HERMITE–HADAMARD TYPE INEQUALITIES FOR INTERVAL-VALUED CO-ORDINATED LR-CONVEXITY VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this paper, we first obtain some new Hermite–Hadamard-type inequalities for interval-valued LR-convex functions. Afterwards, we investigate Hermite–Hadamard-type inequalities for interval-valued co-ordinated LR-convex functions. New results are obtained by making special choices in newly established inequalities in the case of interval-valued LR-convex functions and interval-valued co-ordinated LR-convex functions. It is also shown that the newly established inequalities are extensions of comparable results in the literature.

1. Introduction

The Hermite–Hadamard inequality, as discovered by C. Hermite and J. Hadamard (as presented in references such as [12], [34, p.137]), stands as one of the most firmly established principles within the realm of convex function theory. This inequality not only possesses a geometric interpretation but also finds numerous practical applications. These inequalities articulate that when considering a convex function $f : I \to \mathbb{R}$ defined on a real number interval I, and selecting two distinct points a and b within I such that $a < b$, the following relationships hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f takes a concave form, the inequalities exhibit an inverse correlation. A multitude of mathematicians have played a role in solidifying the Hermite–Hadamard inequalities. It's worth observing that the Hermite–Hadamard inequality could be contemplated as a honing of the concept of convexity, effortlessly stemming from Jensen's inequality. The Hermite–Hadamard inequality concerning convex functions has undergone a revitalization in recent times, fostering a notable array of enhancements and extensions that have been explored extensively (refer, for instance, to works such as [2], [9], [13], [33], [36], [39]).

Interval analysis, which is utilized in mathematics and computer models as one of the ways for resolving interval uncertainty, is an important material. Despite the fact that this theory has a lengthy history dating back to Archimedes' estimate of the circumference of a circle, substantial research on this topic was not published until the 1950s. In 1966, Ramon E. Moore, the pioneer of interval calculus, released the first book [29] on interval analysis. Following that, a slew of researchers delved into the theory and applications of interval analysis.

In the context of this article, we introduce the notation $\mathbb{R}_{\mathcal{I}}^{+}$ to represent the collection of all positive intervals within the real numbers. The set comprising all interval-valued functions that are Riemann integrable and real-valued functions on the interval $[a,b]$ is denoted as $\mathcal{IR}_{([a,b])}$ and $\mathcal{R}_{([a,b])}$, respectively. The subsequent theorem establishes a connection between functions that are integrable in the sense of (IR) and functions that are Riemann integrable ($\mathbb{R}$-integrable). Moreover, for $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right]$ and $\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]$ belonging to $\mathbb{R}_{\mathcal{I}}^{+}$, the symbol "$\subseteq$" is employed to indicate the inclusion relationship, where $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right]$ is considered to be a subset of $\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]$. This inclusion holds true if and only if the condition $\underline{\mathcal{V}} \leq \underline{\mathcal{U}}$ and

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$\overline{\mathcal{U}} \leq \overline{\mathcal{V}}$ is satisfied. Many authors have recently focused on integral inequalities derived from interval-valued functions. Sadowska [35] discovered the Hermite-Hadamard inequality for set-valued functions, which is a more general form of interval-valued mappings:

Theorem 1. *[35] Suppose that $F : [a, b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is interval–valued convex function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. Then, we have the inequalities:*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{2}. \tag{1.2}$$

Furthermore, well-known inequalities such as Ostrowski, Minkowski and Beckenbach and their some applications were provided by considering interval-valued functions in [7, 8, 14, 32]. In addition, some inequalities involving interval-valued Riemann-Liouville fractional integrals were derived by Budak et al. in [4]. In [26], Liu et al. gave the definition of interval-valued harmonically convex functions, and so they obtain some Hermite–Hadamard type inequalities including interval fractional integrals. In [10] and [11], the authors gave the variant of Jensen's inequality for interval-valued functions via fuzzy integrals and proved different integral inequalities. Mitroi et al. proved Hermite–Hadamard type inequalities for set-valued functions in [28] and in [16, 31], the authors used general forms of interval-valued convex functions to prove Hermite-Hadamard type inequalities. Some Gronwal type inequalities for interval-valued functions were obtained by Román Flores et al. in [15]. In [42,43], Zhao et al proved different types of integral inequalities for interval-valued functions.

Jleli and Samet obtained new Hermite-Hadamard type inequalities involving fractional integrals with respect to another function in [17]. In [38], Tunç introduced firstly fractional integrals of a function with respect to the another function. Katugompala established a new fractional integration, which generalizes the Riemann–Liouville and Hadamard fractional integrals into a single form. Budak and Agarwal established the Hermite–Hadamard-type inequalities for co-ordinated convex function via generalized fractional integrals, which generalize some important fractional integrals such as the Riemann–Liouville fractional integrals, the Hadamard fractional integrals, and Katugampola fractional integrals in [3]. Kara et al. [18] defined interval-valued left-sided and right-sided generalized fractional double integrals. In recent years, many authors have focused on interval-valued functions. In [44], the authors gave a new concept of interval-valued general convex functions to prove several new variants of Hermite-Hadamard type inequalities. Moreover, in [5], the authors gave a fractional version of Hermite-Hadamard type inequalities for interval-valued harmonically convex functions. Recently, in [20–23, 37], several researchers extended the concept of interval-valued convexity and defined different kinds of LR-convexity for interval-valued functions. They also obtained many Hermite–Hadmard type inequalities for LR-interval-valued convex functions.

Inspired by the on going studies, we give the notions about generalized fractional integrals for the two variables interval-valued functions to prove Hermite–Hadmard type inequalities for convex and co-ordinated convex functions. The main advantage of the newly established inequalities is that these can be turned into Riemann-Liouville fractional Hermite-Hadamard integral inequalities, Hadamard fractional Hermite-Hadamard integral inequalities, Katugampola fractional Hermite-Hadamard inequalities and classical Hermite–Hadamard integral inequalities for LR-convex and coordinated LR-convex interval-valued functions without having to prove each one separately.

The following is the structure of this paper: Section 2 provides a brief overview of the fundamentals of interval-valued calculus as well as other related studies in this field. We give some generalized fractional integrals for two variables interval-valued functions in Section 3. In Section 4, we establish a new Hermite–Hadamard type inequality for interval-valued LR-convex functions. For Interval-valued coordinated LR-convex functions, several Hermite-Hadamard type inequalities are parented in Section 5. The relationship between the findings reported here and similar findings in the literature are also taken into account. Section 6 concludes with some recommendations for future research.

2. Preliminaries

In this section we recall some basic definitions, results, notions and properties, which are used throughout the paper. A positive interval is an interval that tells you that the left and right endpoints of the interval are also positive. We denote $\mathbb{R}_{\mathcal{I}}^{+}$ the family of all positive intervals of $\mathbb{R}$. The Hausdorff distance between $[\underline{X},\overline{X}]$ and $[\underline{Y},\overline{Y}]$ is defined as

$$d([\underline{X},\overline{X}],[\underline{Y},\overline{Y}])=\max\left\{\left|\underline{X}-\underline{Y}\right|,\left|\overline{X}-\overline{Y}\right|\right\}.$$

The $(\mathbb{R}_{\mathcal{I}},d)$ is a complete metric space. For more details and basic notations on interval-valued functions see ([30,41]).

We now give the properties of fundamental interval analysis operations for the intervals $\mathcal{U}$ and $\mathcal{V}$ as follows:

$$\begin{aligned}
\mathcal{U}+\mathcal{V} &= \left[\underline{\mathcal{U}}+\underline{\mathcal{V}},\overline{\mathcal{U}}+\overline{\mathcal{V}}\right],\\
\mathcal{U}-\mathcal{V} &= \left[\underline{\mathcal{U}}-\overline{\mathcal{V}},\overline{\mathcal{U}}-\underline{\mathcal{V}}\right],\\
\mathcal{U}.\mathcal{V} &= [\min\Lambda,\max\Lambda] \text{ where } \Lambda=\left\{\underline{\mathcal{U}}\,\underline{\mathcal{V}},\underline{\mathcal{U}}\,\overline{\mathcal{V}},\ \overline{\mathcal{U}}\underline{\mathcal{V}},\overline{\mathcal{U}}\,\overline{\mathcal{V}}\right\},\\
\mathcal{U}/\mathcal{V} &= [\min\Delta,\max\Delta] \text{ where } \Delta=\left\{\underline{\mathcal{U}}/\underline{\mathcal{V}},\underline{\mathcal{U}}/\overline{\mathcal{V}},\overline{\mathcal{U}}/\underline{\mathcal{V}},\overline{\mathcal{U}}/\overline{\mathcal{V}}\right\} \text{ and } 0\notin\mathcal{V}.
\end{aligned}$$

Scalar multiplication of the interval $\mathcal{U}$ is indicated by

$$\theta\mathcal{U}=\theta\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right]=\begin{cases}\left[\theta\underline{\mathcal{U}},\theta\overline{\mathcal{U}}\right], & \theta>0\\ \{0\}, & \theta=0\\ \left[\theta\overline{\mathcal{U}},\theta\underline{\mathcal{U}}\right], & \theta<0,\end{cases}$$

where $\theta\in\mathbb{R}$.

For $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right],\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]\in\mathbb{R}_{\mathcal{I}}^{+}$, the inclusion "$\subseteq$" is defined by $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right]\subseteq\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]$, and only if, $\underline{\mathcal{V}}\leq\underline{\mathcal{U}}$, $\overline{\mathcal{U}}\leq\overline{\mathcal{V}}$.

(1) The relation "$\leq_p$" defined on $\mathbb{R}_{\mathcal{I}}$ by $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right]\leq_p\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]$ if and only if $\underline{\mathcal{U}}\leq\underline{\mathcal{V}}$, $\overline{\mathcal{U}}\leq\overline{\mathcal{V}}$, for all $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right],\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]\in\mathbb{R}_{\mathcal{I}}^{+}$, it is an pseudo order relation. For given $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right],\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]\in\mathbb{R}_{\mathcal{I}}$, we say that $\left[\underline{\mathcal{U}},\overline{\mathcal{U}}\right]\leq_p\left[\underline{\mathcal{V}},\overline{\mathcal{V}}\right]$ if and only if $\underline{\mathcal{U}}\leq\underline{\mathcal{V}}$, $\overline{\mathcal{U}}\leq\overline{\mathcal{V}}$.

(2) It can be easily seen that "$\leq_p$" looks like "left and right" on real line $\mathbb{R}$, so we call "$\leq_p$" is "left and right "(or "LR" order, in short).

It is remarkable that Moore [29] introduced the Riemann integral for the interval-valued functions. The set of all Riemann integrable interval-valued functions and real-valued functions on $[a,b]$ are denoted by $\mathcal{IR}_{([a,b])}$ and $\mathcal{R}_{([a,b])}$, respectively. The following theorem gives relation between (IR)–integrable and Riemann integrable (R–integrable) (see [30], pp. 131):

Theorem 2. *Let $F:[a,b]\to\mathbb{R}_{\mathcal{I}}$ be an interval–valued function such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$. $F\in\mathcal{IR}_{([a,b])}$ if and only if $\underline{F}(t)$, $\overline{F}(t)\in\mathcal{R}_{([a,b])}$ and*

$$(IR)\int_a^b F(t)dt=\left[(R)\int_a^b\underline{F}(t)dt,(R)\int_a^b\overline{F}(t)dt\right].$$

In [41,42], Zhao et al. introduced a kind of convex interval–valued function as follows:

Definition 1. *Let $h:[c,d]\to\mathbb{R}$ be a non–negative function, $(0,1)\subseteq[c,d]$ and $h\neq0$. We say that $F:[a,b]\to\mathbb{R}_{\mathcal{I}}^{+}$ is a h–convex interval–valued function, if for all $x,y\in[a,b]$ and $t\in(0,1)$, we have*

$$h(t)F(x)+h(1-t)F(y)\subseteq F(tx+(1-t)y). \tag{2.1}$$

With $SX(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$ will show the set of all h–convex interval–valued functions.

The usual notion of convex interval–valued function corresponds to relation (2.1) with $h(t)=t$, see [35]. Also, if we take $h(t)=t^s$ in (2.1), then Definition 1 gives the other convex interval–valued function defined by Breckner, see [1].

Otherwise, Zhao et al. obtained the following Hermite–Hadamard inequality for interval–valued functions by using h–convex:

Theorem 3. *[41] Let $F : [a,b] \to \mathbb{R}^+_\mathcal{I}$ be an interval–valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $F \in \mathcal{IR}_{([a,b])}$, $h : [0,1] \to \mathbb{R}$ be a non–negative function and $h\left(\frac{1}{2}\right) \neq 0$. If $F \in SX(h, [a,b], \mathbb{R}^+_\mathcal{I})$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq [F(a)+F(b)]\int_0^1 h(t)dt. \tag{2.2}$$

Remark 1. (i) *If $h(t) = t$, then (2.2) reduces to the following result:*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{2},$$

which is obtained by [35].

(ii) *If $h(t) = t^s$, then (2.2) reduces to the following result:*

$$2^{s-1} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{s+1},$$

which is obtained by [16].

In [27] Lupulescu defined the following interval-valued left-sided Riemann–Liouville fractional integral.

Definition 2. *Let $F : [a,b] \to \mathbb{R}_\mathcal{I}$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and let $\alpha > 0$. The interval-valued left-sided Riemann–Liouville fractional integral of function f is defined by*

$$\mathcal{J}^\alpha_{a+} F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_a^x (x-s)^{\alpha-1} F(t)dt, \quad x > a$$

where Γ is Euler Gamma function.

Based on the definition of Lupulescu, Budak et al. in [4] gave the definition of interval-valued right-sided Riemann–Liouville fractional integral of function F by

$$\mathcal{J}^\alpha_{b-} F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_x^b (s-x)^{\alpha-1} F(t)dt, \quad x < b.$$

where Γ is Euler Gamma function.

In [38], Tunç gave following fractional integrals for interval-valued functions and corresponding inequalities of Hermite–Hadamard type as follows:

Definition 3. *Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b) and $F \in \mathcal{IR}_{([a,b])}$. The interval-valued left-sided ($\mathbb{J}^\alpha_{a^+;g}F(x)$) and right-sided ($\mathbb{J}^\alpha_{b^-;g}F(x)$) fractional integral of F with respect to the function g on $[a,b]$ of order $\alpha > 0$ are defined by*

$$\mathbb{J}^\alpha_{a^+;g}F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_a^x \frac{g'(t)}{[g(x)-g(t)]^{1-\alpha}} F(t)dt, \quad x > a$$

and

$$\mathbb{J}^\alpha_{b^-;g}F(x) = \frac{1}{\Gamma(\alpha)}(IR)\int_x^b \frac{g'(t)}{[g(t)-g(x)]^{1-\alpha}} F(t)dt, \quad x < b$$

respectively.

Remark 2. (i) *If we choose $g(t) = \ln t$ in Definition 3, the operators $\mathbb{J}^\alpha_{a^+;g}F(x)$ and $\mathbb{J}^\alpha_{b^-;g}F(x)$ reduce to Hadamard interval-valued fractional integrals $\mathfrak{J}^\alpha_{a^+;g}F(x)$ and $\mathfrak{J}^\alpha_{b^-;g}F(x)$, respectively.*

(ii) *Considering $g(t) = \frac{t^\rho}{\rho}$, $\rho > 0$ in Definition 3, the operators $\mathbb{J}^\alpha_{a^+;g}F(x)$ and $\mathbb{J}^\alpha_{b^-;g}F(x)$ reduce to Katugampola interval-valued fractional integrals ${}^\rho\mathbb{I}^\alpha_{a^+;g}F(x)$ and ${}^\rho\mathbb{I}^\alpha_{b^-;g}F(x)$, respectively.*

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Theorem 4. *[38] Let $F : [a, b] \to \mathbb{R}^+_{\mathcal{I}}$ be an interval-valued convex function on $[a, b]$ such that $F(x) = \left[\underline{F}(x), \overline{F}(x)\right]$ for all $x \in [a, b]$ with and $g : [a, b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a, b]$ having a continuous derivative $g'(x)$ on (a, b), then we obtain the following relation*

$$\mathbb{J}^{\alpha}_{a+;g} F(x) = \left[J^{\alpha}_{a+;g} \underline{F}(x), J^{\alpha}_{a+;g} \overline{F}(x)\right]$$

$$\mathbb{J}^{\alpha}_{b-;g} F(x) = \left[J^{\alpha}_{b-;g} \underline{F}(x), J^{\alpha}_{b-;g} \overline{F}(x)\right]$$

where

$$J^{\alpha}_{a^+;g} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)}{[g(x) - g(t)]^{1-\alpha}} f(t)dt, \quad x > a$$

$$J^{\alpha}_{b^-;g} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)}{[g(t) - g(x)]^{1-\alpha}} f(t)dt, \quad x < b$$

which are defined by Kilbas et al. in [25].

Theorem 5. *[17] Let $g : [a, b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b) and let $\alpha > 0$. If f is a convex function on $[a, b]$, then*

$$f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{4\left[g(b) - g(a)\right]^{\alpha}} \left[J^{\alpha}_{a+;g}\zeta(b) + J^{\alpha}_{b-;g}\zeta(a)\right] \leq \frac{f(a) + f(b)}{2} \tag{2.3}$$

where $\zeta(x) = f(x) + f(a + b - x)$ for $x \in [a, b]$.

Theorem 6. *[38] Let $g : [a, b] \to \mathbb{R}_{\mathcal{I}}$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b) and let $\alpha > 0$. If F is an interval-valued convex function on $[a, b]$, then*

$$F\left(\frac{a + b}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{4\left[g(b) - g(a)\right]^{\alpha}} \left[\mathbb{J}^{\alpha}_{a+;g}\Phi(b) + \mathbb{J}^{\alpha}_{b-;g}\Phi(a)\right] \supseteq \frac{F(a) + F(b)}{2} \tag{2.4}$$

where $\Phi(x) = F(x) + \widetilde{F}(x)$ and $\widetilde{F}(x) = F(a + b - x)$ for $x \in [a, b]$.

3. Interval-valued double integral and Co-ordinated Convexity

A set of numbers $\{t_{i-1}, \xi_i, t_i\}_{i=1}^m$ is called tagged partition P_1 of $[a, b]$ if

$$P_1 : a = t_0 < t_1 < \ldots < t_n = b$$

and if $t_{i-1} \leq \xi_i \leq t_i$ for all $i = 1, 2, 3, \ldots, m$. Moreover if we have $\Delta t_i = t_i - t_{i-1}$, then P_1 is said to be $\delta-$fine if $\Delta t_i < \delta$ for all i. Let $\mathcal{P}(\delta, [a, b])$ denote the set of all $\delta-$fine partitions of $[a, b]$. If $\{t_{i-1}, \xi_i, t_i\}_{i=1}^m$ is a $\delta-$fine P_1 of $[a, b]$ and if $\{s_{j-1}, \eta_j, s_j\}_{j=1}^n$ is $\delta-$fine P_2 of $[c, d]$, then rectangles

$$\Delta_{i,j} = [t_{i-1}, t_i] \times [s_{j-1}, s_j]$$

are the partition of the rectangle $\Delta = [a, b] \times [c, d]$ and the points (ξ_i, η_j) are inside the rectangles $[t_{i-1}, t_i] \times [s_{j-1}, s_j]$. Further, by $\mathcal{P}(\delta, \Delta)$ we denote the set of all $\delta-$fine partitions P of Δ with $P_1 \times P_2$, where $P_1 \in \mathcal{P}(\delta, [a, b])$ and $P_2 \in \mathcal{P}(\delta, [c, d])$. Let $\Delta A_{i,j}$ be the area of rectangle $\Delta_{i,j}$. In each rectangle $\Delta_{i,j}$, where $1 \leq i \leq m$, $1 \leq j \leq n$, choose arbitrary (ξ_i, η_j) and get

$$S(F, P, \delta, \Delta) = \sum_{i=1}^{m} \sum_{j=1}^{n} F(\xi_i, \eta_j) \Delta A_{i,\ j}.$$

We call $S(F, P, \delta, \Delta)$ is integral sum of F associated with $P \in \mathcal{P}(\delta, \Delta)$.
Now we recall the concept of interval-valued double integral given by Zhao et al. in [42].

Theorem 7. *[42]Let $F : \Delta \to \mathbb{R}_{\mathcal{I}}$. Then F is called $ID-$integrable on Δ with $ID-$integral $U = (ID) \iint\limits_{\Delta} F(t, s)dA$, if for any $\epsilon > 0$ there exist $\delta > 0$ such that*

$$d(S(F, P, \delta, \Delta)) < \epsilon$$

for any $P \in \mathcal{P}(\delta, \Delta)$. The collection of all $ID-$integrable functions on Δ will be denoted by $\mathcal{ID}_{(\Delta)}$.

Theorem 8. *[42]Let $\Delta = [a,b] \times [c,d]$. If $F : \Delta \to \mathbb{R}_{\mathcal{I}}$ is $ID-$integrable on Δ, then we have*

$$(ID) \iint_{\Delta} F(s,t)dA = (IR) \int_a^b (IR) \int_c^d F(s,t)dsdt.$$

Definition 4. *[6]Let $F \in L_1([a,b] \times [c,d])$. The Riemann-Lioville integrals $\mathcal{J}^{\alpha,\beta}_{a+,c+}, \mathcal{J}^{\alpha,\beta}_{a+,d-}, +\mathcal{J}^{\alpha,\beta}_{b-,c+}$ and $\mathcal{J}^{\alpha,\beta}_{b-,d-}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by*

$$\begin{aligned}
\mathcal{J}^{\alpha,\beta}_{a+,c+}F(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}F(t,s)\,dsdt, \quad x>a,\ y>c,\\
\mathcal{J}^{\alpha,\beta}_{a+,d-}F(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}F(t,s)\,dsdt, \quad x>a,\ y>d,\\
\mathcal{J}^{\alpha,\beta}_{b-,c+}F(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}F(t,s)\,dsdt, \quad x<b,\ y>c,\\
\mathcal{J}^{\alpha,\beta}_{b-,d-}F(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}F(t,s)\,dsdt, \quad x<b,\ y<d,
\end{aligned}$$

respectively.

Definition 5. *Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b) and let $w : [c,d] \to \mathbb{R}$ be an increasing and positive monotone function on $(c,d]$, having a continuous derivative $w'(y)$ on (c,d) and $F \in \mathcal{IR}_{([a,b]\times[c,d])}$. The interval-valued left sided and right sided fractional integral operators for functions of two variables are defined by*

$$\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}F(x,y) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_c^y \frac{g'(t)}{[g(x)-g(t)]^{1-\alpha}}\frac{w'(s)}{[w(y)-w(s)]^{1-\beta}}F(t,s)dsdt, \quad x>a,\ y>c,$$

$$\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}F(x,y) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_y^d \frac{g'(t)}{[g(x)-g(t)]^{1-\alpha}}\frac{w'(s)}{[w(s)-w(y)]^{1-\beta}}F(t,s)dsdt, \quad x>a,\ y<d,$$

$$\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}F(x,y) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_c^y \frac{g'(t)}{[g(t)-g(x)]^{1-\alpha}}\frac{w'(s)}{[w(y)-w(s)]^{1-\beta}}F(t,s)dsdt, \quad x<b,\ y>c,$$

and

$$\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}F(x,y) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_y^d \frac{g'(t)}{[g(t)-g(x)]^{1-\alpha}}\frac{w'(s)}{[w(s)-w(y)]^{1-\beta}}F(t,s)dsdt, \quad x<b,\ y<d$$

for $\alpha, \beta > 0$.

Similar the above definitions, we can give the following interval-valued integrals:

$$\mathbb{J}^{\alpha}_{a+;g}F\left(x,\frac{c+d}{2}\right) := \frac{1}{\Gamma(\alpha)}(IR)\int_a^x \frac{g'(t)}{[g(x)-g(t)]^{1-\alpha}}F\left(t,\frac{c+d}{2}\right)dt, \quad x>a,$$

$$\mathbb{J}^{\alpha}_{b-;g}F\left(x,\frac{c+d}{2}\right) := \frac{1}{\Gamma(\alpha)}(IR)\int_x^b \frac{g'(t)}{[g(t)-g(x)]^{1-\alpha}}F\left(t,\frac{c+d}{2}\right)dt, \quad x<b,$$

$$\mathbb{J}^{\beta}_{c+;w}F\left(\frac{a+b}{2},y\right):=\frac{1}{\Gamma(\beta)}(IR)\int_{c}^{y}\frac{w'(t)}{[w(y)-w(s)]^{1-\beta}}F\left(\frac{a+b}{2},s\right)ds,\quad y>c,$$

and

$$\mathbb{J}^{\beta}_{d-;w}F\left(\frac{a+b}{2},y\right):=\frac{1}{\Gamma(\beta)}(IR)\int_{c}^{y}\frac{w'(t)}{[w(s)-w(y)]^{1-\beta}}F\left(\frac{a+b}{2},s\right)ds,\quad y<d.$$

Remark 3. (i) *If we choose $g(t)=\ln t$, $w(s)=\ln s$ in Definition 5, the operators $\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}F(x,y)$, $\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}F(x,y)$, $\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}F(x,y)$ and $\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}F(x,y)$ reduce to Hadamard interval-valued fractional integrals $\mathfrak{J}^{\alpha,\beta}_{a+,c+}F(x,y)$, $\mathfrak{J}^{\alpha,\beta}_{a+,d-}F(x,y)$, $\mathfrak{J}^{\alpha,\beta}_{b-,c+}F(x,y)$ and $\mathfrak{J}^{\alpha,\beta}_{b-,d-}F(x,y)$, respectively.*

(ii) *Considering $g(t)=\frac{t^{\rho}}{\rho}$ and $w(s)=\frac{s^{\sigma}}{\sigma}$, $\rho,\sigma>0$, in Definition 5, the operators $\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}F(x,y)$, $\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}F(x,y)$, $\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}F(x,y)$ and $\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}F(x,y)$ reduce to Katugampola interval-valued fractional integrals ${}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{a+,c+}F(x,y)$, ${}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{a+,d-}F(x,y)$, ${}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{b-,c+}F(x,y)$ and ${}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{b-,d-}F(x,y)$, respectively.*

Now we recall the concept of interval-valued co-ordinated convex functions that is given by Zhao et al. in [45] as follows:

Definition 6. *A function $F:\Delta=[a,b]\times[c,d]\to\mathbb{R}^{+}_{\mathcal{I}}$ is said to be interval-valued co-ordinated convex function, if the following inequality holds:*

$$\begin{aligned}&F(tx+(1-t)y,su+(1-s)w)\\&\supseteq tsF(x,u)+t(1-s)F(x,w)+s(1-t)F(y,u)+(1-s)(1-t)F(y,w),\end{aligned}$$

for all $(x,u),(y,w)\in\Delta$ and $s,t\in[0,1]$.

Lemma 1. *A function $F:\Delta=[a,b]\times[c,d]\to\mathbb{R}^{+}_{\mathcal{I}}$ is interval-valued convex on co-ordinates if and only if there exists two functions $F_x:[c,d]\to\mathbb{R}^{+}_{\mathcal{I}}$, $F_x(w)=F(x,w)$ and $F_y:[a,b]\to\mathbb{R}^{+}_{\mathcal{I}}$, $F_y(u)=F(u,y)$ are interval-valued convex.*

Definition 7. *[40] The interval-valued function $F:I\to\mathbb{R}^{+}_{\mathcal{I}}$ is said to be LR-convex interval-valued function on convex set I if for all $a,b\in I$ and $t\in[0,1]$ we have*

$$F\left(ta+(1-t)\,b\right)\leq_{p}\ tF\left(a\right)+(1-t)\,F\left(b\right),\tag{3.1}$$

if inequality (3.1) is reversed, then F is said to be LR-concave on I. F is affine if and only if, it is both LR-convex and LR-concave.

Theorem 9. *[40] Let I be an convex set and $F:I\to\mathbb{R}^{+}_{\mathcal{I}}$ be an interval-valued function such that*

$$F\left(t\right)=\left[\underline{F}\left(t\right),\overline{F}\left(t\right)\right],\quad\forall t\in I.$$

Then F is LR-convex interval-valued function on I, if and only if, $\underline{F}(t)$ and $\overline{F}(t)$ both are convex functions.

Definition 8. *[19] A function $F:\Delta=[a,b]\times[c,d]\to\mathbb{R}^{+}_{\mathcal{I}}$ is said to be interval-valued co-ordinated LR-convex function, if the following inequality holds:*

$$\begin{aligned}&F(ta+(1-t)b,sc+(1-s)d)\\&\leq_{p}tsF(a,c)+t(1-s)F(a,d)+s(1-t)F(b,c)+(1-s)(1-t)F(b,d),\end{aligned}$$

for all $(a,b),(c,d)\in\Delta$ and $s,t\in[0,1]$.

4. Hermite-Hadamard Inequalities for LR-Convex Interval-Valued Function

In this section, we obtain some new Hermite-Hadamard-type inequalities for interval-valued LR-convex functions.

Theorem 10. *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be a LR-convex interval-valued function on $[a,b]$ and given by $F(x) = [\underline{F}(x), \overline{F}(x)]$ for all $x \in [a,b]$. If $F \in L([a,b], \mathbb{R}_{\mathcal{I}}^{+})$, then*

$$F\left(\frac{a+b}{2}\right) \leq_p \frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\Psi(b) + \mathbb{J}_{b-,w}^{\alpha}\Psi(a)\right] \leq_p \frac{F(a)+F(b)}{2}$$

where $\Psi(x) = F(x) + F(a+b-x)$.

Proof. Since F is a LR-convex interval-valued function and Theorem 5, then $\underline{F}$ and $\overline{F}$ are convex we have,

$$\frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\underline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\underline{\Psi}(a)\right] \leq \frac{\underline{F}(a)+\underline{F}(b)}{2} \tag{4.1}$$

and

$$\frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\overline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\overline{\Psi}(a)\right] \leq \frac{\overline{F}(a)+\overline{F}(b)}{2}. \tag{4.2}$$

From the (4.1) and (4.2) inequalities we get the following expression,

$$\begin{aligned}&\frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\underline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\underline{\Psi}(a),\ \left(\mathbb{J}_{a+,w}^{\alpha}\overline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\overline{\Psi}(a)\right)\right]\\ &\leq_p \left[\frac{\underline{F}(a)+\underline{F}(b)}{2},\ \frac{\overline{F}(a)+\overline{F}(b)}{2}\right].\end{aligned}$$

From here we get,

$$\frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\Psi(b) + \mathbb{J}_{b-,w}^{\alpha}\Psi(a)\right] \leq_p \frac{F(a)+F(b)}{2}. \tag{4.3}$$

On the other hand, since F is a LR-convex interval-valued function and using Theorem 5, we have

$$\underline{F}\left(\frac{a+b}{2}\right) \leq \frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\underline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\underline{\Psi}(a)\right] \tag{4.4}$$

and

$$\overline{F}\left(\frac{a+b}{2}\right) \leq \frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\overline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\overline{\Psi}(a)\right]. \tag{4.5}$$

From the (4.4) and (4.5) inequalities we get the following expression,

$$\begin{aligned}&\left[\underline{F}\left(\frac{a+b}{2}\right),\ \overline{F}\left(\frac{a+b}{2}\right)\right]\\ &\leq_p \frac{1}{4[w(b)-w(a)]}\left[\left(\mathbb{J}_{a+,w}^{\alpha}\underline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\underline{\Psi}(a)\right),\ \left(\mathbb{J}_{a+,w}^{\alpha}\overline{\Psi}(b) + \mathbb{J}_{b-,w}^{\alpha}\overline{\Psi}(a)\right)\right].\end{aligned}$$

From here we get,

$$F\left(\frac{a+b}{2}\right) \leq_p \frac{1}{4[w(b)-w(a)]}\left[\mathbb{J}_{a+,w}^{\alpha}\Psi(b) + \mathbb{J}_{b-,w}^{\alpha}\Psi(a)\right]. \tag{4.6}$$

The required result is obtained from the 4.3 and 4.6 inequalities. The proof is completed. □

Remark 4. *If we choose $w(t) = t$ in Theorem 10, then we have the following inequalities for Riemann–Liouville interval-valued fractional integrals*

$$F\left(\frac{a+b}{2}\right) \leq_p \frac{1}{4(b-a)}\left[\mathcal{J}_{a+,w}^{\alpha}\Psi(b) + \mathcal{J}_{b-,w}^{\alpha}\Psi(a)\right] \leq_p \frac{F(a)+F(b)}{2}$$

which is given by Khan et al. in [21].

Corollary 1. *If we choose $w(t) = \ln t$ in Theorem 10, then we have the following inequalities for Hadamard interval-valued fractional integrals*

$$F\left(\frac{a+b}{2}\right) \leq_p \frac{1}{4\left[\ln\frac{b}{a}\right]}\left[\mathfrak{J}_{a+,w}^{\alpha}\Psi(b) + \mathfrak{J}_{b-,w}^{\alpha}\Psi(a)\right] \leq_p \frac{F(a)+F(b)}{2}.$$

Corollary 2. *If we choose $w(t) = \frac{t^\rho}{\rho}$, $\rho > 0$ in Theorem 10, then we have the following inequalities for Katugampola interval-valued fractional integrals*

$$F\left(\frac{a+b}{2}\right) \leq_p \frac{\rho}{4\left[b^\rho - a^\rho\right]} \left[{}^{\rho}\mathbb{I}^{\alpha}_{a+,w}\Psi(b) + {}^{\rho}\mathbb{I}^{\alpha}_{b-,w}\Psi(a) \right] \leq_p \frac{F(a) + F(b)}{2}.$$

5. Hermite-Hadamard Inequalities for Co-ordinated LR-Convex Interval-Valued Function

In this section, we establish some new Hermite-Hadamard-type inequalities for interval-valued co-ordinated LR-convex functions.

Let $F \in \mathcal{IR}_{([a,b]\times[c,d])}$. Firstly, we define the following functions which will be used frequently:

$$\begin{aligned}
\widetilde{F_1}(x,y) &= F(a+b-x,y),\\
\widetilde{F_2}(x,y) &= F(x,c+d-y),\\
\widetilde{F_3}(x,y) &= F(a+b-x,c+d-y),\\
\mathcal{G}(x,y) &= F(x,y) + \widetilde{F_2}(x,y)\\
\mathcal{H}(x,y) &= F(x,y) + \widetilde{F_1}(x,y)\\
\mathcal{K}(x,y) &= \widetilde{F_1}(x,y) + \widetilde{F_3}(x,y)\\
\mathcal{L}(x,y) &= \widetilde{F_2}(x,y) + \widetilde{F_3}(x,y)\\
\mathcal{F}(x,y) &= \widetilde{F_1}(x,y) + \widetilde{F_2}(x,y) + \widetilde{F_3}(x,y) + F(x,y)\\
&= \frac{\mathcal{G}(x,y) + \mathcal{H}(x,y) + \mathcal{K}(x,y) + \mathcal{L}(x,y)}{2}
\end{aligned} \tag{5.1}$$

for $(x,y) \in [a,b] \times [c,d]$. Throughout this section, let us note that $g : [a,b] \to \mathbb{R}$ is an increasing and positive monotone function on $(a,b]$ and this function also have a continuous derivative $g'(x)$ on (a,b). Furthermore, $w : [c,d] \to \mathbb{R}$ is an increasing and positive monotone function on $(c,d]$, having a continuous derivative $w'(y)$ on (c,d).

Theorem 11. *Let $\Delta = [a,b] \times [c,d]$, if $F \in L\left(\Delta, \ \mathbb{R}^+_{\mathcal{I}}\right)$ be a interval-valued co-ordinated LR-convex function, then for $\alpha, \beta > 0$ the following Hermite-Hadamard type inequality holds:*

$$\begin{aligned}
&F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{16\left[g(b)-g(a)\right]^\alpha \left[w(d)-w(c)\right]^\beta}\\
&\times \left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{F}(b,d) + \mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{F}(b,c) + \mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{F}(a,d) + \mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{F}(a,c)\right]\\
&\leq_p \frac{F(a,c) + F(a,d) + F(b,c) + F(b,d)}{4},
\end{aligned} \tag{5.2}$$

where the function $\mathcal{F}$ is defined as in (5.1).

Proof. Since F is an interval-valued co-ordinated LR-convex mapping on Δ, we have

$$F\left(\frac{u+v}{2},\frac{\rho+q}{2}\right)\leq_{p}\frac{F(u,\rho)+F(u,q)+F(v,\rho)+F(v,q)}{4} \tag{5.3}$$

for $(u,\rho),(v,q)\in\Delta$. Now, for $t,s\in[0,1]$, let $u=ta+(1-t)b$, $v=(1-t)a+tb$, $\rho=cs+(1-s)d$ and $q=(1-s)c+sd$. Then we have

$$\begin{aligned} &F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ &\leq_{p}\frac{1}{4}F(ta+(1-t)b,cs+(1-s)d)+\frac{1}{4}F(ta+(1-t)b,(1-s)c+sd) \\ &+\frac{1}{4}F((1-t)a+tb,cs+(1-s)d)+\frac{1}{4}F((1-t)a+tb,(1-s)c+sd). \end{aligned} \tag{5.4}$$

Multiplying both sides of (5.4) by

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}$$

and integrating the resulting inequality with respect to t,s over $[0,1]\times[0,1]$, we get

$$\begin{aligned} &\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)(IR)\int_0^1\int_0^1\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right. \\ &\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}\right]dsdt \\ &\leq_{p}\frac{(b-a)(d-c)}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_0^1\int_0^1\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right. \\ &\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F(ta+(1-t)b,cs+(1-s)d)\right]dsdt \\ &+\frac{(b-a)(d-c)}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_0^1\int_0^1\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right. \\ &\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F(ta+(1-t)b,(1-s)c+sd)\right]dsdt \\ &+\frac{(b-a)(d-c)}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_0^1\int_0^1\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right. \\ &\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F((1-t)a+tb,cs+(1-s)d)\right]dsdt \\ &+\frac{(b-a)(d-c)}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_0^1\int_0^1\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right. \\ &\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F((1-t)a+tb,(1-s)c+sd)\right]dsdt. \end{aligned}$$

By a simple calculations, we have

$$\int_0^1\int_0^1 \frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}dsdt=\frac{[g(b)-g(a)]^{\alpha}[w(d)-w(c)]^{\beta}}{\alpha\beta(b-a)(d-c)}.$$

Using the change of variables $\tau=(1-t)a+tb$ and $\eta=(1-s)c+sd$, we obtain

$$\begin{aligned}
&\frac{[g(b)-g(a)]^{\alpha}[w(d)-w(c)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq_p \frac{1}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^b\int_c^d \frac{g'(\tau)}{[g(b)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(d)-w(\eta)]^{1-\beta}}F(a+b-\tau,c+d-\eta)d\eta d\tau\\
&+\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^b\int_c^d \frac{g'(\tau)}{[g(b)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(d)-w(\eta)]^{1-\beta}}F(a+b-\tau,\eta)d\eta d\tau\\
&+\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^b\int_c^d \frac{g'(\tau)}{[g(b)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(d)-w(\eta)]^{1-\beta}}F(\tau,c+d-\eta)d\eta d\tau\\
&+\frac{1}{4\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^b\int_c^d \frac{g'(\tau)}{[g(b)-g(\tau)]^{1-\alpha}}\frac{w'(\eta)}{[w(d)-w(\eta)]^{1-\beta}}F(\tau,\eta)d\eta d\tau\\
&=\frac{1}{4}\left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\widetilde{F_3}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\widetilde{F_1}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\widetilde{F_2}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}F(b,d)\right]\\
&=\frac{1}{4}\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{F}(b,d).
\end{aligned}$$

That is, we have

$$\frac{[g(b)-g(a)]^{\alpha}[w(d)-w(c)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq_p \frac{1}{4}\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{F}(b,d). \tag{5.5}$$

Similarly, multiplying both sides of (5.4) by

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\frac{w'((1-s)c+sd)}{[w((1-s)c+sd)-w(c)]^{1-\beta}}$$

and integrating the obtained inequality with respect to t,s over $[0,1]\times[0,1]$, we obtain

$$\frac{[g(b)-g(a)]^{\alpha}[w(d)-w(c)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq_p \frac{1}{4}\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{F}(b,c). \tag{5.6}$$

Moreover, multiplying both sides of (5.4) by

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{[g((1-t)a+tb)-g(a)]^{1-\alpha}}\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}$$

and

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{[g((1-t)a+tb)-g(a)]^{1-\alpha}}\frac{w'((1-s)c+sd)}{[w((1-s)c+sd)-w(c)]^{1-\beta}}$$

then integrating the established inequalities with respect to t,s over $[0,1]\times[0,1]$, we have the following inequalities

$$\frac{[g(b)-g(a)]^{\alpha}[w(d)-w(c)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq_p \frac{1}{4}\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{F}(a,d) \tag{5.7}$$

and

$$\frac{[g(b)-g(a)]^{\alpha}[w(d)-w(c)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq_{p}\frac{1}{4}\mathbb{J}_{b-,d-;g,w}^{\alpha,\beta}\mathcal{F}(a,c), \tag{5.8}$$

respectively.

Summing the inequalities (5.5)-(5.8), we get

$$\begin{aligned}&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\leq_{p}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{16\,[g(b)-g(a)]^{\alpha}[w(d)-w(c)]^{\beta}}\\&\times\left[\mathbb{J}_{a+,c+;g,w}^{\alpha,\beta}\mathcal{F}(b,d)+\mathbb{J}_{a+,d-;g,w}^{\alpha,\beta}\mathcal{F}(b,c)+\mathbb{J}_{b-,c+;g,w}^{\alpha,\beta}\mathcal{F}(a,d)+\mathbb{J}_{b-,d-;g,w}^{\alpha,\beta}\mathcal{F}(a,c)\right].\end{aligned}$$

This completes the proof of first inequality in (5.2).

For the proof of the second inequality in (5.2), since F is a co-ordinated LR-convex, we have

$$\begin{aligned}&F(ta+(1-t)b,cs+(1-s)d)+F(ta+(1-t)b,(1-s)c+sd)\\&+F((1-t)a+tb,cs+(1-s)d)+F((1-t)a+tb,(1-s)c+sd)\\&\leq_{p}F(a,c)+F(a,d)+F(b,c)+F(b,d).\end{aligned} \tag{5.9}$$

Multiplying both sides of (5.9) by

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}$$

and integrating the resulting inequality with respect to t,s over $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{0}^{1}\int_{0}^{1}\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right.\\&\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F(ta+(1-t)b,cs+(1-s)d)\right]dsdt\\&+\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{0}^{1}\int_{0}^{1}\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right.\\&\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F(ta+(1-t)b,(1-s)c+sd)\right]dsdt\\&+\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_{0}^{1}\int_{0}^{1}\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right.\\&\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F((1-t)a+tb,cs+(1-s)d)\right]dsdt\\&+\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\int_{0}^{1}\int_{0}^{1}\left[\frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}\right.\\&\left.\times\frac{w'((1-s)c+sd)}{[w(d)-w((1-s)c+sd)]^{1-\beta}}F((1-t)a+tb,(1-s)c+sd)\right]dsdt\end{aligned}$$

$$\leq_p \frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\left[F(a,c)+F(a,d)+F(b,c)+F(b,d)\right]$$
$$\times (IR)\int_0^1\int_0^1 \frac{g'((1-t)a+tb)}{\left[g(b)-g((1-t)a+tb)\right]^{1-\alpha}}\frac{w'((1-s)c+sd)}{\left[w(d)-w((1-s)c+sd)\right]^{1-\beta}}dsdt.$$

Then, we get

$$\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\widetilde{F_3}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\widetilde{F_1}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\widetilde{F_2}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}F(b,d)$$
$$\leq_p \left[F(a,c)+F(a,d)+F(b,c)+F(b,d)\right]\frac{\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)},$$

that is,

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{F}(b,d)\leq_p F(a,c)+F(a,d)+F(b,c)+F(b,d). \tag{5.10}$$

Similarly, multiplying both sides of (5.9) by

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{\left[g(b)-g((1-t)a+tb)\right]^{1-\alpha}}\frac{w'((1-s)c+sd)}{\left[w((1-s)c+sd)-w(c)\right]^{1-\beta}},$$

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{\left[g((1-t)a+tb)-g(a)\right]^{1-\alpha}}\frac{w'((1-s)c+sd)}{\left[w(d)-w((1-s)c+sd)\right]^{1-\beta}}$$

and

$$\frac{(b-a)(d-c)}{\Gamma(\alpha)\Gamma(\beta)}\frac{g'((1-t)a+tb)}{\left[g((1-t)a+tb)-g(a)\right]^{1-\alpha}}\frac{w'((1-s)c+sd)}{\left[w((1-s)c+sd)-w(c)\right]^{1-\beta}}$$

integrating the resulting inequalities with respect to t,s over $[0,1]\times[0,1]$, we establish the following inequalities

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{F}(b,c)\leq_p F(a,c)+F(a,d)+F(b,c)+F(b,d), \tag{5.11}$$

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{F}(a,d)\leq_p F(a,c)+F(a,d)+F(b,c)+F(b,d), \tag{5.12}$$

and

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{F}(a,c)\leq_p F(a,c)+F(a,d)+F(b,c)+F(b,d), \tag{5.13}$$

respectively.

By adding the inequalities (5.10)-(5.13), we have the inequality

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}} \tag{5.14}$$
$$\times\left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{F}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{F}(b,c)+\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{F}(a,d)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{F}(a,c)\right]$$
$$\leq_p 4\left[F(a,c)+F(a,d)+F(b,c)+F(b,d)\right].$$

If we divide the both sides of inequality (5.14) by 16, then we have the second inequality in (5.2).

This completes the proof. □

Corollary 3. *If we choose $g(t) = t$ and $w(s) = s$ in Theorem 11, then we have the following inequalities for Riemann–Liouville interval-valued fractional integrals*

$$\begin{aligned}
&F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\,(b-a)^{\alpha}\,(d-c)^{\beta}} \left[\mathcal{J}^{\alpha,\beta}_{a+,c+}F(b,d) + \mathcal{J}^{\alpha,\beta}_{a+,d-}F(b,c) + \mathcal{J}^{\alpha,\beta}_{b-,c+}F(a,d) + \mathcal{J}^{\alpha,\beta}_{b-,d-}F(a,c)\right]\\
&\leq_p \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.
\end{aligned}$$

Corollary 4. *Under assumption of Theorem 11 with $g(t) = \ln t$ and $w(s) = \ln s$, then we have the following inequalities for Hadamard interval-valued fractional integrals*

$$\begin{aligned}
&F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{16\left[\ln\frac{b}{a}\right]^{\alpha}\left[\ln\frac{d}{c}\right]^{\beta}} \left[\mathfrak{J}^{\alpha,\beta}_{a+,c+}\mathcal{F}(b,d) + \mathfrak{J}^{\alpha,\beta}_{a+,d-}\mathcal{F}(b,c) + \mathfrak{J}^{\alpha,\beta}_{b-,c+}\mathcal{F}(a,d) + \mathfrak{J}^{\alpha,\beta}_{b-,d-}\mathcal{F}(a,c)\right]\\
&\leq_p \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.
\end{aligned}$$

Corollary 5. *Under assumption of Theorem 11 with $g(t) = \frac{t^{\rho}}{\rho}$ and $w(s) = \frac{s^{\sigma}}{\sigma}$, $\rho, \sigma > 0$, then we have the following inequalities for interval-valued Katugampola fractional integrals*

$$\begin{aligned}
&F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\rho^{\alpha}\sigma^{\beta}}{16\left[b^{\rho}-a^{\rho}\right]^{\alpha}\left[d^{\sigma}-c^{\sigma}\right]^{\beta}} \left[{}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{a+,c+}\mathcal{F}(b,d) + {}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{a+,d-}\mathcal{F}(b,c) + {}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{b-,c+}\mathcal{F}(a,d) + {}^{\rho,\sigma}\mathbb{I}^{\alpha,\beta}_{b-,d-}\mathcal{F}(a,c)\right]\\
&\leq_p \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.
\end{aligned}$$

Theorem 12. *Let $\Delta = [a,b] \times [c,d]$, if $F : \Delta \to \mathbb{R}^{+}_{\mathcal{I}}$ be an interval-valued co-ordinated LR-convex on Δ, then for $\alpha, \beta > 0$ the following Hermite-Hadamard type inequality holds:*

$$\begin{aligned}
&F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (5.15)\\
&\leq_p \frac{\Gamma(\alpha+1)}{8\left[g(b)-g(a)\right]^{\alpha}} \left[\mathbb{J}^{\alpha}_{a+;g}\mathcal{H}\left(b, \frac{c+d}{2}\right) + \mathbb{J}^{\alpha}_{b-;g}\mathcal{H}\left(a, \frac{c+d}{2}\right)\right]\\
&+\frac{\Gamma(\beta+1)}{8\left[w(d)-w(c)\right]^{\beta}} \left[\mathbb{J}^{\beta}_{c+;w}\mathcal{G}\left(\frac{a+b}{2}, d\right) + \mathbb{J}^{\beta}_{d-;w}\mathcal{G}\left(\frac{a+b}{2}, c\right)\right]\\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{16\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\\
&\times\left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{F}(b,d) + \mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{F}(b,c) + \mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{F}(a,d) + \mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{F}(a,c)\right]\\
&\leq_p \frac{\Gamma(\alpha+1)}{16\left[g(b)-g(a)\right]^{\alpha}} \left[\mathbb{J}^{\alpha}_{a+;g}\mathcal{H}(b,c) + \mathbb{J}^{\alpha}_{a+;g}\mathcal{H}(b,d) + \mathbb{J}^{\alpha}_{b-;g}\mathcal{H}(a,c) + \mathbb{J}^{\alpha}_{b-;g}\mathcal{H}(a,d)\right]\\
&+\frac{\Gamma(\beta+1)}{16\left[w(d)-w(c)\right]^{\beta}} \left[\mathbb{J}^{\beta}_{c+;w}\mathcal{G}(a,d) + \mathbb{J}^{\beta}_{c+;w}\mathcal{G}(b,d) + \mathbb{J}^{\beta}_{d-;w}\mathcal{G}(a,c) + \mathbb{J}^{\beta}_{d-;w}\mathcal{G}(b,c)\right]\\
&\leq_p \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}
\end{aligned}$$

where the function $\mathcal{H}, \mathcal{F}$ and $\mathcal{G}$ are defined as in (5.1).

Proof. Since F is an interval valued co-ordinated LR-convex on Δ, if we define the mapping $h_x^1 : [c,d] \to \mathbb{R}$, $h_x^1(y) = F(x,y)$, then $h_x^1(y)$ is convex for all $x \in [a,b]$ and $\mathcal{H}_x^1(y) = h_x^1(y) + \widetilde{h_x^1}(y) = F(x,y) + \widetilde{F_2}(x,y) = \mathcal{G}(x,y)$. If we apply the inequalities (2.4) for the convex function $h_x^1(y)$, then we have

$$h_x^1\left(\frac{c+d}{2}\right) \leq_p \frac{\Gamma(\beta+1)}{4\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{c+;w}^{\beta}\mathcal{H}_x^1(d)+\mathbb{J}_{d-;w}^{\beta}\mathcal{H}_x^1(c)\right] \leq_p \frac{h_x^1(c)+h_x^1(d)}{2},$$

that is,

$$\begin{aligned} &F\left(x,\frac{c+d}{2}\right) \\ &\leq_p \frac{\beta}{4\left[w(d)-w(c)\right]^{\beta}}\left[(IR)\int_c^d \frac{w'(y)}{\left[w(d)-w(y)\right]^{1-\beta}}\mathcal{G}(x,y)dy+(IR)\int_c^d \frac{w'(y)}{\left[w(y)-w(c)\right]^{1-\beta}}\mathcal{G}(x,y)dy\right] \\ &\leq_p \frac{F(x,c)+F(x,d)}{2}. \end{aligned} \tag{5.16}$$

Multiplying the inequalities (5.16) by

$$\frac{\alpha}{\left[g(b)-g(a)\right]^{\alpha}}\frac{g'(x)}{\left[g(b)-g(x)\right]^{1-\alpha}},$$

and

$$\frac{\alpha}{\left[g(b)-g(a)\right]^{\alpha}}\frac{g'(x)}{\left[g(x)-g(a)\right]^{1-\alpha}},$$

then by integrating the obtained results with respect to x from a to b, we get

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{\left[g(b)-g(a)\right]^{\alpha}}\mathbb{J}_{a+;g}^{\alpha}F\left(b,\frac{c+d}{2}\right) \\ &\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{a+,c+;g,w}^{\alpha,\beta}\mathcal{G}(b,d)+\mathbb{J}_{a+,d-;g,w}^{\alpha,\beta}\mathcal{G}(b,c)\right] \\ &\leq_p \frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}_{a+;g}^{\alpha}F(b,c)+\mathbb{J}_{a+;g}^{\alpha}F(b,d)\right] \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{\left[g(b)-g(a)\right]^{\alpha}}\mathbb{J}_{b-;g}^{\alpha}F\left(a,\frac{c+d}{2}\right) \\ &\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{b-,c+;g,w}^{\alpha,\beta}\mathcal{G}(a,d)+\mathbb{J}_{b-,d-;g,w}^{\alpha,\beta}\mathcal{G}(a,c)\right] \\ &\leq_p \frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}_{b-;g}^{\alpha}F(a,c)+\mathbb{J}_{b-;g}^{\alpha}F(a,d)\right], \end{aligned} \tag{5.18}$$

respectively.

On the other hand, since F is a co-ordinated LR-convex on Δ, if we define the mapping $h_x^2 : [c,d] \to \mathbb{R}$, $h_x^2(y) = \widetilde{F_1}(x,y)$, then $h_x^2(y)$ is convex for all $x \in [a,b]$ and $\mathcal{H}_x^2(y) = h_x^2(y) + \widetilde{h_x^2}(y) = \widetilde{F_1}(x,y) + \widetilde{F_3}(x,y) = \mathcal{K}(x,y)$. If we apply the inequalities (2.4) for the convex function $h_x^2(y)$, then we have

$$h_x^2\left(\frac{c+d}{2}\right) \leq_p \frac{\Gamma(\beta+1)}{4\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{c+;w}^{\beta}\mathcal{H}_x^2(d)+\mathbb{J}_{d-;w}^{\beta}\mathcal{H}_x^2(c)\right] \leq_p \frac{h_x^2(c)+h_x^2(d)}{2},$$

i.e.

$$\widetilde{F_1}\left(x,\frac{c+d}{2}\right)$$

(5.19)

$$\leq_p \frac{\beta}{4\left[w(d)-w(c)\right]^{\beta}}\left[(IR)\int_c^d \frac{w'(y)}{\left[w(d)-w(y)\right]^{1-\beta}}\mathcal{K}(x,y)dy+(IR)\int_c^d \frac{w'(y)}{\left[w(y)-w(c)\right]^{1-\beta}}\mathcal{K}(x,y)dy\right]$$
$$\leq_p \frac{\widetilde{F_1}(x,c)+\widetilde{F_1}(x,d)}{2}.$$

Similarly, multiplying the inequalities (5.19) by

$$\frac{\alpha}{\left[g(b)-g(a)\right]^{\alpha}}\frac{g'(x)}{\left[g(b)-g(x)\right]^{1-\alpha}},$$

and

$$\frac{\alpha}{\left[g(b)-g(a)\right]^{\alpha}}\frac{g'(x)}{\left[g(x)-g(a)\right]^{1-\alpha}},$$

then by integrating the obtained results with respect to x from a to b, we get

$$\frac{\Gamma(\alpha+1)}{\left[g(b)-g(a)\right]^{\alpha}}\mathbb{J}^{\alpha}_{a+;g}\widetilde{F_1}\left(b,\frac{c+d}{2}\right) \tag{5.20}$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{K}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{K}(b,c)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}^{\alpha}_{a+;g}\widetilde{F_1}(b,c)+\mathbb{J}^{\alpha}_{a+;g}\widetilde{F_1}(b,d)\right]$$

and

$$\frac{\Gamma(\alpha+1)}{\left[g(b)-g(a)\right]^{\alpha}}\mathbb{J}^{\alpha}_{b-;g}\widetilde{F_1}\left(a,\frac{c+d}{2}\right) \tag{5.21}$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{K}(a,d)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{K}(a,c)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}^{\alpha}_{b-;g}\widetilde{F_1}(a,c)+\mathbb{J}^{\alpha}_{b-;g}\widetilde{F_1}(a,d)\right],$$

respectively.

Moreover, if we define the mapping $h^1_y:[a,b]\to\mathbb{R}$, $h^1_y(x)=F(x,y)$, then $h^1_y(x)$ is convex for all $y\in[c,d]$ and $\mathcal{H}^1_y(x)=h^1_y(x)+\widetilde{h^1_y}(x)=F(x,y)+\widetilde{F_1}(x,y)=\mathcal{H}(x,y)$. Applying the inequalities (2.4) for the convex function $h^1_y(x)$, then we have

$$h^1_y\left(\frac{a+b}{2}\right)\leq_p \frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}^{\alpha}_{a+;g}\mathcal{H}^1_y(b)+\mathbb{J}^{\alpha}_{b-;w}\mathcal{H}^1_y(a)\right]\leq_p \frac{h^1_y(a)+h^1_y(b)}{2},$$

that is,

$$F\left(\frac{a+b}{2},y\right) \tag{5.22}$$
$$\leq_p \frac{\alpha}{4\left[g(b)-g(a)\right]^{\alpha}}\left[(IR)\int_a^b \frac{g'(x)}{\left[g(b)-g(x)\right]^{1-\alpha}}\mathcal{H}(x,y)dx+(IR)\int_a^b \frac{g'(x)}{\left[g(x)-g(a)\right]^{1-\alpha}}\mathcal{H}(x,y)dx\right]$$
$$\leq_p \frac{F(a,y)+F(b,y)}{2}.$$

Multiplying the inequalities (5.22) by

$$\frac{\beta}{\left[w(d)-w(c)\right]^{\beta}}\frac{w'(y)}{\left[w(d)-w(y)\right]^{1-\beta}}$$

and

$$\frac{\beta}{[w(d)-w(c)]^{\beta}}\frac{w'(y)}{[w(y)-w(c)]^{1-\beta}}$$

then integrating the established results with respect to y from c to d, we obtain the following inequalities

$$\begin{aligned}(5.23)\quad &\frac{\Gamma(\beta+1)}{[w(d)-w(c)]^{\beta}}\mathbb{J}^{\beta}_{c+;w}F\left(\frac{a+b}{2},d\right)\\ &\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{H}(b,d)+\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{H}(a,d)\right]\\ &\leq_p \frac{\Gamma(\beta+1)}{2\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\beta}_{c+;w}F\left(a,d\right)+\mathbb{J}^{\beta}_{c+;w}F\left(b,d\right)\right]\end{aligned}$$

and

$$\begin{aligned}(5.24)\quad &\frac{\Gamma(\beta+1)}{[w(d)-w(c)]^{\beta}}\mathbb{J}^{\beta}_{d-;w}F\left(\frac{a+b}{2},c\right)\\ &\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{H}(b,c)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{H}(a,c)\right]\\ &\leq_p \frac{\Gamma(\beta+1)}{2\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\beta}_{d-;w}F\left(a,c\right)+\mathbb{J}^{\beta}_{d-;w}F\left(b,c\right)\right],\end{aligned}$$

respectively.

Furthermore, if we define the mapping $h_y^2:[a,b]\to\mathbb{R}$, $h_y^2(x)=\widetilde{F_2}(x,y)$, then $h_y^2(x)$ is convex for all $y\in[c,d]$ and $\mathcal{H}_y^2(x)=h_y^2(x)+\widetilde{h_y^2}(x)=\widetilde{F_2}(x,y)+\widetilde{F_3}(x,y)=\mathcal{L}(x,y)$. Applying the inequalities (2.4) for the convex function $h_y^2(x)$, then we have

$$h_y^2\left(\frac{a+b}{2}\right)\leq_p \frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}^{\alpha}_{a+;g}\mathcal{H}_y^2(b)+\mathbb{J}^{\alpha}_{b-;w}\mathcal{H}_y^2(a)\right]\leq_p \frac{h_y^2\left(a\right)+h_y^2\left(b\right)}{2},$$

i.e.

$$\begin{aligned}(5.25)\quad &\widetilde{F_2}\left(\frac{a+b}{2},y\right)\\ &\leq_p \frac{\alpha}{4\left[g(b)-g(a)\right]^{\alpha}}\left[(IR)\int_a^b\frac{g'(x)}{[g(b)-g(x)]^{1-\alpha}}\mathcal{L}(x,y)dx+(IR)\int_a^b\frac{g'(x)}{[g(x)-g(a)]^{1-\alpha}}\mathcal{L}(x,y)dx\right]\\ &\leq_p \frac{\widetilde{F_2}(a,y)+\widetilde{F_2}\left(b,y\right)}{2}.\end{aligned}$$

Similarly, multiplying the inequalities (5.25) by

$$\frac{\beta}{[w(d)-w(c)]^{\beta}}\frac{w'(y)}{[w(d)-w(y)]^{1-\beta}}$$

and

$$\frac{\beta}{[w(d)-w(c)]^{\beta}}\frac{w'(y)}{[w(y)-w(c)]^{1-\beta}},$$

then integrating the obtained results with respect to y from c to d, we obtain the following inequalities

$$\begin{aligned}(5.26)\quad &\frac{\Gamma(\beta+1)}{[w(d)-w(c)]^{\beta}}\mathbb{J}^{\beta}_{c+;w}\widetilde{F_2}\left(\frac{a+b}{2},d\right)\\ &\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{L}(b,d)+\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{L}(a,d)\right]\end{aligned}$$

$$\leq_p \frac{\Gamma(\beta+1)}{2\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\beta}_{c+;w}\widetilde{F_2}\left(a,d\right)+\mathbb{J}^{\beta}_{c+;w}\widetilde{F_2}\left(b,d\right)\right]$$

and

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{\left[w(d)-w(c)\right]^{\beta}}\mathbb{J}^{\beta}_{d-;w}\widetilde{F_2}\left(\frac{a+b}{2},c\right) \\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{L}(b,c)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{L}(a,c)\right] \\
&\leq_p \frac{\Gamma(\beta+1)}{2\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\beta}_{d-;w}\widetilde{F_2}\left(a,c\right)+\mathbb{J}^{\beta}_{d-;w}\widetilde{F_2}\left(b,c\right)\right],
\end{aligned} \tag{5.27}$$

respectively.

Summing the inequalities (5.17), (5.18), (5.20), (5.21), (5.23), (5.24), (5.26) and (5.27), we have the following inequalities

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}^{\alpha}_{a+;g}F\left(b,\frac{c+d}{2}\right)+\mathbb{J}^{\alpha}_{b-;g}F\left(a,\frac{c+d}{2}\right)\right. \\
&\left.+\mathbb{J}^{\alpha}_{a+;g}\widetilde{F_1}\left(b,\frac{c+d}{2}\right)+\mathbb{J}^{\alpha}_{b-;g}\widetilde{F_1}\left(a,\frac{c+d}{2}\right)\right] \\
&+\frac{\Gamma(\beta+1)}{\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\beta}_{c+;w}F\left(\frac{a+b}{2},d\right)+\mathbb{J}^{\beta}_{d-;w}F\left(\frac{a+b}{2},c\right)\right. \\
&\left.+\mathbb{J}^{\beta}_{c+;w}\widetilde{F_2}\left(\frac{a+b}{2},d\right)+\mathbb{J}^{\beta}_{d-;w}\widetilde{F_2}\left(\frac{a+b}{2},c\right)\right] \\
&\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}} \\
&\times\left[\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{G}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{G}(b,c)+\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{G}(a,d)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{G}(a,c)\right. \\
&+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{K}(b,d)+\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{K}(b,c)+\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{K}(a,d)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{K}(a,c) \\
&+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{H}(b,d)+\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{H}(a,d)+\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{H}(b,c)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{H}(a,c) \\
&\left.+\mathbb{J}^{\alpha,\beta}_{a+,c+;g,w}\mathcal{L}(b,d)+\mathbb{J}^{\alpha,\beta}_{b-,c+;g,w}\mathcal{L}(a,d)+\mathbb{J}^{\alpha,\beta}_{a+,d-;g,w}\mathcal{L}(b,c)+\mathbb{J}^{\alpha,\beta}_{b-,d-;g,w}\mathcal{L}(a,c)\right] \\
&\leq_p \frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}^{\alpha}_{a+;g}F\left(b,c\right)+\mathbb{J}^{\alpha}_{a+;g}F\left(b,d\right)+\mathbb{J}^{\alpha}_{b-;g}F\left(a,c\right)+\mathbb{J}^{\alpha}_{b-;g}F\left(a,d\right)\right. \\
&\left.+\mathbb{J}^{\alpha}_{a+;g}\widetilde{F_1}\left(b,c\right)+\mathbb{J}^{\alpha}_{a+;g}\widetilde{F_1}\left(b,d\right)+\mathbb{J}^{\alpha}_{b-;g}\widetilde{F_1}\left(a,c\right)+\mathbb{J}^{\alpha}_{b-;g}\widetilde{F_1}\left(a,d\right)\right] \\
&+\frac{\Gamma(\beta+1)}{\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}^{\beta}_{c+;w}F\left(a,d\right)+\mathbb{J}^{\beta}_{c+;w}F\left(b,d\right)+\mathbb{J}^{\beta}_{d-;w}F\left(a,c\right)+\mathbb{J}^{\beta}_{d-;w}F\left(b,c\right)\right. \\
&\left.+\mathbb{J}^{\beta}_{c+;w}\widetilde{F_2}\left(a,d\right)+\mathbb{J}^{\beta}_{c+;w}\widetilde{F_2}\left(b,d\right)+\mathbb{J}^{\beta}_{d-;w}\widetilde{F_2}\left(a,c\right)+\mathbb{J}^{\beta}_{d-;w}\widetilde{F_2}\left(b,c\right)\right].
\end{aligned}$$

That is, we have

$$\frac{\Gamma(\alpha+1)}{\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}^{\alpha}_{a+;g}\mathcal{H}\left(b,\frac{c+d}{2}\right)+\mathbb{J}^{\alpha}_{b-;g}\mathcal{H}\left(a,\frac{c+d}{2}\right)\right]$$

$$+\frac{\Gamma(\beta+1)}{[w(d)-w(c)]^{\beta}}\left[\mathbb{J}_{c+;w}^{\beta}\mathcal{G}\left(\frac{a+b}{2},d\right)+\mathbb{J}_{d-;w}^{\beta}\mathcal{G}\left(\frac{a+b}{2},c\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\left[g(b)-g(a)\right]^{\alpha}\left[w(d)-w(c)\right]^{\beta}}$$
$$\times\left[\mathbb{J}_{a+,c+;g,w}^{\alpha,\beta}\mathcal{F}(b,d)+\mathbb{J}_{a+,d-;g,w}^{\alpha,\beta}\mathcal{F}(b,c)+\mathbb{J}_{b-,c+;g,w}^{\alpha,\beta}\mathcal{F}(a,d)+\mathbb{J}_{b-,d-;g,w}^{\alpha,\beta}\mathcal{F}(a,c)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}_{a+;g}^{\alpha}\mathcal{H}(b,c)+\mathbb{J}_{a+;g}^{\alpha}\mathcal{H}(b,d)+\mathbb{J}_{b-;g}^{\alpha}\mathcal{H}(a,c)+\mathbb{J}_{b-;g}^{\alpha}\mathcal{H}(a,d)\right]$$
$$+\frac{\Gamma(\beta+1)}{2\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{c+;w}^{\beta}\mathcal{G}(a,d)+\mathbb{J}_{c+;w}^{\beta}\mathcal{G}(b,d)+\mathbb{J}_{d-;w}^{\beta}\mathcal{G}(a,c)+\mathbb{J}_{d-;w}^{\beta}\mathcal{G}(b,c)\right]$$

which completes the proof of the second and third inequalities in (5.15).

On the other hand, from the first inequality in (2.4), we have

$$F\left(\frac{a+b}{2}\right) \leq_p \frac{\alpha}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\int_a^b \frac{g'(x)}{\left[g(b)-g(x)\right]^{\alpha}}\left[F(x)+F(a+b-x)\right]dx + \int_a^b \frac{g'(x)}{\left[g(x)-g(a)\right]^{\alpha}}\left[F(x)+F(a+b-x)\right]dx\right]. \tag{5.28}$$

Since F is interval-valued co-ordinated convex on Δ, by using the inequality (5.28), we obtain

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq_p \frac{\alpha}{4\left[g(b)-g(a)\right]^{\alpha}}\left[(IR)\int_a^b \frac{g'(x)}{\left[g(b)-g(x)\right]^{\alpha}}\left[F\left(x,\frac{c+d}{2}\right)+F\left(a+b-x,\frac{c+d}{2}\right)\right]dx + (IR)\int_a^b \frac{g'(x)}{\left[g(x)-g(a)\right]^{\alpha}}\left[F\left(x,\frac{c+d}{2}\right)+F\left(a+b-x,\frac{c+d}{2}\right)\right]dx\right] = \frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}_{a+;g}^{\alpha}\mathcal{H}\left(b,\frac{c+d}{2}\right)+\mathbb{J}_{b-;g}^{\alpha}\mathcal{H}\left(a,\frac{c+d}{2}\right)\right], \tag{5.29}$$

and similarly we have

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq_p \frac{\beta}{4\left[w(d)-w(c)\right]^{\beta}}\left[(IR)\int_c^d \frac{w'(y)}{\left[w(d)-w(y)\right]^{\alpha}}\left[F\left(\frac{a+b}{2},y\right)+F\left(\frac{a+b}{2},c+d-y\right)\right]dy + (IR)\int_c^d \frac{w'(y)}{\left[w(y)-w(c)\right]^{\alpha}}\left[F\left(\frac{a+b}{2},y\right)+F\left(\frac{a+b}{2},c+d-y\right)\right]dy\right] = \frac{\Gamma(\beta+1)}{4\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{c+;w}^{\beta}\mathcal{G}\left(\frac{a+b}{2},d\right)+\mathbb{J}_{d-;w}^{\beta}\mathcal{G}\left(\frac{a+b}{2},c\right)\right]. \tag{5.30}$$

Combining the inequalities (5.29) and (5.30), we obtain the first inequality in (5.15).

From the second inequality in (2.4), we have

$$\frac{\alpha}{4\left[g(b)-g(a)\right]^{\alpha}}\left[(IR)\int\limits_{a}^{b}\frac{g'(x)}{\left[g(b)-g(x)\right]^{\alpha}}\left[F(x)+F(a+b-x)\right]dx\right. \tag{5.31}$$
$$\left.+(IR)\int\limits_{a}^{b}\frac{g'(x)}{\left[g(x)-g(a)\right]^{\alpha}}\left[F(x)+F(a+b-x)\right]dx\right]$$
$$\leq_{p}\frac{F(a)+F(b)}{2}.$$

By using the inequality (5.31), we obtain the following inequalities

$$\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}_{a+;g}^{\alpha}\mathcal{H}\left(b,c\right)+\mathbb{J}_{b-;g}^{\alpha}\mathcal{H}\left(a,c\right)\right]\leq_{p}\frac{F(a,c)+F(b,c)}{2}, \tag{5.32}$$

$$\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\mathbb{J}_{a+;g}^{\alpha}\mathcal{H}\left(b,d\right)+\mathbb{J}_{b-;g}^{\alpha}\mathcal{H}\left(a,d\right)\right]\leq_{p}\frac{F(a,d)+F(b,d)}{2}, \tag{5.33}$$

$$\frac{\Gamma(\beta+1)}{4\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{c+;w}^{\beta}\mathcal{G}\left(a,d\right)+\mathbb{J}_{d-;w}^{\beta}\mathcal{G}\left(a,c\right)\right]\leq_{p}\frac{F(a,c)+F(a,d)}{2} \tag{5.34}$$

and

$$\frac{\Gamma(\beta+1)}{4\left[w(d)-w(c)\right]^{\beta}}\left[\mathbb{J}_{c+;w}^{\beta}\mathcal{G}\left(b,d\right)+\mathbb{J}_{d-;w}^{\beta}\mathcal{G}\left(b,c\right)\right]\leq_{p}\frac{F(b,c)+F(b,d)}{2}. \tag{5.35}$$

Combining the inequalities (5.32)-(5.35), we obtain the last inequality in (5.15).

This completes the proof completely. □

Corollary 6. *If we choose $g(t)=t$ and $w(s)=s$ in Theorem 12, then we have the following inequalities for Reimann-Liouville interval-valued fractional integrals*

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq_{p}\frac{\Gamma(\alpha+1)}{4\left(b-a\right)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha}F\left(b,\frac{c+d}{2}\right)+\mathcal{J}_{b-}^{\alpha}F\left(a,\frac{c+d}{2}\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{4\left(d-c\right)^{\beta}}\left[\mathcal{J}_{c+}^{\beta}F\left(\frac{a+b}{2},d\right)+\mathcal{J}_{d-}^{\beta}F\left(\frac{a+b}{2},c\right)\right]$$
$$\leq_{p}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\left(b-a\right)^{\alpha}\left(d-c\right)^{\beta}}\left[\mathcal{J}_{a+,c+}^{\alpha,\beta}F(b,d)+\mathcal{J}_{a+,d-}^{\alpha,\beta}F(b,c)+\mathcal{J}_{b-,c+}^{\alpha,\beta}F(a,d)+\mathcal{J}_{b-,d-}^{\alpha,\beta}F(a,c)\right]$$
$$\leq_{p}\frac{\Gamma(\alpha+1)}{8\left(b-a\right)^{\alpha}}\left[\mathcal{J}_{a+}^{\alpha}F\left(b,c\right)+\mathcal{J}_{a+}^{\alpha}F\left(b,d\right)+\mathcal{J}_{b-}^{\alpha}F\left(a,c\right)+\mathcal{J}_{b-}^{\alpha}F\left(a,d\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{8\left(d-c\right)^{\beta}}\left[\mathcal{J}_{c+}^{\beta}F\left(a,d\right)+\mathcal{J}_{c+}^{\beta}F\left(b,d\right)+\mathcal{J}_{d-}^{\beta}F\left(a,c\right)+\mathcal{J}_{d-}^{\beta}F\left(b,c\right)\right]$$
$$\leq_{p}\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.$$

Corollary 7. *Under assumption of Theorem 12 with $g(t)=\ln t$ and $w(s)=\ln s$, then we have the following inequalities for Hadamard interval-valued fractional integrals*

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leq_p \frac{\Gamma(\alpha+1)}{8\left[\ln\frac{b}{a}\right]^{\alpha}}\left[\mathfrak{J}_{a+}^{\alpha}\mathcal{H}\left(b,\frac{c+d}{2}\right)+\mathfrak{J}_{b-}^{\alpha}\mathcal{H}\left(a,\frac{c+d}{2}\right)\right]$$
$$+\frac{\Gamma(\beta+1)}{8\left[\ln\frac{d}{c}\right]^{\beta}}\left[\mathfrak{J}_{c+}^{\beta}\mathcal{G}\left(\frac{a+b}{2},d\right)+\mathfrak{J}_{d-}^{\beta}\mathcal{G}\left(\frac{a+b}{2},c\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{16\left[\ln\frac{b}{a}\right]^{\alpha}\left[\ln\frac{d}{c}\right]^{\beta}}\left[\mathfrak{J}_{a+,c+}^{\alpha,\beta}\mathcal{F}(b,d)+\mathfrak{J}_{a+,d-}^{\alpha,\beta}\mathcal{F}(b,c)+\mathfrak{J}_{b-,c+}^{\alpha,\beta}\mathcal{F}(a,d)+\mathfrak{J}_{b-,d-}^{\alpha,\beta}\mathcal{F}(a,c)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)}{16\left[\ln\frac{b}{a}\right]^{\alpha}}\left[\mathfrak{J}_{a+}^{\alpha}\mathcal{H}(b,c)+\mathfrak{J}_{a+}^{\alpha}\mathcal{H}(b,d)+\mathfrak{J}_{b-}^{\alpha}\mathcal{H}(a,c)+\mathfrak{J}_{b-}^{\alpha}\mathcal{H}(a,d)\right]$$
$$+\frac{\Gamma(\beta+1)}{16\left[\ln\frac{d}{c}\right]^{\beta}}\left[\mathfrak{J}_{c+}^{\beta}\mathcal{G}(a,d)+\mathfrak{J}_{c+}^{\beta}\mathcal{G}(b,d)+\mathfrak{J}_{d-}^{\beta}\mathcal{G}(a,c)+\mathfrak{J}_{d-}^{\beta}\mathcal{G}(b,c)\right]$$
$$\leq_p \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.$$

Corollary 8. *Under assumption of Theorem 11 with $g(t)=\frac{t^\rho}{\rho}$ and $w(s)=\frac{s^\sigma}{\sigma}$, then we have the following inequalities for interval-valued Katugampola fractional integrals*

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq_p \frac{\Gamma(\alpha+1)\rho^{\alpha}}{8\left[b^\rho-a^\rho\right]^{\alpha}}\left[{}^{\rho}\mathbb{I}_{a+}^{\alpha}\mathcal{H}\left(b,\frac{c+d}{2}\right)+{}^{\rho}\mathbb{I}_{b-}^{\alpha}\mathcal{H}\left(a,\frac{c+d}{2}\right)\right]$$
$$+\frac{\Gamma(\beta+1)\sigma^{\beta}}{8\left[d^\sigma-c^\sigma\right]^{\beta}}\left[{}^{\sigma}\mathbb{I}_{c+}^{\beta}\mathcal{G}\left(\frac{a+b}{2},d\right)+{}^{\sigma}\mathbb{I}_{d-}^{\beta}\mathcal{G}\left(\frac{a+b}{2},c\right)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\rho^{\alpha}\sigma^{\beta}}{16\left[b^\rho-a^\rho\right]^{\alpha}\left[d^\sigma-c^\sigma\right]^{\beta}}\left[{}^{\rho,\sigma}\mathbb{I}_{a+,c+}^{\alpha,\beta}\mathcal{F}(b,d)+{}^{\rho,\sigma}\mathbb{I}_{a+,d-}^{\alpha,\beta}\mathcal{F}(b,c)+{}^{\rho,\sigma}\mathbb{I}_{b-,c+}^{\alpha,\beta}\mathcal{F}(a,d)+{}^{\rho,\sigma}\mathbb{I}_{b-,d-}^{\alpha,\beta}\mathcal{F}(a,c)\right]$$
$$\leq_p \frac{\Gamma(\alpha+1)\rho^{\alpha}\sigma^{\beta}}{16\left[b^\rho-a^\rho\right]^{\alpha}}\left[{}^{\rho}\mathbb{I}_{a+}^{\alpha}\mathcal{H}(b,c)+{}^{\rho}\mathbb{I}_{a+}^{\alpha}\mathcal{H}(b,d)+{}^{\rho}\mathbb{I}_{b-}^{\alpha}\mathcal{H}(a,c)+{}^{\rho}\mathbb{I}_{b-}^{\alpha}\mathcal{H}(a,d)\right]$$
$$+\frac{\Gamma(\beta+1)\sigma^{\beta}}{16\left[d^\sigma-c^\sigma\right]^{\beta}}\left[{}^{\sigma}\mathbb{I}_{c+}^{\beta}\mathcal{G}(a,d)+{}^{\sigma}\mathbb{I}_{c+}^{\beta}\mathcal{G}(b,d)+{}^{\sigma}\mathbb{I}_{d-}^{\beta}\mathcal{G}(a,c)+{}^{\sigma}\mathbb{I}_{d-}^{\beta}\mathcal{G}(b,c)\right]$$
$$\leq_p \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.$$

6. Concluding Remarks

In this research, authors established Hermite-Hadamard type inequalities for interval-valued LR-convex functions and co-ordinated interval-valued LR-convex functions. The results in this paper are the extension of several previously obtained results. Interested author can find more new integral inequalities another type co-ordinated interval-valued convexity.

Competing Interests
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Author contributions HK worked in the field of spelling. He did a resource search for the article. He took part in the conceptualization of the article and in the field of software. MAA did the calculations about the problem. He took part in the writing phase. Included in the Methodology of the article. HB posed the problem. He conducted an investigation on this subject and acted as the supervisor of the article. All authors read and approved the final manuscript.
Data Availability

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

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