# On uniformly $S$-coherent rings

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**Abstract**

In this paper, we introduce and study the notions of uniformly $S$-finitely presented modules and uniformly $S$-coherent rings (resp., modules) which are "uniform" versions of ($c$-)$S$-finitely presented modules and ($c$-)$S$-coherent rings (resp., modules) introduced by Bennis and Hajoui [3]. Among the results, the uniform $S$-versions of the Chase result, the Chase theorem, and the Matlis theorem are obtained.

*Key Words:* uniformly $S$-coherent ring; uniformly $S$-finitely presented module; uniformly $S$-coherent module; uniformly $S$-flat module; uniformly $S$-injective module.
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## 1. Introduction

Throughout this paper, all rings are commutative with identity. Let $R$ be a ring. For a subset $U$ of an $R$-module $M$, we denote by $\langle U\rangle$ the submodule of $M$ generated by $U$. A subset $S$ of $R$ is called a multiplicative subset of $R$ if $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$.

The study of commutative rings in terms of multiplicative sets began with Anderson and Dumitrescu [1], who introduced the notion of $S$-Noetherian rings. Recall that a ring $R$ is called an $S$-Noetherian ring if for any ideal $I$ of $R$, there is a finitely generated sub-ideal $K$ of $I$ such that $sI \subseteq K$ for some $s \in S$. Cohen's theorem, Eakin-Nagata theorem and Hilbert basis theorem for $S$-Noetherian rings are also given in [1]. However, the element $s \in S$ in the definition of $S$-Noetherian rings is not "uniform" in general. This situation make it difficult to study $S$-Noetherian rings via module-theoretic methods. To overcome this difficulty, Qi et al. [16] defined uniformly $S$-Noetherian rings as $S$-Noetherian rings in which the choice of $s$ is fixed. Then they characterized uniformly $S$-Noetherian rings using $u$-$S$-injective modules.

Recall from [7] that a ring $R$ is said to be a coherent ring provided that any finitely generated ideal is finitely presented. The notion of coherent rings, which is a generalization of Noetherian rings, is another important rings defined by finiteness condition. Many algebraists studied coherent rings in terms of various of modules. Early in 1960, Chase [5, Theorem 2.1] showed that a ring is coherent exactly when the class of flat modules is closed under the direct product. In 1970 Stenström [19, Theorem 3.2] obtained that coherent rings are exactly rings over which every direct limit of absolutely pure modules is absolutely pure. In 1982, Matlis [14, Theorem 1] proved that a ring $R$ is coherent if and only if $\mathrm{Hom}_R(M, E)$ is flat for any injective modules $M$ and $E$.

To extend coherent rings by multiplicative sets, Bennis et al. [3] introduced the notions of $S$-coherent rings and $c$-$S$-coherent rings. They also gave an $S$-version of Chase's result to characterize $S$-coherent rings using ideals. Recently, the authors in paper et al.[17] characterized $S$-coherent rings in terms of $S$-Mittag-Leffler modules and $S$-flat modules (which can be seen as flat modules by localizing at $S$).

The main motivation of this paper is to introduce and study the "uniform" version of $S$-coherent rings for extending uniformly $S$-Noetherian rings. The organization of the paper is as follows: In Section 2, we introduce and study uniformly $S$-finitely presented modules and their connections with $u$-$S$-flat modules and $u$-$S$-projective modules (see Proposition 2.8). In Section 3, we introduce uniformly $S$-coherent modules and uniformly $S$-coherent rings. In particular, we study ideal-theoretic characterizations of uniformly $S$-coherent rings (see Proposition 3.11). Moreover examples of $S$-coherent rings and $c$-$S$-coherent rings which are not uniformly $S$-coherent of are provided (see Example 3.15). In Section 4, the Chase theorem and the Matlis theorem are obtained for uniformly $S$-coherent rings (see Theorem 4.4 and Theorem 4.7).

Since the paper involves uniformly torsion theory, we give a quick review (see [21] for more details). An $R$-module $T$ is called $u$-$S$-torsion (with respect to $s$) provided that there exists $s \in S$ such that $sT = 0$. An $R$-sequence $\cdots \to A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \to \cdots$ is $u$-$S$-exact if for any $n$ there is an element $s \in S$ such that $s\mathrm{Ker}(f_{n+1}) \subseteq \mathrm{Im}(f_n)$ and $s\mathrm{Im}(f_n) \subseteq \mathrm{Ker}(f_{n+1})$. An $R$-sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a short $u$-$S$-exact sequence (with respect to $s$) if $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$ for some $s \in S$. An $R$-homomorphism $f : M \to N$ is a *$u$-$S$-monomorphism* (resp., *$u$-$S$-epimorphism*, *$u$-$S$-isomorphism*) (with respect to $s$) provided $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$ ) is $u$-$S$-exact (with respect to $s$). Let $M$ and $N$ be $R$-modules. We say $M$ is $u$-$S$-isomorphic to $N$ if there exists a $u$-$S$-isomorphism $f : M \to N$. A family $\mathcal{C}$ of $R$-modules is said to be closed under $u$-$S$-isomorphisms if $M$ is $u$-$S$-isomorphic to $N$ and $M$ is in $\mathcal{C}$, then $N$



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is also in $\mathcal{C}$. One can deduce from the following [24, Lemma 2.1] that the existence of $u$-$S$-isomorphisms of two $R$-modules is actually an equivalence relation.

## 2. UNIFORMLY $S$-FINITELY PRESENTED MODULES

Recall from [1] that an $R$-module $M$ is called $S$-finite (with respect to $s$) provided that there exist an element $s \in S$ and a finitely generated $R$-module $F$ such that $sM \subseteq F \subseteq M$. Trivially, $S$-finite modules are generalizations of finitely generated modules. For generalizing finitely presented $R$-modules, Bennis et al. [3] introduced the notions of $S$-finitely presented modules and $c$-$S$-finitely presented modules. Following [3, Definition 2.1] that an $R$-module $M$ is called $S$-finitely presented provided that there exists an exact sequence of $R$-modules $0 \to K \to F \to M \to 0$ with $K$ $S$-finite and $F$ finitely generated free. Certainly, an $R$-module $M$ is $S$-finitely presented if and only if there exists an exact sequence of $R$-modules $0 \to T_1 \to N \to M \to 0$ with $N$ finitely presented and $sT_1 = 0$ for some $s \in S$. Following [3, Definition 4.1] that an $R$-module $M$ is called $c$-$S$-finitely presented provided that there exists a finitely presented submodule $N$ of $M$ such that $sM \subseteq N \subseteq M$ for some $s \in S$. Trivially, an $R$-module $M$ is called $c$-$S$-finitely presented if and only if there exists an exact sequence of $R$-modules $0 \to N \to M \to T_2 \to 0$ with $N$ finitely presented and $sT_2 = 0$ for some $s \in S$. Next we will give the notion of uniformly $S$-finitely presented modules which generalize both $S$-finitely presented modules and $c$-$S$-finitely presented modules.

**Definition 2.1.** Let $R$ be a ring, $S$ be a multiplicative subset of $R$ and $s \in S$. An $R$-module $M$ is called *$u$-$S$-finitely presented* (abbreviates *uniformly $S$-finitely presented*) (with respect to $s$) provided that there is an exact sequence

$$0 \to T_1 \to F \xrightarrow{f} M \to T_2 \to 0$$

with $F$ finitely presented and $sT_1 = sT_2 = 0$.

Trivially, $S$-finitely presented modules and $c$-$S$-finitely presented modules are all $u$-$S$-finitely presented. Certainly, every $u$-$S$-finitely presented $R$-module is $S$-finite. Indeed, since in Definition 2.1 we have $sT_2 = 0$, so $sM \subseteq \mathrm{Im}(f)$. Note that the fact that $\mathrm{Im}(f)$ is finitely generated implies $M$ is $S$-finite.

By [24, Lemma 2.1], an $R$-module $M$ is $u$-$S$-finitely presented if and only if there is an exact sequence $0 \to T_1 \to M \xrightarrow{g} F \to T_2 \to 0$ with $F$ finitely presented and $s'T_1 = s'T_2 = 0$ for some $s' \in S$. So an $R$-module $M$ is $u$-$S$-finitely presented if and only if it is $u$-$S$-isomorphic to a finitely presented $R$-module.



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**Theorem 2.2.** *Let $\Phi : 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ be a $u$-$S$-exact sequence of $R$-modules. The following statements hold.*

1. *The class of $u$-$S$-finitely presented modules is closed under $u$-$S$-isomorphisms.*
2. *If $M$ and $L$ are $u$-$S$-finitely presented, so is $N$.*
3. *Any finite direct sum of $u$-$S$-finitely presented modules is $u$-$S$-finitely presented.*
4. *If $N$ is $u$-$S$-finitely presented, then $L$ is $u$-$S$-finitely presented if and only if $M$ is $S$-finite.*

*Moreover, if $\Phi$ is an exact sequence, the both sides of the conditions in (2) and (4) can be taken to be "uniform" with respect to the same $s \in S$.*

*Proof.* (1) It follows from the fact that an $R$-module $M$ is $u$-$S$-finitely presented if and only if it is $u$-$S$-isomorphic to a finitely presented $R$-module.

(2) Since $u$-$S$-finitely presented modules are closed under $u$-$S$-isomorphisms, we may assume $\Phi$ is an exact sequence by (1). Consider the following push-out:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \| & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{m} & X & \xrightarrow{n} & L & \longrightarrow & 0. \end{array}$$

with $F_2$ finitely presented, $\mathrm{Ker}(h)$ and $\mathrm{Coker}(h)$ $u$-$S$-torsion. So $l$ is also a $u$-$S$-isomorphism. Consider the following pull-back:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{m} & X & \xrightarrow{n} & L & \longrightarrow & 0 \\ & & \| & & {\scriptstyle k}\uparrow & & {\scriptstyle j}\uparrow & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & Y & \longrightarrow & F_2 & \longrightarrow & 0. \end{array}$$

with $F_2$ finitely presented, $\mathrm{Ker}(j)$ and $\mathrm{Coker}(j)$ $u$-$S$-torsion. So $k$ is also a $u$-$S$-isomorphism. Since $F_1$ and $F_2$ are finitely presented, $Y$ is also finitely presented. Hence $N$ is $u$-$S$-isomorphic to a finitely presented $R$-module, and thus is $u$-$S$-finitely presented.

(3) This follows from (2).

(4) Since $u$-$S$-finitely presented modules and $S$-finite modules are closed under $u$-$S$-isomorphisms respectively, we may assume $\Phi$ is an exact sequence by (1). Suppose $M$ is $S$-finite. Since $N$ is $u$-$S$-finitely presented, there is an exact sequence $0 \to T_1 \to F \xrightarrow{l} N \to T_2 \to 0$ with $F$ finitely presented and $sT_1 = sT_2 = 0$ for some $s \in S$.

Consider the following pull-back of $f$ an $l$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & L & \longrightarrow & 0 \\
 & & {\scriptstyle s}\uparrow & & {\scriptstyle l}\uparrow & & {\scriptstyle t}\uparrow & & \\
0 & \longrightarrow & Z & \longrightarrow & F & \longrightarrow & K & \longrightarrow & 0.
\end{array}$$

Since $l$ is a $u$-$S$-isomorphism, $s$ and $t$ are both $u$-$S$-isomorphisms. So $Z$ is also $S$-finite. Note that $L$ is $u$-$S$-isomorphic to $K$ which is $u$-$S$-finitely presented (see [3, Theorem 2.4(4)]). So $L$ is $u$-$S$-finitely presented. Suppose $L$ is $u$-$S$-finitely presented. Considering the above pull-back, we have $K$ is also $S$-finitely presented. Hence $Z$ is $S$-finite by [3, Theorem 2.4(5)], which implies that $M$ is also $S$-finite.

The "Moreover" part can be checked by the proof of (2) and (4). $\square$

Recall from [4] that an $R$-module $M$ is said to be $S$-Noetherian provided that any submodule of $M$ is $S$-finite. A ring $R$ is called $S$-Noetherian if $R$ itself is an $S$-Noetherian $R$-module.

**Proposition 2.3.** *Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Then a ring $R$ is $S$-Noetherian if and only if any $S$-finite module is $u$-$S$-finitely presented.*

*Proof.* For necessity, let $M$ be an $S$-finite module. Then there is a $u$-$S$-epimorphism $f : F \to M$ with $F$ finitely generated free. Since $R$ is an $S$-Noetherian ring, we have $F$ is also $S$-Noetherian (see [4]). Hence $M$ is $u$-$S$-finitely presented by Theorem 2.2(4). For sufficiency, let $I$ be an ideal of $R$. Then $R/I$ is $S$-finite, and thus $u$-$S$-finitely presented. By Theorem 2.2(4) again, $I$ is $S$-finite. $\square$

**Proposition 2.4.** *Let $R$ be a ring, $S$ a multiplicative subset of $R$ consisting of finite elements. Then an $R$-module $M$ is a $u$-$S$-finitely presented $R$-module if and only if $M_S$ is a finitely presented $R_S$-module.*

*Proof.* Suppose $M$ is a $u$-$S$-finitely presented $R$-module. Then there is an exact sequence $0 \to T_1 \to N \xrightarrow{f} M \to T_2 \to 0$ with $N$ finitely presented and $sT_1 = sT_2 = 0$. Localizing at $S$, we have $0 \to (T_1)_S \to N_S \xrightarrow{f} M_S \to (T_2)_S \to 0$. Since $sT_1 = sT_2 = 0$, $(T_1)_S = (T_2)_S = 0$. So $M_S \cong N_S$ is a finitely generated $R_S$-module. On the other hand, suppose $M_S$ is a finitely generated $R_S$-module. Let $S = \{s_1, \dots, s_n\}$ and set $s = s_1 \cdots s_n$. We may assume that $M_S$ is generated by $\{\frac{m_1}{s}, \dots, \frac{m_n}{s}\}$. Consider the $R$-homomorphism $f : R^n \to M$ satisfying $f(e_i) = m_i$ for each $i = 1, \dots, n$. It is easy to verify that $f$ is a $u$-$S$-epimorphism. Consider the exact sequence $0 \to \mathrm{Ker}(f_S) \to R_S^n \xrightarrow{f_S} M_S \to 0$. Then $\mathrm{Ker}(f_S)$ is a finitely generated $R_S$-module, and thus $\mathrm{Ker}(f)$ is $S$-finite. By Theorem 2.2(2), $M$ is $u$-$S$-finitely presented. $\square$



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Let $\mathfrak{p}$ be a prime ideal of $R$. We say an $R$-module $M$ is (simply) $\mathfrak{p}$-*finite* provided $M$ is $(R \setminus \mathfrak{p})$-finite. We always denote by $\mathrm{Spec}(R)$ the spectrum of all prime ideals of $R$, and $\mathrm{Max}(R)$ the set of all maximal ideals of $R$, respectively.

**Lemma 2.5.** *Let $R$ be a ring, $S$ be a multiplicative subset of $R$ and $M$ be an $R$-module. The following statements are equivalent:*

(1) *$M$ is finitely generated $R$-module;*
(2) *$M$ is $\mathfrak{p}$-finite for any $\mathfrak{p} \in \mathrm{Spec}(R)$;*
(3) *$M$ is $\mathfrak{m}$-finite for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) For each $\mathfrak{m} \in \mathrm{Max}(R)$, there exist an element $s^{\mathfrak{m}} \in R \setminus \mathfrak{m}$ and a finitely generated submodule $F^{\mathfrak{m}}$ of $M$ such that $s^{\mathfrak{m}}M \subseteq F^{\mathfrak{m}}$. Since $\{s^{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}(R)\}$ generated $R$, there exist finite elements $\{s^{\mathfrak{m}_1}, \dots, s^{\mathfrak{m}_n}\}$ such that $\langle s^{\mathfrak{m}_1}, \dots, s^{\mathfrak{m}_n} \rangle = R$. So $M = \langle s^{\mathfrak{m}_1}, \dots, s^{\mathfrak{m}_n} \rangle M \subseteq F^{\mathfrak{m}_1} + \cdots + F^{\mathfrak{m}_n} \subseteq M$. Hence $M = F^{\mathfrak{m}_1} + \cdots + F^{\mathfrak{m}_n}$. It follows that $M$ is finitely generated. □

Let $\mathfrak{p}$ be a prime ideal of $R$. We say an $R$-module $M$ is (simply) $u$-$\mathfrak{p}$-*finitely presented* provided $M$ is $u$-$(R \setminus \mathfrak{p})$-finitely presented.

**Proposition 2.6.** *Let $R$ be a ring, $S$ be a multiplicative subset of $R$ and $M$ be an $R$-module. The following statements are equivalent:*

(1) *$M$ is a finitely presented $R$-module;*
(2) *$M$ is $u$-$\mathfrak{p}$-finitely presented for any $\mathfrak{p} \in \mathrm{Spec}(R)$;*
(3) *$M$ is $u$-$\mathfrak{m}$-finitely presented for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) By Lemma 2.5, $M$ is finitely generated. Consider the exact sequence $0 \to K \to F \to M \to 0$ with $F$ finitely generated free. By Theorem 2.2, $K$ is $\mathfrak{m}$-finite for any $\mathfrak{m} \in \mathrm{Max}(R)$. So $K$ is also finitely generated, and thus $M$ is finitely presented. □

Let $\{M_j\}_{j\in\Gamma}$ be a family of $R$-modules and $N_j$ be a submodule of $M_j$ generated by $\{m_{i,j}\}_{i\in\Lambda_j} \subseteq M_j$ for each $j \in \Gamma$. Recall from [21] that a family of $R$-modules $\{M_j\}_{j\in\Gamma}$ is *$u$-$S$-generated* (with respect to $s$) by $\{\{m_{i,j}\}_{i\in\Lambda_j}\}_{j\in\Gamma}$ provided that there exists an element $s \in S$ such that $sM_j \subseteq N_j$ for each $j \in \Gamma$, where $N_j = \langle\{m_{i,j}\}_{i\in\Lambda_j}\rangle$. We say that a family of $R$-modules $\{M_j\}_{j\in\Gamma}$ is *$u$-$S$-finite* (with respect to $s$) if the set $\{m_{i,j}\}_{i\in\Lambda_j}$ can be chosen as a finite set for each $j \in \Gamma$, that is, there is $s \in S$ such that $\{M_j\}_{j\in\Gamma}$ are all $S$-finite with respect to $s$. Recall from [16] that an $R$-module $M$ is called a *$u$-$S$-Noetherian module* provided the set of all submodules of $M$ is

$u$-$S$-finite. A ring $R$ is called to be a *$u$-$S$-Noetherian ring* provided that $R$ itself is a $u$-$S$-Noetherian $R$-module.

**Theorem 2.7.** *Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Then the following statements are equivalent:*

(1) *A ring $R$ is $u$-$S$-Noetherian;*
(2) *Any $S$-finite module is $u$-$S$-Noetherian;*
(3) *Any finitely generated module is $u$-$S$-Noetherian;*
(4) *There is $s \in S$ such that any finitely generated module is $u$-$S$-finitely presented with respect to $s$.*

*Proof.* (1) $\Rightarrow$ (2) Let $M$ be an $S$-finite module. Then there is a $u$-$S$-epimorphism $f : F \to M$ with $F$ finitely generated free. Since $R$ is $u$-$S$-Noetherian, we have $F$ is also $u$-$S$-Noetherian, and so is $M$ (see [16, Proposition 2.13]).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) Let $I$ be an ideal of $R$. Then $R/I$ is $u$-$S$-finitely presented with respect to $s$. So $I$ is $S$-finite with respect to $s$ by Theorem 2.2(4), which implies that $R$ is $u$-$S$-Noetherian. $\square$

Recall from [21, 24] that an $R$-module $P$ is called *$u$-$S$-projective* (resp., *$u$-$S$-flat)* provided that the induced sequence $0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0$ (resp., $0 \to P \otimes_R A \to P \otimes_R B \to P \otimes_R C \to 0$) is $u$-$S$-exact for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$. It was proved in [24, Proposition 2.9] that any $u$-$S$-projective module is $u$-$S$-flat.

**Proposition 2.8.** *Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Then the following statements hold.*

(1) *Every $S$-finite $u$-$S$-projective module is $u$-$S$-finitely presented.*
(2) *Every $u$-$S$-finitely presented $u$-$S$-flat module is $u$-$S$-projective.*

*Proof.* (1) Let $P$ be an $S$-finite $u$-$S$-projective module, then there is a $u$-$S$-exact sequence $\Psi : 0 \to \mathrm{Ker}(f) \xrightarrow{i} F \xrightarrow{f} P \to 0$ with $F$ finitely generated free. Since $P$ is $u$-$S$-projective, the sequence $\Psi$ is $u$-$S$-split by [24, Theorem 2.7]. So there is a $u$-$S$-epimorphism $i' : F \to \mathrm{Ker}(f)$ such that $i' \circ i = s\mathrm{Id}_{\mathrm{Ker}(f)}$ for some $s \in S$. Hence $\mathrm{Ker}(f)$ is $S$-finite, and so $P$ is $u$-$S$-finitely presented by Theorem 2.2.

(2) Let $M$ be a $u$-$S$-finitely presented $u$-$S$-flat module. Then there is a $u$-$S$-exact sequence $\Upsilon : 0 \to \mathrm{Ker}(f) \xrightarrow{i} F \xrightarrow{f} M \to 0$ with $F$ finitely generated free and $\mathrm{Ker}(f)$ $S$-finite. Since $M$ is $u$-$S$-flat, $\Upsilon$ is $u$-$S$-pure by [22, Proposition 2.4]. It follows from [22, Theorem 2.2] that $\Upsilon$ is $u$-$S$-split. Thus $M$ is $u$-$S$-projective. $\square$



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## 3. UNIFORMLY $S$-COHERENT MODULES AND UNIFORMLY $S$-COHERENT RINGS

Recall that an $R$-module is said to be a *coherent module* if it is finitely generated and any finitely generated submodule is finitely presented. A ring $R$ is said to be a *coherent ring* if $R$ is a coherent $R$-module. In this section, we will introduce a "uniform" version of coherent rings and coherent modules.

**Definition 3.1.** Let $R$ be a ring and $S$ be a multiplicative subset of $R$. An $R$-module $M$ is called a *$u$-$S$-coherent module* (abbreviates uniformly $S$-coherent) (with respect to $s$) provided that there is $s \in S$ such that it is $S$-finite with respect to $s$ and any finitely generated submodule of $M$ is $u$-$S$-finitely presented with respect to $s$.

**Theorem 3.2.** *Let $\Phi : 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ be a $u$-$S$-exact sequence of $R$-modules. The following statements hold.*

1. *The class of $u$-$S$-coherent modules is closed under $u$-$S$-isomorphisms.*
2. *If $L$ is $u$-$S$-coherent, then $M$ is $u$-$S$-coherent if and only if $N$ is $u$-$S$-coherent.*
3. *Any finite direct sum of $u$-$S$-coherent modules is $u$-$S$-coherent.*
4. *If $N$ is $u$-$S$-coherent and $M$ is $S$-finite, then $L$ is $u$-$S$-coherent.*

*Proof.* (1) Let $h : A \to B$ be a $u$-$S$-isomorphism with $s_1\mathrm{Ker}(h) = s_1\mathrm{Coker}(h) = 0$. Suppose $B$ is $u$-$S$-coherent with respect to $s_2$. Then one can check $A$ is $u$-$S$-coherent with respect to $s_1s_2$. Similarly, if $A$ is $u$-$S$-coherent, then $B$ is also $u$-$S$-coherent (see [24, Lemma 2.1]).

(2) By (1), we can assume that $\Phi$ is an exact sequence. Suppose $M$ and $L$ are $u$-$S$-coherent with respect to $s$. Then one can check $N$ is $u$-$S$-coherent with respect to $s$ from the proof of Theorem 2.2(2). Suppose $N$ and $L$ are $u$-$S$-coherent with respect to $s$. Then $M$ is $S$-finite with respect to some $s \in S$ by Theorem 2.2(4). Since $N$ is $u$-$S$-coherent with respect to $s$, $M$ is $u$-$S$-coherent with respect to $s$.

(3) This follows by (2).

(4) Assume that $\Phi$ is an exact sequence. Suppose $N$ is $u$-$S$-coherent with respect to $s$ and $M$ is $S$-finite with respect to $s$ for some $s \in S$. Then $L$ is also $S$-finite with respect to $s$. Let $K$ be a finitely generated submodule of $L$. Then the sequence $0 \to M \to g^{-1}(K) \to K \to 0$ is exact. So $g^{-1}(K)$ is $S$-finite. Consider the following



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commutative diagram with rows and columns exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(m) & \longrightarrow & \mathrm{Ker}(l) & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^n & \longrightarrow & R^{n+e} & \longrightarrow & R^e & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle l} & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & g^{-1}(K) & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

where $m$ and $l$ are $u$-$S$-epimorphisms. Since $N$ is $u$-$S$-coherent, $\mathrm{Ker}(l)$ is $S$-finite, and so is $K_1$. Thus $L$ is $u$-$S$-coherent (with respect to $s$). □

**Corollary 3.3.** *Let $f : M \to N$ be an $R$-homomorphism of $u$-$S$-coherent modules $M$ and $N$. Then* $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ *and* $\mathrm{Coker}(f)$ *are also $u$-$S$-coherent.*

*Proof.* Use Theorem 3.2 and the exact sequences $0 \to \mathrm{Ker}(f) \to M \to \mathrm{Im}(f) \to 0$ and $0 \to \mathrm{Im}(f) \to N \to \mathrm{Coker}(f) \to 0$. □

**Corollary 3.4.** *Let $M$ and $N$ be $u$-$S$-coherent sub-modules of a $u$-$S$-coherent module. Then $M + N$ is $u$-$S$-coherent if and only if so is $M \cap N$.*

*Proof.* This follows by Theorem 3.2 and the exact sequence $0 \to M \cap N \to M \oplus N \to M + N \to 0$. □

Let $\mathfrak{p}$ be a prime ideal of $R$. We say that an $R$-module $M$ is (simply) *$u$-$\mathfrak{p}$-coherent* provided $M$ is $u$-$(R \setminus \mathfrak{p})$-coherent.

**Proposition 3.5.** *Let $R$ be a ring, $S$ be a multiplicative subset of $R$ and $M$ be an $R$-module. The following statements are equivalent:*

1. *$M$ is a coherent $R$-module;*
2. *$M$ is $u$-$\mathfrak{p}$-coherent for any $\mathfrak{p} \in \mathrm{Spec}(R)$;*
3. *$M$ is $u$-$\mathfrak{m}$-coherent for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) By Lemma 2.5, $M$ is finitely generated. Let $N$ be a finitely generated of $M$. Then $M$ is $u$-$\mathfrak{m}$-finitely presented for any $\mathfrak{m} \in \mathrm{Max}(R)$. So $M$ is finitely presented by Proposition 2.6. □

**Definition 3.6.** Let $R$ be a ring, $S$ be a multiplicative subset of $R$ and $s \in S$. Then $R$ is called a *$u$-$S$-coherent ring* (abbreviates uniformly $S$-coherent ring) (with



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respect to $s$) provided that $R$ itself is a uniformly $S$-coherent $R$-module with respect to $s$.

Trivially, every coherent ring is $u$-$S$-coherent for any multiplicative set $S$. And if $S$ is composed of units, then $u$-$S$-coherent rings are exactly coherent rings.

The proof of the following result is easy and direct, so we omit it.

**Lemma 3.7.** *Let $R = R_1 \times R_2$ be direct product of rings $R_1$ and $R_2$, $S = S_1 \times S_2$ be a multiplicative subset of $R$. Then $R$ is $u$-$S$-coherent if and only if $R_i$ is $u$-$S_i$-coherent for any $i = 1, 2$.*

The following example shows that not every $u$-$S$-coherent rings is coherent.

**Example 3.8.** Let $R_1$ be a coherent ring and $R_2$ be a non-coherent ring, $S_1 = \{1\}$ and $S_2 = \{0\}$. Set $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Then $R$ is a $u$-$S$-coherent non-coherent ring.

Let $\mathfrak{p}$ be a prime ideal of $R$. We say a ring $R$ is (simply) $u$-$\mathfrak{p}$-*coherent* provided $R$ is $u$-$(R \setminus \mathfrak{p})$-coherent.

**Proposition 3.9.** *Let $R$ be a ring and $S$ be a multiplicative subset of $R$ . The following statements are equivalent:*

(1) *$R$ is a coherent ring;*
(2) *$R$ is a $u$-$\mathfrak{p}$-coherent ring for any $\mathfrak{p} \in \mathrm{Spec}(R)$;*
(3) *$R$ is a $u$-$\mathfrak{m}$-coherent ring for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

*Proof.* This follows by Proposition 3.5. □

**Proposition 3.10.** *Let $R$ be a ring and $S$ be a multiplicative subset of $R$. If $R$ is a $u$-$S$-Noetherian ring, then $R$ is $u$-$S$-coherent.*

*Proof.* This follows from Theorem 2.7. □

Trivially, $u$-$S$-coherent rings are not $u$-$S$-Noetherian in general. Indeed, we can find a non-Noetherian coherent ring in the case that $S = \{1\}$.

In 1960, Chase characterized coherent rings by considering annihilator of elements and intersection of finitely generated ideals in [5, Theorem 2.2]. Now, we give a "uniform" version of Chase's result.

**Proposition 3.11.** (***Chase's result for $u$-$S$-coherent rings***) *Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Then the following statements are equivalent:*

(1) *$R$ is a $u$-$S$-coherent ring;*

(2) *there is* $s \in S$ *such that* $(0 :_R r)$ *is* $S$*-finite with respect to* $s$ *for any* $r \in R$*, and the intersection of two finitely generated ideals of* $R$ *is* $S$*-finite with respect to* $s$*;*

(3) *there is* $s \in S$ *such that* $(I :_R b)$ *is* $S$*-finite with respect to* $s$ *for any element* $b \in R$ *and any finitely generated ideal* $I$ *of* $R$*.*

*Proof.* (1) ⇒ (2): Suppose $R$ is $u$-$S$-coherent with respect to $s$. Considering the exact sequence $0 \to (0 :_R r) \to R \to Rr \to 0$, we have $(0 :_R r)$ is $S$-finite with respect to $s$ by Theorem 2.2. For any two finitely generated ideals $I, J$ of $R$, we have $I \cap J$ is $S$-finite with respect to $s$ by Corollary 3.4 and Theorem 2.2.

(2) ⇒ (1): Let $I = \langle a_1, \ldots, a_n \rangle$ be a finitely generated ideal of $R$. We claim that $I$ is $u$-$S$-finitely presented with respect to $s$ by induction on $n$. Suppose $n = 1$. the claim follows by the exact sequence $0 \to (0 :_R r) \to R \to Rr \to 0$. Suppose $n = k$. Then the claim holds. Suppose $n = k + 1$. the claim holds by the exact sequence $0 \to \langle a_1, \ldots, a_k \rangle \cap \langle a_{k+1} \rangle \to \langle a_1, \ldots, a_k \rangle \oplus \langle a_{k+1} \rangle \to \langle a_1, \ldots, a_{k+1} \rangle \to 0$. So the claim holds for all $n$.

(1) ⇒ (3): Suppose $R$ is $u$-$S$-coherent with respect to $s$. Let $I$ be a finitely generated ideal of $R$ and $b$ be an element in $R$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & Rb + I & \longrightarrow & (Rb + I)/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & (I :_R b) & \longrightarrow & R & \longrightarrow & R/(I :_R b) & \longrightarrow & 0.
\end{array}$$

Since $R$ is $u$-$S$-coherent with respect to $s$, we have $Rb + I$ is $u$-$S$-finitely presented with respect to $s$. Since $I$ is finitely generated, $(Rb + I)/I$ is $u$-$S$-finitely presented with respect to $s$ by Theorem 2.2. Thus $(I :_R b)$ is $S$-finite is with respect to $s$ by Theorem 2.2 again.

(3) ⇒ (1): Let $I$ be a finitely generated ideal of $R$ generated by $\{a_1, \ldots, a_n\}$ . We will show that $I$ is $u$-$S$-finitely presented by induction on $n$. The case $n = 1$ follows from the exact sequence $0 \to (0 :_R a_1) \to R \to Ra_1 \to 0$. For $n \geq 2$, let $L = \langle a_1, \ldots, a_{n-1} \rangle$. Consider the exact sequence $0 \to (L :_R a_n) \to R \to (Ra_n + L)/L \to 0$. Then $(Ra_n + L)/L = I/L$ is $u$-$S$-finitely presented with respect to $s$ by (3) and Theorem 2.2. Consider the exact sequence $0 \to L \to I \to I/L \to 0$. Since $L$ is finitely presented by induction and $I/L$ is $u$-$S$-finitely presented with respect to $s$ , $I$ is also $u$-$S$-finitely presented with respect to $s$ by Theorem 2.2. □

Recall from [3] that a ring $R$ is $S$*-coherent* (resp., $c$-$S$*-coherent*) provided that any finitely generated ideal is $S$-finitely presented (resp., $c$-$S$-finitely presented).



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**Proposition 3.12.** *Let $R$ be a ring, $S$ be a multiplicative subset of $R$. If $R$ is a $u$-$S$-coherent ring, then $R$ is both $S$-coherent and $c$-$S$-coherent.*

*Proof.* Let $I$ be a finitely generated ideal and $0 \to K \to F \to I \to 0$ be an exact sequence with $F$ finitely generated free. Then $K$ is $S$-finite by Theorem 2.2(4). Thus $I$ is $S$-finitely presented, and so $R$ is $S$-coherent. Consider the exact sequence $0 \to T_1 \to N \xrightarrow{f} I \to T_2 \to 0$ with $N$ finitely presented and $sT_1 = sT_2 = 0$. Note that since $sT_2 = 0$, we have $sI \subseteq \mathrm{Im}(f) \cong N/T_1$. Since $sT_1 = 0$, $s^2 I$ can be seen as a submodule of $N$. Hence $I$ is $c$-$S$-finitely presented. Consequently, $R$ is $c$-$S$-coherent. □

**Proposition 3.13.** *Let $R$ be a ring and $S$ a multiplicative subset of $R$ consisting of finite elements. Then the following statements are equivalent:*

(1) *$R$ is a $u$-$S$-coherent ring;*
(2) *$R$ is an $S$-coherent ring;*
(3) *$R$ is a $c$-$S$-coherent ring.*

*Proof.* Suppose $S = \{s_1, \dots, s_n\}$ and set $s = s_1 \cdots s_n$.

(1) ⇒ (2) and (1) ⇒ (3) These follow by Proposition 3.12.

(2) ⇒ (1) Let $I$ be a finitely generated ideal of $R$. Then there is an exact sequence $0 \to K \to F \to I \to 0$ with $F$ finitely generated free and $K$ $S$-finite. Let $X$ be a submodule of $K$ such that $s_i K \subseteq X$ for some $s_i \in S$. So $sK/X = 0$ Then the exact sequence $0 \to K/X \to F/X \to I \to 0$ makes $I$ $u$-$S$-finitely presented with respect to $s$. So $R$ is a $u$-$S$-coherent ring.

(3) ⇒ (1) Let $I$ be a finitely generated ideal of $R$. Then there is a finitely presented sub-ideal $J$ of $R$ such that $s_i I \subseteq J = 0$. So $s(I/J) = 0$. Then the exact sequence $0 \to I \to J \to I/J \to 0$ makes $I$ $u$-$S$-finitely presented with respect to $s$. So $R$ is a $u$-$S$-coherent ring. □

Let $R$ be a ring, $M$ be an $R$-module and $S$ be a multiplicative subset of $R$. For any $s \in S$, there is a multiplicative subset $S_s = \{1, s, s^2, ....\}$ of $S$. We denote by $M_s$ the localization of $M$ at $S_s$. Certainly, $M_s \cong M \otimes_R R_s$

**Proposition 3.14.** *Let $R$ be a ring and $S$ a multiplicative subset of $R$. If $R$ is a $u$-$S$-coherent ring with respect to some $s \in S$, then $R_s$ is a coherent ring.*

*Proof.* Suppose $R$ is a $u$-$S$-coherent ring with respect to $s \in S$. Let $J$ be a finitely generated ideal of $R_s$. Then $J \cong I_s$ for some finitely generated ideal $I$ of $R$. So there is an exact sequence $0 \to T_1 \to K \to I \to T_2 \to 0$ with $K$ finitely presented and $sT_1 = sT_2 = 0$. Localizing at $S_s$, we have $(T_1)_s = (T_2)_s = 0$. So $J \cong I_s \cong K_s$ that is finitely presented over $R_s$. So $R_s$ is a coherent ring. □



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Next, we will give an example of a ring which is both $S$-coherent and $c$-$S$-coherent, but not $u$-$S$-coherent.

**Example 3.15.** Let $R$ be a domain. Set $S = R - \{0\}$. First, we will show $R$ is $c$-$S$-coherent. Let $I$ be a nonzero finitely generated ideal of $R$. Suppose $0 \neq r \in I$. Then we have $rI \subseteq Rr \subseteq I$. Since $Rr \cong R$ is finitely presented, $R$ is a $c$-$S$-coherent ring.

Next we will show that $R$ is $S$-coherent. Let $I$ be a nonzero finitely generated ideal of $R$ generated by nonzero elements $\{a_1, \ldots, a_n\}$. Set $a = a_1 \cdots a_n$. Consider the natural exact sequence $0 \to K \to R^n \xrightarrow{f} I \to 0$ satisfying $f(e_i) = a_i$ for each $i$. We claim that $K$ is $S$-finite with respect to $a$ by induction on $n$. Set $I_k = \langle a_1, \ldots, a_k \rangle$. Suppose $n = 1$. Then $K = 0$ as $a_1$ is a non-zero-divisor. So the claim trivially holds. Suppose the claim holds for $n = k$. Now let $n = k + 1$. Consider the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_k & \longrightarrow & R^k & \longrightarrow & I_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{k+1} & \longrightarrow & R^{k+1} & \longrightarrow & I_k + Ra_{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I_k :_R Ra_{k+1}) & \longrightarrow & R & \longrightarrow & (I_k + Ra_{k+1})/I_k & \longrightarrow & 0,
\end{array}$$

Since $a(I_k :_R Ra_{k+1}) \subseteq aR \subseteq (I_k :_R Ra_{k+1})$, it follows that $(I_k :_R Ra_{k+1})$ is $S$-finite with respect to $a$. By induction, $K_k$ is $S$-finite with respect to $a$. It is easy to check $K_{k+1}$ is also $S$-finite with respect to $a$. So the claim holds. Consequently, $R$ is $S$-coherent.

Now, let $R$ is a domain such that $R_s$ is not coherent for any $s \neq 0$. For example, $R = \mathbb{Q} + x\mathbb{R}[[x]]$ be the subring of formal power series ring $T = \mathbb{R}[[x]]$ with constants in $\mathbb{R}$ of real numbers, where $\mathbb{Q}$ is the set of all rational numbers. Indeed, let $0 \neq s = a + xf(x) \in R$. We divide it into two cases.

Case I: $a \neq 0$. In this case, $s$ is a unit in $R$, and so $R_s \cong R$, which is not coherent by [7, Theorem 5.2.3] .

Case II: $a = 0$. In this case, $R_s \cong \mathbb{Q} + (x\mathbb{R}[[x]])_{xf(x)} \cong \mathbb{Q} + (x\mathbb{R}[[x]])_x$. So $R_s$ can fit into a Milnor square of type II:

$$\begin{array}{ccc}
R_s & \hookrightarrow & \mathbb{R}[[x]][x^{-1}] \\
\downarrow & & \downarrow \\
\mathbb{Q} & \hookrightarrow & \mathbb{R}.
\end{array}$$



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Hence $R_s$ is not a coherent domain by [20, Theorem 8.5.17]. We will show that $R$ is not a $u$-$S$-coherent ring. On the contrary, suppose $R$ is $u$-$S$-coherent. Then there is a $s \neq 0$ such that $R_s$ is a coherent ring by Proposition 3.14, which is a contradiction.

## 4. Module-theoretic characterizations of uniformly $S$-coherent rings

In this section, we will characterize uniformly $S$-coherent rings in terms of $u$-$S$-flat modules and $u$-$S$-injective modules. The following lemma is basic and of independent interest.

**Lemma 4.1.** *Let $R$ be a ring, $r \in R$ and $M$ be an $R$-module. Suppose $N$ is a pure submodule of $M$. Then we have the following natural isomorphism*

$$\frac{rM}{rN} \cong r(\frac{M}{N}).$$

*Consequently, suppose $\{M_i \mid i \in \Lambda\}$ is a direct system of $R$-modules. Then*

$$r \varinjlim M_i \cong \varinjlim (rM_i).$$

*Proof.* Consider the surjective map $f : \frac{rM}{rN} \to r(\frac{M}{N})$ defined by $f(rm + rN) = r(m + N)$. It is certainly $R$-linear. We will check it is also well defined. Indeed, $f(rn+rN) = r(n+N) = r(0+N) = 0$. So $f$ is an $R$-epimorphism. Let $rm+rN \in \mathrm{Ker}(f)$. Then $rm \in N$. Since $N$ is a pure submodule of $M$, there is $n \in N$ such that $rm = rn$. So $rm + rN = rn + rN = 0$. Hence $f$ is an isomorphism. Suppose $\{(M_i, f_{ij}) \mid i, j \in \Lambda\}$ is a direct system of $R$-modules. Then there is a pure exact sequence $0 \to K \to \bigoplus M_i \to \varinjlim M_i \to 0$, where $K = \langle x - f_{ij}(x) \mid x \in M_i, i \leq j \in I\rangle$ (see [8, (2.1.1)]). Note that $\{(rM_i, f_{ij}) \mid i, j \in \Lambda\}$ is also a direct system of $R$-modules. We have the following equivalence

$$\varinjlim (rM_i) \cong \frac{\bigoplus rM_i}{K'} = \frac{r\bigoplus M_i}{rK} \cong r\frac{\bigoplus M_i}{K} \cong r \varinjlim M_i$$

where $K' = \langle rx - f_{ij}(rx) \mid rx \in rM_i, i \leq j \in I\rangle$. ∎

**Lemma 4.2.** *Let $E$ be an injective cogenerator. Then the following statements are equivalent:*

1. *$T$ is uniformly $S$-torsion with respect to $s$;*
2. *$\mathrm{Hom}_R(T, E)$ is uniformly $S$-torsion with respect to $s$.*

*Proof.* (1) ⇒ (2): This follows from [16, Lemma 4.2].

(2) ⇒ (1): Let $f : sT \to E$ be an $R$-homomorphism and $i : sT \to T$ be the embedding map. Since $E$ is injective, there exists an $R$-homomorphism $g : T \to E$ such

that $f = gi$. Let $st \in sT$. Then we have $f(st) = sg(t) = 0$ since $s\mathrm{Hom}_R(T, E) = 0$ . So $\mathrm{Hom}_R(sT, E) = 0$. Hence $sT = 0$ since $E$ is an injective cogenerator. □

Let $R$ be a ring and $S$ a multiplicative subset of $R$. Recall from [16, 22] that an $R$-module $E$ is said to be *u-S-injective* provided that $\mathrm{Ext}^1_R(M, E)$ is uniformly $S$-torsion for any $R$-module $M$; and is said to be *u-S-absolutely pure* provided that there exists an element $s \in S$ satisfying that for any finitely presented $R$-module $N$, $\mathrm{Ext}^1_R(N, E)$ is $u$-$S$-torsion with respect to $s$. A multiplicative subset $S$ of $R$ is said to be *regular* if it is composed of non-zero-divisors. Next, we give some new characterizations of $u$-$S$-flat modules

**Proposition 4.3.** *Let $R$ be a ring and $S$ be a multiplicative subset of $R$. Then the following statements are equivalent:*

(1) *$F$ is u-$S$-flat;*
(2) *there exists an element $s \in S$ such that $\mathrm{Tor}^R_1(N, F)$ is uniformly $S$-torsion with respect to $s$ for any finitely presented $R$-module $N$;*
(3) *$\mathrm{Hom}_R(F, E)$ is u-$S$-injective for any injective module $E$;*
(4) *$\mathrm{Hom}_R(F, E)$ is u-$S$-absolutely pure for any injective module $E$;*
(5) *if $E$ is an injective cogenerator, then $\mathrm{Hom}_R(F, E)$ is u-$S$-injective;*
(6) *if $E$ is an injective cogenerator, then $\mathrm{Hom}_R(F, E)$ is u-$S$-absolutely pure.*

*Moreover, if $S$ is regular, then all above are equivalent to the following statements:*

(7) *there exists $s \in S$ such that $\mathrm{Tor}^R_1(R/I, F)$ is uniformly $S$-torsion with respect to $s$ for any ideal $I$ of $R$;*

(8) *there exists $s \in S$ such that, for any ideal $I$ of $R$, the natural homomorphism $\sigma_I : I \otimes_R F \to IF$ is a u-$S$-isomorphism with respect to $s$;*

(9) *there exists $s \in S$ such that $\mathrm{Tor}^R_1(R/K, F)$ is uniformly $S$-torsion with respect to $s$ for any finitely generated ideal $K$ of $R$;*

(10) *there exists $s \in S$ such that, for any finitely generated ideal $K$ of $R$, the natural homomorphism $\sigma_K : K \otimes_R F \to KF$ is a u-$S$-isomorphism with respect to $s$.*

*Proof.* (1) $\Rightarrow$ (2): Set the set $\Gamma = \{(K, R^n) \mid K$ is a finitely generated submodule of $R^n$ and $n < \infty\}$. Define $M = \bigoplus\limits_{(K,R^n)\in\Gamma} R^n/K$. Then $s\mathrm{Tor}^R_1(M, F) = s \bigoplus\limits_{(K,R^n)\in\Gamma} \mathrm{Tor}^R_1(R^n/K, F) = 0$ for some $s \in S$. Let $N$ be a finitely presented $R$-module. Then $N \cong R^n/K$ for some $(K, R^n) \in \Gamma$. Hence $\mathrm{Tor}^R_1(N, F) = 0$ is uniformly $S$-torsion with respect to $s$.

(2) $\Rightarrow$ (1): Let $M$ be an $R$-module. Then $M = \varinjlim N_i$ for some direct system of finitely presented $R$-modules $\{N_i\}$. So $s\mathrm{Tor}_1^R(M, F) = s\mathrm{Tor}_1^R(\varinjlim N_i, F) \cong s(\varinjlim \mathrm{Tor}_1^R(N_i, F)) \cong \varinjlim(s\mathrm{Tor}_1^R(N_i, F)) = 0$ by Lemma 4.1. Hence $F$ is $u$-$S$-flat by [21, Theorem 3.2]

(1) $\Rightarrow$ (3): Let $M$ be an $R$-module and $E$ be an injective $R$-module. Since $M$ is $u$-$S$-flat, $\mathrm{Tor}_1^R(M, F)$ is uniformly $S$-torsion. Thus $\mathrm{Ext}_R^1(M, \mathrm{Hom}_R(F, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(M, F), E)$ is also uniformly $S$-torsion by [16, Lemma 4.2]. Thus $\mathrm{Hom}_R(F, E)$ is $u$-$S$-injective by [16, Theorem 4.3].

(3) $\Rightarrow$ (4) $\Rightarrow$ (6) and (3) $\Rightarrow$ (5) $\Rightarrow$ (6): Trivial.

(6) $\Rightarrow$ (2): Let $E$ be an injective cogenerator. Since $\mathrm{Hom}_R(F, E)$ is $u$-$S$-absolutely pure, there exists $s \in S$ such that $\mathrm{Hom}_R(\mathrm{Tor}_1^R(N, F), E) \cong \mathrm{Ext}_R^1(N, \mathrm{Hom}_R(F, E))$ is uniformly $S$-torsion with respect to $s$ for any finitely presented $R$-module $N$. Since $E$ is an injective cogenerator, $\mathrm{Tor}_1^R(N, F)$ is uniformly $S$-torsion with respect to $s$ for any finitely presented $R$-module $N$ by Lemma 4.2.

(2) $\Rightarrow$ (9), (7) $\Rightarrow$ (9), (7) $\Leftrightarrow$ (8) and (9) $\Leftrightarrow$ (10): Obvious.

(10) $\Rightarrow$ (8): Let $\sum\limits_{i=1}^{n} a_i \otimes x_i \in \mathrm{Ker}(\sigma_I)$. Let $K$ be the finitely generated ideal generated by $\{a_i \mid i = 1, \dots, n\}$. Consider the following commutative diagram:

$$\begin{array}{ccc} K \otimes_R F & \xrightarrow{i\otimes 1} & I \otimes_R F \\ \downarrow{\scriptstyle \sigma_K} & & \downarrow{\scriptstyle \sigma_I} \\ KF & \xrightarrow{i'} & IF \end{array}$$

Let $\sum\limits_{i=1}^{n} a_i \otimes x_i$ be the element in $K \otimes_R F$ such that $i \otimes 1(\sum\limits_{i=1}^{n} a_i \otimes x_i) = \sum\limits_{i=1}^{n} a_i \otimes x_i \in I \otimes_R F$. Since $i'\sigma_K(\sum\limits_{i=1}^{n} a_i \otimes x_i) = \sigma_I(i \otimes 1(\sum\limits_{i=1}^{n} a_i \otimes x_i)) = \sigma_I(\sum\limits_{i=1}^{n} a_i \otimes x_i) = 0$, we have $\sum\limits_{i=1}^{n} a_i \otimes x_i \in \mathrm{Ker}(\sigma_K)$ since $i'$ is a monomorphism. Then $s \sum\limits_{i=1}^{n} a_i \otimes x_i = 0 \in K \otimes_R F$. So $s \sum\limits_{i=1}^{n} a_i \otimes x_i = si \otimes 1(\sum\limits_{i=1}^{n} a_i \otimes x_i) = i \otimes 1(s \sum\limits_{i=1}^{n} a_i \otimes x_i) = 0 \in I \otimes_R F$. Hence $s\mathrm{Ker}(\sigma_I) = 0$.

Now assume the multiplicative subset $S$ is regular.

(7) $\Rightarrow$ (5) Let $E$ be an injective cogenerator. Since $\mathrm{Tor}_1^R(R/I, F)$ is uniformly $S$-torsion with respect to $s$, we have $\mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, F), E) \cong \mathrm{Ext}_R^1(R/I, \mathrm{Hom}_R(F, E))$ is uniformly $S$-torsion with respect to $s$ by Lemma 4.2. Since $s$ is regular and $E$ is injective, we have $E$ is $s$-divisible, i.e., $sE = E$. So $\mathrm{Hom}_R(F, E)$ is also $s$-divisible. Hence $\mathrm{Hom}_R(F, E)$ is $u$-$S$-injective by [16, Proposition 4.9]. $\square$



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In 1960, Chase also characterized coherent rings in terms of flat modules (see [5, Theorem 2.1]). Now, we are ready to give a "uniform" $S$-version of the Chase Theorem.

**Theorem 4.4.** (***Chase theorem for*** $u$-$S$***-coherent rings***) *Let $R$ be a ring and $S$ be a regular multiplicative subset of $R$. Then the following statements are equivalent:*

1. *$R$ is a $u$-$S$-coherent ring;*
2. *there is $s \in S$ such that any direct product of flat modules is $u$-$S$-flat with respect to $s$;*
3. *there is $s \in S$ such that any direct product of projective modules is $u$-$S$-flat with respect to $s$;*
4. *there is $s \in S$ such that any direct product of $R$ is $u$-$S$-flat with respect to $s$.*

*Proof.* (2) $\Rightarrow$ (3) $\Rightarrow$ (4) Trivial.

(1) $\Rightarrow$ (2) Suppose $R$ is $u$-$S$-coherent with respect to some $s \in S$. Let $\{F_i \mid i \in \Lambda\}$ be a family of flat $R$-modules and $I$ a finitely generated ideal of $R$. Then $I$ is $u$-$S$-finitely presented with respect to $s$. So we have an exact sequence $0 \to T' \to K \xrightarrow{f} I \to T \to 0$ with $K$ finitely presented and $sT = sT' = 0$. Set $\mathrm{Im}(f) = K'$. Consider the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc}
 & & T' \otimes_R \prod_{i\in I} F_i & \longrightarrow & K \otimes_R \prod_{i\in I} F_i & \longrightarrow & K' \otimes_R \prod_{i\in I} F_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma\ \cong} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \prod_{i\in I}(T' \otimes_R F_i) & \longrightarrow & \prod_{i\in I}(K \otimes_R F_i) & \longrightarrow & \prod_{i\in I}(K' \otimes_R F_i) & \longrightarrow & 0,
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & K' \otimes_R \prod_{i\in I} F_i & \longrightarrow & I \otimes_R \prod_{i\in I} F_i & \longrightarrow & T \otimes_R \prod_{i\in I} F_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\
0 & \longrightarrow & \prod_{i\in I}(K' \otimes_R F_i) & \longrightarrow & \prod_{i\in I}(I \otimes_R F_i) & \longrightarrow & \prod_{i\in I}(T \otimes_R F_i) & \longrightarrow & 0.
\end{array}$$

By [8, Lemma 3.8(2)], $\gamma$ is an isomorphism. Then $\mathrm{Ker}(\beta) \cong \mathrm{Coker}(\alpha)$ which is $u$-$S$-torsion with respect to $s$. Since $K'$ is finitely generated, we have $\beta$ is an epimorphism by [8, Lemma 3.8(1)]. Since $T \otimes_R \prod_{i\in I} F_i$ and $\mathrm{Ker}(\beta)$ are all $u$-$S$-torsion with respect to $s$, so $\mathrm{Ker}(\theta)$ is also $u$-$S$-torsion with respect to $s$.

Now we consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Tor}_1^R(R/I, \prod_{i\in I} F_i) & \longrightarrow & I \otimes_R \prod_{i\in I} F_i & \longrightarrow & R \otimes_R \prod_{i\in I} F_i \\
 & & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow \\
0 & = & \prod_{i\in I} \mathrm{Tor}_1^R(R/I, F_i) & \longrightarrow & \prod_{i\in I}(I \otimes_R F_i) & \longrightarrow & \prod_{i\in I}(R \otimes_R F_i),
\end{array}$$

Note $\mathrm{Tor}_1^R(R/I, \prod_{i\in I} F_i) \subseteq \mathrm{Ker}(\theta)$. So $\mathrm{Tor}_1^R(R/I, \prod_{i\in I} F_i)$ is $u$-$S$-torsion with respect to $s$, Hence $\prod_{i\in I} F_i$ is $u$-$S$-flat (with respect to $s$) by Proposition 4.3.

(4) $\Rightarrow$ (1) Let $I$ be a finitely generated ideal of $R$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & I \otimes_R \prod_{i\in I} R & \xrightarrow{f} & R \otimes_R \prod_{i\in I} R & \longrightarrow & R/I \otimes_R \prod_{i\in I} R & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \prod_{i\in I}(I \otimes_R R) & \longrightarrow & \prod_{i\in I}(R \otimes_R R) & \longrightarrow & \prod_{i\in I}(R/I \otimes_R R) & \longrightarrow & 0.
\end{array}$$

Since $\prod_{i\in I} R$ is a $u$-$S$-flat module with respect to $s$, it follows that $f$ is a $u$-$S$-monomorphism. So $g$ is also a $u$-$S$-monomorphism with respect to $s$.

Let $0 \to L \to F \to I \to 0$ be an exact sequence with $F$ finitely generated free. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & L \otimes_R \prod_{i\in I} R & \longrightarrow & F \otimes_R \prod_{i\in I} R & \longrightarrow & I \otimes_R \prod_{i\in I} R & \longrightarrow & 0 \\
 & & \downarrow h & & \downarrow \cong & & \downarrow g & & \\
0 & \longrightarrow & \prod_{i\in I}(L \otimes_R R) & \longrightarrow & \prod_{i\in I}(F \otimes_R R) & \longrightarrow & \prod_{i\in I}(I \otimes_R R) & \longrightarrow & 0.
\end{array}$$

Since $g$ is a $u$-$S$-monomorphism with respect to $s$, $h$ is a $u$-$S$-epimorphism with respect to $s$. Set $\Lambda$ to be equal to the cardinal of $L$. We will show $L$ is $S$-finite with respect to $s$. Indeed, consider the following exact sequence

$$\begin{array}{ccccccc}
L \otimes_R R^{\Lambda} & & \xrightarrow{\quad h \quad} & & L^{\Lambda} \longrightarrow T \longrightarrow 0 \\
 & \searrow & & \nearrow & \\
 & & \mathrm{Im}h & &
\end{array}$$

with $T$ a $u$-$S$-torsion module with respect to $s$. Let $x = (m)_{m\in L} \in L^{\Lambda}$. Then $sx \subseteq \mathrm{Im}h$. Subsequently, there exist $m_j \in L, r_{j,i} \in R, i \in L, j = 1, \dots, n$ such that for each $t = 1, \dots, k$, we have

$$sx = h(\sum_{j=1}^{n} m_j \otimes (r_{j,i})_{i\in L}) = (\sum_{j=1}^{n} m_j r_{j,i})_{i\in L}.$$

Set $U = \langle m_j \mid j = 1, \dots, n\rangle$ be the finitely generated submodule of $L$. Now, for any $m \in L$, $sm \in \langle \sum_{j=1}^{n} m_j r_{j,m}\rangle \subseteq U$, thus the embedding map $U \hookrightarrow L$ is a $u$-$S$-isomorphism with respect to $s$ and so $L$ is $S$-finite with respect to $s$. Consequently, $I$ is $u$-$S$-finitely presented with respect to $s$. Hence, $R$ is $u$-$S$-coherent with respect to $s$. $\square$

In 1982, Matlis [14, Theorem 1] showed that a ring $R$ is coherent if and only if $\mathrm{Hom}_R(M,E)$ is flat for any injective modules $M$ and $E$. The rest of this paper is devoted to obtain a "uniform" $S$-version of this result.

**Lemma 4.5.** *Let $R$ be a ring, $S$ be a regular multiplicative subset of $R$ and $E$ be an injective cogenerator over $R$. Suppose $\mathrm{Hom}_R(E,E)$ is $u$-$S$-flat with respect to $s\in S$. Then $\mathrm{Hom}_R(E,E)/R$ is also $u$-$S$-flat with respect to $s$.*

*Proof.* Let $I$ be an ideal of $R$. Set $H=\mathrm{Hom}_R(E,E)$. Let $i: R\rightarrowtail H$ be the multiplication map. Suppose $H$ is $u$-$S$-flat with respect to $s\in S$. Then there is a long exact sequence

$$\mathrm{Tor}_1^R(R/I,H)\to \mathrm{Tor}_1^R(R/I,H/R)\to R/I\otimes_R R\xrightarrow{R/I\otimes i} R/I\otimes H.$$

Note that $\mathrm{Ker}(R/I\otimes i)\cong (HI\cap R)/I=0$ by [14, Proposition 1(2)]. Since $\mathrm{Tor}_1^R(R/I,H)$ is $u$-$S$-torsion with respect to $s\in S$, $\mathrm{Tor}_1^R(R/I,H/R)$ is $u$-$S$-torsion with respect to $s\in S$, which implies that $H/R$ is also $u$-$S$-flat with respect to $s$. □

**Lemma 4.6.** *Let $R$ be a ring and $S$ be a regular multiplicative subset of $R$. Suppose that $\{A_\lambda \mid \lambda\in\Lambda\}$ is a family of $u$-$S$-flat modules with respect to $s\in S$, and that $B_\lambda$ is a submodule of $A_\lambda$ such that $A_\lambda/B_\lambda$ is $u$-$S$-flat with respect to $s$ for each $\lambda\in\Lambda$. Then $\prod\limits_{\lambda\in\Lambda} A_\lambda$ is $u$-$S$-flat with respect to $s$ if and only if so are $\prod\limits_{\lambda\in\Lambda} B_\lambda$ and $\prod\limits_{\lambda\in\Lambda} A_\lambda/B_\lambda$.*

*Proof.* Let $I$ be a finitely generated ideal of $R$. Then there is an exact sequence

$$\mathrm{Tor}_2^R(R/I,\prod_{\lambda\in\Lambda} A_\lambda/B_\lambda)\to \mathrm{Tor}_1^R(R/I,\prod_{\lambda\in\Lambda} B_\lambda)\to \mathrm{Tor}_1^R(R/I,\prod_{\lambda\in\Lambda} A_\lambda).$$

By [21, Theorem 3.2], we only need to show $\prod\limits_{\lambda\in\Lambda} A_\lambda/B_\lambda$ is $u$-$S$-flat with respect to $s$. Consider the following exact sequence

$$\mathrm{Tor}_1^R(R/I,\prod_{\lambda\in\Lambda} A_\lambda)\to \mathrm{Tor}_1^R(R/I,\prod_{\lambda\in\Lambda} A_\lambda/B_\lambda)\to R/I\otimes_R\prod_{\lambda\in\Lambda} B_\lambda\xrightarrow{f} R/I\otimes_R\prod_{\lambda\in\Lambda} A_\lambda.$$

Since $\mathrm{Tor}_1^R(R/I,\prod\limits_{\lambda\in\Lambda} A_\lambda)$ is $u$-$S$-torsion with respect to $s$, to show $\prod\limits_{\lambda\in\Lambda} B_\lambda$ is $u$-$S$-flat with respect to $s$, we only need to show $\mathrm{Ker}(f)$ is $u$-$S$-torsion with respect to $s$. Note that $\mathrm{Ker}(f)\cong(\prod_{\lambda\in\Lambda} B_\lambda\cap I(\prod_{\lambda\in\Lambda} A_\lambda))/I\prod_{\lambda\in\Lambda} B_\lambda\cong\prod_{\lambda\in\Lambda}(B_\lambda\cap IA_\lambda)/IB_\lambda$ as $I$ is finitely generated. Consider the following exact sequence $\mathrm{Tor}_1^R(R/I,A_\lambda)\to \mathrm{Tor}_1^R(R/I,A_\lambda/B_\lambda)\to R/I\otimes_R B_\lambda\xrightarrow{f_\lambda} R/I\otimes_R\prod A_\lambda$. We have $\mathrm{Ker}(f_\lambda)\cong(B_\lambda\cap IA_\lambda)/IB_\lambda$ is $u$-$S$-torsion with respect to $s$ since $A_\lambda/B_\lambda$ is $u$-$S$-flat with respect to $s$ . So $\mathrm{Ker}(f)\cong\prod_{\lambda\in\Lambda}\mathrm{Ker}(f_\lambda)$ is $u$-$S$-torsion with respect to $s$. □

**Theorem 4.7.** (***Matlis theorem for $u$-$S$-coherent rings***) *Let $R$ be a ring and $S$ be a regular multiplicative subset of $R$. Then the following statements are equivalent:*

(1) *$R$ is a u-$S$-coherent ring;*
(2) *there are $s_1, s_2 \in S$ such that $\mathrm{Hom}_R(M, E)$ is u-$S$-flat with respect to $s_1$ for any u-$S$-absolutely pure module $M$ with respect to $s_2$ and any injective module $E$;*
(3) *there are $s_1, s_2 \in S$ such that $\mathrm{Hom}_R(M, E)$ is u-$S$-flat with respect to $s_1$ for any u-$S$-injective module $M$ with respect to $s_2$ and any injective module $E$;*
(4) *there is $s_1, s_2 \in S$ such that if $E$ is an injective cogenerator, then $\mathrm{Hom}_R(M, E)$ is u-$S$-flat with respect to $s_1$ for any u-$S$-injective module $M$ with respect to $s_2$;*
(5) *there are $s_1, s_2 \in S$ such that $\mathrm{Hom}_R(\mathrm{Hom}_R(M, E_1), E_2)$ is u-$S$-flat with respect to $s_1$ for any u-$S$-flat module $M$ with respect to $s_2$ and any injective modules $E_1$, $E_2$;*
(6) *there are $s_1, s_2 \in S$ such that if $E_1$ and $E_2$ are injective cogenerators, then $\mathrm{Hom}_R(\mathrm{Hom}_R(M, E_1), E_2)$ is u-$S$-flat with respect to $s_1$ for any u-$S$-flat module $M$ with respect to $s_2$;*
(7) *there is $s \in S$ such that if $E_1$ is an injective cogenerator, then $\mathrm{Hom}_R(E_1, E_2)$ is u-$S$-flat with respect to $s$ for any injective cogenerator $E_2$.*

*Proof.* (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (7) and (5) $\Rightarrow$ (6): Trivial.

(3) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6): This follows from Proposition 4.3.

(1) $\Rightarrow$ (2): Suppose $R$ is a uniformly $S$-coherent ring with respect to some element $s \in S$. Let $I$ be a finitely generated ideal of $R$. Then we have an exact sequence $0 \to T' \to K \xrightarrow{f} I \to T \to 0$ with $K$ finitely presented and $sT = sT' = 0$. Set $\mathrm{Im}(f) = K'$. Consider the following commutative diagrams with exact rows ($(-,-)$ is instead of $\mathrm{Hom}_R(-,-)$):

$$\begin{array}{ccccccc}
(M,E)\otimes_R T' & \longrightarrow & (M,E)\otimes_R K & \longrightarrow & (M,E)\otimes_R K' & \longrightarrow & 0 \\
\downarrow \psi^1_{T'} & & \psi_K \downarrow \cong & & \psi_{K'} \downarrow & & \\
((T',M),E) & \longrightarrow & ((K,M),E) & \longrightarrow & ((K',M),E) & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Tor}^R_1((M,E),R/K') & \longrightarrow & (M,E)\otimes_R K' & \longrightarrow & (M,E)\otimes_R R & \longrightarrow & (M,E)\otimes_R R/K' & \longrightarrow 0 \\
& \downarrow \psi^1_{R/K'} & & \psi_{K'} \downarrow & & \psi_R \downarrow \cong & & \psi_{R/K'} \downarrow & \\
0 \longrightarrow & (\mathrm{Ext}^1_R(R/K',M),E) & \longrightarrow & ((K',M),E) & \longrightarrow & ((R,M),E) & \longrightarrow & ((R/K',M),E) & \longrightarrow 0
\end{array}$$

and

$$\begin{array}{ccccccc}
\mathrm{Tor}_1^R((M,E),T) & \longrightarrow & \mathrm{Tor}_1^R((M,E),R/K') & \longrightarrow & \mathrm{Tor}_1^R((M,E),R/I) & \longrightarrow & (M,E)\otimes_R T \\
\downarrow & & \downarrow{\scriptstyle \psi^1_{R/K'}} & & {\scriptstyle \psi^1_{R/I}}\downarrow & & \downarrow \\
(\mathrm{Ext}_R^1(T,M),E) & \longrightarrow & (\mathrm{Ext}_R^1(R/K',M),E) & \longrightarrow & (\mathrm{Ext}_R^1(R/I,M),E) & \longrightarrow & ((T,M),E)
\end{array}$$

Since $\psi_K$ is an isomorphism by [2, Proposition 8.14(1)] and [10, Theorem 2], $\psi_{K'}$ is a $u$-$S$-isomorphism with respect to $s$, and so is $\psi^1_{R/K'}$. Then $\psi^1_{R/I}$ is a $u$-$S$-isomorphism with respect to $s^3$ (see the proof of [23, Theorem 1.2]). Since $M$ is $u$-$S$-absolutely pure, $\mathrm{Ext}_R^1(R/I,M)$ is $u$-$S$-torsion with respect to $s_2$ ($s_2$ is independent of $I$). Then $\mathrm{Tor}_1^R(\mathrm{Hom}_R(M,E),R/I)$ is $u$-$S$-torsion with respect to $s_1 := s^3s'$, and thus $\mathrm{Hom}_R(M,E)$ is $u$-$S$-flat with respect to $s_1$ by Proposition 4.3.

(7) ⇒ (1): Let $E$ be an injective cogenerator and set $H = \mathrm{Hom}_R(E,E)$. Then $H$ is $u$-$S$-flat with respect to $s$ by assumption. Since $R \subseteq H$, we have that $H/R$ is $u$-$S$-flat with respect to $s$ by Lemma 4.5. Let $\Lambda$ be an index set. Set $H_\lambda = H$, $R_\lambda = R$ and $E_\lambda = E$ for any $\lambda \in \Lambda$. Since $\prod\limits_{\lambda\in\Lambda} E_\lambda$ is also a injective cogenerator, $\prod\limits_{\lambda\in\Lambda} H_\lambda \cong \mathrm{Hom}_R(E_\lambda, \prod\limits_{\lambda\in\Lambda} E_\lambda)$ is $u$-$S$-flat with respect to $s$ by assumption. Hence $\prod\limits_{\lambda\in\Lambda} R_\lambda$ is $u$-$S$-flat with respect to $s$ by Lemma 4.6. So $R$ is a $u$-$S$-coherent ring by Theorem 4.4. □

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