h = G_0 S_0 h = G_0 S_{-1}\ell = 0,$$

as (6.38) shows. Thus any solution h of (6.36) must be zero if $S(k)$ is regular at k_0, and so the hypothesis of the lemma implies that $S(k)$ has a pole at k_0.

For the next lemma we assume that a Rellich-type theorem holds for the differential equation

$$(6.39)\qquad (-\Delta + q(x) - k_0^2)u(x) = 0, \qquad x \in R^3,$$

i.e. we assume that a solution u of (6.39) which satisfies

$$u(x) = o(e^{-ik_0|x|}/|x|) \quad \text{as } |x| \to \infty$$

vanishes identically outside of some sphere. This is exactly the Rellich theorem in the case q has compact support, and is easy to prove if q is eventually radially symmetric. We believe it holds for the class of potentials q under consideration here.

LEMMA 6.5. *The $\{s_j(\theta)\}$ are linearly independent.*

For the proof see [**6**].

The next lemma follows easily from an application of Green's formula; see [**6**].

LEMMA 6.6. *If $u(x)$ is a resonant state at k_0^2, its radiation pattern* $(B_{k_0}^+u)(\theta)$ *is orthogonal to the* $\{s_j(\theta)\}$ *in* $L^2(S^2)$.

THEOREM 6.2. *Consider k_0 with* $\operatorname{Im} k_0 < 0$ *and k_0^2 an eigenvalue of H.*

(i) *Suppose* $h \neq 0$ *satisfies* (6.36). *Define* $\beta_j(h)$ *(uniquely by Lemma* 6.5) *by*

$$(S_0h)(\theta) = \frac{k_0}{2\pi i}\sum \beta_j(h)s_j(-\theta), \tag{6.40}$$

and set

$$u(x) = \int_{|\omega|=1} h(\omega)\phi_0(x, k_0, \omega)d\omega - \sum_{j=1}^{N}[\alpha_j(h) + \beta_j(h)]\phi_j(x), \tag{6.41}$$

with

$$\alpha_j(h) = -\frac{1}{2k_0}\int_{|\omega|=1}\sigma_j(\omega)h(\omega)d\omega.$$

Then $u(x)$ is a resonant state at k_0, and

$$u(r\theta) \sim \frac{2\pi}{ik_0r}h(\theta)e^{ik_0r}. \tag{6.42}$$

(ii) *Suppose k_0 is a pole of $S(k)$. Then there exist resonant states at k_0. Let u be a resonant state at k_0, and define $h(\theta)$ by* (6.42). *Then h satisfies* (6.36) *and $u(x)$ is given by* (6.41), *so in particular* $h \neq 0$.

PROOF OF (i). Set

$$U(x) = \int_{|\omega|=1} h(\omega)\phi_0(x, k_0, \omega)d\omega,$$

and write

$$(I-T_{k_0}^+)U(x) = (I - T_{k_0}^-)U(x) - (T_{k_0}^+ - T_{k_0}^-)U(x).$$

Note that $S_{-1}\,h = 0$ since h is orthogonal to the $\{s_j\}$, so (6.33), (6.35) and the definition (6.31) of S_0 yield

$$(I-T_{k_0}^+)U(x) = \int_{|\theta|=1} e^{ik_0x\cdot\theta}\left[S_0h(\theta)^+ \frac{k_0}{2\pi i}\sum\alpha_j(h)s_j(-\theta)\right]d\theta. \tag{6.43}$$

Since ϕ_j is an eigenfunction of H at k_0^2, $\phi_j = T_{k_0}^{-}\phi_j$ so it follows that

$$(I - T_{k_0}^{+})\phi_j = (T_{k_0}^{-} - T_{k_0}^{+})\phi_j = -\frac{k_0}{2\pi i}\int_{|\theta|=1} e^{ik_0x\cdot\theta} s_j(-\theta)d\theta. \tag{6.44}$$

(6.40), (6.43) and (6.44) combine to show that $(I - T_{k_0}^{+})u(x) = 0$, i.e. u defined by (6.41) is a resonant state. (6.42) follows from Lemma 6.3, (6.35) and (6.44).

Proof of (ii). The existence of resonant states at k_0 follows from Lemma 6.1. Let $u(x)$ be a resonant state at k_0, and define $h(\theta)$ by (6.42). Lemma 6.6 shows that $S_{-1}h = 0$.

Since u is a resonant state, $u = T_{k_0}^{+}u$, and so

$$(I - T_{k_0}^{-})u(x) = (T_{k_0}^{+} - T_{k_0}^{-})u(x) = \int_{|\omega|=1} e^{ik_0x\cdot\omega}h(\omega)d\omega \tag{6.45}$$

by (6.15). If we set

$$U'(x) = \int_{|\omega|=1} h(\omega)\phi_0(x, k_0, \omega)d\omega,$$

then (6.33) shows that

$$(I - T_{k_0}^{-})\,U'(x) = \int_{|\omega|=1} e^{ik_0x\cdot\omega}h(\omega)d\omega. \tag{6.46}$$

(6.45) and (6.46) combine to show that $u - U' = T_{k_0}^{-}(u - U')$, which shows that $u - U' \in L^2(R^3)$ and $u - U' \in$ Null space $(k_0^2 - H)$. Thus

$$u = U'(x) + \sum_{j=1}^{N} \gamma_j\phi_j(x). \tag{6.47}$$

Since u is a resonant state

$$0 = (I - T_{k_0}^{+})u = (I - T_{k_0}^{+})U' - (I - T_{k_0}^{+})\sum \gamma_j\phi_j,$$

and so by (6.43) and (6.44),

$$0 = \int_{|\theta|=1} e^{ik_0x\cdot\omega}\left[S_0h(\theta) + \frac{k_0}{2\pi i}\sum(\alpha_j(h) - \gamma_j)s_j(-\theta)\right]d\theta. \tag{6.48}$$

Apply $(\partial/\partial x)^\alpha$ to (6.48) and set $x = 0$ to conclude that

$$0 = \int_{|\theta|=1} (ik_0\theta)^\alpha \left[S_0h(\theta) + \frac{k_0}{2\pi i}\sum(\alpha_j(h) - \gamma_j)s_j(-\theta)\right]d\theta$$

for any triple $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of nonnegative integers. We conclude that

$$S_0 h(\theta) = \frac{k_0}{2\pi i} \sum (\gamma_j - \alpha_j(h)) s_j(-\theta),$$

which shows $S_0 h \in \text{Range } S_{-1}$, and so by Lemma 6.4, k_0 is a pole of $S(k)$.

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