Strongly ϕ-flat modules, strongly nonnil-injective modules and their homological dimensions

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Abstract

In this paper, we first introduce and study the notions of strongly ϕ-flat modules and strongly nonnil-injective modules. And then, we investigate the homological dimensions of modules and rings in terms of these two notions. Finally, we give some new homological characterizations of ϕ-Dedekind rings and ϕ-Prüfer rings.

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Throughout this paper, all rings are commutative with identity and all modules are unitary. First, we recall some notions on ϕ-rings, which are good generalizations of integral domains, originated from [5]. A ring R is called an NP-*ring* if the nilpotent radical $\mathrm{Nil}(R)$ is a prime ideal; and a ZN-*ring* if $\mathrm{Z}(R) = \mathrm{Nil}(R)$ where $\mathrm{Z}(R)$ is the set of all zero-divisors of R. A prime ideal $\mathfrak{p}$ of R is called *divided prime* if $\mathfrak{p} \subsetneq (x)$, for every $x \in R - \mathfrak{p}$. A ring R is a ϕ-*ring* if $\mathrm{Nil}(R)$ is a divided prime ideal of R. Moreover, a ZN ϕ-ring is said to be a *strong* ϕ-*ring*. Many well-known notions of integral domains have the corresponding analogues in the class of ϕ-rings, such as valuation domains, Dedekind domains, Prüfer domains, Noetherian domains, coherent domains, Bezout domains and Krull domains (see [1, 2, 4, 6, 7]).

The studies of ϕ-rings from the moduletic viewpoint started from Yang [21], who introduced the notion of nonnil-injective modules by replacing the ideals in Baer's criterion for injective modules with nonnil ideals. Dually, Zhao et al. [26] defined the ϕ-flat modules in terms of nonnil ideals and Tor-functors. They also gave the conceptions of ϕ-*von Neumann rings* over which any module is ϕ-flat, and then showed that a ϕ-ring R is ϕ-von Neumann if and only if its Krull dimension is 0, if and only if $R/\mathrm{Nil}(R)$ is a von Neumann regular ring. In 2018, Zhao [25] gave a homological characterization of ϕ-Prüfer rings: a strong ϕ-ring R is ϕ-Prüfer if and only if each submodule of a ϕ-flat module is ϕ-flat, if and only if each nonnil ideal of R is ϕ-flat. Recently, the first author and Qi [22] characterized ϕ-von Neumann rings and ϕ-Dedekind rings in terms of nonnil-injective modules.

Let R be a ring. Recall that a class of R-modules is said to be resolving if it contains all projective R-modules and is closed under direct summands, extensions and kernels of surjective homomorphisms; and to be coresolving if it contains all injective modules and is closed under direct summands, extensions and cokernels of injective homomorphisms. It is well-known that the class of flat (resp., injective) modules is resolving (resp., coresolving). These properties of a given class of R-modules are very crucial to study the homology dimensions (see [9]). So it is natural to ask that: Is the class of ϕ-flat (resp., nonnil-injective) modules also resolving (resp., coresolving)? The original motivation of this paper is to investigate this question. Actually, we deny these for both ϕ-flat modules and nonnil-injective modules (see Examples 1.1 and 1.2). So we introduce the notions of strongly ϕ-flat modules and strongly nonnil-injective modules to fill this gap (see Definition 1.4). The new notions and the old ones are consistent over a ZN-ring (see Theorem 1.6). It is proved in [24] that a ϕ-ring R is an integral domain if and only if every ϕ-flat module is flat. However, it does not hold for strongly ϕ-flat modules (see Example 1.12). We introduce the ϕ-flat dimensions and ϕ-injective dimensions of R-modules, investigate the ϕ-weak global dimensions and ϕ-global dimensions of rings, and characterize ϕ-rings with ϕ-weak global dimensions and ϕ-global dimensions at most 1, respectively.

1. STRONGLY ϕ-FLAT MODULES AND STRONGLY NONNIL-INJECTIVE MODULES

Let R be an NP-ring. Then the set of all nonnil ideals of R, denoted by $\mathrm{NN}(R)$, is closed under multiplication. From now on, we always suppose R is an NP-ring. Let M be an R-module. Set

$$\phi\text{-}tor(M) = \{x \in M \mid Ix = 0 \text{ for some } I \in \mathrm{NN}(R)\}.$$

An R-module M is said to be *ϕ-torsion* (resp., *ϕ-torsion free*) provided that ϕ-$tor(M) = M$ (resp., ϕ-$tor(M) = 0$). Then the classes of ϕ-torsion modules and ϕ-torsion free modules constitute a hereditary torsion theory of finite type.

Recall from [26, 27] that an R-module M is called *ϕ-flat* if $\mathrm{Tor}_1^R(T, M) = 0$ for any ϕ-torsion module T; and M is called *nonnil-injective* if $\mathrm{Ext}_R^1(T, M) = 0$ for any ϕ-torsion module T. It is shown in [26, Theorem 3.2] and [27, Theorem 1.7] that an R-module M is ϕ-flat if and only if $\mathrm{Tor}_1^R(R/I, M) = 0$ for any (finitely generated) nonnil ideal I of R; and M is nonnil-injective if and only if $\mathrm{Ext}_R^1(R/I, M) = 0$ for any nonnil ideal I of R.

It is well-known that the class of flat modules is resolving; and the class of injective modules is coresolving. And so it is ubiquitous to study modules and rings by using flats and injectives. So it is natural to ask that:

Is the class of all ϕ-flat (resp., nonnil-injective) modules resolving (resp., coresolving)?

Before we give a negative answer for above question, we need to recall the trivial extension of rings. Let R be a ring and M an R-module. As in [3], let $R(+)M$ be an R-module isomorphic to $R \oplus M$, and define

(1) $(r, m)+(s, n)=(r+s, m+n)$,
(2) $(r, m)(s, n)=(rs, sm+rn)$.

Then $R(+)M$ become a commutative ring with identity $(1, 0)$.

Now, we are ready to give the example to show if $0 \to A \to B \to C \to 0$ is an exact sequence with B and C ϕ-flat, then A is not necessarily ϕ-flat. Specially, the class of all ϕ-flat modules is not resolving.

Example 1.1. Let $\mathbb{Z}$ be the ring of all integers with $\mathbb{Q}$ its quotients field, and $\mathbb{Z}(\mathfrak{p}^\infty) := \{\frac{n}{\mathfrak{p}^k} + \mathbb{Z} \mid \frac{n}{\mathfrak{p}^k} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}\}$ the $\mathfrak{p}$-Prüfer group with $\mathfrak{p}$ a prime in $\mathbb{Z}$. Set $R = \mathbb{Z}(+)\mathbb{Z}(\mathfrak{p}^\infty)$ the trivial extension of $\mathbb{Z}$ with $\mathbb{Z}(\mathfrak{p}^\infty)$. Since $\mathbb{Z}(\mathfrak{p}^\infty)$ is a divisible module, we have R is a ϕ-ring by [13, Corollary 2.4] where $\mathrm{Nil}(R) = 0(+)\mathbb{Z}(\mathfrak{p}^\infty)$, and so $R/\mathrm{Nil}(R)$ is ϕ-flat since $\mathrm{Tor}_1^R(R/I, R/\mathrm{Nil}(R)) = (I \cap \mathrm{Nil}(R))/I\mathrm{Nil}(R) = 0$ for any nonnil ideal I of R. However, we claim that $\mathrm{Nil}(R)$ is not ϕ-flat. Indeed, let $I = \langle(\mathfrak{p}, 0)\rangle$. Then I is nonnil. The claim follows by the following isomorphisms (see [12, Proposition 1]):

$$\begin{aligned}
&\mathrm{Tor}_1^R(R/I, \mathrm{Nil}(R)) \\
\cong&\{(0, m) \in 0(+)\mathbb{Z}(\mathfrak{p}^\infty) \mid (\mathfrak{p}, 0)(0, m) = 0\}/(0 :_R (\mathfrak{p}, 0)) \cdot 0(+)\mathbb{Z}(\mathfrak{p}^\infty) \\
\cong&0(+)\mathbb{Z}(\mathfrak{p}^1)/0(+)\mathbb{Z}(\mathfrak{p}^1) \cdot 0(+)\mathbb{Z}(\mathfrak{p}^\infty) \\
\cong&0(+)\mathbb{Z}(\mathfrak{p}^1) \neq 0,
\end{aligned}$$

where $\mathbb{Z}(\mathfrak{p}^1) := \{\frac{n}{p} + \mathbb{Z} \in \mathbb{Z}(\mathfrak{p}^\infty) \mid n \text{ is an integer}\}$ is a subgroup of $\mathbb{Z}(\mathfrak{p}^\infty)$.

The following example also shows that if $0 \to A \to B \to C \to 0$ is an exact sequence with A and B nonnil-injective, then C is not necessarily nonnil-injective. Specially, the class of all nonnil-injective modules is not coresolving.

Example 1.2. Consider the above Example 1.1. Let $E := \mathrm{Hom}_{\mathbb{Z}}(R/\mathrm{Nil}(R), \mathbb{Q}/\mathbb{Z})$. Then E is nonnil-injective by [22, Proposition 1.4]. However, we claim the quotient $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Nil}(R), \mathbb{Q}/\mathbb{Z})$ of the injective module $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ by E is not nonnil-injective. Indeed,

$$\mathrm{Ext}_R^1(R/I, \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Nil}(R), \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_1^R(R/I, \mathrm{Nil}(R)), \mathbb{Q}/\mathbb{Z}) \neq 0$$

by Example 1.1. Hence, $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Nil}(R), \mathbb{Q}/\mathbb{Z})$ is not nonnil-injective.

In view of the above examples, the class of all ϕ-flat modules is not resolving, and the class of all nonnil-injective is not coresolving in general. To obtain the resolving or coresolving property similar to flatness and injectivity in NP-rings, we introduce the following "strong version" of ϕ-flat modules and nonnil-injective modules using higher derived functors.

Definition 1.3. Let R be an NP-ring and M an R-module. Then

(1) M is called *strongly ϕ-flat* if $\mathrm{Tor}_n^R(T, M) = 0$ for any ϕ-torsion module T and any $n \geq 1$.

(2) M is called *strongly nonnil-injective* if $\mathrm{Ext}_R^n(T, M) = 0$ for any ϕ-torsion module T and any $n \geq 1$.

Lemma 1.4. *Let R be a ϕ-ring and M an R-module. Then the following statements hold.*

(1) M is strongly ϕ-flat if and only if $\mathrm{Tor}_n^R(R/I, M) = 0$ for any (finitely generated) nonnil ideal I of R and any $n \geq 1$.

(2) M is strongly nonnil-injective if and only if $\mathrm{Ext}_R^n(R/I, M) = 0$ for any nonnil ideal I of R and any $n \geq 1$.

Proof. One can easily verify that an R-module M is strongly ϕ-flat (resp., strongly nonnil-injective) if and only if each syzygies $\Omega^n(M)$ (resp., co-syzygies $\Omega^{-n}(M)$) of M is ϕ-flat (resp., nonnil-injective) and that each $\Omega^n(M)$ (resp., $\Omega^{-n}(M)$) is ϕ-flat (resp., nonnil-injective) if and only if $\mathrm{Tor}_1^R(R/I, \Omega^n(M)) = 0$ for any nonnil ideal I of R. (resp., $\mathrm{Ext}_R^1(R/I, \Omega^{-n}(M)) = 0$ for any (finitely generated) nonnil ideal I of R.) □

Proposition 1.5. *Let R be a ϕ-ring and $0 \to A \to B \to C \to 0$ a short exact sequence of R-modules. Then the following statements hold.*

(1) The class of strongly ϕ-flat modules (resp., strongly nonnil-injective modules) is closed under direct limits (resp., direct products), direct summands, and extensions.

(2) If B and C are strongly ϕ-flat modules, so is A.

(3) If A and B are strongly nonnil-injective modules, so is C.

Proof. We only prove (2), since the proof of (1) is easy and the proof of (3) is similar to that of (2). Let T be a ϕ-torsion module. Then we have an exact sequence $\cdots \to \mathrm{Tor}_{n+1}^R(T, C) \to \mathrm{Tor}_n^R(T, A) \to \mathrm{Tor}_n^R(T, B) \to \cdots \to \mathrm{Tor}_2^R(T, C) \to \mathrm{Tor}_1^R(T, A) \to \mathrm{Tor}_1^R(T, B) \to \mathrm{Tor}_1^R(T, C)$. Since B and C are strongly ϕ-flat modules, $\mathrm{Tor}_n^R(T, B) = \mathrm{Tor}_n^R(T, C) = 0$ for any $n \geq 1$. Hence $\mathrm{Tor}_n^R(T, A) = 0$ for any $n \geq 1$, whence A is strongly ϕ-flat. □

Obviously, every strongly ϕ-flat module is ϕ-flat, and every strongly nonnil-injective module is nonnil-injective. By Lemma 1.5, Example 1.1 and 1.2, ϕ-flat modules are not necessarily strongly ϕ-flat, and nonnil-injective modules are also not necessarily strongly nonnil-injective. But the following result gives that over ZN rings, ϕ-flat modules are exactly strongly ϕ-flat and nonnil-injective modules are exactly strongly nonnil-injective.

Theorem 1.6. *Let R be a* ZN *ring. Then the following statements hold.*

(1) An R-module M is ϕ-flat if and only if it is strongly ϕ-flat.

(2) An R-module M is nonnil-injective if and only if it is strongly nonnil-injective.

Proof. (1) Suppose M is a ϕ-flat R-module. Let J be a nonnil ideal of R. Then there exists a nonnilpotent element $a \in J$. Since a is a non-zero-divisor of R, $\mathrm{Tor}_n^R(R/\langle a\rangle, M) = 0$ for any positive integer n. It follows by [8, Proposition 4.1.1] that

$$\mathrm{Tor}_1^{R/\langle a\rangle}(R/J, M/aM) \cong \mathrm{Tor}_1^{R/\langle a\rangle}(R/J, M \otimes_R R/\langle a\rangle) \cong \mathrm{Tor}_1^R(R/J, M) = 0.$$

Hence M/Ma is a flat $R/\langle a\rangle$-module. Consequently, for any $n \geq 1$ we have

$$\mathrm{Tor}_n^R(R/J, M) \cong \mathrm{Tor}_n^{R/\langle a\rangle}(R/J, M \otimes_R R/\langle a\rangle) \cong \mathrm{Tor}_n^{R/\langle a\rangle}(R/J, M/aM) = 0.$$

It follows that M is a strongly ϕ-flat R-module.

(2) Now suppose M is a nonnil-injective R-module. Let J be a nonnil ideal of R. Then there exists a nonnilpotent element $a \in J$. Since a is a non-zero-divisor of R, $\mathrm{Ext}_R^n(R/\langle a\rangle, M) = 0$ for any positive integer n. It follows by [8, Proposition 4.1.4] that

$$\mathrm{Ext}^1_{R/\langle a\rangle}(R/J, \mathrm{Hom}_R(R/\langle a\rangle, M)) \cong \mathrm{Ext}^1_R(R/J, M) = 0.$$

Hence $\mathrm{Hom}_R(R/\langle a\rangle, M)$ is an injective $R/\langle a\rangle$-module by Baer criterion. Consequently, for any $n \geq 1$ we have

$$\mathrm{Ext}^n_R(R/J, M) \cong \mathrm{Ext}^n_{R/\langle a\rangle}(R/J, \mathrm{Hom}_R(R/\langle a\rangle, M)) = 0.$$

It follows that M is a strongly nonnil-injective R-module. □

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Remark 1.7. Recall from [16, 20] that an R-module M is called to be regular flat (resp., regular injective) if $\mathrm{Tor}_1^R(R/I, M) = 0$ (resp., $\mathrm{Ext}_R^1(R/I, M) = 0$) for any regular ideal (i.e., an ideal that contains a non-zero-divisor) I of R. Similar with the proof of Theorem 1.6, one can show that an R-module M is regular flat (resp., regular injective) if and only if $\mathrm{Tor}_n^R(R/I, M) = 0$ (resp., $\mathrm{Ext}_R^n(R/I, M) = 0$) for any regular ideal I of R and any $n \geq 1$.

It is known that a ZN ϕ-ring is exactly a strong ϕ-ring. The following result is devoted to the converse of Theorem 1.6 under some assumptions.

Theorem 1.8. *Let R be a ϕ-ring such that either* $\mathrm{Nil}(R)$ *is nilpotent or* $(0 :_R a)$ *is finitely generated for any non-nilpotent element a (e.g. R is a nonnil-coherent ring). If one of the following two statements holds:*

(1) every ϕ-flat R-module is strongly ϕ-flat;

(2) every nonnil-injective R-module is strongly nonnil-injective,

then R is a strong ϕ-ring.

Proof. (1) Let R be a ϕ-ring and a a non-nilpotent element in R. Suppose every ϕ-flat R-module is strongly ϕ-flat. It follows by the proof of [24, Proposition 1] that $R/\mathrm{Nil}(R)$ is a ϕ-flat R-module, and so is strongly ϕ-flat. Hence,

$$\mathrm{Tor}_2^R(R/Ra, R/\mathrm{Nil}(R)) \cong \mathrm{Tor}_1^R(R/(0 :_R a), R/\mathrm{Nil}(R)) \cong \frac{(0 :_R a) \cap \mathrm{Nil}(R)}{(0 :_R a)\mathrm{Nil}(R)} = 0.$$

Since R is a ϕ-ring, $(0 :_R a) \subseteq \mathrm{Nil}(R)$, and so $(0 :_R a) \cap \mathrm{Nil}(R) = (0 :_R a)$. So $\mathrm{Tor}_2^R(R/Ra, R/\mathrm{Nil}(R)) \cong \frac{(0:_R a)}{(0:_R a)\mathrm{Nil}(R)} = 0$. And hence $(0 :_R a) = (0 :_R a)\mathrm{Nil}(R)$.

(a) Suppose $(0 :_R a)$ is finitely generated. By Nakayama's lemma, we have $(0 :_R a) = 0$, that is, a is a nonzero-divisor. So R is a strong ϕ-ring.

(b) Suppose $\mathrm{Nil}(R)$ is nilpotent. Assume $\mathrm{Nil}(R)^m = 0$. Then $(0 :_R a) = (0 :_R a)\mathrm{Nil}(R) = \cdots = (0 :_R a)\mathrm{Nil}(R)^m = 0$. So R is a strong ϕ-ring.

(2) Let R be a ϕ-ring and a a non-nilpotent element in R. Suppose every nonnil-injective R-module is strongly nonnil-injective. It follows by the proof of [22, Theorem 1.6] that $(R/\mathrm{Nil}(R))^+ := \mathrm{Hom}_{\mathbb{Z}}(R/\mathrm{Nil}(R), \mathbb{Q}/\mathbb{Z})$ is a nonnil-injective R-module, and so is strongly nonnil-injective. Hence,

$$\mathrm{Ext}_R^2(R/Ra, (R/\mathrm{Nil}(R))^+) \cong \mathrm{Tor}_2^R(R/Ra, R/\mathrm{Nil}(R))^+ = 0.$$

So $\mathrm{Tor}_2^R(R/Ra, R/\mathrm{Nil}(R)) = 0$, and hence $(0 :_R a) = (0 :_R a)\mathrm{Nil}(R)$. The rest is the same with that of (1). □

Proposition 1.9. *Let R be an* NP*-ring. Then the following statements are equivalent.*

(1) M is strongly ϕ-flat.

(2) $\mathrm{Hom}_R(M, E)$ is strongly nonnil-injective for any injective module E.

(3) If E is an injective cogenerator, then $\mathrm{Hom}_R(M, E)$ is strongly nonnil-injective.

Proof. (1) $\Rightarrow$ (2): Let T be a ϕ-torsion R-module and E an injective R-module. Since M is strongly ϕ-flat,

$$\mathrm{Ext}_R^n(T, \mathrm{Hom}_R(M, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_n^R(T, M), E) = 0$$

for any positive integer n. Thus $\mathrm{Hom}_R(M, E)$ is strongly nonnil-injective.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let I be a nonnil ideal of R and E an injective cogenerator. Since $\mathrm{Hom}_R(M, E)$ is strongly nonnil-injective,

$$\mathrm{Hom}_R(\mathrm{Tor}_n^R(R/I, M), E) \cong \mathrm{Ext}_R^n(R/I, \mathrm{Hom}_R(M, E)) = 0$$

for any positive integer n. Since E is an injective cogenerator, $\mathrm{Tor}_n^R(R/I, M) = 0$ for any positive integer n. Thus M is strongly ϕ-flat by Lemma 1.4. □

Remark 1.10. By linking Proposition 1.9 and [17, Proposition 1.8], one can deduce that every nonnil-injective R-module is strongly nonnil-injective implies that every ϕ-flat R-module is strongly ϕ-flat.

Let R be an NP-ring. Then every flat R-module is strongly ϕ-flat, and every injective R-module is strongly nonnil-injective. The converses are trivially true for integral domains, but not in general.

Example 1.11. It is obvious that all flat (resp., injective) modules are strongly ϕ-flat (resp., strongly nonnil-injective). However, the converse does not hold in general. Indeed, let R be a strong ϕ-ring which is not an integral domain (e.g. $R = D(+)Q$ with D a domain and Q its quotient field). Then every strongly ϕ-flat (resp., strongly nonnil-injective) module is ϕ-flat (resp., nonnil-injective) by Theorem 1.6. However, there exist ϕ-flat (resp., nonnil-injective) modules which are not flat (resp., injective), (see [24, Proposition 1] and [22, Theorem 1.6]).

It is proved in [24, Proposition 1] and [22, Theorem 1.6] that a ϕ-ring R is an integral domain if and only if every ϕ-flat R-module is flat, if and only if every nonnil-injective R-module is injective. The following example shows that all strongly ϕ-flat (resp., strongly nonnil-injective ϕ-torision-free) modules can be flat (resp., injective) over ϕ-rings which are not domains.

Example 1.12. Let $R = \mathbb{Z}(+)\mathbb{Z}(\mathfrak{p}^\infty)$ be the ring in Example 1.1. Then the following statements hold.

(1) Every strongly ϕ-flat R-module is flat.
(2) Every strongly nonnil-injective ϕ-torision-free R-module is injective.

Proof. Let I be an ideal of R. Then, by [3, Corollary 3.4], I is of the following two forms:

(a) $I := \langle (n,0)\rangle = \langle n\rangle(+)\mathbb{Z}(\mathfrak{p}^\infty)$ where $0 \neq n \in \mathbb{Z}$;
(b) $I := 0(+)N$, where N is a subgroup of $\mathbb{Z}(\mathfrak{p}^\infty)$.

The ideal I in case (a) is a nonnil ideal of R. Now we consider the ideal in case (b). Then N is of the form $\mathbb{Z}(\mathfrak{p}^k) := \{\frac{n}{\mathfrak{p}^k} + \mathbb{Z} \in \mathbb{Z}(\mathfrak{p}^\infty) \mid n \text{ is an integer}\}$ with k a non-negative integer or $\mathbb{Z}(\mathfrak{p}^\infty)$. Set $I_k := \langle (0, \frac{1}{\mathfrak{p}^k})\rangle = 0(+)\mathbb{Z}(\mathfrak{p}^k)$ for each positive integer k. Note that there is a short exact sequence $0 \to I_k \to R \to (\mathfrak{p}^k, 0)R \to 0$ for each non-negative integer k. Note that $\mathbb{Z}(\mathfrak{p}^\infty) = \bigcup \mathbb{Z}(\mathfrak{p}^k) = \varinjlim \mathbb{Z}(\mathfrak{p}^k)$. Set $I_\infty = 0(+)\mathbb{Z}(\mathfrak{p}^\infty)$, then $I_\infty = \varinjlim I_k$.

(1) Suppose that M is a strongly ϕ-flat R-module. It follows that

$$\mathrm{Tor}_1^R(R/I_k, M) \cong \mathrm{Tor}_1^R(\langle(\mathfrak{p}^k, 0)\rangle, M) \cong \mathrm{Tor}_2^R(R/\langle(\mathfrak{p}^k, 0)\rangle, M) = 0$$

for each positive integer k. And so each natural homomorphism $f_k : I_k \otimes_R M \to R \otimes_R M$ is a monomorphism. Now consider the case I_∞. Then the natural map $f_\infty : I_\infty \otimes_R M \to R \otimes_R M$, which can be seen as the direct limits of f_k, is also a monomorphism. So $\mathrm{Tor}_1^R(R/I_\infty, M) = 0$. In conclusion, $\mathrm{Tor}_1^R(R/I, M) = 0$ for any ideal I of R. It follows that M is a flat R-module.

(2) Suppose that M is a strongly nonnil-injective ϕ-torision-free R-module. Then

$$\mathrm{Ext}_R^1(R/I_k, M) \cong \mathrm{Ext}_R^1((\mathfrak{p}^k, 0)R, M) \cong \mathrm{Ext}_R^2(R/(\mathfrak{p}^k, 0)R, M) = 0$$

for each non-negative integer k. Now, consider the the case I_∞. Let

$$0 \to \mathrm{Hom}_R(R/I_k, M) \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I_k, M) \to 0$$

be the natural exact sequence. Taking inverse limits, we have the following exact sequence:

$$0 \to \varprojlim \mathrm{Hom}_R(R/I_k, M) \to \varprojlim \mathrm{Hom}_R(R, M) \to \varprojlim \mathrm{Hom}_R(I_k, M) \to \varprojlim{}^1 \mathrm{Hom}_R(R/I_k, M) \to 0$$

by [18, 1.2.2]. Considering the R-exact sequence $0 \to I_{k+1}/I_k \to R/I_k \to R/I_{k+1} \to 0$, we have an exact sequence

$$0 \to \mathrm{Hom}_R(R/I_{k+1}, M) \to \mathrm{Hom}_R(R/I_k, M) \to \mathrm{Hom}_R(I_{k+1}/I_k, M) \to 0.$$

Since $(0 :_R I_{k+1}/I_k) = (0 :_R I_1) = \langle (p, 0)\rangle$, we have

$$\mathrm{Hom}_R(I_{k+1}/I_k, M) \cong \mathrm{Hom}_R(R/\langle (p, 0)\rangle, M) = 0$$

because M is ϕ-torsion-free. So we have a natural isomorphism $\mathrm{Hom}_R(R/I_{k+1}, M) \cong \mathrm{Hom}_R(R/I_k, M)$ for each non-negative integer k, and hence the inverse system $\{\mathrm{Hom}_R(R/I_k, M) \mid k \geq 0\}$ is Mittag-Leffler. It follows by [18, 1.2.3] that

$$\varprojlim{}^1 \mathrm{Hom}_R(R/I_k, M) = 0.$$

Consequently, the natural map

$$\varprojlim \mathrm{Hom}_R(R, M) \cong \mathrm{Hom}_R(R, M) \twoheadrightarrow \varprojlim \mathrm{Hom}_R(I_k, M) \cong \mathrm{Hom}_R(I_\infty, M)$$

is an epimorphism and so $\mathrm{Ext}^1_R(R/I_\infty, M) = 0$. In conclusion, $\mathrm{Ext}^1_R(R/I, M) = 0$ for any ideal I of R. It follows that M is an injective R-module. □

2. On ϕ-flat dimensions of modules and ϕ-weak global dimensions of rings

It is well known that the flat dimension of an R-module M is defined as the length of the shortest flat resolutions of M and the weak global dimension of R is the supremum of the flat dimensions of all R-modules. We now introduce the notion of ϕ-flat dimension of an R-module as follows.

Definition 2.1. Let R be a ring and M an R-module. We write $\phi\text{-fd}_R(M) \leq n$ (ϕ-fd abbreviates *ϕ-flat dimension*) if there is an exact sequence of R-modules

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0 \qquad (\diamondsuit)$$

where each F_i is strongly ϕ-flat for $i = 0, \ldots, n$. The exact sequence $(\diamondsuit)$ is said to be a ϕ-flat resolution of length n of M. If such finite resolution does not exist, then we say $\phi\text{-fd}_R(M) = \infty$; otherwise, define $\phi\text{-fd}_R(M) = n$ if n is the length of the shortest ϕ-flat resolution of M.

It is obvious that an R-module M is strongly ϕ-flat if and only if $\phi\text{-fd}_R(M) = 0$. Certainly, $\phi\text{-fd}_R(M) \leq \mathrm{fd}_R(M)$. If R is an integral domain, then $\phi\text{-fd}_R(M) = \mathrm{fd}_R(M)$.

Proposition 2.2. *Let R be an* NP*-ring. Then the following statements are equivalent for an R-module M.*

(1) $\phi\text{-fd}_R(M) \leq n$.
(2) $\mathrm{Tor}^R_{n+k}(T, M) = 0$ *for all ϕ-torsion R-modules T and all positive integers k.*
(3) $\mathrm{Tor}^R_{n+k}(R/I, M) = 0$ *for all nonnil ideals I and all positive integers k.*
(4) $\mathrm{Tor}^R_{n+k}(R/I, M) = 0$ *for all finitely generated nonnil ideals I and all positive integers k.*
(5) If $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is an exact sequence, where $F_0, F_1, \ldots, F_{n-1}$ are strongly ϕ-flat R-modules, then F_n is strongly ϕ-flat.

(6) If $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is an exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are flat* R*-modules, then* F_n *is strongly* ϕ*-flat.*

(7) There exists an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$*, where* $F_0, F_1, \ldots, F_{n-1}$ *are flat* R*-modules and* F_n *is a strongly* ϕ*-flat* R*-module.*

Proof. (1) ⇒ (2): We prove (2) by induction on n. For the case $n = 0$, (2) trivially holds because M is strongly ϕ-flat. If $n > 0$, then there is an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is strongly ϕ-flat for $i = 0, \ldots, n$. Set $K_0 = \ker(F_0 \to M)$. Then both $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to K_0 \to 0$ are exact. So $\phi\text{-fd}_R(K_0) \leq n - 1$. By induction, $\mathrm{Tor}^R_{n-1+k}(T, K_0) = 0$ for all ϕ-torsion R-modules T and all positive integers k. Thus, it follows from the exact sequence

$$0 = \mathrm{Tor}^R_{n+k}(T, F_0) \to \mathrm{Tor}^R_{n+k}(T, M) \to \mathrm{Tor}^R_{n-1+k}(T, K_0) \to \mathrm{Tor}^R_{n-1+k}(T, F_0) = 0$$

that $\mathrm{Tor}^R_{n+k}(T, M) \cong \mathrm{Tor}^R_{n-1+k}(T, K_0) = 0$.

(2) ⇒ (3) ⇒ (4) and (5) ⇒ (6): Trivial.

(4) ⇒ (5): Let $K_0 = \ker(F_0 \to M)$ and $K_i = \ker(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Then $K_{n-1} \cong F_n$. Since all $F_0, F_1, \ldots, F_{n-1}$ are strongly ϕ-flat, $\mathrm{Tor}^R_k(R/I, F_n) \cong \mathrm{Tor}^R_{1+k}(R/I, K_{n-2}) \cong \cdots \cong \mathrm{Tor}^R_{n+k}(R/I, M) = 0$ for all finitely generated nonnil ideal I and any positive integer k by dimensional shift. Hence F_n is strongly ϕ-flat by Lemma 1.4.

(6) ⇒ (7): Since the class of flat modules is covering, we can construct an exact sequence $\cdots \to F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \to \cdots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \ldots, F_{n-1}$ are flat R-modules, then $F_n := \mathrm{Ker}(d_{n-1})$ is strongly ϕ-flat by (6).

(7) ⇒ (1): Trivial. □

The proofs of the following two results are similar with the classical ones, and so we omit their proofs.

Corollary 2.3. *Let* R *be an* NP*-ring and* $0 \to A \to B \to C \to 0$ *be an exact sequence of* R*-modules. Then the following statements hold.*

(1) $\phi\text{-fd}_R(C) \leq 1 + \max\{\phi\text{-fd}_R(A), \phi\text{-fd}_R(B)\}$.

(2) If $\phi\text{-fd}_R(B) < \phi\text{-fd}_R(C)$*, then* $\phi\text{-fd}_R(A) = \phi\text{-fd}_R(C) - 1 \geq \phi\text{-fd}_R(B)$.

Corollary 2.4. *Let* R *be an* NP*-ring and* $\{M_i \mid i \in \Gamma\}$ *be a direct system of* R*-modules. Then*

$$\phi\text{-fd}_R(\varinjlim M_i) = \sup\{\phi\text{-fd}_R(M_i)\}.$$

Now, we are ready to introduce the ϕ-weak global dimension of a ring in terms of ϕ-flat dimensions.

Definition 2.5. The *ϕ-weak global dimension* of a ring R is defined by

$$\phi\text{-w.gl.dim}(R) = \sup\{\phi\text{-fd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Obviously, by definition, ϕ-w.gl.dim(R) $\leq$w.gl.dim(R). Notice that if R is an integral domain, then ϕ-w.gl.dim(R) =w.gl.dim(R). The following result can easily be deduced by Proposition 2.2 and so we omit its proof.

Theorem 2.6. *Let R be an* NP*-ring. Then the following statements are equivalent for R.*

(1) ϕ-w.gl.dim$(R) \leq n$.
(2) ϕ-fd$_R(M) \leq n$ *for all R-modules M.*
(3) $\mathrm{Tor}^R_{n+k}(T, M) = 0$ *for all R-modules M, all ϕ-torsion modules T and all positive integers k.*
(4) $\mathrm{Tor}^R_{n+k}(R/I, M) = 0$ *for all R-modules M, all nonnil ideals I of R and all positive integers k.*
(5) $\mathrm{Tor}^R_{n+k}(R/I, M) = 0$ *for all R-modules M, all finitely generated nonnil ideals I of R and all positive integers k.*
(6) $\mathrm{Tor}^R_{n+1}(T, M) = 0$ *for all R-modules M and all ϕ-torsion modules T.*
(7) $\mathrm{Tor}^R_{n+1}(R/I, M) = 0$ *for all R-modules M and all nonnil ideals I of R.*
(8) $\mathrm{Tor}^R_{n+1}(R/I, M) = 0$ *for all R-modules M and all finitely generated nonnil ideals I of R.*
(9) $\mathrm{fd}_R(R/I) \leq n$ *for all nonnil ideals I of R.*
(10) $\mathrm{fd}_R(R/I) \leq n$ *for all finitely generated nonnil ideals I of R.*

Consequently, the ϕ-weak global dimension of R is determined by the formulas:

$$\begin{aligned}\phi\text{-w.gl.dim}(R) &= \sup\{\mathrm{fd}_R(R/I) \mid I \text{ is a nonnil ideal of } R\}\\ &= \sup\{\mathrm{fd}_R(R/I) \mid I \text{ is a finitely generated nonnil ideal of } R\}.\end{aligned}$$

Theorem 2.7. *Let R be a strong ϕ-ring. Then the following statements hold.*

(1) w.gl.dim$(R/\mathrm{Nil}(R)) \leq \phi$-w.gl.dim$(R)$.
(2) ϕ-w.gl.dim$(R) - \mathrm{fd}_R(R/\mathrm{Nil}(R)) \leq$ w.gl.dim$(R/\mathrm{Nil}(R))$.

Proof. (1) Suppose w.gl.dim$(R/\mathrm{Nil}(R)) = n$. Then there exists a nonnil ideal I of R and an $R/\mathrm{Nil}(R)$-module M such that

$$\mathrm{Tor}^{R/\mathrm{Nil}(R)}_n(R/I \otimes_R R/\mathrm{Nil}(R), M) \cong \mathrm{Tor}^{R/\mathrm{Nil}(R)}_n(R/I, M) \neq 0.$$

Note that $R/\mathrm{Nil}(R)$ is ϕ-flat, and then, by Theorem 1.6, we have $\mathrm{Tor}^R_n(R/I, R/\mathrm{Nil}(R)) = 0$ for all $n \geq 1$. So

$$\mathrm{Tor}^R_n(R/I, M) \cong \mathrm{Tor}^{R/\mathrm{Nil}(R)}_n(R/I \otimes_R R/\mathrm{Nil}(R), M) \neq 0,$$

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and hence $\mathrm{fd}_R(R/I) \geq n$. It follows by Theorem 2.6 that ϕ-w.gl.dim$(R) \geq n$.

(2) It immediately follows by [19, Theorem 3.8.5] and Theorem 2.6. □

It is natural to ask the question:

Question 2.8. Let R be a strong ϕ-ring. Does the following equation hold?

$$\text{w.gl.dim}(R/\text{Nil}(R)) = \phi\text{-w.gl.dim}(R).$$

We can verify it in the following case.

Proposition 2.9. *Let D be an integral domain, Q its quotient field and V a Q-linear space. Then ϕ-w.gl.dim$(D(+)V)$ =w.gl.dim(D).*

Proof. Set $R = D(+)V$. Assume w.gl.dim$(D) \leq n$. Let M be an R-module, Then M is naturally a D-module. Let J be a nonnil ideal of R. Then by [3, Corollary3.4], we have $J = I(+)V$ with I a nonzero ideal of D. Note that R is a flat D-module. By [8, Proposition 4.1.2] we have

$$\text{Tor}^R_{n+1}(R/J, M) \cong \text{Tor}^R_{n+1}(D/I \otimes_D R, M) \cong \text{Tor}^D_{n+1}(D/I, M) = 0.$$

So ϕ-w.gl.dim$(D(+)V) \leq$w.gl.dim(D). The result follows by Theorem 2.7. □

It is well known that a ring R with weak global dimension 0 is exactly a *von Neumann regular ring*, equivalently $a \in (a^2)$ for any $a \in R$. Recall from [26] that a ϕ-ring R is said to be *ϕ-von Neumann regular* provided that every R-module is ϕ-flat. A ϕ-ring R is ϕ-von Neumann regular, if and only if for any non-nilpotent element $a \in R$ there is an element $x \in R$ such that $a = xa^2$, if and only if $R/\text{Nil}(R)$ is a von Neumann regular ϕ-ring, i.e., $R/\text{Nil}(R)$ is a field (see [26, Theorem 4.1]). Now, we characterize ϕ-von Neumann regular rings in terms of strongly ϕ-flat modules and ϕ-weak global dimensions.

Theorem 2.10. *Let R be a ϕ-ring. Then the following statements are equivalent for R.*

(1) ϕ-w.gl.dim$(R) = 0$.

(2) Every R-module is strongly ϕ-flat.

(3) R is a ϕ-von Neumann regular ring.

Proof. (1) ⇔ (2): By definition.

(2) ⇒ (3): It follows by [26, Theorem 4.1] and [14, Corollary 4.5].

(3) ⇒ (2): Suppose R is a ϕ-von Neumann regular ring. Then we claim that R is a ZN-ring. Indeed, let a be a non-nilpotent element in R. Since $R/\text{Nil}(R)$ is a field by [26, Theorem 4.1], we have $(1 - ab)^n = 0$ for some $b \in R$ and positive integer

n. So a is a unit, and thus a non-zero-divisor. Now (2) follows by Theorem 1.6 and [26, Theorem 4.1]. □

For a ϕ-ring R, there is a ring homomorphism $\phi : \mathrm{T}(R) \to R_{\mathrm{Nil}(R)}$ such that $\phi(a/b) = a/b$ where $a \in R$ and b is a regular element. Denote by the ring $\phi(R)$ the image of ϕ restricted to R. Then $\phi(R)$ is a strong ϕ-ring. Recall that a regular ideal I of R is called *invertible* if $II^{-1} = R$ where $I^{-1} = \{x \in \mathrm{T}(R) \mid Ix \subseteq R\}$. Recall from [1] that a nonnil ideal I of a ϕ-ring R is said to be *ϕ-invertible* provided that $\phi(I)$ is an invertible ideal of $\phi(R)$.

Following [1], a ϕ-ring R is said to be a ϕ-Prüfer *ring* if every finitely generated nonnil ideal I is ϕ-invertible, i.e., $\phi(I)\phi(I^{-1}) = \phi(R)$. A ϕ-ring R is said to be a *ϕ-chain ring* (ϕ-CR for short) if for any $a, b \in R - \mathrm{Nil}(R)$, either $a|b$ or $b|a$ in R. It follows from [1, Corollary 2.10] that a ϕ-ring R is ϕ-Prüfer, if and only if $R_{\mathfrak{m}}$ is a ϕ-CR for any maximal ideal $\mathfrak{m}$ of R, if and only if $R/\mathrm{Nil}(R)$ is a Prüfer domain, if and only if $\phi(R)$ is Prüfer. For a strong ϕ-ring R, Zhao [25, Theorem 4.3] showed that R is a ϕ-Prüfer ring if and only if all ϕ-torsion free R-modules are ϕ-flat, if and only if each submodule of a ϕ-flat R-module is ϕ-flat, if and only if each nonnil ideal of R is ϕ-flat.

Theorem 2.11. *Let R be a ϕ-ring. Then the following statements are equivalent for R.*

(1) ϕ-w.gl.dim$(R) \leq 1$.
(2) Every submodule of flat R-module is strongly ϕ-flat.
(3) Every submodule of strongly ϕ-flat R-module is strongly ϕ-flat.
(4) R is a ϕ-Prüfer *strong ϕ-ring.*

Proof. (1) ⇔ (2) ⇔ (3): By Theorem 2.6.

(4) ⇒ (2): It follows by [26, Theorem 4.1].

(2) ⇒ (4): Since every submodule of a flat R-module is strongly ϕ-flat, every ideal of R is ϕ-flat. It follows by [15, Corollary 2.8] that R is a strong ϕ-ring. Hence the result follows by [25, Theorem 4.3] and Theorem 1.6. □

Note that when w.gl.dim$(R/\mathrm{Nil}(R)) \leq 1$, Question 2.8 holds by Theorem 2.10 and Theorem 2.11.

Corollary 2.12. *Let D be an integral domain, Q its quotient field and V a Q-linear space. Then $D(+)V$ is a* ϕ-Prüfer *ring if and only if D is a* Prüfer *domain.*

Proof. Note that $D(+)V$ is a strong ϕ-ring. So the result follows by Proposition 2.9 and Theorem 2.10. □

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The following example shows that the ϕ-weak global dimensions of ϕ-Prüfer rings can be sufficiently large, and so the condition "R is a strong ϕ-ring" in Theorem 2.10(4) cannot be removed.

Example 2.13. Let R be the ring in Example 1.1. Then R is a ϕ-Prüfer rings since $R/\mathrm{Nil}(R) \cong \mathbb{Z}$ is a Prüfer domain. It is easy to verify that there is a projective resolution of $\langle p\rangle(+)\mathbb{Z}(\mathfrak{p}^\infty)$

$$\cdots \to R \xrightarrow{d_4} R \xrightarrow{d_3} R \xrightarrow{d_2} R \xrightarrow{d_1} R \xrightarrow{d_0} \langle p\rangle(+)\mathbb{Z}(\mathfrak{p}^\infty) \to 0,$$

where d_n is a multiplication by $(\mathfrak{p}, 0)$ when n is even, and a multiplication by $(0, \frac{1}{\mathfrak{p}}+\mathbb{Z})$ when n is odd. Note that the above projective resolution is not split. So the global dimension, and hence the weak global dimension of R is infinite. By Example 1.12, every strongly ϕ-flat R-module is flat. Hence the ϕ-weak global dimension of R is also infinite.

3. ON ϕ-INJECTIVE DIMENSIONS OF MODULES AND ϕ-GLOBAL DIMENSIONS OF RINGS

It is well known that the injective dimension of an R-module M is defined as the length of the shortest injective resolutions of M and the global dimension of R is the supremum of the injective dimensions of all R-modules. We now introduce the notion of ϕ-injective dimension of an R-module as follows.

Definition 3.1. Let R be a ring and M an R-module. We write $\phi\text{-id}_R(M) \leq n$ (ϕ-id abbreviates *ϕ-injective dimension*) if there is an exact sequence of R-modules

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0 \qquad (\heartsuit)$$

where each E_i is strongly nonnil-injective for $i = 0, \dots, n$. The exact sequence ($\heartsuit$) is said to be a ϕ-injective resolution of length n of M. If such finite resolution does not exist, then we say $\phi\text{-id}_R(M) = \infty$; otherwise, define $\phi\text{-id}_R(M) = n$ if n is the length of the shortest ϕ-injective resolution of M.

It is obvious that an R-module M is strongly nonnil-injective if and only if ϕ-$\mathrm{id}_R(M) = 0$. Certainly, $\phi\text{-id}_R(M) \leq \mathrm{id}_R(M)$. If R is an integral domain, then $\phi\text{-id}_R(M) = \mathrm{id}_R(M)$

Proposition 3.2. *Let R be an* NP*-ring. Then the following statements are equivalent for an R-module M.*

(1) ϕ-$\mathrm{id}_R(M) \leq n$.
(2) $\mathrm{Ext}_R^{n+k}(T, M) = 0$ *for all ϕ-torsion R-modules T and all positive integers k.*
(3) $\mathrm{Ext}_R^{n+k}(R/I, M) = 0$ *for all nonnil ideals I and all positive integers k.*

(4) If $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ *is an exact sequence, where* $E_0, E_1, \ldots, E_{n-1}$ *are strongly nonnil-injective* R*-modules, then* E_n *is strongly nonnil-injective.*

(5) If $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ *is an exact sequence, where* $E_0, E_1, \ldots, E_{n-1}$ *are injective* R*-modules, then* E_n *is strongly nonnil-injective.*

(6) There exists an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$*, where* $E_0, E_1, \ldots, E_{n-1}$ *are injective* R*-modules and* E_n *is a strongly nonnil-injective* R*-module.*

Proof. (1) ⇒ (2): We prove (2) by induction on n. For the case $n = 0$, (2) trivially holds because M is strongly nonnil-injective. If $n > 0$, then there is an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$, where each E_i is strongly nonnil-injective for $i = 0, \ldots, n$. Set $K_0 = \mathrm{Coker}(E_0 \to M)$. Then both $0 \to M \to E_0 \to K_0 \to 0$ and $0 \to K_0 \to E_1 \to \cdots \to E_n \to 0$ are exact. So $\phi\text{-id}_R(K_0) \le n - 1$. By induction, $\mathrm{Ext}_R^{n-1+k}(T, K_0) = 0$ for all ϕ-torsion R-modules T and all positive integers k. Thus, it follows from the exact sequence

$$0 = \mathrm{Ext}_R^{n+k-1}(T, E_0) \to \mathrm{Ext}_R^{n+k-1}(T, K_0) \to \mathrm{Ext}_R^{n+k}(T, M) \to \mathrm{Ext}_R^{n+k}(T, E_0) = 0$$

that $\mathrm{Ext}_R^{n+k}(M, T) \cong \mathrm{Ext}_R^{n-1+k}(T, K_0) = 0$.

(2) ⇒ (3) and (4) ⇒ (5): Trivial.

(3) ⇒ (4): Let $K_0 = \mathrm{Coker}(M \to E_0)$ and $K_i = \mathrm{Coker}(E_{i-1} \to E_i)$, where $i = 1, \ldots, n-1$. Then $K_{n-1} \cong E_n$. Since all $E_0, E_1, \ldots, E_{n-1}$ are strongly nonnil-injective, $\mathrm{Ext}_R^k(R/I, E_n) \cong \mathrm{Ext}_R^{1+k}(R/I, K_{n-2}) \cong \cdots \cong \mathrm{Ext}_R^{n+k}(R/I, M) = 0$ for all nonnil ideal I and any positive integer k by dimensional shift. Hence E_n is strongly nonnil-injective by Lemma 1.4.

(5) ⇒ (6): Consider the injective resolution of M: $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-2} \xrightarrow{d_{n-2}} E_{n-1} \to \cdots$, where $E_0, E_1, \ldots, E_{n-1}$ are injective R-modules. Then $E_n := \mathrm{Coker}(d_{n-2})$ is strongly nonnil-injective by (5).

(6) ⇒ (1): Trivial. □

Corollary 3.3. *Let* R *be an* NP*-ring,* M *an* R*-module and* E *an injective cogenerator of the category of all* R*-modules. Then* $\phi\text{-fd}_R(M) = \phi\text{-}id_R(\mathrm{Hom}_R(M, E))$*.*

Proof. It follows by Proposition 2.2, Proposition 3.2 and the adjoint isomorphism: $\mathrm{Ext}_R^n(T, \mathrm{Hom}_R(M, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_n^R(T, M), E) = 0$. □

The proofs of the following two results are similar with the classical ones, and so we omit their proofs.

Corollary 3.4. *Let* R *be an* NP*-ring and* $0 \to A \to B \to C \to 0$ *be an exact sequence of* R*-modules. Then the following statements hold.*

(1) ϕ*-id*$_R(A) \leq 1 + \max\{\phi\text{-}id_R(B), \phi\text{-}id_R(C)\}$.

(2) If ϕ*-id*$_R(B) < \phi$*-id*$_R(A)$*, then* ϕ*-id*$_R(C) = \phi$*-id*$_R(A) - 1 \geq \phi$*-id*$_R(B)$.

Corollary 3.5. *Let R be an* NP*-ring and $\{M_i \mid i \in \Gamma\}$ be a family of R-modules. Then*

$$\phi\text{-}id_R(\prod_{i\in\Gamma} M_i) = \sup\{\phi\text{-}id_R(M_i)\}.$$

Now, we are ready to introduce the ϕ-global dimension of a ring in terms of nonnil-injective dimensions.

Definition 3.6. The *ϕ-global dimension* of a ring R is defined by

$$\phi\text{-gl.dim}(R) = \sup\{\phi\text{-id}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Obviously, by definition, ϕ-gl.dim$(R) \leq$gl.dim(R). Notice that if R is an integral domain, then ϕ-gl.dim$(R) =$gl.dim(R). The following result can easily be deduced by Proposition 3.2 and so we omit its proof.

Theorem 3.7. *Let R be an* NP*-ring. Then the following statements are equivalent for R.*

(1) ϕ-gl.dim$(R) \leq n$.

(2) ϕ*-id*$_R(M) \leq n$ *for all R-modules M.*

(3) $\mathrm{Ext}_R^{n+k}(T, M) = 0$ *for all R-modules M, all ϕ-torsion modules T and all positive integers k.*

(4) $\mathrm{Ext}_R^{n+k}(R/I, M) = 0$ *for all R-modules M, all nonnil ideals I of R and all positive integers k.*

(5) $\mathrm{Ext}_R^{n+1}(T, M) = 0$ *for all R-modules M and all ϕ-torsion modules T.*

(6) $\mathrm{Ext}_R^{n+1}(R/I, M) = 0$ *for all R-modules M and all nonnil ideals I of R.*

Consequently, the ϕ-global dimension of R is determined by the formulas:

$$\phi\text{-gl.dim}(R) = \sup\{\mathrm{pd}_R(R/I) | I \textit{ is a nonnil ideal of } R\}.$$

It follows by Theorem 2.6 and Theorem 3.6 that ϕ-gl.dim$(R) \leq \phi$-w.gl.dim(R) for any NP-ring R. Recall from [10] that an R-module M is called super finitely presented if there exists an exact sequence of R-modules

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with each P_i finitely generated and projective. It is well-known that the weak global dimensions and global dimensions coincide over Noetherian rings. For ϕ-dimensions, we have the following result.

Corollary 3.8. *Let R be a* NP*-ring such that every nonnil ideal of R is super finitely presented. Then* ϕ-gl.dim$(R) = \phi$-w.gl.dim(R).

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Proof. Let I be a nonnil ideal of R. Since R/I is super finitely presented, $\mathrm{pd}_R(R/I) = \mathrm{fd}_R(R/I)$ because finitely presented flat modules are projective. Hence ϕ-gl.dim$(R) = \phi$-w.gl.dim(R) by Theorem 2.6 and Theorem 3.6. □

Remark 3.9. Recall from [6] that a ϕ-ring R is said to be nonnil-Noetherian if every nonnil ideal is finitely generated. Trivially, if every nonnil ideal of R is super finitely presented, then R is nonnil-Noetherian. However, the converse does not hold in general. Indeed, let $R = \mathbb{Z}(+)\bigoplus_{i=1}^{\infty}(\mathbb{Q}/\mathbb{Z})$. Then R is nonnil-Noetherian. But there exists a nonnil ideal of R which is not finitely presented (see [17, Remark 1.1] or [11, Example 4.11]). However, we do not have an example in hand to distinguish ϕ-gl.dim(R) and ϕ-w.gl.dim(R) over a nonnil-Noetherian ring R.

Theorem 3.10. *Let R be an* NP*-ring. Then the following statements hold.*

(1) If R is a strong ϕ-ring, then $\mathrm{gl.dim}(R/\mathrm{Nil}(R)) \leq \phi\text{-gl.dim}(R)$.

(2) $\phi\text{-gl.dim}(R) - \mathrm{fd}_R(R/\mathrm{Nil}(R)) \leq \mathrm{gl.dim}(R/\mathrm{Nil}(R))$.

Proof. Suppose $\mathrm{gl.dim}(R/\mathrm{Nil}(R)) = n$. So there exists a nonnil ideal I of R and an $R/\mathrm{Nil}(R)$-module M such that

$$\mathrm{Ext}^n_{R/\mathrm{Nil}(R)}(R/I, M) \cong \mathrm{Ext}^n_{R/\mathrm{Nil}(R)}(R/I \otimes_R R/\mathrm{Nil}(R), M) \neq 0.$$

Note that $\mathrm{Tor}^R_n(R/I, R/\mathrm{Nil}(R)) = 0$ for all $n \geq 1$. So

$$\mathrm{Ext}^n_R(R/I, M) \cong \mathrm{Ext}^n_{R/\mathrm{Nil}(R)}(R/I \otimes_R R/\mathrm{Nil}(R), M) \neq 0,$$

and hence $\mathrm{pd}_R(R/I) \geq n$. It follows by Theorem 3.7 that ϕ-gl.dim$(R) \geq n$.

(2) It immediately follows from [19, Theorem 3.8.1] and Theorem 3.7. □

It is natural to ask the question:

Question 3.11. Let R be a strong ϕ-ring. Does the following equation hold?

$$\mathrm{gl.dim}(R/\mathrm{Nil}(R)) = \phi\text{-gl.dim}(R).$$

We can verify it in the following case.

Proposition 3.12. *Let D be an integral domain with quotient field Q and let V be a linear space over Q. Then* ϕ-gl.dim$(D(+)V)$ =gl.dim(D).

Proof. Set $R = D(+)V$. Assume $\mathrm{gl.dim}(D) \leq n$. Let M be an R-module. Then M is naturally a D-module. Let J be a nonnil ideal of R. Then by [3, Corollary 3.4], we have $J = I(+)V$ with I a nonzero ideal of D. Note that R is a flat D-module. By [8, Proposition 4.1.3] we have

$$\mathrm{Ext}^{n+1}_R(R/J, M) \cong \mathrm{Ext}^{n+1}_R(D/I \otimes_D R, M) \cong \mathrm{Ext}^{n+1}_D(D/I, M) = 0.$$

So ϕ-gl.dim$(D(+)V)$ $\leq$gl.dim(D). The result follows by Theorem 3.10. □

It is proved in [22, Theorem 1.7] that a ϕ-ring R is a ϕ-von Neumann regular ring if and only if every R-module is nonnil-injective. Moreover, we have the following result.

Theorem 3.13. *Let R be a ϕ-ring. Then the following statements are equivalent for R.*

(1) ϕ-gl.dim$(R) = 0$.

(2) Every R-module is strongly nonnil-injective.

(3) R is a ϕ-von Neumann regular ring.

Proof. (1) ⇔ (2): Clearly.

(2) ⇒ (3): It follows by [22, Theorem 1.7].

(3) ⇒ (2): Suppose R is a ϕ-von Neumann regular ring. Then R is a ZN-ring by the proof of Theorem 2.10. Now (2) follows by Theorem 1.6 and [22, Theorem 1.7]. □

Recall from [2] that a ϕ-ring R is called a *ϕ-Dedekind* ring provided that any nonnil ideal of R is ϕ-invertible. It is proved in [2, Theorem 2.5] that a ϕ-ring R is a ϕ-Dedekind ring if and only if $R/\text{Nil}(R)$ is a Dedekind domain.

Theorem 3.14. *Let R be a ϕ-ring. Then the following statements are equivalent for R.*

(1) ϕ-gl.dim$(R) \leq 1$.

(2) Every quotient module of injective R-module is strong ϕ-injective.

(3) Every quotient module of strong ϕ-injective R-module is strong ϕ-injective.

*(4) R is a ϕ-*Dedekind *strong ϕ-ring.*

Proof. (1) ⇔ (2) ⇔ (3): Clearly.

(4) ⇒ (2): It follows by [22, Theorem 1.7].

(2) ⇒ (4): Suppose every quotient module of injective R-module is strong ϕ-injective. We claim every ideal of R is strongly ϕ-flat. Indeed, let I be an ideal of R. Then for any ϕ-torsion R-module T and positive integer n, we have $\text{Hom}_{\mathbb{Z}}(\text{Tor}_n^R(T, I), \mathbb{Q}/\mathbb{Z}) \cong \text{Ext}_R^n(T, \text{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})) = 0$ since $\text{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a quotient module of the injective R-module $\text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. Hence $\text{Tor}_n^R(T, I) = 0$, whence I is strongly ϕ-flat. It follows by [15, Corollary 2.8] that R is a strong ϕ-ring. Hence the result follows by [22, Theorem 1.7] and Theorem 1.6. □

Corollary 3.15. *Let D be an integral domain with quotient field Q and let V be a linear space over Q. Then $D(+)V$ is a ϕ-Dedekind ring if and only if D is a Dedekind domain.*

Proof. Note that $D(+)V$ is a strong ϕ-ring. So the result immediately follows by Proposition 3.12 and Theorem 3.13. $\square$

Remark 3.16. When $\mathrm{gl.dim}(R/\mathrm{Nil}(R)) \leq 1$, Question 3.11 holds by Theorem 3.13 and Theorem 3.14. The ϕ-global dimensions of ϕ-Dedekind rings can be large than 1. Indeed, let D be a Dedekind domain and Q its quotient field. Then $R = D(+)Q/D$ is a ϕ-Dedekind ring since $R/\mathrm{Nil}(R) \cong \mathbb{Z}$ is a Dedekind domain. However, since R is not a strong ϕ-ring, we have ϕ-$\mathrm{gl.dim}(R) > 1$.

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