

# The negative degree $q$ -Bernstein bases and the multirational $q$ -blossom

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## Abstract

We investigate algebraic properties of the negative degree  $q$ -Bernstein bases. Our fundamental tool in this investigation is a recently introduced variant of the blossom, the multirational  $q$ -blossom, which provides the dual functionals for the negative degree  $q$ -Bernstein basis functions. By applying the dual functional property of the multirational  $q$ -blossom, we are readily able to generate several fundamental identities involving the negative degree  $q$ -Bernstein bases, including a new variant of Marsden's identity, a partition of unity property, a reparametrization formula, and a formula for representing monomials. We also show how to use the homogeneous variant of the multirational  $q$ -blossom to convert between the  $q$ -Taylor bases and the negative degree  $q$ -Bernstein bases.

*Keywords:* negative degree  $q$ -Bernstein bases; multirational  $q$ -blossom; divided difference;  $q$ -Marsden identity; homogeneous multirational  $q$ -blossom

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## 1 Introduction

### 1.1 Bernstein bases

The Bernstein basis functions for degree  $n \geq 0$  on the interval  $[0, 1]$  are defined by

$$B_k^n(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, \dots, n. \quad (1.1)$$

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These basis functions were introduced by Bernstein and used to provide a constructive proof of the Weierstrass approximation theorem for uniform polynomial approximation of continuous functions on the interval  $[0, 1]$  [6]. Along with roles in different classical fields such as approximation theory, operator theory, and probability theory, the Bernstein basis functions today play a central role in the construction and analysis of Bézier curves and surfaces which are essential to a wide variety of applications in Computer Graphics, Geometric Modeling, and Computer Aided Geometric Design (CAGD) [8].

The Bernstein basis functions (1.1) are the terms appearing in the binomial expansion

$$((1-t) + t)^n = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k}.$$

Many properties of the Bernstein basis functions can be derived directly from this expansion. Since the binomial theorem is valid for negative integer exponents where

$$((1-t) + t)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} t^k (1-t)^{-n-k},$$

it is natural to consider the Bernstein basis functions of negative degree  $-n \leq 0$  [11]:

$$B_k^{-n}(t) = \binom{-n}{k} t^k (1-t)^{-n-k}, \quad k = 0, 1, \dots \quad (1.2)$$

Most of the standard identities and properties of the Bernstein basis functions of positive degree extend naturally to their negative degree counterparts. A comprehensive analysis of the negative degree Bernstein bases in the context of CAGD and the associated blossoming theory are provided in [11, 12]. These functions are known in Approximation Theory as the Baskakov basis functions since they were first introduced by V.A. Baskakov in [5].

## 1.2 Blossoming

Blossoming is one of the fundamental tools for studying Bézier and B-spline curves and surfaces [7, 13, 28–30]. Many important identities and algorithms such as Marsden’s identity, evaluation, subdivision, and knot insertion algorithms can all be derived from blossoming [13].

The multiaffine blossom of a polynomial  $p(t)$  of degree  $n$  is the unique, symmetric, multiaffine polynomial  $p(u_1, \dots, u_n)$  that reduces to  $p(t)$  along the diagonal, i.e.,  $p(t, \dots, t) = p(t)$  [13]. This multiaffine blossom is equivalent to the classical polar form [30]. The blossom evaluated at the endpoints of the parameter interval provides the dual functionals for the Bernstein basis functions of positive degree.

A new kind of blossom called *the multirational blossom* is introduced in [11], associated with Bernstein basis functions of negative degree. The multirational blossom of degree  $-n \leq -1$  and order  $k \geq 0$  of a continuous function  $f(t)$  is the multivariate function  $f(u_1, \dots, u_k/v_1, \dots, v_{n+k})$  uniquely characterized by the following four axioms: bisymmetry in the  $u$  and  $v$  parameters, multiaffine in the  $u$  parameters, satisfies a cancellation property, and reduces to  $f(t)$  along the diagonal [11, 12, 35]. The multirational blossom evaluated at the endpoints 0, 1 provides the dual functionals for the negative degree Bernstein basis functions.

## 1.3 Motivation

In addition to the classical Bernstein bases (1.1), there are two quantum versions of Bernstein bases of positive degree: the  $q$ -Bernstein bases and the  $h$ -Bernstein bases. The  $h$ -Bernstein bases

were introduced in Approximation Theory by Stancu [33, 34] as a generalization of the classical Bernstein bases and later investigated by Goldman [10], and Goldman and Barry [14, 15] in the context of Probability Theory and CAGD. The  $q$ -Bernstein bases were introduced and studied by Phillips [24–27] and Phillips and Oruç [22, 23] for the interval  $[0, 1]$  and extended to arbitrary parameter intervals  $[a, b]$  by Lewanowicz and Woźny [21]. The  $h$ -Bernstein bases reduce to the classical Bernstein bases when  $h = 0$ , and the  $q$ -Bernstein bases reduce to the classical Bernstein bases when  $q = 1$ .

Recently new properties and algorithms for these quantum Bernstein bases and associated quantum Bézier curves have been derived by introducing quantum versions of the multiaffine blossom [31, 32]. The  $q$ -blossom and the  $h$ -blossom are much the same as the multiaffine blossom but with the diagonal property replaced by the  $q$ -diagonal property  $p(t, qt, \dots, q^{n-1}t; q) = p(t)$  or by the  $h$ -diagonal property  $p(t, t - h, \dots, t - (n - 1)h; h) = p(t)$ . These two quantum blossoming theories have been unified into a single theory by introducing and investigating  $(q, h)$ -Bernstein bases and  $(q, h)$ -Bézier curves [17].

As in the case of Bernstein bases of positive degree, the Bernstein bases of negative degree have also been generalized to quantum version as the  $q$ -Baskakov bases. Approximation properties of the corresponding  $q$ -Baskakov operators are investigated in [2]. Another type of  $q$ -Baskakov operator more suitable for studying the  $q$ -derivatives and their applications is proposed in [3]. Many properties of these  $q$ -Baskakov bases and  $q$ -Baskakov operators have been investigated in the context of Approximation Theory [3, 9, 18, 37]. Moreover using the  $q$ -analogue of the Baskakov operators defined in [2], geometric and analytic properties of these  $q$ -Baskakov bases, as well as the corresponding  $q$ -Baskakov curves and  $q$ -Baskakov surfaces have been studied in [38].

Recently axioms for a new kind of multirational blossom, *the multirational  $q$ -blossom*, along with an explicit formula for this  $q$ -blossom has been presented in [36]. The homogeneous variant of this multirational  $q$ -blossom can be used to compute  $q$ -derivatives of continuous functions.

*The purpose of this paper is to show that the multirational  $q$ -blossom provides the dual functionals for the negative degree  $q$ -Bernstein bases, which makes this blossom a powerful tool for analyzing the negative degree  $q$ -Bernstein bases. We derive several important properties and identities involving the negative degree  $q$ -Bernstein bases including a new variant of Marsden's identity, a partition of unity property, a reparametrization formula, and a formula for representing monomials. Finally, using the homogeneous multirational  $q$ -blossom, we show how to represent  $q$ -derivatives and to convert between the  $q$ -Taylor bases and the negative degree  $q$ -Bernstein bases.*

This paper is organized as follows. Section 2 provides some basic definitions and properties from  $q$ -calculus and divided differences. In Section 3, we introduce the negative degree  $q$ -Bernstein bases and derive several basic properties of these bases, including a dual functional property. The multirational  $q$ -blossom is briefly reviewed in Section 4, where the dual functional property of this blossom is derived. In Section 5, Marsden's identity, the partition of unity property, the monomial representation, and the reparametrization formula are derived from this dual functional property and in Section 6, we use the homogeneous multirational  $q$ -blossom to convert between the  $q$ -Taylor bases and the negative degree  $q$ -Bernstein bases.

This paper presents a comprehensive analysis of the negative degree  $q$ -Bernstein basis functions based on the multirational  $q$ -blossom. But there is also the subject of negative degree  $q$ -Bernstein-Bézier curves with these blending functions. One of our future goals is to investigate geometric properties of these curves using the identities for the negative degree  $q$ -Bernstein bases derived in this paper.

## 2 Preliminaries

In this section we establish notation, terminology, and preliminary results. A brief review of two of our main tools, *the  $q$ -calculus and the divided difference*, is provided in the following two subsections. For more details and proofs related to the  $q$ -calculus, see [19] and [4]. For further information about divided differences, see [13].

### 2.1 $q$ -integers, $q$ -factorials, and $q$ -binomial coefficients

Fix  $q \in (0, 1)$ . The  $q$ -integers  $[k]_q$  are defined by

$$[k]_q = \frac{1 - q^k}{1 - q}, \quad k \in \mathbb{N}.$$

The  $q$ -analogue of the ordinary factorial is given by

$$[0]_q! = 1, \quad [k]_q! = [k]_q [k-1]_q \cdots [1]_q, \quad k = 1, 2, \dots,$$

and the  $q$ -binomial coefficients are defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q!}{[k]_q! [m-k]_q!}, \quad k = 0, 1, \dots, m.$$

The  $q$ -binomial coefficients satisfy the following two recurrence relations

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = q^k \begin{bmatrix} m-1 \\ k \end{bmatrix}_q + \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} m-1 \\ k \end{bmatrix}_q + q^{m-k} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q, \quad 0 \leq k \leq m. \quad (2.1)$$

We shall make use of the following identity for the  $q$ -binomial coefficients

$$\begin{bmatrix} m+k \\ k \end{bmatrix}_q = \sum_{i=0}^k q^{m(k-i)} \begin{bmatrix} m+i-1 \\ i \end{bmatrix}_q, \quad (2.2)$$

which is easily proved using (2.1) and induction on  $k$ . The following property of the generalized  $q$ -binomial coefficients will also be useful in this paper

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = q^{k(m-k)} \begin{bmatrix} m \\ k \end{bmatrix}_{1/q}. \quad (2.3)$$

We shall need the  $q$ -shifted factorial notation  $(a; q)_0 = 1$  and  $(a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a)$ ,  $n \in \mathbb{N}$ .

Then  $(a; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a)$  is well-defined since  $q \in (0, 1)$ . Moreover

$$(a; q)_n = \frac{(a; q)_\infty}{(q^n a; q)_\infty}. \quad (2.4)$$

We shall make use of the following theorem.

**Theorem 2.1** (Euler's formula [19, Theorem 12.2.6]).

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1, \quad (2.5)$$

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} q^{\binom{n}{2}} = (-z; q)_\infty. \quad (2.6)$$

We shall also use the  ${}_r\phi_s$  basic hypergeometric series [19, (12.1.6)]

$${}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} (-q^{(k-1)/2})^{k(s+1-r)} z^k, \quad (2.7)$$

where  $(a_1, \dots, a_m; q)_n = \prod_{j=0}^m (a_j; q)_n$  for  $n \geq 0$ .

The following formula is the special case  $b \rightarrow \infty$  of the  $q$ -analogue of Gauss' theorem [19, (12.2.18)].

$${}_1\phi_1 \left( \begin{matrix} a \\ c \end{matrix} \middle| q, \frac{c}{a} \right) = \frac{(c/a; q)_{\infty}}{(c; q)_{\infty}}. \quad (2.8)$$

## 2.2 $q$ -derivatives, $q$ -Taylor expansions, and $q$ -antiderivatives

For a fixed  $q \neq 1$ , the  $q$ -derivatives of a function  $f(t)$  are defined recursively by

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad D_q^k f(t) = D_q(D_q^{k-1} f(t)), \quad k \geq 2. \quad (2.9)$$

It follows easily from (2.9) that

$$D_q(t; q)_n = -[n]_q (qt; q)_{n-1}, \quad D_q^k(t; q)_n = (-1)^k \frac{[n]_q!}{[n-k]_q!} q^{\binom{k}{2}} (q^k t; q)_{n-k}. \quad (2.10)$$

A product rule for the  $q$ -derivative is provided by the formula

$$D_q(f(t)g(t)) = f(t)D_q g(t) + g(qt)D_q f(t). \quad (2.11)$$

More generally, a  $q$ -version of the Leibniz rule for differentiating products is given by

$$D_q^n(fg)(t) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k(f)(q^{n-k}t) D_q^{n-k}(g)(t). \quad (2.12)$$

The  $q$ -version of the Taylor expansion of a function  $f(t)$  analytic at  $t = a$  is [1, 20]

$$f(t) = \sum_{k=0}^{\infty} \frac{D_q^k f(a)}{[k]_q!} (t-a)_q^k, \quad (2.13)$$

where  $(t-a)_q^0 = 1$  and  $(t-a)_q^k = (t-a)(t-qa)\cdots(t-q^{k-1}a)$  for  $k \geq 1$ .

The value  $t = 0$  is special. Let  $f(t)$  be a function that is differentiable at  $t = 0$ . By L'Hôpital's rule,  $D_q(f)(0) = f'(0)$ . Moreover it follows by induction on  $j$  that

$$D_q^j(f)(0) = [j]_q! f^{(j)}(0)/j!. \quad (2.14)$$

Therefore for a function  $f(t)$  that is analytic at  $t = 0$ , the  $q$ -Taylor expansion and the standard Taylor expansion agree at  $a = 0$ .

Finally for the explicit representation of the multirational  $q$ -blossom defined in Section 4, we shall need the  $q$ -antiderivative. A function  $F(t)$  is called a  $q$ -antiderivative of  $f(t)$  if  $D_q F(t) = f(t)$ . We shall denote a  $q$ -antiderivative of  $f(t)$  by  $I_q f(t)$ . The  $q$ -antiderivative of a continuous function exists and is itself a continuous function, so that higher order  $q$ -antiderivatives of continuous functions also exist and are defined recursively by

$$I_q^m f(t) = I_q(I_q^{m-1} f(t)). \quad (2.15)$$

**Lemma 2.2** ([36]). *Let  $f(t)$  be a continuous function in a neighborhood of zero. Then any two  $q$ -antiderivatives of  $f(t)$  can differ only by a constant.*

There is a simple relation between divided differences and  $q$ -derivatives (2.9):

$$D_q F(t) = F[t, qt], \quad D_q^j F(t) = [j]_q! F[t, qt, \dots, q^j t], \quad (2.16)$$

which is easy to prove by induction on  $j$ . Let  $v^{(n)} = v$  denote  $v$  repeated  $n$  times. From (2.14) and (2.16) it follows that

$$F[0^{(j+1)}] = F^{(j)}(0)/j! = D_q^j F(0)/[j]_q! \quad (2.17)$$

### 3 Negative degree $q$ -Bernstein bases

Let  $q \in (0, 1)$ . The  $q$ -Bernstein basis functions of degree  $n > 0$  on  $[0, 1]$  are defined by [25]

$$B_k^n(t; q) = \begin{bmatrix} n \\ k \end{bmatrix}_q t^k (t; q)_{n-k}, \quad k = 0, \dots, n, \quad (3.1)$$

and they reduce to the classical Bernstein basis functions (1.1) when  $q \rightarrow 1^-$ .

The  $q$ -Bernstein basis functions of negative degree on the interval  $(-\infty, 1)$  are defined by

$$B_k^{-n}(t; q) = \beta_{n,k}(q) \frac{t^k}{(t; q)_{n+k}}, \quad k \geq 0, n \geq 0, \quad (3.2)$$

$$\beta_{n,k}(q) = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q. \quad (3.3)$$

Notice that the  $q$ -Bernstein basis functions of negative degree reduce to the Bernstein basis functions of negative degree (1.2) when  $q \rightarrow 1^-$ . To distinguish the cases  $n = 0$  of definitions (3.1) and (3.2)-(3.3), we will write  $B_k^{-0}(t; q)$  when  $n = 0$  in (3.2)-(3.3).

It is straightforward to verify the following recursive relations:

$$B_k^0(t; q) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0, \end{cases}$$

$$B_k^{n+1}(t; q) = (1 - q^{n-k}t)B_k^n(t; q) + q^{n-k+1}tB_{k-1}^n(t; q), \quad (3.4)$$

$$B_k^{-n+1}(t; q) = (1 - q^{n+k-1}t)B_k^{-n}(t; q) + q^{n+k-2}tB_{k-1}^{-n}(t; q). \quad (3.5)$$

Solving (3.5) for  $B_k^{-n}(t; q)$  we get

$$B_k^{-n}(t; q) = \frac{1}{1 - q^{n+k-1}t} B_k^{-n+1}(t; q) - \frac{q^{n+k-2}t}{1 - q^{n+k-1}t} B_{k-1}^{-n}(t; q).$$

Another recursive relation similar to (3.5) is

$$B_k^{-n+1}(t; q) = q^{-k}(1 - q^{n+k-1}t)B_k^{-n}(t; q) + q^{-1}tB_{k-1}^{-n}(t; q).$$

These relations follow directly from (3.2)-(3.3) and (2.1).

The functions  $B_k^{-n}(t; q)$  are non-negative over the interval  $(-\infty, 0]$ . Figure 1 shows the graphs of the first four  $q$ -Bernstein basis functions of degree  $-3$  for  $q = 2/3$ .

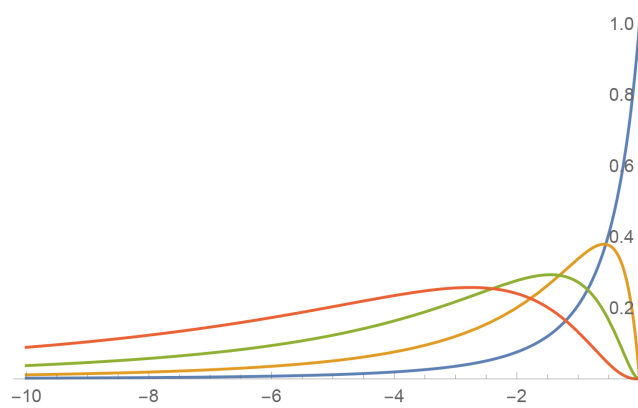


Figure 1: The first four  $q$ -Bernstein basis functions of degree  $-3$  for  $q = 2/3$ .

The  $q$ -Bernstein basis functions of positive degree can be built up from the  $q$ -Bernstein basis functions of degree 1, and the  $q$ -Bernstein basis functions of negative degree can be built up from the  $q$ -Bernstein basis functions of degree  $-1$ . To show how we need two propositions.

**Proposition 3.1.**

$$B_h^n(t; q) = \sum_{k+l=h} B_k^1(t; q) B_l^{n-1}(q^{1-k}t; q), \quad (3.6)$$

$$B_h^{-n}(t; q) = \sum_{k+l=h} B_k^1(q^{n+l}t; q) B_l^{-n-1}(t; q). \quad (3.7)$$

*Proof.* We prove only (3.7) since the proof of (3.6) is similar. Replace  $n$  by  $n + 1$  in (3.5). Then by (3.1)

$$\begin{aligned} B_h^{-n}(t; q) &= (1 - q^{n+h}t)B_h^{-n-1}(t; q) + q^{n+h-1}tB_{h-1}^{-n-1}(t; q) \\ &= B_0^1(q^{n+h}t; q)B_h^{-n-1}(t; q) + B_1^1(q^{n+h-1}t; q)B_{h-1}^{-n-1}(t; q) \\ &= \sum_{k+l=h} B_k^1(q^{n+l}t; q)B_l^{-n-1}(t; q). \end{aligned}$$

□

Equation (3.7) implies that

$$B_h^0(t; q) = \sum_{k+l=h} B_k^1(q^l t; q) B_l^{-1}(t; q) = \begin{cases} 1, & h = 0, \\ 0, & h \neq 0. \end{cases} \quad (3.8)$$

**Proposition 3.2.**

$$B_h^{-n-1}(t; q) = \sum_{p+r=h} B_p^{-n}(t; q) B_r^{-1}(q^{n+p}t; q), \quad (3.9)$$

$$B_h^{-n-1}(t; q) = \sum_{p+r=h} B_p^{-1}(t; q) B_r^{-n}(q^{p+1}t; q). \quad (3.10)$$

*Proof.* The proofs of these two identities are very similar, so we shall prove only (3.9).  
 From (3.2)-(3.3) and (2.2)

$$\begin{aligned} \sum_{p+r=h} B_p^{-n}(t; q) B_r^{-1}(q^{n+pt}; q) &= \sum_{p+r=h} (-1)^p \begin{bmatrix} n+p-1 \\ p \end{bmatrix}_q \frac{q^{\binom{p}{2}} t^p}{(t; q)_{n+p}} (-1)^r \frac{q^{(n+p)r} q^{\binom{r}{2}} t^r}{(q^{n+pt}; q)_{r+1}} \\ &= (-1)^h \frac{t^h}{(t; q)_{n+h+1}} q^{\binom{h}{2}} \sum_{p=0}^h q^{n(h-p)} \begin{bmatrix} n+p-1 \\ p \end{bmatrix}_q \\ &= (-1)^h \frac{t^h}{(t; q)_{n+h+1}} q^{\binom{h}{2}} \begin{bmatrix} n+h \\ h \end{bmatrix}_q = B_h^{-n-1}(t; q). \end{aligned}$$

□

**Proposition 3.3.** *Let  $\varepsilon \in \{\pm 1\}$  and  $n \geq 1$ . Then*

$$B_h^{\varepsilon n}(t; q) = \sum_{p_1+\dots+p_n=h} B_{p_1}^{\varepsilon}(t; q) B_{p_2}^{\varepsilon}(q^{1-\varepsilon p_1} t; q) B_{p_3}^{\varepsilon}(q^{2-\varepsilon(p_1+p_2)} t; q) \cdots B_{p_n}^{\varepsilon}(q^{n-1-\varepsilon(h-p_n)} t; q). \quad (3.11)$$

*Proof.* The case  $n = 1$  is trivial. We proceed by induction on  $n$ . By (3.11) with  $n$  replaced by  $n - 1$  and the inductive hypothesis,

$$\sum_{p_2+\dots+p_n=h-p_1} \prod_{j=2}^n B_{p_j}^{\varepsilon}(q^{j-2-\varepsilon \sum_{\ell=2}^{j-1} p_\ell} t; q) = B_{h-p_1}^{\varepsilon(n-1)}(t; q). \quad (3.12)$$

On the other hand by (3.6) if  $\varepsilon = 1$  and by (3.10) with  $n$  replaced by  $n - 1$  if  $\varepsilon = -1$  we have

$$B_h^{\varepsilon n}(t; q) = \sum_{p_1=0}^h B_{p_1}^{\varepsilon}(t; q) B_{h-p_1}^{\varepsilon(n-1)}(q^{1-\varepsilon p_1} t; q). \quad (3.13)$$

Substituting (3.12) with  $t := q^{1-\varepsilon p_1} t$  into (3.13) we obtain (3.11). □

From Propositions 3.2 and 3.3 it follows that

$$B_h^{-m-n}(t; q) = \sum_{k+l=h} B_k^{-m}(t; q) B_l^{-n}(q^{m+k} t; q). \quad (3.14)$$

Next we list some basic properties of the negative degree  $q$ -Bernstein bases.

**Proposition 3.4.** *The negative degree  $q$ -Bernstein basis functions satisfy the following properties.*

1. *Non-negativity:*

$$B_k^{-n}(t; q) \geq 0, \quad t \in (-\infty, 0]. \quad (3.15)$$

2. *Interpolation:*

$$B_k^{-n}(0; q) = \delta_{k,0}, \quad \lim_{t \rightarrow -\infty} B_k^{-n}(t; q) = 0, \quad \lim_{t \rightarrow 1^-} (-1)^k B_k^{-n}(t; q) = \infty. \quad (3.16)$$



3. *Differentiation:*

$$D_q B_k^{-n}(t; q) = q^{k-1} [n]_q (q B_k^{-n-1}(t; q) - B_{k-1}^{-n-1}(t; q)), \quad (3.17)$$

$$D_q B_k^{-n}(t; q) = [n]_q (B_k^{-n-1}(t; q) - B_{k-1}^{-n-1}(qt; q)). \quad (3.18)$$

4. *Degree elevation:*

$$B_k^{-n}(t; q) = \frac{[n+k-1]_q}{[n-1]_q} B_k^{-n+1}(t; q) - \frac{q^{n-1}[k+1]_q}{[n-1]_q} B_{k+1}^{-n+1}(t; q). \quad (3.19)$$

5. *Degree reduction:*

$$B_k^{-n}(t; q) = \sum_{j=0}^{\infty} q^{jn} \left\{ \begin{matrix} [n+k-1]_q \\ k \end{matrix} \middle/ \begin{matrix} [n+k+j]_q \\ k+j \end{matrix} \right\} B_{k+j}^{-(n+1)}(t; q). \quad (3.20)$$

*Proof.* Formulas (3.15) and (3.16) follow easily from definitions (3.2)-(3.3).

Formulas (3.17), (3.18), and (3.19) can also be derived directly from (3.2)-(3.3) and the product rule (2.11).

To prove the degree reduction formula (3.20), we will use (2.8). Letting  $a = q$  and  $c = q^{n+k+1}t$  in (2.8), and using (2.4) and (2.7) yields

$$\sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}}}{(q^{n+k+1}t; q)_j} q^{(n+k)j} t^j = {}_1\phi_1 \left( \begin{matrix} q \\ q^{n+k+1}t \end{matrix} \middle| q, q^{n+k}t \right) = 1 - q^{n+k}t.$$

Multiplying this equation by  $\beta_{n,k}(q)t^k/(t; q)_{n+k+1}$  and using (3.2)-(3.3), we obtain

$$\sum_{j=0}^{\infty} (-1)^j q^{\binom{j}{2}+jn+jk} \frac{\beta_{n,k}(q)}{\beta_{n+1,k+j}(q)} B_{k+j}^{-n-1}(t; q) = B_k^{-n}(t; q),$$

which by (3.3) reduces to (3.20). □

In the following proposition, we derive a formula for higher-order derivatives of the negative degree  $q$ -Bernstein basis functions.

**Proposition 3.5.**

$$D_q^m B_k^{-n}(t; q) = q^{mk} [m]_q! \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q \sum_{j=0}^{\min\{m,k\}} \begin{bmatrix} m \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}-mj} B_{k-j}^{-(n+m)}(t; q). \quad (3.21)$$

*Proof.* For  $m = 1$ , (3.21) agrees with (3.17). We will use the identities

$$D_q t^k = [k]_q t^{k-1}, \quad D_q^j t^k = \frac{[k]_q!}{[k-j]_q!} t^{k-j}, \quad j = 0, \dots, k,$$

and

$$D_q(t; q)_s^{-1} = [s]_q(t; q)_{s+1}^{-1}, \quad D_q^j(t; q)_s^{-1} = \frac{[s+j-1]_q!}{[s-1]_q!} (t; q)_{s+j}^{-1}.$$

Applying the  $q$ -Leibniz rule (2.12) and these identities, we derive

$$D_q^m B_k^{-n}(t; q) = \beta_{n,k}(q) D_q^m (t^k(t; q)_{n+k}^{-1})$$

$$\begin{aligned}
&= \beta_{n,k}(q) \sum_{j=0}^{\min\{m,k\}} \begin{bmatrix} m \\ j \end{bmatrix}_q D_q^{m-j}(t; q)_{n+k}^{-1} (D_q^j z^k) \Big|_{z=q^{m-j}t} \\
&= \beta_{n,k}(q) \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{[n+k+m-j-1]_q!}{[n+k-1]_q!} \frac{[k]_q!}{[k-j]_q!} q^{(k-j)(m-j)} \frac{t^{k-j}}{(t; q)_{n+k+m-j}} \\
&= \beta_{n,k}(q) \sum_{j=0}^m \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{[n+m-1]_q! [k]_q!}{[n+k-1]_q!} (-1)^{k-j} q^{(k-j)(m-j) - \binom{k-j}{2}} B_{k-j}^{-(n+m)}(t; q),
\end{aligned}$$

where to get the last line we used (3.2)-(3.3). Applying (3.3) for  $\beta_{n,k}(q)$  and simplifying the last line, we obtain (3.21).  $\square$

**Theorem 3.6** (Dual functional property). *Let  $F(t)$  be a function that is analytic at  $t = 0$ . Then in a neighborhood of  $t = 0$*

$$F(t) = \sum_{k=0}^{\infty} F_k B_k^{-n}(t; q), \quad F_k = \sum_{j=0}^k \frac{\begin{bmatrix} k \\ j \end{bmatrix}_q}{\begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q} (-1)^j q^{\binom{k-j}{2} - \binom{k}{2}} \frac{D_q^j F(0)}{[j]_q!}. \quad (3.22)$$

*Proof.* It follows from (3.16) that  $F_0 = F(0)$ . Using (3.21), we get

$$D_q^k F(t) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q [k]_q! \sum_{j=0}^{\infty} F_j q^{kj} \sum_{i=0}^{\min\{k,j\}} \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2} - ki} B_{j-i}^{-n-k}(t; q).$$

Setting  $t = 0$  in this equation and using (3.16), we find that

$$D_q^k F(0) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q [k]_q! \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q F_j. \quad (3.23)$$

To see how (3.22) follows from (3.23), we set

$$A_j = q^{\binom{j}{2}} F_j / [j]_q!, \quad B_k = (-1)^k D_q^k F(0) / \left( ([k]_q!)^2 \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \right). \quad (3.24)$$

Then (3.23) is equivalent to  $B_k = \sum_{j=0}^k \frac{(-1)^{k-j}}{[k-j]_q!} A_j$ . Define  $A(z) = \sum_{j=0}^{\infty} A_j z^j$  and  $B(z) = \sum_{k=0}^{\infty} B_k z^k$ .

Both  $A(z)$  and  $B(z)$  converge at  $z = 0$  by our assumptions on  $F$ . From (2.5) we have

$$B(z) = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^k \frac{(-1)^{k-j}}{[k-j]_q!} A_j \right\} z^k = \sum_{j=0}^{\infty} A_j z^j \sum_{k=j}^{\infty} \frac{(-1)^{k-j}}{[k-j]_q!} z^{k-j} = \frac{A(z)}{(-(1-q)z; q)_{\infty}}.$$

Therefore by (2.6)

$$A(z) = -(1-q)z; q)_{\infty} B(z) = \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}}{[j]_q!} z^j \sum_{k=0}^{\infty} B_k z^k.$$

Comparing coefficients of  $z^k$  in the last equation yields  $A_k = \sum_{j=0}^k \frac{q^{\binom{k-j}{2}}}{[k-j]_q!} B_j$ . So

$$F_k = \sum_{j=0}^k \frac{[k]_q!}{[k-j]_q!} q^{\binom{k-j}{2} - \binom{k}{2}} B_j,$$

which by (3.24) reduces to (3.22).  $\square$

## 4 The multirational $q$ -blossom

The multirational  $q$ -blossom of degree  $-n \leq -1$  of a continuous function  $F(t)$  is a sequence of multivariable functions  $f(u_1, \dots, u_k/v_1, \dots, v_{k+n}; q)$ ,  $k \geq 0$ , where  $k$  is the blossom order, satisfying the following four multirational  $q$ -blossoming axioms:

A.1 *Bisymmetry in the  $u$  and  $v$  parameters:*

$$f(u_1, \dots, u_k/v_1, \dots, v_{k+n}; q) = f(u_{\sigma(1)}, \dots, u_{\sigma(k)}/v_{\tau(1)}, \dots, v_{\tau(k+n)}; q)$$

for all  $\sigma \in S_k$  and  $\tau \in S_{k+n}$ , where  $S_p$  denotes the set of all permutations of  $\{1, \dots, p\}$ .

A.2 *Multiaffine in the  $u$  parameters:*

$$\begin{aligned} f(u_1, \dots, (1 - \alpha)w_1 + \alpha w_2, \dots, u_k/v_1, \dots, v_{k+n}; q) \\ = (1 - \alpha)f(u_1, \dots, w_1, \dots, u_k/v_1, \dots, v_{k+n}; q) \\ + \alpha f(u_1, \dots, w_2, \dots, u_k/v_1, \dots, v_{k+n}; q) \end{aligned}$$

A.3 *Cancellation property:*

$$f(u_1, \dots, u_{k-1}, w/v_1, \dots, v_{k+n-1}, w; q) = f(u_1, \dots, u_{k-1}/v_1, \dots, v_{k+n-1}; q).$$

A.4  *$q$ -Diagonal property:*

$$f(-/t, qt, \dots, q^{n-1}t; q) = F(t).$$

If  $F(t)$  is continuous at  $t = 0$ , we also require that  $F(t)$  is  $(n - 1)$ -times continuously differentiable at  $t = 0$ . Existence and uniqueness of the multirational  $q$ -blossom are established in [36]. Our goal here is to establish the dual functional property of this blossom. But first we need to recall the main result of [36].

**Theorem 4.1** ([36]). *Let  $F(t)$  be a continuous function. Then*

$$f(u_1, \dots, u_k/v_1, \dots, v_{k+n}; q) = [n - 1]_q! \{(t - u_1) \cdots (t - u_k) I_q^{n-1} F(t)\} [v_1, \dots, v_{k+n}] \quad (4.1)$$

*is the unique multirational  $q$ -blossom of  $F(t)$ .*

**Theorem 4.2** (Dual functional property of the multirational  $q$ -blossom). *Let  $F(t)$  be a function that is analytic at  $t = 0$  and let  $f(u_1, \dots, u_k/v_1, \dots, v_{n+k}; q)$  be the multirational  $q$ -blossom of  $F(t)$ . Then*

$$F(t) = \sum_{k=0}^{\infty} f(1, q^{-1}, \dots, q^{-(k-1)}/0^{(n+k)}; q) B_k^{-n}(t; q), \quad (4.2)$$

*Proof.* From (3.22) it is enough to show that

$$f(1, q^{-1}, \dots, q^{-(k-1)}/0^{(n+k)}; q) = \sum_{j=0}^k \frac{[j]_q}{[n+j-1]_q} (-1)^j q^{\binom{k-j}{2} - \binom{k}{2}} \frac{D_q^j F(0)}{[j]_q!}.$$

By (4.1), (2.17) and the  $q$ -Leibniz rule (2.12)

$$\begin{aligned} f(1, q^{-1}, \dots, q^{-(k-1)}/0^{(n+k)}; q) \\ = \{[n - 1]_q! (-1)^k q^{-\binom{k}{2}}(t; q)_k I_q^{n-1} F(t)\} [0^{(n+k)}] \end{aligned}$$

$$= \frac{[n-1]_q!}{[n+k-1]_q!} \left\{ \sum_{i=0}^{n+k-1} \begin{bmatrix} n+k-1 \\ i \end{bmatrix}_q (-1)^k q^{-\binom{k}{2}} D_q^i(z; q)_k \Big|_{z=q^{n+k-1-t}} D_q^{n+k-1-i} I_q^{n-1} F(t) \right\} (0).$$

Applying (2.10) to the last equation and setting  $j = k - i$  we get

$$\begin{aligned} f(1, q^{-1}, \dots, q^{-(k-1)} / 0^{(n+k)}; q) &= \sum_{i=0}^k \frac{[n-1]_q!}{[n+k-1-i]_q!} \begin{bmatrix} k \\ i \end{bmatrix}_q (-1)^{k+i} q^{\binom{i}{2} - \binom{k}{2}} D_q^{k-i} F(0) \\ &= \sum_{j=0}^k (-1)^j \left\{ \begin{bmatrix} k \\ j \end{bmatrix}_q / \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q \right\} q^{\binom{k-j}{2} - \binom{k}{2}} \frac{D_q^j F(0)}{[j]_q!}. \end{aligned}$$

□

To make effective use of the dual functional property, we need explicit formulas for the multirational  $q$ -blossom of some standard functions.

**Example 4.3** (The negative power basis [36]). *The multirational  $q$ -blossom of the function*

$$F(t) = \prod_{j=0}^{n-1} \frac{1}{x - q^j t} = \frac{1}{x^n (t/x; q)_n}$$

is given by

$$f(u_1, \dots, u_k / v_1, \dots, v_{n+k}; q) = \frac{(x - u_1) \cdots (x - u_k)}{(x - v_1) \cdots (x - v_{n+k})}, \quad (4.3)$$

since the right-hand side of (4.3) satisfies axioms A.1-A.4.

**Example 4.4** (The monomial basis [36]). *Let  $P(t) = t^m$ . The multirational  $q$ -blossom of  $P(t)$  is given by*

$$f(u_1, \dots, u_k / v_1, \dots, v_{n+k}; q) = \frac{\sum (-1)^\beta u_{i_1} \cdots u_{i_\alpha} v_{j_1} \cdots v_{j_\beta}}{(-1)^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q}, \quad (4.4)$$

where the sums are taken over all collections of indices  $\{i_1, \dots, i_\alpha\}$  and  $\{j_1, \dots, j_\beta\}$  such that (i) the  $i$  indices are distinct, (ii) the  $j$  indices need not be distinct, and (iii)  $\alpha + \beta = m$ .

**Proposition 4.5** (The negative degree  $q$ -Bernstein bases). *The multirational  $q$ -blossom of order  $k$  and degree  $-n$  of  $B_i^{-n}(t; q)$  is given by*

$$\begin{aligned} & \tilde{b}_i^{-n}(u_1, \dots, u_k / v_1, \dots, v_{n+k}; q) \\ &= \sum B_{p_1}^1(q^{i-p_1} u_1; q) B_{p_2}^1(q^{i-p_1-p_2} u_2; q) \cdots B_{p_k}^1(q^{i-\sum p_m} u_k; q) \\ & \quad \times B_{r_1}^{-1}(v_1; q) B_{r_2}^{-1}(q^{r_1} v_2; q) B_{r_3}^{-1}(q^{r_1+r_2} v_3; q) \cdots B_{r_{n+k}}^{-1}(q^{\sum r_s - r_{n+k}} v_{n+k}; q), \end{aligned} \quad (4.5)$$

where the sum is taken over all  $\{p_1, \dots, p_k\}$  and  $\{r_1, \dots, r_{n+k}\}$  such that  $\sum p_m + \sum r_s = i$ .

*Proof.* To establish (4.5), we need to show that the right-hand side of (4.5) satisfies axioms A.1-A.4. We shall denote the right-hand side of (4.5) by  $\tilde{b}_i^{-n}(u_1, \dots, u_k / v_1, \dots, v_{n+k})$ . To begin we will show that  $\tilde{b}_i^{-n}$  satisfies A.1 for the  $u$ -variables only, since the proof of A.1 is similar for the  $v$ -variables. We proceed by induction on  $k$ . For  $0 \leq p \leq \min\{i, k\}$  set

$$S_{k,i,p} = S_{k,i,p}(u_1, \dots, u_k; q) = \sum_{\sum p_j = p} \prod_{j=1}^k B_{p_j}^1(q^{i - \sum_{\nu=1}^j p_\nu} u_j; q).$$

Then  $S_{1,i,p} = B_p^1(q^{i-p}u_1; q)$ ,  $S_{2,i,0} = (1 - q^i u_1)(1 - q^i u_2)$ ,  $S_{2,i,1} = q^{i-1}(u_1 + u_2) - q^{2i-2}(1+q)u_1 u_2$ ,  $S_{2,i,2} = q^{2i-3}u_1 u_2$ , which verifies A.1 for the cases  $k = 1, 2$ . Next, for  $k \geq 3$ , from the induction hypothesis it follows that

$$S_{k,i,p} = \sum_{p_1=0}^1 B_{p_1}^1(q^{i-p_1}u_1; q) S_{k-1,i-p_1,p-p_1}(u_2, \dots, u_k; q)$$

is symmetric in  $u_2, \dots, u_k$ , and

$$S_{k,i,p} = \sum_{p_k=0}^1 B_{p_k}^1(q^{i-p}u_k; q) S_{k-1,i,p-p_k}(u_1, \dots, u_{k-1}; q)$$

is symmetric in  $u_1, \dots, u_{k-1}$ . Therefore  $S_{k,i,p}$  is symmetric in  $u_1, \dots, u_k$ . To continue with the other axioms, clearly  $\tilde{b}_i^{-n}$  satisfies axiom A.2. To show that axiom A.3 is satisfied, set  $v_{n+k} = u_k = w$ . Then

$$\begin{aligned} & \tilde{b}_i^{-n}(u_1, \dots, u_{k-1}, w/u_1, \dots, v_{n+k-1}, w; q) \\ &= \sum B_{p_1}^1(q^{i-p_1}u_1; q) B_{p_2}^1(q^{i-p_1-p_2}u_2; q) \cdots B_{p_k}^1(q^{i-\sum p_m}w; q) \\ & \quad \times B_{r_1}^{-1}(v_1; q) B_{r_2}^{-1}(q^{r_1}v_2; q) B_{r_3}^{-1}(q^{r_1+r_2}v_3; q) \cdots B_{r_{n+k}}^{-1}(q^{\sum r_s-r_{n+k}}w; q) \\ &= \sum_{h=0}^i \sum_{p_k+r_{n+k}=h} B_{p_k}^1(q^{r_{n+k}}(q^{\sum r_s-r_{n+k}}w); q) B_{r_{n+k}}^{-1}(q^{\sum r_s-r_{n+k}}w; q) \\ & \quad \times \left\{ \sum B_{p_1}^1(q^{i-p_1}u_1; q) \cdots B_{p_{k-1}}^1(q^{i-\sum p_m+p_k}u_{k-1}; q) \right. \\ & \quad \left. \times B_{r_1}^{-1}(v_1; q) B_{r_2}^{-1}(q^{r_1}v_2; q) \cdots B_{r_{n+k-1}}^{-1}(q^{\sum r_s-r_{n+k}-r_{n+k-1}}v_{n+k-1}; q) \right\} \\ &= \sum B_{p_1}^1(q^{i-p_1}u_1; q) \cdots B_{p_{k-1}}^1(q^{i-\sum p_m+p_k}u_{k-1}; q) \\ & \quad \times B_{r_1}^{-1}(v_1; q) B_{r_2}^{-1}(q^{r_1}v_2; q) \cdots B_{r_{n+k-1}}^{-1}(q^{\sum r_s-r_{n+k}-r_{n+k-1}}v_{n+k-1}; q) \\ &= \tilde{b}_i^{-n}(u_1, \dots, u_{k-1}/v_1 \cdots, v_{n+k-1}; q), \end{aligned}$$

where in line 4 we applied (3.8) with  $t = q^{\sum r_s-r_{n+k}}w$ . Hence axiom A.3 holds. Finally it follows from (3.11) that

$$\begin{aligned} \tilde{b}_i^{-n}(-/t, qt \dots, q^{n-1}t; q) &= \sum_{r_1+\dots+r_n=i} B_{r_1}^{-1}(t; q) B_{r_2}^{-1}(q^{r_1}t; q) \cdots B_{r_n}^{-1}(q^{i-r_n}t; q) \\ &= \sum_{r_1+\dots+r_n=i} B_{r_1}^{-1}(t; q) B_{r_2}^{-1}(q^{1+r_1}t; q) \cdots B_{r_n}^{-1}(q^{n-1+i-r_n}t; q) = B_i^{-n}(t; q), \end{aligned}$$

which establishes axiom A.4. By the uniqueness of the multirational  $q$ -blossom  $b_i^{-n} = \tilde{b}_i^{-n}$ .  $\square$

To give a simple example to Proposition 4.5, consider the multirational  $q$ -blossom of  $B_1^{-1}(t; q)$ . For  $k = 1$  and  $n = 1$ ,

$$\begin{aligned} b_1^{-1}(u_1/v_1, v_2; q) &= \sum_{p_1+r_1+r_2=1} B_{p_1}^1(q^{1-p_1}u_1; q) B_{r_1}^{-1}(v_1; q) B_{r_2}^{-1}(q^{r_1}v_2; q) \\ &= B_1^1(u_1; q) B_0^{-1}(v_1; q) B_0^{-1}(v_2; q) + B_0^1(qu_1; q) B_1^{-1}(v_1; q) B_0^{-1}(qv_2; q) \end{aligned}$$

$$\begin{aligned}
& + B_0^1(qu_1; q)B_0^{-1}(v_1; q)B_1^{-1}(v_2; q) \\
& = \frac{(1 - qv_1v_2)u_1 + (1 + q)v_1v_2 - (v_1 + v_2)}{(1 - v_1)(1 - qv_1)(1 - v_2)(1 - qv_2)}.
\end{aligned}$$

It is easy to derive the same blossoming formula for  $B_1^{-1}(t; q)$  from equation (4.1).

## 5 Identities for negative degree $q$ -Bernstein basis functions based on the multirational $q$ -blossom

In this section we derive identities for the negative degree  $q$ -Bernstein bases using the dual functional property of the multirational  $q$ -blossom. As expected, when  $q \rightarrow 1^-$  each of these identities yields a known identity for the standard negative degree Bernstein bases [11].

The following theorem provides a new  $q$ -variant of Marsden's identity.

**Theorem 5.1** (Marsden's identity).

$$\prod_{i=0}^{n-1} \frac{1}{(x - q^i t)} = \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (x; q)_k}{x^{k+n}} B_k^{-n}(t; q). \quad (5.1)$$

*Proof.* By Example 4.3 the multirational  $q$ -blossom of the left-hand side of (5.1) is

$$f(u_1, \dots, u_k/v_1, \dots, v_{n+k}; q) = \frac{(x - u_1) \cdots (x - u_k)}{(x - v_1) \cdots (x - v_{n+k})}.$$

Hence by the dual functional property (4.2)

$$\prod_{i=0}^{n-1} \frac{1}{(x - q^i t)} = \sum_{k=0}^{\infty} f(1, q^{-1}, \dots, q^{-(k-1)}/0^{(n+k)}) B_k^{-n}(t; q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (x; q)_k}{x^{k+n}} B_k^{-n}(t; q).$$

□

Next we give a very simple proof of the partition of unity property.

**Theorem 5.2** (Partition of unity).

$$1 = \sum_{k=0}^{\infty} B_k^{-n}(t; q). \quad (5.2)$$

*Proof.* The multirational  $q$ -blossom of the function  $F(t) = 1$  is  $f(u_1, \dots, u_k/v_1, \dots, v_{n+k}; q) = 1$ , so  $f(1, q^{-1}, \dots, q^{-(k-1)}/0^{(n+k)}; q) = 1$ . Now (5.2) follows immediately from the dual functional property (4.2). □

**Theorem 5.3** (Monomial representation).

$$t^m = \sum_{k=m}^{\infty} \frac{\begin{bmatrix} k \\ m \end{bmatrix}_q}{\begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q} (-1)^m q^{\binom{k-m}{2} - \binom{k}{2}} B_k^{-n}(t; q). \quad (5.3)$$

*Proof.* By Example 4.4 the multirational  $q$ -blossom of  $F(t) = t^m$  is given by (4.4). Denote the numerator of the right-hand side of (4.4) by

$$\phi(u_1, \dots, u_k/v_1, \dots, v_{n+k}) = \sum (-1)^\beta u_{i_1} \cdots u_{i_\alpha} v_{j_1} \cdots v_{j_\beta}.$$

Setting all the  $v$  parameters to zero yields

$$\phi_{k,m}(u_1, \dots, u_k) := \phi(u_1, \dots, u_k/0^{(n+k)}) = \sum_{1 \leq i_1 < \cdots < i_m \leq k} u_{i_1} \cdots u_{i_m}.$$

By [31, Lemma 3.2],  $\phi_{k,m}(1, q, \dots, q^{k-1}) = q^{\binom{m}{2}} \begin{bmatrix} k \\ m \end{bmatrix}_q$ , which together with (2.3) yields

$$\phi_{k,m}(1, q^{-1}, \dots, q^{-(k-1)}) = q^{-\binom{m}{2}} \begin{bmatrix} k \\ m \end{bmatrix}_{1/q} = \begin{bmatrix} k \\ m \end{bmatrix}_q q^{\binom{k-m}{2} - \binom{k}{2}}.$$

Finally (5.3) follows from the last equation, the dual functional property (4.2), and (4.4).  $\square$

The last identity in this section is a reparametrization formula for negative degree  $q$ -Bernstein basis functions. For the proof of this formula we need the following lemma.

**Lemma 5.4.** *Let  $b_i^{-n}(u_1, \dots, u_k/v_1, \dots, v_{n+k}; q)$  denote the multirational  $q$ -blossom of order  $k$  and degree  $-n$  of  $B_i^{-n}(t; q)$  and let  $k \geq i$ . Then*

$$b_i^{-n}(u_1, \dots, u_k/0^{(n+k)}; q) = b_i^k(u_1, \dots, u_k; 1/q),$$

where  $b_i^k$  is the  $q$ -blossom of the  $q$ -Bernstein basis function  $B_i^k(t; q)$ .

*Proof.* Setting  $v_j = 0$ ,  $j = 1, \dots, n+k$  in (4.5), and using (3.1) and axiom A.1 yields

$$\begin{aligned} b_i^{-n}(u_1, \dots, u_k/0^{(n+k)}; q) &= \sum_{p_1 + \cdots + p_k = i} B_{p_1}^1(q^{i-p_1}u_1; q) B_{p_2}^1(q^{i-p_1-p_2}u_2; q) \cdots B_{p_k}^1(u_k; q) \\ &= \sum_{p_1 + \cdots + p_k = i} B_{p_1}^1(u_1; q) B_{p_2}^1(q^{p_1}u_2; q) \cdots B_{p_k}^1(q^{i-p_k}u_k; q) \\ &= \sum_{p_1 + \cdots + p_k = i} B_{p_1}^1(u_1; 1/q) B_{p_2}^1((1/q)^{-p_1}u_2; 1/q) \cdots B_{p_k}^1((1/q)^{-i+p_k}u_k; 1/q). \end{aligned}$$

By definition  $b_i^{-n}(u_1, \dots, u_k/0^{(n+k)}; q)$  is symmetric and multiaffine. Set  $u_m = (1/q)^{m-1}t$ ,  $m = 1, \dots, k$ . Then the diagonal property follows directly from Proposition 3.3. Therefore  $b_i^{-n}(u_1, \dots, u_k/0^{(n+k)}; q)$  is the  $1/q$ -blossom of  $B_i^k(t; 1/q)$ .  $\square$

**Theorem 5.5** (Reparametrization formula).

$$B_i^{-n}(rt; q) = \sum_{k=i}^{\infty} B_i^k(r; 1/q) B_k^{-n}(t; q). \quad (5.4)$$

*Proof.* Take a function  $F$  with multirational  $q$ -blossom  $f$ . Then  $f(ru_1, \dots, ru_k/rv_1, \dots, rv_{n+k}; q)$  is the multirational  $q$ -blossom of  $F(rt)$ , since it satisfies axioms A.1-A.4 for  $F(rt)$ . In particular  $b_i^{-n}(ru_1, \dots, ru_k/rv_1, \dots, rv_{n+k}; q)$  is the multirational  $q$ -blossom of  $B_i^{-n}(rt; q)$ . Now by the dual functional property (4.2), Lemma 5.4, and axiom A.4

$$\begin{aligned} B_i^{-n}(rt; q) &= \sum_{k=0}^{\infty} b_i^{-n}(r, rq^{-1}, \dots, rq^{-(k-1)}/0^{(n+k)}; q) B_k^{-n}(t; q) \\ &= \sum_{k=i}^{\infty} b_i^k(r, rq^{-1}, \dots, rq^{-(k-1)}; 1/q) B_k^{-n}(t; q) = \sum_{k=i}^{\infty} B_i^k(r; 1/q) B_k^{-n}(t; q). \end{aligned}$$

$\square$

## 6 Conversion between the $q$ -Taylor bases and the negative degree $q$ -Bernstein bases

Homogeneous blossoms are primarily used to represent derivatives, and to derive formulas and algorithms for derivatives of Bézier curves both in the classical and in the quantum settings [13, 16]. Previously we have shown that the homogeneous multirational  $q$ -blossom can be used to represent  $q$ -derivatives [36]. We shall now use the homogeneous multirational  $q$ -blossom to convert between the  $q$ -Taylor bases and the negative degree  $q$ -Bernstein bases.

The degree  $-n$  homogeneous multirational  $q$ -blossom of a degree  $-n$  homogeneous function  $F(t, w)$  is a sequence of functions  $f((u_1, r_1), \dots, (u_k, r_k)/(v_1, w_1), \dots, (v_{n+k}, w_{n+k}))$  that satisfies the following four axioms [36]:

1. *Bisymmetry in the  $(u, r)$  and  $(v, w)$  parameters:*

$$\begin{aligned} & f((u_1, r_1), \dots, (u_k, r_k)/(v_1, w_1), \dots, (v_{k+n}, w_{k+n}); q) \\ &= f((u_{\sigma(1)}, r_{\sigma(1)}), \dots, (u_{\sigma(k)}, r_{\sigma(k)})/(v_{\tau(1)}, w_{\tau(1)}), \dots, (v_{\tau(k+n)}, w_{\tau(k+n)}); q), \end{aligned}$$

for all permutations  $\sigma \in S_k$  and  $\tau \in S_{k+n}$ .

2. *Multilinearity in the  $(u, r)$  parameters:*

$$\begin{aligned} & f((u_1, r_1), \dots, c(u_i, r_i) + d(p_i, s_i), \dots, (u_k, r_k)/(v_1, w_1), \dots, (v_{k+n}, w_{k+n}); q) \\ &= cf((u_1, r_1), \dots, (u_i, r_i), \dots, (u_k, r_k)/(v_1, w_1), \dots, (v_{k+n}, w_{k+n}); q) \\ &\quad + df((u_1, r_1), \dots, (p_i, s_i), \dots, (u_k, r_k)/(v_1, w_1), \dots, (v_{k+n}, w_{k+n}); q). \end{aligned}$$

3. *Cancellation property:*

$$\begin{aligned} & f((u_1, r_1), \dots, (u_{k-1}, r_{k-1}), (p, s)/(v_1, w_1), \dots, (v_{n+k-1}, w_{n+k-1}), (p, s); q) \\ &= f((u_1, r_1), \dots, (u_{k-1}, r_{k-1})/(v_1, w_1), \dots, (v_{n+k-1}, w_{n+k-1}); q). \end{aligned}$$

4.  *$q$ -Diagonal property:*

$$f(-/(t, w), (qt, w), \dots, (q^{n-1}t, w); q) = F(t, w).$$

In [36] the existence and uniqueness of the homogeneous multirational  $q$ -blossom are established. In addition, the following explicit formula for this blossom is provided:

$$\begin{aligned} & f((u_1, r_1), \dots, (u_k, r_k)/(v_1, w_1), \dots, (v_{n+k}, w_{n+k}); q) \\ &= \frac{1}{w_1 \cdots w_{n+k}} \{[n-1]_q! (r_1 t - u_1) \cdots (r_k t - u_k) I_q^{n-1} F(t)\} \left[ \frac{v_1}{w_1}, \dots, \frac{v_{n+k}}{w_{n+k}} \right]. \end{aligned} \quad (6.1)$$

It is also shown in [36] that  $q$ -derivatives of differentiable functions can be computed by

$$D_q^k F(t) = (-1)^k \frac{[n+k-1]_q!}{[n-1]_q!} f(\delta^{(k)}/t, qt, \dots, q^{n+k-1}t; q), \quad (6.2)$$

where  $\delta = (1, 0)$  and  $q^j t$  represents  $(q^j t, 1)$ .

Let  $F(t)$  be an analytic function at  $t = 0$ . To convert from a negative degree  $q$ -Bernstein basis of degree  $-n$  to the  $q$ -Taylor form, consider the  $q$ -Taylor basis

$$M_k^n(t; q) = (-1)^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k, \quad k = 0, 1, \dots$$



For  $k \geq 0$  let  $\{Q_k^0\}$  be the homogenized coefficients in (4.2), that is

$$Q_k^0 = F_k = f(\hat{1}, \hat{q}^{-1}, \dots, \hat{q}^{-(k-1)} / \hat{0}^{\langle n+k \rangle}; q),$$

where  $\hat{0} = (0, 1)$ ,  $\hat{1} = (1, 1)$ , and  $\hat{q}^{-i} = (q^{-i}, 1)$ . For  $d \geq 1$  define recursively

$$Q_k^d = q^k(Q_{k+1}^{d-1} - Q_k^{d-1}).$$

Then it is straightforward to show that

$$Q_k^d = f(\delta^{\langle d \rangle}, \hat{1}, \hat{q}^{-1}, \dots, \hat{q}^{-(k-1)} / \hat{0}^{\langle n+k+d \rangle}; q).$$

In particular  $Q_0^k = f(\delta^{\langle k \rangle} / \hat{0}^{\langle n+k \rangle}; q)$ , which by (6.2) and the  $q$ -Taylor expansion (2.13) are the coefficients of  $F(t)$  in the  $q$ -Taylor basis  $\{M_k^n(t; q)\}$ .

Conversely, it is also possible to convert  $F(t)$  from the  $q$ -Taylor basis  $\{M_k^n(t; q)\}$  to the negative degree  $q$ -Bernstein form of degree  $-n$ . Let  $R_k^0 = f(\delta^{\langle k \rangle} / \hat{0}^{\langle n+k \rangle}; q)$  be the coefficients of  $F(t)$  in the basis  $\{M_k^n(t; q)\}$  and for  $d \geq 1$  define recursively

$$R_k^d = q^{-(d-1)} R_{k+1}^{d-1} + R_k^{d-1}.$$

It is straightforward to verify that

$$R_k^d = f(\delta^{\langle k \rangle}, \hat{1}, \hat{q}^{-1}, \dots, \hat{q}^{-(d-1)} / \hat{0}^{\langle n+k+d \rangle}; q).$$

In particular

$$R_0^k = f(\hat{1}, \hat{q}^{-1}, \dots, \hat{q}^{-(k-1)} / \hat{0}^{\langle n+k \rangle}; q) = F_k,$$

which by the dual functional property are the coefficients of  $F(t)$  in the basis  $\{B_k^{-n}(t; q)\}$ .

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