EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR IMPULSIVE DISTRIBUTED DELAY LOTKA-VOLTERRA DYNAMIC SYSTEM ON TIME SCALES

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Abstract. In this manuscript, we study the positivity and periodicity of solutions for a nonlinear impulsive distributed delay Lotka-Volterra dynamic system on time scales. We provide sufficient conditions for the existence of positive periodic solutions by using Krasnoselskii's fixed point theorem. The findings from this study extend the findings of Benhadri and Caraballo [8].

1. Introduction

The idea of time scale analysis, which unites the conventional fields of continuous and discrete analysis into one theory, was first suggested in 1988 by German mathematician Stefan Hilger in his doctoral thesis [16]. Following the release of two textbooks in this subject matter by Bohner and Peterson [10] and [11], an increasing number of researchers began to work in this rapidly expanding area of mathematics.

The conventional fields of differential and difference equations are combined in the study of dynamic equations. It enables one to manage these two study fields concurrently, illuminating the causes of their apparent discrepancies. In fact, by analyzing the more general time scale example, see [1]–[7], [9]–[17] and the references therein, numerous novel results for the continuous and discrete situations have been achieved.

Benhadri and Caraballo [8] investigated the positivity and periodicity of solutions for the impulsive distributed delay Lotka-Volterra differential system

$$\begin{aligned}\psi_k'(\tau) = \psi_k(\tau)\Bigg[v_k(\tau) - \sum_{i=1}^{N} a_{ki}(\tau)\, h_i(\psi_i(\tau)) - \sum_{i=1}^{N} b_{ki}(\tau)\, f_i(\psi_i(\tau - r_i(\tau)))\\ - \sum_{i=1}^{N} c_{ki}(\tau)\int_{-\infty}^{\tau} D_{ki}(\tau,\varsigma)\, g_i(\psi_i(\varsigma))\, d\varsigma\Bigg],\ \tau \neq \tau_l,\ l \in \mathbb{N}^* = \{1,2,...\},\\ \psi_k(\tau_l^+) - \psi_k(\tau_l^-) = I_{kl}(\tau_l, \psi_k(\tau_l)),\ \tau = \tau_l,\ l \in \mathbb{N}^*. \qquad (1.1)\end{aligned}$$

By applying Krasnoselskii's fixed point theorem, the authors demonstrated the positivity and periodicity of solutions and improved the findings in [19].

Let $\mathbb{S}$ be a periodic time scale such that $0 \in \mathbb{S}$. In this manuscript, we extend the findings in [8] by proving the positivity and periodicity of solutions for the nonlinear

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impulsive distributed delay Lotka-Volterra dynamic system

$$\psi_k^{\Delta}(\tau)=\psi_k(\tau)\left[\upsilon_k(\tau)-\sum_{i=1}^{N}a_{ki}(\tau)h_i(\psi_i(\tau))-\sum_{i=1}^{N}b_{ki}(\tau)f_i(\psi_i(\tau-r_i(\tau)))\right.$$
$$\left.-\sum_{i=1}^{N}c_{ki}(\tau)\int_{-\infty}^{\tau}D_{ki}(\tau,\varsigma)g_i(\psi_i(\varsigma))\Delta\varsigma\right],\ \tau\neq\tau_l,\ l\in\mathbb{N}^*,\ \tau\in\mathbb{S},$$
$$\psi_k(\tau_l^+)-\psi_k(\tau_l^-)=-\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\upsilon_k(\tau_l)},\ \tau=\tau_l,\ l\in\mathbb{N}^*. \tag{1.2}$$

for $k=1,...,N$, where $\tau_l\in\mathbb{S}$, $h_i,f_i,g_i\in C(\mathbb{R}_+,\mathbb{R}_+)$, $D_{ki}\in C_{rd}(\mathbb{S}\times\mathbb{S},\mathbb{R}_+)$, $a_{ki},b_{ki},c_{ki}\in C_{rd}(\mathbb{S},\mathbb{R}_+)$, $r_i\in C_{rd}(\mathbb{S},\mathbb{S})$ and $\upsilon_k\in\mathcal{R}^+(\mathbb{S},\mathbb{R}_+)$ for $k,i=1,\dots,N$. Moreover, the expression $\psi_k(\tau_l^+)-\psi_k(\tau_l^-)$ denotes the impulse at moment τ_l, $\psi_k(\tau_l^-)$ and $\psi_k(\tau_l^+)$ stand for the the left-hand and right-hand limits of $\psi_k(\tau)$ at the impulsive moment τ_l respectively, and $I_{kl}\in C_{rd}(\mathbb{S}\times\mathbb{R}_+,\mathbb{R}_+)$, $l\in\mathbb{N}^*$. We suppose that there exists $q\in\mathbb{N}^*$ such that

$$\tau_{l+q+1}=\tau_l+\varpi,\ I_{k(l+q+1)}(\tau_{l+q+1},\psi_k(\tau_{l+q+1}))=I_{kl}(\tau_l,\psi_k(\tau_l)),\ l\in\mathbb{N}^*,$$

where $0<\tau_1<\tau_2<\ldots<\tau_q<\varpi$. Since we are looking for the existence of periodic solutions to (1.2), it makes sense to suppose that

$$\begin{aligned}a_{ki}(\tau+\varpi)&=a_{ki}(\tau),\ r_i(\tau+\varpi)=r_i(\tau),\ D_{ki}(\tau+\varpi,\iota+\varpi)=D_{ki}(\tau,\iota),\\ b_{ki}(\tau+\varpi)&=b_{ki}(\tau),\ c_{ki}(\tau+\varpi)=c_{ki}(\tau),\ \upsilon_k(\tau+\varpi)=\upsilon_k(\tau),\ k,i=1,2,\dots,N,\end{aligned}$$

with $\tau-r_i(\tau)\in\mathbb{S}$ and $r_i(\tau)\geq r_i^*>0$ for all $\tau\in\mathbb{S}$, $i=1,...,N$.

To demonstrate the positivity and periodicity of solutions for (1.2), we transform (1.2) into an equivalent integral system and then use Krasnoselskii's fixed point theorem. Finally, to demonstrate the reliability of our findings, we provide an example.

2. Preliminaries

Definition 1 ([10])**.** *A time scale $\mathbb{S}$ is an arbitrary nonempty closed subset of $\mathbb{R}$.*

Definition 2 ([10])**.** *Let $\mathbb{S}$ be a time scale. The forward and backward jump mappings $\sigma,\rho:\mathbb{S}\to\mathbb{S}$ and the graininess mapping $\mu:\mathbb{S}\to\mathbb{R}_+$ are defined, respectively, by*

$$\sigma(\tau)=\inf\{\iota\in\mathbb{S}:\iota>\tau\},\ \rho(\tau)=\sup\{\iota\in\mathbb{S}:\iota<\tau\},\ \mu(\tau)=\sigma(\tau)-\tau.$$

A point $\tau\in\mathbb{S}$ is called left-dense if $\tau>\inf\mathbb{S}$ and $\rho(\tau)=\tau$, left-scattered if $\rho(\tau)<\tau$, right-dense if $\tau<\sup\mathbb{S}$ and $\sigma(\tau)=\tau$, and right-scattered if $\sigma(\tau)>\tau$. If $\mathbb{S}$ has a left-scattered maximum M, then $\mathbb{S}^\kappa=\mathbb{S}\backslash\{M\}$. Otherwise, we define $\mathbb{S}^\kappa=\mathbb{S}$. If $\mathbb{S}$ has a right scattered minimum m, we define $\mathbb{S}_\kappa=\mathbb{S}\backslash\{m\}$. Otherwise, we define $\mathbb{S}_\kappa=\mathbb{S}$.

The next two definitions are borrowed from [7] and [17].

Definition 3 ([17])**.** *A time scale $\mathbb{S}$ is ω-periodic ($\omega>0$) if $\tau\pm\omega\in\mathbb{S}$ for all $\tau\in\mathbb{S}$. For $\mathbb{S}\neq\mathbb{R}$, the smallest positive ω is the period of $\mathbb{S}$.*

Remark 1 ([17])**.** *The ω-periodic time scale $\mathbb{S}$ is unbounded below and above.*

Definition 4 ([17])**.** *Let $\mathbb{S}\neq\mathbb{R}$ be an ω-periodic time scale. The function $\psi:\mathbb{S}\to\mathbb{R}$ is ω-periodic ($\omega>0$) if there is a smallest $N_0\in\mathbb{N}^*$ such that $\varpi=N_0\omega$, $\psi(\tau\pm\varpi)=\psi(\tau)$ for all $\tau\in\mathbb{S}$.*

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Remark 2 ([17]). *If* $\mathbb{S}$ *is an* ω*-periodic time scale, then* $\sigma(\tau \pm N_0\omega) = \sigma(\tau) \pm N_0\omega$ *and* $\mu(\tau \pm N_0\omega) = \mu(\tau)$.

Definition 5 ([10]). *The function* $\psi : \mathbb{S} \to \mathbb{R}$ *is rd-continuous if it is continuous at every right-dense point* $\tau \in \mathbb{S}$ *and its left-sided limits exist, and is finite at every left-dense point* $\tau \in \mathbb{S}$. *The set of rd-continuous functions* $\psi : \mathbb{S} \to \mathbb{R}$ *is denoted by*

$$C_{rd} = C_{rd}(\mathbb{S}) = C_{rd}(\mathbb{S}, \mathbb{R}).$$

Definition 6 ([10]). *Let* $\psi : \mathbb{S} \to \mathbb{R}$ *and* $\tau \in \mathbb{S}^\kappa$. *We define* $\psi^\Delta(\tau)$ *(if it exists) with the property that for every* $\varepsilon > 0$ *there is a neighborhood* U *of* τ *such that*

$$\left|\psi(\sigma(\tau)) - \psi(\iota) - \psi^\Delta(\tau)(\sigma(\tau) - \iota)\right| \leq \varepsilon |\sigma(\tau) - \iota| \text{ for all } \iota \in \mathrm{U}.$$

We call $\psi^\Delta(\tau)$ *the* Δ*-derivative of* ψ *at* τ. *We say that* ψ *is* Δ*-differentiable in* $\mathbb{S}^l$ *if* $\psi^\Delta(\tau)$ *exists for all* $\tau \in \mathbb{S}^\kappa$. *The function* $\psi^\Delta : \mathbb{S} \to \mathbb{R}$ *is said to be the* Δ*-derivative of* ψ *in* $\mathbb{S}^\kappa$.

Now, we state some properties of the Δ-derivative of ψ. Note $\psi^\sigma(\tau) = \psi(\sigma(\tau))$.

Theorem 1 ([10]). *Assume* $\psi, \varphi : \mathbb{S} \to \mathbb{R}$ *are* Δ*-differentiable at* $\tau \in \mathbb{S}^\kappa$ *and let* α *be a scalar.*

(1) $(\psi + \varphi)^\Delta(\tau) = \psi^\Delta(\tau) + \varphi^\Delta(\tau)$.
(2) $(\alpha\psi)^\Delta(\tau) = \alpha\psi^\Delta(\tau)$.
(3) *The product rules*

$$\begin{aligned}(\psi\varphi)^\Delta(\tau) &= \psi^\Delta(\tau)\varphi(\tau) + \psi^\sigma(\tau)\varphi^\Delta(\tau), \\ (\psi\varphi)^\Delta(\tau) &= \varphi^\Delta(\tau)\psi(\tau) + \varphi^\sigma(\tau)\psi^\Delta(\tau).\end{aligned}$$

(4) *If* $\varphi(\tau)\varphi^\sigma(\tau) \neq 0$ *then*

$$\left(\frac{\psi}{\varphi}\right)^\Delta(\tau) = \frac{\psi^\Delta(\tau)\varphi(\tau) - \psi(\tau)\varphi^\Delta(\tau)}{\varphi(\tau)\varphi^\sigma(\tau)}.$$

Definition 7 ([10]). *A function* $\upsilon : \mathbb{S} \to \mathbb{R}$ *is said to be regressive provided* $1 + \mu(\tau)\upsilon(\tau) \neq 0$ *for all* $\tau \in \mathbb{S}^\kappa$. *The set of all regressive rd-continuous function* $\upsilon : \mathbb{S} \to \mathbb{R}$ *is denoted by*

$$\mathcal{R} = \mathcal{R}(\mathbb{S}) = \mathcal{R}(\mathbb{S}, \mathbb{R}).$$

The set of all positively regressive functions $\mathcal{R}^+$, *is given by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{S}, \mathbb{R}) = \{\upsilon \in \mathcal{R} : 1 + \mu(\tau)\upsilon(\tau) > 0 \text{ for all } \tau \in \mathbb{S}\}.$$

Definition 8 ([10]). *Let* $\upsilon \in \mathcal{R}$. *The exponential function on* $\mathbb{S}$ *is defined by*

$$e_\upsilon(\tau, \iota) = \exp\left(\int_\iota^\tau \xi_{\mu(s)}(\upsilon(s))\,\Delta s\right) \text{ for } \iota, \tau \in \mathbb{S},$$

where $\xi_h(z)$ *is a cylinder transformation introduced in* [10, Definition 2.21].

Remark 3 ([10]). *If* $\upsilon \in \mathcal{R}^+$, *then* $e_\upsilon(\tau, \iota) > 0$ *for all* $\tau \in \mathbb{S}$. *Also, the exponential function* $\psi(\tau) = e_\upsilon(\tau, \iota)$ *is the solution to the initial value problem* $\psi^\Delta(\tau) = \upsilon(\tau)\psi(\tau)$, $\psi(\iota) = 1$.

We give the exponential function properties in the next theorem.

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Theorem 2 ([10]). *Let $\upsilon \in \mathcal{R}$. Then*
(1) $e_0(\tau,\iota) = 1$ *and* $e_\upsilon(\tau,\tau) = 1$,
(2) $e_\upsilon(\sigma(\tau),\iota) = (1+\mu(\tau)\upsilon(\tau))\, e_\upsilon(\tau,\iota)$,
(3) $\frac{1}{e_\upsilon(\tau,\iota)} = e_{\ominus\upsilon}(\tau,\iota)$ *where* $\ominus\upsilon(\tau) = -\frac{\upsilon(\tau)}{1+\mu(\tau)\upsilon(\tau)}$,
(4) $e_\upsilon(\tau,\iota) = \frac{1}{e_\upsilon(\iota,\tau)} = e_{\ominus\upsilon}(\iota,\tau)$,
(5) $e_\upsilon(\tau,\iota)\, e_\upsilon(\iota,s) = e_\upsilon(\tau,s)$,
(6) $e_\upsilon^\Delta(.,\iota) = \upsilon e_\upsilon(.,\iota)$ *and* $\left(\frac{1}{e_\upsilon(.,\tau)}\right)^\Delta = -\frac{\upsilon(\tau)}{e_\upsilon^\sigma(.,\tau)}$.

Theorem 3 ([9]). *Let $\mathbb{S}$ be an ω-periodic time scale. If $\upsilon \in C_{rd}(\mathbb{S})$ is an ϖ-periodic function ($\varpi = N_0\omega$), then*

$$\int_{a+\varpi}^{b+\varpi} \upsilon(s)\,\Delta s = \int_a^b \upsilon(s)\,\Delta s,\ \ e_\upsilon(b+\varpi, a+\varpi) = e_\upsilon(b,a) \ \text{ if } \upsilon \in \mathcal{R},$$

and $e_\upsilon(\tau+\varpi,\tau)$ is independent of $\tau \in \mathbb{S}$ whenever $\upsilon \in \mathcal{R}$.

Lemma 1 ([1]). *If $\upsilon \in \mathcal{R}^+$, the*

$$0 < e_\upsilon(\tau,\iota) \le \exp\left(\int_\iota^\tau \upsilon(s)\,\Delta s\right),\ \forall \tau \in \mathbb{S}.$$

Corollary 1 ([1]). *If $\upsilon \in \mathcal{R}^+$ and $\upsilon(\tau) < 0$ for all $\tau \in \mathbb{S}$, then for all $\iota \in \mathbb{S}$ with $\iota \le \tau$ we have*

$$0 < e_\upsilon(\tau,\iota) \le \exp\left(\int_\iota^\tau \upsilon(s)\,\Delta s\right) < 1.$$

Next, we give the cone definition and Krasnoselskii's fixed point theorem.

Definition 9 ([18]). *Let Ω be a closed nonempty subset of a Banach space $\mathbb{E}$. We call that Ω is a cone if*
(1) $\alpha\psi + \beta\varphi \in \Omega$ *for all* $\psi,\varphi \in \Omega$ *and all* $\alpha,\beta \ge 0$,
(2) $\psi, -\psi \in \Omega$ *imply* $\psi = 0$.

Theorem 4 ([18]). *Let $\mathbb{E}$ be a Banach space, and let $\Omega \subset \mathbb{E}$ be a cone in $\mathbb{E}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathbb{E}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\Phi : \Omega \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to \Omega,$$

be a completely continuous mapping such that either

$$\|\Phi\psi\| \le \|\psi\| \ \text{ for } \psi \in \Omega \cap \partial\Omega_1 \text{ and } \|\Phi\psi\| \ge \|\psi\| \ \text{ for } \psi \in \Omega \cap \partial\Omega_2,$$

or

$$\|\Phi\psi\| \ge \|\psi\| \ \text{ for } \psi \in \Omega \cap \partial\Omega_1 \text{ and } \|\Phi\psi\| \le \|\psi\| \ \text{ for } \psi \in \Omega \cap \partial\Omega_2.$$

Then Φ has a fixed point in $\Omega \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

3. Existence of positive periodic solutions

Let $\varpi > 0$ with $\varpi \in \mathbb{S}$ and if $\mathbb{S} \ne \mathbb{R}$, $\varpi = N_0\omega$ for $N_0 \in \mathbb{N}^*$. We denote

$$[a,b] = \{\tau \in \mathbb{S} : a \le \tau \le b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly.

The next lemma is fundamental to our findings.

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Lemma 2. *The function ψ is an ϖ-periodic solution of (1.2) if and only if ψ is an ϖ-periodic solution of the following system*

$$\psi_k(\tau) = \int_\tau^{\tau+\varpi} G_k(\tau,\iota)\,\psi_k(\iota)\left[\sum_{i=1}^N a_{ki}(\iota)\,h_i(\psi_i(\iota)) + \sum_{i=1}^N b_{ki}(\iota)\,f_i(\psi_i(\iota - r_i(\iota)))\right.$$
$$\left. + \sum_{i=1}^N c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right]\Delta\iota$$
$$+ \sum_{\tau\le\tau_l<\tau+\varpi} G_k(\tau,\tau_l)\,I_{kl}(\tau_l,\psi_k(\tau_l)),\ k = 1,\ldots,N, \tag{3.1}$$

where

$$G_k(\tau,\iota) = \frac{e_{\ominus\upsilon_k}(\sigma(\iota),\tau)}{1-e_{\ominus\upsilon_k}(\varpi,0)},\ \tau\le\iota\le\tau+\varpi,\ k=1,\ldots,N, \tag{3.2}$$

such that

$$1 \ne e_{\ominus\upsilon_k}(\varpi,0),\ k=1,\ldots,N.$$

Proof. Let ψ be an ϖ-periodic solution of (1.2). For any $\tau\in\mathbb{S}$, there exists $l\in\mathbb{N}^*$ such that τ_l is the first impulsive point after τ. Multiplying both sides of (1.2) by $e_{\ominus\upsilon_k}(\sigma(\tau),0)$ and then integrating from τ to $u\in[\tau,\tau_l]$, we get

$$\int_\tau^u \left[\psi_k(\iota)\,e_{\ominus\upsilon_k}(\iota,0)\right]^\Delta \Delta\iota$$
$$= \int_\tau^u e_{\ominus\upsilon_k}(\sigma(\iota),0)\,\psi_k(\iota)\left\{-\sum_{i=1}^N a_{ki}(\iota)\,h_i(\psi_i(\iota)) - \sum_{i=1}^N b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))\right.$$
$$\left. -\sum_{i=1}^N c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right\}\Delta\iota,$$

or

$$\psi_k(u) = \psi_k(\tau)\,e_{\ominus\upsilon_k}(\tau,u) + \int_\tau^u e_{\ominus\upsilon_k}(\sigma(\iota),u)\,\psi_k(\iota)\left[-\sum_{i=1}^N a_{ki}(\iota)\,h_i(\psi_i(\iota))\right.$$
$$\left. -\sum_{i=1}^N b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota))) - \sum_{i=1}^N c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right]\Delta\iota,$$

hence

$$\psi_k(\tau_l) = \psi_k(\tau)\,e_{\ominus\upsilon_k}(\tau,\tau_l) + \int_\tau^{\tau_l} e_{\ominus\upsilon_k}(\sigma(\iota),\tau_l)\,\psi_k(\iota)\left[-\sum_{i=1}^N a_{ki}(\iota)\,h_i(\psi_i(\iota))\right.$$
$$\left. -\sum_{i=1}^N b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota))) - \sum_{i=1}^N c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right]\Delta\iota, \tag{3.3}$$

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for $k=1,\ldots,N$. Similarly, for $u\in(\tau_l,\tau_{l+1}]$, we obtain

$$\begin{aligned}\psi_k(u)&=\psi_k(\tau_l^+)\,e_{\ominus\upsilon_k}(\tau_l,u)+\int_{\tau_l}^{u}e_{\ominus\upsilon_k}(\sigma(\iota),u)\,\psi_k(\iota)\Bigg[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\\&\quad-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota\\&=\psi_k(\tau_l^-)\,e_{\ominus\upsilon_k}(\tau_l,u)+\int_{\tau_l}^{u}e_{\ominus\upsilon_k}(\sigma(\iota),u)\,\psi_k(\iota)\Bigg[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\\&\quad-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota\\&\quad-e_{\ominus\upsilon_k}(\tau_l,u)\,\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\,\upsilon_k(\tau_l)}\\&=\psi_k(\tau_l)\,e_{\ominus\upsilon_k}(\tau_l,u)+\int_{\tau_l}^{u}e_{\ominus\upsilon_k}(\sigma(\iota),u)\,\psi_k(\iota)\Bigg[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\\&\quad-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota\\&\quad-e_{\ominus\upsilon_k}(\tau_l,u)\,\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\,\upsilon_k(\tau_l)},\ k=1,\ldots,N.\end{aligned}$$

By replacing (3.3) in the above equality, we have

$$\begin{aligned}\psi_k(u)&=\psi_k(\tau)\,e_{\ominus\upsilon_k}(\tau,u)+\int_{\tau}^{u}e_{\ominus\upsilon_k}(\sigma(\iota),u)\,\psi_k(\iota)\Bigg[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\\&\quad-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,d\varsigma\Bigg]\Delta\iota\\&\quad-e_{\ominus\upsilon_k}(\tau_l,u)\,\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\,\upsilon_k(\tau_l)},\ k=1,\ldots,N.\end{aligned}$$

By repeating the above process for $u\in[\tau,\tau+\varpi]$, we get

$$\begin{aligned}\psi_k(u)&=\psi_k(\tau)\,e_{\ominus\upsilon_k}(\tau,u)+\int_{\tau}^{u}e_{\ominus\upsilon_k}(\sigma(\iota),u)\,\psi_k(\iota)\Bigg[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\\&\quad-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota\\&\quad-\sum_{\tau\le\tau_l<\tau+\varpi}e_{\ominus\upsilon_k}(\tau_l,u)\,\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\,\upsilon_k(\tau_l)},\ k=1,\ldots,N.\end{aligned}$$

Let $u=\tau+\varpi$ in the above equality, then we have

$$\begin{aligned}\psi_k(\tau+\varpi) &= \psi_k(\tau)\,e_{\upsilon_k}(\tau+\varpi,\tau)\\ &\quad+\int_{\tau}^{\tau+\varpi} e_{\ominus\upsilon_k}(\sigma(\iota),\tau+\varpi)\,\psi_k(\iota)\left[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\right.\\ &\quad\left.-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right]\Delta\iota\\ &\quad-\sum_{\tau\le\tau_l<\tau+\varpi} e_{\ominus\upsilon_k}(\tau_l,\tau+\varpi)\,\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\,\upsilon_k(\tau_l)},\ k=1,\dots,N.\end{aligned}$$

By noticing that $\psi_k(\tau+\varpi)=\psi_k(\tau)$ and $e_{\ominus\upsilon_k}(\tau+\varpi,\tau)=e_{\ominus\upsilon_k}(\varpi,0)$, we get

$$\begin{aligned}&(1-e_{\upsilon_k}(\varpi,0))\,\psi_k(\tau)\\ &=\int_{\tau}^{\tau+\varpi} e_{\ominus\upsilon_k}(\sigma(\iota),\tau+\varpi)\,\psi_k(\iota)\left[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\right.\\ &\left.-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right]\Delta\iota\\ &-\sum_{\tau\le\tau_l<\tau+\varpi} e_{\ominus\upsilon_k}(\tau_l,\tau+\varpi)\,\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\,\upsilon_k(\tau_l)},\ k=1,\dots,N. \qquad (3.4)\end{aligned}$$

Thus

$$\begin{aligned}\psi_k(\tau) &= \int_{\tau}^{\tau+\varpi}\frac{e_{\ominus\upsilon_k}(\sigma(\iota),\tau+\varpi)}{1-e_{\upsilon_k}(\varpi,0)}\psi_k(\iota)\left[-\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))\right.\\ &\quad\left.-\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))-\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right]\Delta\iota\\ &\quad-\sum_{\tau\le\tau_l<\tau+\varpi}\frac{e_{\ominus\upsilon_k}(\tau_l,\tau+\varpi)}{1-e_{\upsilon_k}(\varpi,0)}\,\frac{I_{kl}(\tau_l,\psi_k(\tau_l))}{1+\mu(\tau_l)\,\upsilon_k(\tau_l)}\\ &= \int_{\tau}^{\tau+\varpi} G_k(\tau,\iota)\,\psi_k(\iota)\left[\sum_{i=1}^{N}a_{ki}(\iota)\,h_i(\psi_i(\iota))+\sum_{i=1}^{N}b_{ki}(\iota)\,f_i(\psi_i(\iota-r_i(\iota)))\right.\\ &\quad\left.+\sum_{i=1}^{N}c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\right]\Delta\iota\\ &\quad+\sum_{\tau\le\tau_l<\tau+\varpi} G_k(\tau,\tau_l)\,I_{kl}(\tau_l,\psi_k(\tau_l)),\end{aligned}$$

for $k=1,\dots,N$. Since the functions υ_k, a_{ki}, b_{ki}, c_{ki}, r_i are ϖ-periodic with respect to τ, $D_{ki}(\tau+\varpi,\iota+\varpi)=D_{ki}(\tau,\iota)$ and

$$G_k(\tau+\varpi,\iota+\varpi)=G_k(\tau,\iota)\ \text{ for all }\ (\tau,\iota)\in[0,\varpi]\times[0,\varpi],\ k=1,...,N,$$

then, ψ is an ϖ-periodic solution of (3.1).

Next, we prove the converse. It is easy to prove that for all $(\tau,\iota)\in[0,\varpi]\times[0,\varpi]$, we obtain

$$G_k(\sigma(\tau),\tau+\varpi)-G_k(\sigma(\tau),\tau)=-1,\ k=1,...,N,$$

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and

$$(G_k(\tau,\iota))^{\Delta}(\tau) = \upsilon_k(\tau)\, G_k(\tau,\iota), \ \ k = 1,...,N.$$

Let $\psi = (\psi_1, \psi_2, \dots, \psi_N)^T$ be an ϖ-periodic solution of (3.1). Then, if $\tau \neq \tau_l$, $l \in \mathbb{N}^*$, we get

$$\begin{aligned}
\psi_k^{\triangle}(\tau) &= G_k(\sigma(\tau), \tau+\varpi)\, \psi_k(\tau+\varpi) \left[\sum_{i=1}^{N} a_{ki}(\tau+\varpi)\, h_i(\psi_i(\tau+\varpi)) \right. \\
&+ \sum_{i=1}^{N} b_{ki}(\tau+\varpi)\, f_i(\psi_i(\tau+\varpi - r_i(\tau+\varpi))) \\
&\left. + \sum_{i=1}^{N} c_{ki}(\tau+\varpi) \int_{-\infty}^{\tau+\varpi} D_{ki}(\tau+\varpi,\varsigma)\, g_i(\psi_i(\varsigma))\, \Delta\varsigma \right] \\
&- G_k(\sigma(\tau),\tau)\, \psi_k(\tau) \left[\sum_{i=1}^{N} a_{ki}(\tau)\, h_i(\psi_i(\tau)) + \sum_{i=1}^{N} b_{ki}(\tau)\, f_i(\psi_i(\tau - r_i(\tau))) \right. \\
&\left. + \sum_{i=1}^{N} c_{ki}(\tau) \int_{-\infty}^{\tau} D_{ki}(\tau,\varsigma)\, g_i(\psi_i(\varsigma))\, \Delta\varsigma \right] \\
&+ \int_{\tau}^{\tau+\varpi} \upsilon_k(\tau)\, G_k(\tau,\iota)\, \psi_k(\iota) \left[\sum_{i=1}^{N} a_{ki}(\iota)\, h_i(\psi_i(\iota)) \right. \\
&\left. + \sum_{i=1}^{N} b_{ki}(\iota)\, f_i(\psi_i(\iota - r_i(\iota))) + \sum_{i=1}^{N} c_{ki}(\iota) \int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\, g_i(\psi_i(\varsigma))\, \Delta\varsigma \right] \Delta\iota \\
&= (G_k(\sigma(\tau), \tau+\varpi) - G_k(\sigma(\tau),\tau))\, \psi_k(\tau) \left[\sum_{i=1}^{N} a_{ki}(\tau)\, h_i(\psi_i(\tau)) \right. \\
&\left. + \sum_{i=1}^{N} b_{ki}(\tau)\, f_i(\psi_i(\tau - r_i(\tau))) + \sum_{i=1}^{N} c_{ki}(\tau) \int_{-\infty}^{\tau} D_{ki}(\tau,\varsigma)\, g_i(\psi_i(\varsigma))\, \Delta\varsigma \right] \\
&+ \upsilon_k(\tau)\, \psi_k(\tau) \\
&= \psi_k(\tau) \left(\upsilon_k(\tau) - \sum_{i=1}^{N} a_{ki}(\tau)\, h_i(\psi_i(\tau)) - \sum_{i=1}^{N} b_{ki}(\tau)\, f_i(\psi_i(\tau - r_i(\tau))) \right. \\
&\left. - \sum_{i=1}^{N} c_{ki}(\tau) \int_{-\infty}^{\tau} D_{ki}(\iota,\varsigma)\, g_i(\psi_i(\varsigma))\, \Delta\varsigma \right).
\end{aligned}$$

If $\tau = \tau_l$, $l \in \mathbb{N}^*$, we get

$$G_k(\tau_l, \tau_l+\varpi) - G_k(\tau_l,\tau_l) = -\frac{1}{1+\mu(\tau_l)\, \upsilon_k(\tau_l)}$$

$$
\begin{aligned}
&\psi_k\left(\tau_l^+\right)-\psi_k\left(\tau_l^-\right)\\
&=\sum_{\tau_l^+\le\tau_i<\tau_l^++\varpi} G_k\left(\tau_l,\tau_i\right) I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)-\sum_{\tau_l^-\le\tau_i<\tau_l^-+\varpi} G_k\left(\tau_l,\tau_i\right) I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)\\
&=G_k\left(\tau_l,\tau_l+\varpi\right) I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)-G_k\left(\tau_l,\tau_l\right) I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)\\
&=-\frac{I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)}{1+\mu\left(\tau_l\right)\upsilon_k\left(\tau_l\right)},\ k=1,...,N.
\end{aligned}
$$

So, ψ is an ϖ-periodic solution of (1.2). This completes the proof. □

Along this manuscript, we suppose the next assumption.

$(A1)$ There exist positive constants T_i, $\overline{T}_i$, F_i, $\overline{F}_i$, R_i, $\overline{R}_i$, E_{ki}, $\overline{E}_{ki}$ such that for all $\psi\in\mathbb{R}_+$ and all $\tau\in\mathbb{S}$, we have

$$\overline{T}_i\psi\le h_i\left(\psi\right)\le T_i\psi,\ k=1,...,N, \tag{3.5}$$

$$\overline{F}_i\psi\le f_i\left(\psi\right)\le F_i\psi,\ k=1,...,N, \tag{3.6}$$

$$\overline{R}_i\psi\le g_i\left(\psi\right)\le R_i\psi,\ k=1,...,N, \tag{3.7}$$

and

$$\overline{E}_{ki}\le\int_{-\infty}^{\tau} D_{ki}\left(\tau,\iota\right)\Delta\iota\le E_{ki},\ k=1,...,N. \tag{3.8}$$

Now, we introduce the next notations

$$
\begin{aligned}
\widehat{\upsilon}_k &= \frac{1}{\varpi}\int_0^{\varpi}\upsilon_k\left(\iota\right)\Delta\iota>0,\ \widehat{a}_{ki}=\frac{T_i}{\varpi}\int_0^{\varpi}a_{ki}\left(\iota\right)\Delta\iota\ge 0,\\
\widehat{b}_{ki} &= \frac{F_i}{\varpi}\int_0^{\varpi}b_k\left(\iota\right)\Delta\iota\ge 0,\ \widehat{c}_{ki}=\frac{R_iE_{ki}}{\varpi}\int_0^{\varpi}c_{ki}\left(\iota\right)\Delta\iota\ge 0,
\end{aligned}
$$

for $k,i=1,...,N$, where T_i, F_i, R_i, E_{ki} are given in (3.5)-(3.8).

To use Theorem 4, we define

$$
\begin{aligned}
PC\left(\mathbb{S},\mathbb{R}^N\right)=\Big\{&\psi:\mathbb{S}\to\mathbb{R}^N:\psi|_{(\tau_l,\tau_{l+1})}\in C_{rd}\left(\left(\tau_l,\tau_{l+1}\right),\mathbb{R}^N\right),\\
&\psi\left(\tau_l^-\right)\text{ and }\psi\left(\tau_l^+\right)\text{ exist},\ \psi_k\left(\tau_l^-\right)=\psi_k\left(\tau_l\right),\ l\in\mathbb{N}^*\Big\}.
\end{aligned}
$$

So, we let $(\mathbb{E},\|.\|)=(C_{\varpi},\|.\|)$ where

$$C_{\varpi}=\left\{\psi=\left(\psi_1,\psi_2,\ldots,\psi_N\right)^T\in PC\left(\mathbb{S},\mathbb{R}^N\right):\psi\left(\tau+\varpi\right)=\psi\left(\tau\right),\ \forall\tau\in\mathbb{S}\right\}, \tag{3.9}$$

with the norm

$$\|\psi\|=\sum_{k=1}^{N}\left|\psi_k\right|_0,\ \ \left|\psi_k\right|_0=\max_{\tau\in[0,\varpi]}\left|\psi_k\left(\tau\right)\right|,\ k=1,\ldots,N,\ \forall\psi\in C_{\varpi}. \tag{3.10}$$

Then, $(C_{\varpi},\|.\|)$ is a Banach space.

Definition 10. *A vector $\psi=\left(\psi_1,\psi_2,...,\psi_N\right)^T$ is positive if $\psi_l>0$, $l=1,...,N$.*

Define a cone Ω in C_{ϖ} by

$$\Omega=\left\{\psi=\left(\psi_1,\psi_2,\ldots,\psi_N\right)^T\in C_{\varpi}:\psi_k\left(\tau\right)\ge\delta\left|\psi_k\right|_0,\ \forall\tau\in\mathbb{R},\ k=1,\ldots,N\right\},$$

where

$$\delta=\min\left\{e_{\ominus\upsilon_k}\left(\varpi,0\right),\ k=1,\ldots,N\right\}.$$

Use (3.1) to define the operator $\Phi:C_{\varpi}\to C_{\varpi}$ by

$$\left(\Phi\psi\right)\left(\tau\right)=\left(\left(\Phi_1\psi\right)\left(\tau\right),\left(\Phi_2\psi\right)\left(\tau\right),\ldots,\left(\Phi_N\psi\right)\left(\tau\right)\right)^T,$$

where

$$\begin{aligned}(\Phi_k\psi)(\tau) = \int_{\tau}^{\tau+\varpi} G_k(\tau,\iota)\,\psi_k(\iota)\Bigg[\sum_{i=1}^{N} a_{ki}(\tau)\,h_i(\psi_i(\tau)) + \sum_{i=1}^{N} b_{ki}(\iota)\,f_i(\psi_i(\iota - r_i(\iota)))\\ + \sum_{i=1}^{N} c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota\\ + \sum_{\tau\le\tau_l<\tau+\varpi} G_k(\tau,\tau_l)\,I_{kl}(\tau_l,\psi_k(\tau_l)).\end{aligned} \tag{3.11}$$

By (3.1), it is simple to demonstrate that $\psi \in C_\varpi$ is an ϖ-periodic solution of (1.2) as long as ψ is a fixed point of Φ.

Lemma 3. *Suppose that the assumption $(A1)$ holds. Then $\Phi : \Omega \to \Omega$ defined by (3.11) is well defined, namely, $\Phi\Omega \subset \Omega$.*

Proof. According to (3.11) it is easy to demonstrate that $\Phi\psi$ is rd-continuous in (τ_l, τ_{l+1}), $(\Phi\psi)\left(\tau_l^+\right)$ and $(\Phi\psi)\left(\tau_l^-\right)$ exist, $(\Phi\psi)\left(\tau_l^-\right) = (\Phi\psi)(\tau_l)$ for $l \in \mathbb{N}^*$. Furthermore, for $\psi \in \Omega$,

$$\begin{aligned}&(\Phi_k\psi)(\tau+\varpi)\\ &= \int_{\tau+\varpi}^{\tau+2\varpi} G_k(\tau+\varpi,\iota)\,\psi_k(\iota)\Bigg[\sum_{i=1}^{N} a_{ki}(\iota)\,h_i(\psi_i(\iota)) + \sum_{i=1}^{N} b_{ki}(\iota)\,f_i(\psi_i(\iota - r_i(\iota)))\\ &+ \sum_{i=1}^{N} c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota\\ &+ \sum_{\tau+\varpi\le\tau_l<\tau+2\varpi} G_k(\tau+\varpi,\tau_l)\,I_{kl}(\tau_l,\psi_k(\tau_l))\\ &= \int_{\tau}^{\tau+\varpi} G_k(\tau+\varpi,\iota+\varpi)\,\psi_k(\iota+\varpi)\Bigg[\sum_{i=1}^{N} a_{ki}(\iota+\varpi)\,h_i(\psi_i(\iota+\varpi))\\ &+ \sum_{i=1}^{N} b_{ki}(\iota+\varpi)\,f_i(\psi_i(\iota+\varpi) - r_i(\iota+\varpi))\\ &+ \sum_{i=1}^{N} c_{ki}(\iota+\varpi)\int_{-\infty}^{\iota+\varpi} D_{ki}(\iota+\varpi,\varsigma)\,g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota\\ &+ \sum_{\tau\le\tau_l<\tau+\varpi} G_k(\tau,\tau_l)\,I_{kl}(\tau_l,\psi_k(\tau_l))\\ &= \int_{\tau}^{\tau+\varpi} G_k(\tau,\iota)\,\psi_k(\iota)\Bigg[\sum_{i=1}^{N} a_{ki}(\iota)\,h_i(\psi_i(\iota)) + \sum_{i=1}^{N} b_{ki}(\iota)\,f_i(\psi_i(\iota - r_i(\iota)))\\ &+ \sum_{i=1}^{N} c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\,g_i(\psi_i(\varsigma))\,d\varsigma\Bigg]\Delta\iota + \sum_{\tau\le\tau_l<\tau+\varpi} G_k(\tau,\tau_l)\,I_{kl}(\tau_l,\psi_k(\tau_l))\\ &= (\Phi_k\psi)(\tau),\ k = 1,\ldots,N.\end{aligned}$$

Then, $(\Phi_k\psi)(\tau+\varpi)=(\Phi_k\psi)(\tau)$, $\tau\in\mathbb{R}$, $k=1,...,N$. So, $\Phi\psi\in C_\varpi$. According to (3.2), it is easy to show that for $\iota\in[\tau,\tau+\varpi]$, we get

$$A_k=\frac{e_{\ominus v_k}(\varpi,0)}{1-e_{\ominus v_k}(\varpi,0)}\leq G_k(\tau,\iota)\leq\frac{1}{1-e_{\ominus v_k}(\varpi,0)}=B_k,\ k=1,\ldots,N. \tag{3.12}$$

From (3.11) and (3.12), we have for $\psi\in\Omega$,

$$|\Phi_k\psi|_0\leq B_k\int_0^\varpi\psi_k(\iota)\left[\sum_{i=1}^N a_{ki}(\iota)h_i(\psi_i(\iota))+\sum_{i=1}^N b_{ki}(\iota)f_i(\psi_i(\iota-r_i(\iota)))\right.$$
$$\left.+\sum_{i=1}^N c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)g_i(\psi_i(\varsigma))\Delta\varsigma\right]\Delta\iota+B_k\sum_{l=1}^q I_{kl}(\tau_l,\psi_k(\tau_l)),$$

and

$$\begin{aligned}(\Phi_k\psi)(\tau) &\geq A_k\int_0^\varpi\psi_k(\iota)\left[\sum_{i=1}^N a_{ki}(\iota)h_i(\psi_i(\iota))+\sum_{i=1}^N b_{ki}(\iota)f_i(\psi_i(\iota-r_i(\iota)))\right.\\ &\quad\left.+\sum_{i=1}^N c_{ki}(\iota)\int_{-\infty}^{\iota}D_{ki}(\iota,\varsigma)g_i(\psi_i(\varsigma))\Delta\varsigma\right]\Delta\iota+A_k\sum_{l=1}^q I_{kl}(\tau_l,\psi_k(\tau_l))\\ &\geq\frac{A_k}{B_k}|\Phi_k\psi|_0\geq\delta|\Phi_k\psi|_0,\ k=1,...,N.\end{aligned}$$

Consequently, $\Phi\Omega\subset\Omega$. The proof is completed. ☐

We introduce the next notation before establishing the main findings

$$\varphi^M=\max_{\tau\in[0,\varpi]}\{|\varphi(\tau)|\},$$

where the function φ is ϖ-periodic rd-continuous.

3.1. **The case of subquadratic impulses.**

Lemma 4. *Suppose that the assumption* $(A1)$ *holds and*

$(A2)$ *There exist positive functions* $\overline{\lambda}_{kl},\lambda_{kl}\in C_{rd}(\mathbb{S},\mathbb{R}_+)$ *such that for all* $\psi\in\mathbb{R}_+$, $\tau\in\mathbb{S}$, *we have*

$$\overline{\lambda}_{kl}(\tau)\psi^2\leq I_{kl}(\tau,\psi)\leq\lambda_{kl}(\tau)\psi^2,\ l\in\mathbb{N}^*,\ k=1,...,N.$$

Then $\Phi:\Omega\to\Omega$ *defined by (3.11) is completely continuous.*

Proof. Set

$$\Gamma_k(\tau,\psi(\tau))=\psi_k(\tau)\left[\sum_{i=1}^N a_{ki}(\tau)h_i(\psi_i(\tau))+\sum_{i=1}^N b_{ki}(\tau)f_i(\psi_i(\tau-r_i(\tau)))\right.$$
$$\left.+\sum_{i=1}^N c_{ki}(\tau)\int_{-\infty}^{\tau}D_{ki}(\tau,\varsigma)g_i(\psi_i(\varsigma))\Delta\varsigma\right],\ \tau\in\mathbb{R}. \tag{3.13}$$

We first demonstrate that Φ is continuous. Since h_i, f_i, g_i and I are continuous in ψ, it follows that, for any $L_0\geq0$ and $\varepsilon>0$, there exists $\mu_1>0$ such that for $\|\psi\|\leq L_0$, $\|y\|\leq L_0$ and $\|\psi-y\|<\mu_1$, we get

$$|\Gamma_k(\iota,\psi(\iota))-\Gamma_k(\iota,y(\iota))|<\frac{\varepsilon}{2NB\varpi},\ \iota\in\mathbb{R}_+,\ k=1,\ldots,N, \tag{3.14}$$

where $B=\max\limits_{1\leq k\leq N}B_k$. For any $L_0>0$ and $\varepsilon>0$, there exists $\mu_2>0$ such that for $\|\psi\|\leq L_0$, $\|y\|\leq L_0$ and $\|\psi-y\|<\mu_2$, we have

$$\left|I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)-I_{kl}\left(\tau_l,y_k\left(\tau_l\right)\right)\right|<\frac{\varepsilon}{2qB\varpi},\ q\in\mathbb{N}^*,\ k=1,\ldots,N. \tag{3.15}$$

Therefore, if $\psi,y\in C_\varpi$ with $\|\psi\|\leq L_0$, $\|y\|\leq L_0$ and $\|\psi-y\|\leq\mu$, where $\mu=\min\left(\mu_1,\mu_2\right)$ then, from (3.11), (3.12), (3.14) and (3.15), we obtain

$$\begin{aligned}&\left|\Phi_k\psi-\Phi_k y\right|_0\\&\leq B\int_\tau^{\tau+\varpi}\left|\Gamma_k\left(\iota,\psi\left(\iota\right)\right)-\Gamma_k\left(\iota,y\left(\iota\right)\right)\right|\Delta\iota+B\sum_{l=1}^{q}\left|I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)-I_{kl}\left(\tau_l,y_k\left(\tau_l\right)\right)\right|\\&\leq B\frac{\varpi\varepsilon}{2NB\varpi}+Bq\frac{\varepsilon}{2qBN}<\frac{\varepsilon}{N},\ k=1,...,N.\end{aligned}$$

This yields

$$\|\Phi\psi-\Phi y\|=\sum_{k=1}^{N}\left|\Phi_k\psi-\Phi_k y\right|_0<\varepsilon,$$

which implies that Φ is continuous on Ω.

We let

$$\Lambda=\left\{\psi=\left(\psi_1,\psi_2,\ldots,\psi_N\right)^T\in C_\varpi:\|\psi\|\leq L\right\},$$

where $L>0$. By using (3.11) and (3.12), for any $\psi\in\Lambda$, we have

$$\begin{aligned}\left(\Phi_k\psi\right)(\tau)=&\int_\tau^{\tau+\varpi}G_k\left(\tau,\iota\right)\psi_k\left(\iota\right)\left[\sum_{i=1}^{N}a_{ki}\left(\iota\right)h_i\left(\psi_i\left(\iota\right)\right)+\sum_{i=1}^{N}b_{ki}\left(\iota\right)f_i\left(\psi_i\left(\iota-r_i\left(\iota\right)\right)\right)\right.\\&\left.+\sum_{i=1}^{N}c_{ki}\left(\iota\right)\int_{-\infty}^{\iota}D_{ki}\left(\iota,\varsigma\right)g_i\left(\psi_i\left(\varsigma\right)\right)\Delta\varsigma\right]\Delta\iota\\&+\sum_{\tau\leq\tau_l<\tau+\varpi}G_k\left(\tau,\tau_l\right)I_{kl}\left(\tau_l,\psi_k\left(\tau_l\right)\right)\\&\leq B_kL^2\int_0^{\varpi}\left[\sum_{i=1}^{N}T_ia_{ki}\left(\iota\right)+\sum_{i=1}^{N}F_ib_{ki}\left(\iota\right)+\sum_{i=1}^{N}R_iE_{ki}c_{ki}\left(\iota\right)\right]\Delta\iota\\&+B_kL^2\sum_{\tau\leq\tau_l<\tau+\varpi}\lambda_{kl}\left(\tau_l\right)\\&=B_k^*,\ k=1,\ldots,N.\end{aligned}$$

Consequently,

$$\|\Phi\psi\|=\sum_{k=1}^{N}\left|\Phi_k\psi\right|_0\leq\sum_{k=1}^{N}B_k^*,\ \forall\psi\in\Lambda.$$

Hence, $\Phi\left(\Lambda\right)$ is uniformly bounded.

To demonstrate that $\Phi(\Lambda)$ is equicontinuous, let $\psi \in \Lambda$, we calculate $(\Phi_k\psi)^{\Delta}$ and prove that it is uniformly bounded. Indeed, by applying derivative in (3.11) we get

$$\begin{aligned}\left|(\Phi_k\psi)^{\Delta}(\tau)\right| \leq & \left|\upsilon_k(\tau)(\Phi_k\psi)(\tau) - \psi_k(\tau)\left[\sum_{i=1}^{N} a_{ki}(\tau) h_i(\psi_i(\tau))\right.\right. \\ & \left.\left. + \sum_{i=1}^{N} b_{ki}(\iota) f_i(\psi_i(\iota - r_i(\iota))) + \sum_{i=1}^{N} c_{ki}(\iota)\int_{-\infty}^{\tau} D_{ki}(\tau,\varsigma) g_i(\psi_i(\varsigma))\Delta\varsigma\right]\right| \\ \leq & \upsilon_k^M B_k^* + L^2 \sum_{i=1}^{N}\left(T_i a_{ki}^M + F_i b_{ki}^M + E_{ki} R_i c_{ki}^M\right), \ k = 1,\dots,N,\end{aligned}$$

and

$$\left\|(\Phi\psi)^{\Delta}\right\| \leq \sum_{i=1}^{N}\left[\upsilon_k^M B_k^* + L^2 \sum_{i=1}^{N}\left(T_i a_{ki}^M + F_i b_{ki}^M + E_{ki} R_i c_{ki}^M\right)\right].$$

Hence, $\Phi S \subset C_{\varpi}$ is uniformly bounded and equicontinuous. By using the Ascoli-Arzela theorem, the mapping Φ is compact. Therefore, the mapping Φ is completely continuous. The proof is completed. □

Along the following steps, we set $\theta = \min(\theta_1, \theta_2, \theta_3, \theta_4)$, where

$$\begin{aligned}\theta_1 &= \min_{1\leq i\leq N}\left(\frac{\overline{T}_i}{T_i}\right), \ \theta_2 = \min_{1\leq i\leq N}\left(\frac{\overline{F}_i}{F_i}\right), \\ \theta_3 &= \min_{1\leq i\leq N}\left[\min_{1\leq k\leq N}\left(\frac{\overline{E}_{ki}}{E_{ki}}\right)\frac{\overline{R}_i}{R_i}\right], \ \theta_4 = \min_{1\leq l\leq q}\left[\min_{1\leq k\leq N}\left(\frac{\overline{\lambda}_{kl}(\tau_l)}{\lambda_{kl}(\tau_l)}\right)\right]. \end{aligned} \tag{3.16}$$

Now, we state and demonstrate our main findings in this manuscript.

Theorem 5. *Suppose that the assumptions* $(A1)$ *and* $(A2)$ *hold, and*

$(A3)$ *The linear system*

$$\sum_{i=1}^{N}\left(\widehat{a}_{ki} + \widehat{b}_{ki} + \widehat{c}_{ki}\right)\psi_i + \widehat{\beta}_{kl}\psi_k = \widehat{\upsilon}_k, \ l \in \mathbb{N}^*, \ k = 1,\dots,N, \tag{3.17}$$

where

$$\widehat{\beta}_{kl} = \frac{1}{\varpi}\sum_{l=1}^{q}\lambda_{kl}(\tau_l), \ (\lambda_{kl}(\tau_l) \neq 0),$$

admits a unique positive solution. Then, the system (1.2) admits a positive ϖ*-periodic solution.*

Proof. Let

$$\psi^* = (\psi_1^*, \psi_2^*, ..., \psi_N^*)^T,$$

with $\psi_k^* > 0, \ k = 1, ..., N$, be a solution of (3.17). Set

$$m_0 = \min_{1\leq k\leq N}\{\widehat{\upsilon}_k A_k\}, \ M_0 = \min_{1\leq k\leq N}\{\widehat{\upsilon}_k B_k\}.$$

Then $0 < m_0 < M_0 < +\infty$. Choose a constant $M \geq M_0$ such that $\frac{1}{M\varpi} < 1$. Let $\alpha_1 = \frac{1}{M\varpi}$ and

$$\Omega_1 = \left\{\psi = (\psi_1, \psi_2, ..., \psi_N)^T \in C_{\varpi} : |\psi_k|_0 < \alpha_1\psi_k^*, \ k = 1,\dots,N\right\}.$$

If $\psi \in \Omega \cap \partial\Omega_1$, then

$$\delta|\psi_k|_0 \leq \psi_k(\tau) \leq |\psi_k|_0 = \alpha_1\psi_k^*, \ k = 1,\dots,N.$$

and

$$\begin{aligned}
(\Phi_k\psi)(\tau) &\le B_k \int_0^{\varpi} \psi_k(\iota)\Bigg[\sum_{i=1}^{N} a_{ki}(\iota)\, h_i(\psi_i(\iota)) + \sum_{i=1}^{N} b_{ki}(\iota)\, f_i(\psi_i(\iota - r_i(\iota))) \\
&+ \sum_{i=1}^{N} c_{ki}(\iota) \int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\, g_i(\psi_i(\varsigma))\,\Delta\varsigma\Bigg]\Delta\iota \\
&+ B_k \sum_{0\le\tau_l\le\varpi}^{q} I_{kl}(\tau_l, \psi_k(\tau_l)) \\
&\le B_k \int_0^{\varpi} |\psi_k|_0 \sum_{i=1}^{N} T_i a_{ki}(\iota)\, |\psi_i|_0\,\Delta\iota + B_k \int_0^{\varpi} |\psi_k|_0 \sum_{i=1}^{N} F_i b_{ki}(\iota)\, |\psi_i|_0\,\Delta\iota \\
&+ B_k \int_0^{\varpi} |\psi_k|_0 \sum_{i=1}^{N} R_i E_{ki}(\iota)\, |\psi_i|_0\, d\iota + B_k\, |\psi_k|_0 \sum_{0\le\tau_l<\varpi} \lambda_k(\tau_l)\, |\psi_k|_0 \\
&\le \alpha_1 B_k \varpi\, |\psi_k|_0 \sum_{i=1}^{N} \widehat{a}_{ki}\psi_i^* + \alpha_1 B_k \varpi\, |\psi_k|_0 \sum_{i=1}^{N} \widehat{b}_{ki}\psi_i^* \\
&+ \alpha_1 B_k \varpi\, |\psi_k|_0 \sum_{i=1}^{N} \widehat{c}_{ki}\psi_i^* + \alpha_1 B_k \varpi\, |\psi_k|_0 \sum_{i=1}^{N} \widehat{\beta}_{ki}\psi_i^* \\
&= \alpha_k B_k \varpi\, |\psi_k|_0 \left[\sum_{i=1}^{N}\left(\widehat{a}_{ki} + \widehat{b}_{ki} + \widehat{c}_{ki}\right)\psi_i^* + \widehat{\beta}_{ki}\psi_i^*\right] \\
&= (B_k \widehat{\upsilon}_k)\, \alpha_1 \varpi\, |\psi_k|_0 \le \alpha_1 M_0 \varpi\, |\psi_k|_0 \le |\psi_k|_0,\ k = 1, \dots, N.
\end{aligned}$$

Hence for any $\psi \in \Omega \cap \partial\Omega_1$,

$$\|\Phi\psi\| = \sum_{k=1}^{N} |\Phi_k \psi_k|_0 \le \sum_{k=1}^{N} |\psi_k|_0 = \|\psi\|.$$

Also, choose $0 < m \le m_0$ such that $\frac{1}{\delta^2 m\theta\varpi} > 1$. Let $\alpha_2 = \frac{1}{\delta^2 m\theta\varpi}$ and

$$\Omega_2 = \{\psi \in C_{\varpi} : |\psi_k|_0 < \alpha_2 \psi_k^*,\ k = 1, \dots, N\}.$$

If $\psi \in \Omega \cap \partial\Omega_2$, then $\delta\left|\psi_k\right|_0 \leq \psi_k(\tau) \leq \left|\psi_k\right|_0 = \alpha_2\psi_k^*$, $k = 1, \ldots, N$, and consequently

$$
\begin{aligned}
\left(\Phi_k\psi\right)(\tau) \geq A_k \int_0^{\varpi} \psi_k(\iota) & \left[\sum_{i=1}^N a_{ki}(\iota)\, h_i\left(\psi_i(\iota)\right) + \sum_{i=1}^N b_{ki}(\iota)\, f_i\left(\psi_i\left(\iota - r_{ki}(\iota)\right)\right)\right. \\
& \left. + \sum_{i=1}^N c_{ki}(\iota) \int_{-\infty}^{\iota} D_{ki}(\iota, \varsigma)\, g_i\left(\psi_i(\varsigma)\right) \Delta\varsigma\right] \Delta\iota + A_k \sum_{0 \leq \tau_l \leq \varpi} I_{kl}\left(\tau_l, \psi_k\left(\tau_l\right)\right) \\
& \geq \delta^2 A_k \left|\psi_k\right|_0 \sum_{i=1}^N \int_0^{\varpi} a_{ki}(\iota)\, T_i \left[\min_{1 \leq i \leq N}\left(\frac{\overline{T}_i}{T_i}\right)\right] \left|\psi_i\right|_0 \Delta\iota \\
& + \delta^2 A_k \left|\psi_k\right|_0 \sum_{i=1}^N \int_0^{\varpi} F_i \left[\min_{1 \leq i \leq N}\left(\frac{\overline{F}_i}{F_i}\right)\right] b_{ki}(\iota) \left|\psi_i\right|_0 \Delta\iota \\
& + \delta^2 A_k \left|\psi_k\right|_0 \sum_{i=1}^N \int_0^{\varpi} E_{ki}(\iota)\, R_i \left[\min_{1 \leq k \leq N}\left(\frac{\overline{E}_{ki}}{E_{ki}}\right)\right] \frac{\overline{R}_i}{R_i} c_{ki}(\iota) \left|\psi_i\right|_0 \Delta\iota \\
& + \delta^2 A_k \left|\psi_k\right|_0 \sum_{l=1}^q \lambda_{kl}\left(\tau_l\right) \left[\min_{1 \leq k \leq N}\left(\frac{\overline{\lambda}_{kl}\left(\tau_l\right)}{\lambda_{kl}\left(\tau_l\right)}\right)\right] \left|\psi_k\left(\tau_l\right)\right|_0 \\
& \geq \theta_1\delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \sum_{i=1}^N \widehat{a}_{ki}\psi_i^* + \theta_2\delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \sum_{i=1}^N \widehat{b}_{ki}\psi_i^* \\
& + \theta_3\delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \sum_{i=1}^N \widehat{c}_{ki}\psi_i^* + \theta_4\delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \widehat{\beta}_{kl}\psi_i^* \\
& \geq \theta\delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \sum_{i=1}^N \widehat{a}_{ki}\psi_i^* + \theta\delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \sum_{i=1}^N \widehat{b}_{ki}\psi_i^* \\
& + \theta\delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \sum_{i=1}^N \widehat{c}_{ki}\psi_i^* + \theta \times \delta^2 A_k\alpha_2\varpi \left|\psi_k\right|_0 \widehat{\beta}_{kl}\psi_i^* \\
& = \alpha A\varpi\delta^2\alpha_2 \left|\psi_k\right|_0 \left(\sum_{i=1}^N \left(\widehat{a}_{ki} + \widehat{b}_{ki} + \widehat{c}_{ki}\right)\psi_i^* + \widehat{\beta}_{ki}\psi_i^*\right) \\
& = \left(A_k\widehat{\upsilon}_k\right)\theta\varpi\delta^2\alpha_2 \left|\psi_k\right|_0 \geq \alpha_2\theta m_0\varpi\delta^2 \left|\psi_k\right|_0 \geq \left|\psi_k\right|_0, \ k = 1, \ldots, N.
\end{aligned}
$$

and thus

$$
\|\Phi\psi\| = \sum_{k=1}^N \left|\left(\Phi_k\psi_k\right)\right|_0 \geq \sum_{i=1}^N \left|\psi_k\right|_0 = \|\psi\|, \ \forall\psi \in \Omega \cap \partial\Omega_2.
$$

Clearly, Ω_1 and Ω_2 are bounded open subsets of C_ϖ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. So, the mapping $\Phi : \Omega \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \rightarrow \Omega$ is completely continuous and satisfies the first condition in Theorem 4. By applying Krasnoselskii's theorem, there exists a fixed point $\psi \in \Omega \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ such that $\psi = \Phi\psi$. Hence, the system (1.2) admits a positive ϖ-periodic solution. □

3.2. **The case of sublinear impulse functions.** In this case, we demonstrate similar findings for (1.2).

Theorem 6. *Suppose that the assumption* (A1) *holds and*

(*A4*) *There exist positive functions* $\overline{\zeta}_{kl}, \zeta_{kl} \in C_{rd}(\mathbb{S}, \mathbb{R}_+)$ *such that for all* $\psi \in \mathbb{R}^+$, $\tau \in \mathbb{S}$, *we have*

$$\overline{\zeta}_{kl}(\tau)\psi \le I_{kl}(\tau,\psi) \le \zeta_{kl}(\tau)\psi,\ k=1,\dots,N,\ l \in \mathbb{N}^*.$$

(*A5*) *The linear system*

$$\sum_{i=1}^{N}\left(\widehat{a}_{ki}+\widehat{b}_{ki}+\widehat{c}_{ki}\right)\psi_i = \widehat{\upsilon}_k,\ k=1,\dots,N,\ l\in\mathbb{N}^*, \tag{3.18}$$

admits a unique positive solution. Then, the system (1.2) admits a positive ϖ*-periodic solution.*

Proof. By a similar proof of Lemma 4, we can easily demonstrate that the mapping $\Phi : \Omega \to \Omega$ is completely continuous. We only need to demonstrate the first condition in Theorem 4. Let

$$\psi^* = (\psi_1^*, \psi_2^*, \dots, \psi_N^*)^T,$$

with $\psi_k^* > 0$, $k = 1, ..., N$, be a solution of (3.18). We set $\overline{\theta} = \min(\theta_1, \theta_2, \theta_3, \theta_4)$ where

$$\theta_5 = \theta_4 = \min_{1\le l\le q}\left[\min_{1\le k\le N}\left(\frac{\overline{\zeta}_{kl}(\tau_l)}{\zeta_{kl}(\tau_l)}\right)\right],$$

and θ_1, θ_2, θ_3 are given in (3.16). We also set

$$\widehat{\gamma}_{kl} = \frac{1}{\varpi}\sum_{l=1}^{q}\zeta_{kl}(\tau_l),\ (\zeta_{kl}(\tau_l)\neq 0),$$

and

$$\begin{aligned}
\widetilde{m}_0 &= \min\left\{\min_{1\le k\le N}\{\widehat{\upsilon}_k A_k\},\ \min_{1\le k\le N}\{A_k\widehat{\gamma}_{kl}\}\right\},\\
\widetilde{M}_0 &= \min\left\{\min_{1\le k\le N}\{\widehat{\upsilon}_k B_k\},\ \min_{1\le k\le N}\{B_k\widehat{\gamma}_{kl}\}\right\}.
\end{aligned}$$

Choose a positive constant $\widetilde{M} \ge \widetilde{M}_0$ such that $0 < \frac{1-\widetilde{M}\varpi}{\widetilde{M}\varpi} < 1$ where $0 < \widetilde{M}\varpi < 1$. Let $\eta_1 = \frac{1-\widetilde{M}\varpi}{\widetilde{M}\varpi}$ and

$$\widehat{\Omega}_1 = \left\{\psi = (\psi_1, \psi_2, ..., \psi_N)^T \in C_\varpi : |\psi_k|_0 < \eta_1\psi_k^*,\ k=1,\dots,N\right\}.$$

If $\psi \in \Omega \cap \partial\widehat{\Omega}_1$, then

$$\delta|\psi_k|_0 \le \psi_k(\tau) \le |\psi_k|_0 = \eta_1\psi_k^*,\ k=1,\dots,N.$$

and

$$
\begin{aligned}
\left(\Phi_k\psi\right)(\tau) \leq B_k \int_0^{\varpi} \Bigg[& \psi_k(\iota)\sum_{i=1}^{N} a_{ki}(\iota)\, h_i(\psi_i(\iota)) + \psi_k(\iota)\sum_{i=1}^{N} b_{ki}(\iota)\, f_i(\psi_i(\iota - r_i(\iota))) \\
& + \psi_k(\iota)\sum_{i=1}^{N} c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma)\, g_i(\psi_i(\varsigma))\,\Delta\varsigma \Bigg] \Delta\iota + B_k \sum_{0\leq\tau_l\leq\varpi} I_{kl}\left(\tau_l, \psi_k(\tau_l)\right) \\
& \leq B_k \int_0^{\varpi} |\psi_k|_0 \sum_{i=1}^{N} T_i a_{ki}(\iota)\, |\psi_i|_0\, \Delta\iota + B_k \int_0^{\varpi} |\psi_k|_0 \sum_{i=1}^{N} F_i b_{ki}(\iota)\, |\psi_i|_0\, \Delta\iota \\
& + B_k \int_0^{\varpi} |\psi_k|_0 \sum_{i=1}^{N} R_i E_{ki} c_{ki}(\iota)\, |\psi_i|_0\, d\iota + B_k \sum_{0\leq\tau_l<\varpi} \zeta_k(\tau_l)\, |\psi_k|_0 \\
& \leq \eta_1 B_k \varpi\, |\psi_k|_0 \sum_{i=1}^{N} \widehat{a}_{ki}\psi_i^* + \eta_1 B_k \varpi\, |\psi_k|_0 \sum_{i=1}^{N} \widehat{b}_{ki}\psi_i^* \\
& + \eta_1 B_k \varpi\, |\psi_k|_0 \sum_{i=1}^{N} \widehat{c}_{ki}\psi_i^* + \varpi\left(B_k \widehat{\gamma}_{kl}\right)|\psi_k|_0 \\
& = \alpha\eta_1 B_k \varpi\, |\psi_k|_0 \left[\sum_{i=1}^{N}\left(\widehat{a}_{ki} + \widehat{b}_{ki} + \widehat{c}_{ki}\right)\psi_i^*\right] + \varpi\left(B_k \widehat{\gamma}_{kl}\right)|\psi_k|_0 \\
& = \varpi\left(B_k \widehat{\upsilon}_k\right)\eta_1 \varpi\, |\psi_k|_0 + \varpi\left(B_k \widehat{\gamma}_{kl}\right)|\psi_k|_0 \\
& \leq \eta_1 \widetilde{M}_0 \varpi\, |\psi_k|_0 + \widetilde{M}_0 \varpi\, |\psi_k|_0 \leq |\psi_k|_0,\ k = 1, \ldots, N.
\end{aligned}
$$

Hence for any $\psi \in \Omega \cap \partial\widehat{\Omega}_1$,

$$
\|\Phi\psi\| = \sum_{k=1}^{N} |\Phi_k \psi_k|_0 \leq \sum_{i=1}^{N} |\psi_k|_0 = \|\psi\|.
$$

Choose a positive constant $\widetilde{m} \leq \widetilde{m}_0$ such that $\frac{1-\delta\widetilde{m}\theta\varpi}{\delta^2\widetilde{m}\theta\varpi} > 1$ where $0 < \delta\widetilde{m}\theta\varpi < 1$. Let $\eta_2 = \frac{1-\delta\widetilde{m\theta\varpi}}{\delta^2\widetilde{m}\theta\varpi}$ and

$$
\widehat{\Omega}_2 = \left\{\psi \in C_{\varpi} : |\psi_k|_0 < \eta_2 \psi_k^*,\ k = 1, \ldots, N\right\}.
$$

If $\psi \in \Omega \cap \partial\widehat{\Omega}_2$, then

$$
\delta\, |\psi_k|_0 \leq \psi_k(\tau) \leq |\psi_k|_0 = \eta_2 \psi_k^*,\ k = 1, \ldots, N.
$$

and consequently

$$
\begin{aligned}
(\Phi_k\psi)(\tau) \geq A_k \int_0^{\varpi} \Bigg[& \psi_i(\iota)\sum_{i=1}^{N} a_{ki}(\iota) h_i(\psi_i(\iota)) + \psi_i(\iota)\sum_{i=1}^{N} b_{ki}(\iota) f_i(\psi_i(\iota - r_i(\iota))) \\
& +\psi_i(\iota)\sum_{i=1}^{N} c_{ki}(\iota)\int_{-\infty}^{\iota} D_{ki}(\iota,\varsigma) g_i(\psi_i(\varsigma))\Delta\varsigma \Bigg] \Delta\iota \\
& + \sum_{0\leq\tau_l\leq\varpi} G_k(\tau,\tau_l) I_{kl}(\tau_l,\psi_k(\tau_l)) \\
& \geq \delta^2 A_k |\psi_k|_0 \sum_{i=1}^{N}\int_0^{\varpi} a_{ki}(\iota) T_i \left[\min_{1\leq i\leq N}\left(\frac{\overline{T}_i}{T_i}\right)\right] |\psi_i|_0 \Delta\iota \\
& + \delta^2 A_k |\psi_k|_0 \sum_{i=1}^{N}\int_0^{\varpi} F_i \left[\min_{1\leq i\leq N}\left(\frac{\overline{F}_i}{F_i}\right)\right] b_{ki}(\iota) |\psi_i|_0 \Delta\iota \\
& + \delta^2 A_k |\psi_k|_0 \sum_{i=1}^{N}\int_0^{\varpi} E_{ki}(\iota) R_i \left[\min_{1\leq k\leq N}\left(\frac{\overline{E}_{ki}}{E_{ki}}\right)\right] \frac{\overline{R}_i}{R_i} c_{ki}(\iota) |\psi_i|_0 \Delta\iota \\
& + \delta^2 A_k |\psi_k|_0 \sum_{l=1}^{q} \lambda_{kl}(\tau_l) \left[\min_{1\leq i\leq N}\left(\frac{\overline{\lambda}_{kl}(\tau_l)}{\lambda_{kl}(\tau_l)}\right)\right] |\psi_k(\tau_l)|_0 \\
& \geq \theta_1\delta^2 A_k\eta_2\varpi |\psi_k|_0 \sum_{i=1}^{N}\widehat{a}_{ki}\psi_i^* + \theta_2\delta^2 A_k\eta_2\varpi |\psi_k|_0 \sum_{i=1}^{N}\widehat{b}_{ki}\psi_i^* \\
& + \theta_3\delta^2 A_k\eta_2\varpi |\psi_k|_0 \sum_{i=1}^{N}\widehat{c}_{ki}\psi_i^* + \theta_4\delta^2 A_k\eta_2\varpi |\psi_k|_0 \widehat{\gamma}_{kl}\psi_i^* \\
& \geq \overline{\theta}\delta^2 A_k\eta_2\varpi |\psi_k|_0 \sum_{i=1}^{N}\widehat{a}_{ki}\psi_i^* + \overline{\theta}\delta^2 A_k\eta_2\varpi |\psi_k|_0 \sum_{i=1}^{N}\widehat{b}_{ki}\psi_i^* \\
& + \overline{\theta}\times\delta^2 A_k\eta_2\varpi |\psi_k|_0 \sum_{i=1}^{N}\widehat{c}_{ki}\psi_i^* + \overline{\theta}\times\delta A_k\eta_2\varpi |\psi_k|_0 \widehat{\gamma}_{kl} \\
& = \overline{\theta} A\varpi\delta^2\eta_2 |\psi_k|_0 \sum_{i=1}^{N}\left(\widehat{a}_{ki} + \widehat{b}_{ki} + \widehat{c}_{ki}\right)\psi_i^* + \overline{\theta}\delta A_k\eta_2\varpi |\psi_k|_0 \widehat{\gamma}_{kl} \\
& = (A_k\widehat{\upsilon}_k)\theta\varpi\delta^2\alpha_2 |\psi_k|_0 + (A_k\widehat{\gamma}_{kl})\overline{\theta}\varpi |\psi_k|_0 \delta\eta \\
& \geq \eta_2\overline{\theta}\widetilde{m}_0\varpi\delta^2 |\psi_k|_0 + \overline{\theta}\widetilde{m}_0\varpi\delta |\psi_k|_0 \geq |\psi_k|_0,\ k = 1,\ldots,N.
\end{aligned}
$$

and therefore

$$\|\Phi\psi\| = \sum_{k=1}^{N} |(\Phi_k\psi_k)|_0 \geq \sum_{i=1}^{N} |\psi_k|_0 = \|\psi\|,\ \forall\psi\in\Omega\cap\partial\widehat{\Omega}_2.$$

Hence, the mapping $\Phi : \Omega\cap\left(\overline{\widehat{\Omega}}_2\backslash\widehat{\Omega}_1\right) \to \Omega$ is completely continuous and satisfies the first condition in Theorem 4. By applying Krasnoselskii's theorem, there exists a fixed point $\psi\in\Omega\cap\left(\overline{\widehat{\Omega}}_2\backslash\widehat{\Omega}_1\right)$ such that $\psi = \Phi\psi$. Therefore, the system (1.2) admits a positive ϖ-periodic solution. The proof is completed. □

4. An example

In this section, we provide an example to demonstrate the reliability of our findings.

Example 1. *Let* $\mathbb{S} = \frac{1}{4}\mathbb{Z}$ *be a* $\frac{1}{4}$*-periodic time scale. We consider the following dynamic system*

$$\begin{aligned}\psi_k^{\Delta}(\tau) &= \psi_k(\tau)\Bigg[\upsilon_k(\tau) - \sum_{i=1}^{N} b_{ki}(\tau) f_i(\psi_i(\tau - r_i(\tau))) \\ &\quad - \sum_{i=1}^{N} c_{ki}(\tau)\int_{-\infty}^{\tau} D_{ki}(\tau,\varsigma) g_i(\psi_i(\varsigma))\Delta\varsigma\Bigg], \ \tau \neq \tau_l = 2l,\ l \in \mathbb{N}^*,\ \tau \in \mathbb{S}, \\ \psi_k(\tau_l^+) - \psi_k(\tau_l^-) &= -\frac{I_{kl}(\tau_l, \psi_k(\tau_l))}{1 + \mu(\tau_l)\upsilon_k(\tau_l)},\ \tau = \tau_l,\ l \in \mathbb{N}^*, \end{aligned} \tag{4.1}$$

for $k = 1,2,3$. *This system corresponds to (1.2) when* $N = 3$ *and* $\varpi = 8$. *Let*

$$\upsilon_1(\tau) = \frac{4}{3},\ \upsilon_2(\tau) = \frac{4}{5},\ \upsilon_3(\tau) = \frac{4}{7},$$

and the arbitrary positive functions $r_i \in C_{rd}(\mathbb{S},\mathbb{S})$ *satisfy* $r_i(\tau+\varpi) = r_i(\tau)$, $k = 1,2,3$. *Then, we get*

$$\overline{\upsilon}_1 = \frac{4}{3},\ \overline{\upsilon}_2 = \frac{4}{5},\ \overline{\upsilon}_3 = \frac{4}{7}.$$

It is easy to check that $A_k \leq G_k(\tau,\iota) \leq B_k$, *for* $k = 1,2,3$ *where*

$$G_1(\tau,\iota) = \frac{\left(\frac{4}{3}\right)^{4\left(\tau-\iota-\frac{1}{4}\right)}}{1-\left(\frac{4}{3}\right)^{32}},\ G_2(\tau,\iota) = \frac{\left(\frac{6}{5}\right)^{4\left(\tau-\iota-\frac{1}{4}\right)}}{1-\left(\frac{6}{5}\right)^{32}},\ G_3(\tau,\iota) = \frac{\left(\frac{8}{7}\right)^{4\left(\tau-\iota-\frac{1}{4}\right)}}{1-\left(\frac{8}{7}\right)^{32}},$$

and

$$\begin{aligned} A_1 &= \frac{\left(\frac{4}{3}\right)^{32}}{1-\left(\frac{4}{3}\right)^{32}},\ A_2 = \frac{\left(\frac{4}{5}\right)^{32}}{1-\left(\frac{4}{5}\right)^{32}},\ A_3 = \frac{\left(\frac{4}{7}\right)^{32}}{1-\left(\frac{4}{7}\right)^{32}}, \\ B_1 &= \frac{1}{1-\left(\frac{4}{3}\right)^{32}},\ B_2 = \frac{1}{1-\left(\frac{4}{5}\right)^{32}},\ B_3 = \frac{1}{1-\left(\frac{4}{7}\right)^{32}}. \end{aligned}$$

Let

$$\begin{aligned} f_1(\psi) &= \psi(\cos(\psi)+2)^2,\ g_1(\psi) = (\sin(\psi)+2)\frac{\psi}{2}, \\ f_2(\psi) &= \psi\left|\cos(\psi)+3\right|,\ g_2(\psi) = \psi\exp(\sin 2\psi), \\ f_3(\psi) &= \psi\left|\ln(\sin(\psi)+3)\right|,\ g_3(\psi) = \sqrt{\psi^2(3+\sin(2\psi))}, \\ I_1(\tau,\psi) &= \left(\sin\left(\frac{\pi}{4}\tau\right)+2\right)\left(\left|\cos\left(\frac{\pi}{4}\psi\right)\right|+2\right)\psi^2, \\ I_2(\tau,\psi) &= 3\psi^2\left(2\frac{4}{\pi}\left|\sin\left(\frac{\pi}{4}\tau\right)\right|+3\right), \\ I_3(\tau,\psi) &= \psi^2\left(3\frac{4}{\pi}\left|\sin\left(\frac{\pi}{4}\tau\right)\right|+2\right). \end{aligned}$$

Since $|\sin\psi| \leq 1$ *and* $|\cos\psi| \leq 1$*, then for* $\psi \in \mathbb{R}$*, we obtain*

$$\begin{aligned}
\overline{T}_i\psi &\leq f_i(\psi) \leq T_i\psi, \text{ for } i = 1,2,3 \\
\overline{R}_i\psi &\leq g_i(\psi) \leq R_i\psi, \text{ for } i = 1,2,3 \\
\overline{\lambda}_i(\tau)\psi^2 &\leq I_k(\tau,\psi) \leq \lambda_i(\tau)\psi^2, \text{ for } k = 1,2,3 \\
\overline{E}_{ki} &\leq \int_{-\infty}^{\tau} D_{ki}(\tau,\iota)\Delta\iota \leq E_{ki}, \text{ for } k,i = 1,2,3.
\end{aligned}$$

where

$$\begin{aligned}
\overline{T}_1 &= 1,\ T_1 = 9,\ \overline{T}_2 = 2,\ T_2 = 4,\ \overline{T}_3 = \sqrt{2},\ T_3 = 2,\ \overline{R}_1 = \frac{1}{2}, \\
R_1 &= \frac{3}{2},\ \overline{R}_2 = \exp(-1),\ R_2 = \exp(1),\ \overline{R}_3 = \exp(-1),\ R_3 = \exp(1),
\end{aligned}$$

and

$$\begin{aligned}
\overline{\lambda}_1(\tau) &= 2\left(\sin\left(\frac{\pi}{4}\tau\right)+2\right),\ \lambda_1(\tau) = 3\left(\sin\left(\frac{\pi}{4}\tau\right)+2\right), \\
\overline{\lambda}_2(\tau) &= 9,\ \lambda_2(\tau) = 3(2|\tau|+3),\ \overline{\lambda}_3(\tau) = 2,\ \lambda_3(\tau) = 3|\tau|+2.
\end{aligned}$$

Let

$$\begin{aligned}
b_{11}(\tau) &= \frac{\cos\left(\frac{\pi}{4}\tau\right)+3}{\frac{1}{4}T_1},\ b_{12}(\tau) = \frac{\left|\sin\left(\frac{\pi}{4}\tau\right)+3\right|}{\frac{1}{4}T_2},\ b_{13}(\tau) = \frac{\left|\sin\left(\frac{\pi}{4}\tau\right)+1\right|}{T_3}, \\
b_{21}(\tau) &= \frac{9\left(\cos\left(\frac{\pi}{4}\tau\right)+2\right)}{\frac{1}{4}T_1},\ b_{22}(\tau) = \frac{2\sin\left(\frac{\pi}{4}\tau\right)+2}{\frac{1}{4}T_2},\ b_{23}(\tau) = \frac{1}{T_3}, \\
b_{31}(\tau) &= \frac{\left|\left(\cos\left(\frac{\pi}{4}\tau\right)+3\right)\right|}{\frac{1}{4}T_1},\ b_{32}(\tau) = 0,\ b_{33}(\tau) = \frac{3}{T_3},
\end{aligned}$$

and

$$\begin{aligned}
c_{11}(\tau) &= \frac{1+\sin\left(\frac{\pi}{4}\tau\right)}{R_1},\ c_{12}(\tau) = \frac{4}{R_2},\ c_{13}(\tau) = 4+\sin\left(\frac{\pi}{4}\tau\right) \\
c_{21}(\tau) &= \frac{4+\cos\left(\frac{\pi}{4}\tau\right)}{\frac{1}{4}R_1},\ c_{22}(\tau) = 4\left(2+\sin\left(\frac{\pi}{4}\tau\right)\right),\ c_{23}(\tau) = \frac{2}{R_3} \\
c_{31}(\tau) &= 0,\ c_{32}(\tau) = \frac{1+\cos\left(\frac{\pi}{4}\tau\right)}{\frac{1}{4}},\ c_{33}(\tau) = \frac{4}{R_3}.
\end{aligned}$$

Then

$$\begin{aligned}
\overline{b}_{11} &= 12,\ \overline{b}_{12} = 12,\ \overline{b}_{13} = 1, \\
\overline{b}_{21} &= 72,\ \overline{b}_{22} = 8,\ \overline{b}_{23} = 4, \\
\overline{b}_{31} &= 12,\ \overline{b}_{32} = 0,\ \overline{b}_{33} = 12,
\end{aligned}$$

and

$$\begin{aligned}
\overline{c}_{11} &= 3,\ \overline{c}_{12} = 48,\ \overline{c}_{13} = 8, \\
\overline{c}_{21} &= 40,\ \overline{c}_{22} = 8,\ \overline{c}_{23} = 24, \\
\overline{c}_{31} &= 0,\ \overline{c}_{32} = 48,\ \overline{c}_{33} = 112.
\end{aligned}$$

Let

$$
\begin{aligned}
D_{11}(\tau,\iota) &= -\exp\left(\iota-\tau+\frac{1}{4}\right)\left(\left(4e^{-\frac{1}{4}}-4\right)\left(\sin\left(\frac{\pi}{4}\tau\right)+2\right)\right),\\
D_{12}(\tau,\iota) &= -\exp\left(4\iota-4\tau+1\right)\left(\left(4e^{-1}-4\right)\left(\cos\left(\frac{\pi}{4}\tau\right)+2\right)\right),\\
D_{13}(\tau,\iota) &= -\exp\left(8\iota-8\tau+2\right)\left(\left(4e^{-2}-4\right)\frac{1}{\exp(1)}\left(\left|\cos\left(\frac{\pi}{4}\tau\right)\right|+1\right)\right),\\
D_{21}(\tau,\iota) &= -\exp\left(\iota-\tau+\frac{1}{4}\right)\left(\left(2e^{-\frac{1}{4}}-2\right)\left(2\cos\left(\frac{\pi}{4}\tau\right)+3\right)\right),\\
D_{22}(\tau,\iota) &= -\exp\left(\iota-\tau+\frac{1}{4}\right)\left(\left(4e^{-\frac{1}{4}}-4\right)\frac{1}{e\ln(4)}\left(\ln\left(\left|\cos\left(\frac{\pi}{4}\tau\right)\right|+2\right)\right)\right),\\
D_{23}(\tau,\iota) &= -\exp\left(\iota-\tau+\frac{1}{4}\right)\left(\left(4e^{-\frac{1}{4}}-4\right)\left(\left|\sin\left(\frac{\pi}{4}\tau\right)\right|+2\right)\right),\\
D_{31}(\tau,\iota) &= -\exp\left(2\iota-2\tau+\frac{1}{2}\right)\left(4e^{-\frac{1}{2}}-4\right)\exp\left(\sin\left(\frac{\pi}{4}\tau\right)+2\right),\\
D_{32}(\tau,\iota) &= -2\exp\left(2\iota-2\tau+\frac{1}{2}\right)\left(\left(4e^{-\frac{1}{2}}-4\right)\frac{1}{\exp(1)}\left(2\sin\left(\frac{\pi}{4}\tau\right)+4\right)\right),\\
D_{33}(\tau,\iota) &= -\exp\left(2\iota-2\tau+\frac{1}{2}\right)\left(\left(4e^{-\frac{1}{2}}-4\right)\left(\sin\left(\frac{\pi}{4}\tau\right)+2\right)\right),
\end{aligned}
$$

then, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\tau} D_{11}(\tau,\iota)\,\Delta\iota = \sin\left(\frac{\pi}{4}\tau\right)+2,\ \int_{-\infty}^{\tau} D_{12}(\tau,\iota)\,\Delta\iota = \cos\left(\frac{\pi}{4}\tau\right)+2,\\
&\int_{-\infty}^{\tau} D_{13}(\tau,\iota)\,\Delta\iota = e^{-1}\left(\left|\cos\left(\frac{\pi}{4}\tau\right)\right|+1\right),\ \int_{-\infty}^{\tau} D_{21}(\tau,\iota)\,\Delta\iota = \cos\left(\frac{\pi}{4}\tau\right)+\frac{3}{2},\\
&\int_{-\infty}^{\tau} D_{22}(\tau,\iota)\,\Delta\iota = \frac{e^{-1}}{\ln 4}\ln\left(\cos\left(\frac{\pi}{4}\tau\right)+3\right),\ \int_{-\infty}^{\tau} D_{23}(\tau,\iota)\,\Delta\iota = \left|\sin\left(\frac{\pi}{4}\tau\right)\right|+2,\\
&\int_{-\infty}^{\tau} D_{31}(\tau,\iota)\,\Delta\iota = e^{\sin\left(\frac{\pi}{4}\tau\right)+2},\ \int_{-\infty}^{\tau} D_{32}(\tau,\iota)\,\Delta\iota = 4e^{-1}\left(\sin\left(\frac{\pi}{4}\tau\right)+2\right),\\
&\int_{-\infty}^{\tau} D_{33}(\tau,\iota)\,\Delta\iota = 5\left|\sin\left(\frac{\pi}{4}\tau\right)\right|+2,
\end{aligned}
$$

hence

$$
\begin{aligned}
\overline{E}_{11} &= 1,\ E_{11}=3,\ \overline{E}_{12}=1,\ E_{12}=3,\ \overline{E}_{13}=e^{-1},\ E_{13}=2e^{-1},\\
\overline{E}_{21} &= \frac{1}{2},\ E_{21}=\frac{5}{2},\ \overline{E}_{22}=\frac{\ln(2)\,e^{-1}}{\ln(4)},\ E_{22}=e^{-1},\ \overline{E}_{23}=2,\ E_{23}=3,\\
\overline{E}_{31} &= e,\ E_{31}=e^{3},\ \overline{E}_{32}=4e^{-1},\ E_{32}=12e^{-1},\ \overline{E}_{33}=2,\ E_{33}=7.
\end{aligned}
$$

For $q=3$, we have $\tau_{l+q+1}=\tau_l+\varpi$, $I_{k(l+q+1)}(\tau_{l+q+1},\psi_k(\tau_{l+q+1}))=I_{kl}(\tau_l,\psi_k(\tau_l))$, $k=1,2,3$. Then, for $\tau_1=2$, $\tau_2=4$ and $\tau_3=6$, we get

$$
\widehat{\beta}_{1l}=\frac{9}{4},\ \widehat{\beta}_{2l}=\frac{99}{8},\ \widehat{\beta}_{3l}=54.
$$

Furthermore, it is easy to prove that the associated system of (4.1)

$$\begin{cases} \sum_{i=1}^{3}\left(\widehat{b}_{1i}+\widehat{c}_{1i}\right)\psi_i+\widehat{\beta}_{1l}\psi_1=\overline{\upsilon}_1, \\ \sum_{i=1}^{3}\left(\widehat{b}_{2i}+\widehat{c}_{2i}\right)\psi_i+\widehat{\beta}_{2l}\psi_2=\overline{\upsilon}_2, \\ \sum_{i=1}^{3}\left(\widehat{b}_{3i}+\widehat{c}_{3i}\right)\psi_i+\widehat{\beta}_{3l}\psi_3=\overline{\upsilon}_3, \end{cases}$$

admits a unique positive solution $\psi=(\psi_1,\psi_2,\psi_3)=\left(\frac{511292}{51808275},\frac{40024}{2072331},\frac{8219}{17269425}\right)$*. Hence, the assumptions of Theorem 5 are satisfied. So, the system (4.1) admits a positive* 8*-periodic solution.*

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