STRONG CONVERGENCE THEOREM FOR SOLVING FINITE FAMILIES OF EQUILIBRIUM PROBLEMS AND DEMICONTRACTIVE OPERATORS WITH APPLICATIONS

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Abstract. The main purpose of this paper is to introduce a new line search viscosity method for finding a common solution of finite families of fixed-point problems of demicontractive mappings and pseudomonotone equilibrium in a real Hilbert space. We give a strong convergence theorem under mild conditions. We proposed a method that solves a single strongly convex problem for only one iteration and uses an Armijo line searching rule to identify an optimal step size for the next step without computing any half space. We have demonstrated the importance of the applicability of our algorithm to find solutions of a variety of nonlinear analysis issues. Our result enhances many existing results in the literature. Furthermore, we give numerical examples to show its relevance.

Keywords. Line search technique, Contraction operator, Equilibrium problem, Demicontractive operator, Fixed point problem.

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1. Introduction

Equilibrium problem (EP) includes various mathematical problems such as the Kirszbraun problems, optimization problems, fixed point problems, Nash equilibrium of noncooperative games, variational inequality problems (VIP), saddle point problems, complementarity problems and vector minimization problems as its special case (for more details; see [1, 2]). According to our knowledge, Muu and Oettli [3] coined the term equilibrium problem in 1992 and Blum and Oettli [4] popularized it in 1994. Several authors have established and extended a number of results concerning the existence of an equilibrium problem solution (e.g. see [1, 5, 6]). In addition, various numerical approaches for solving equilibrium problems in both finite and infinite-dimensional spaces have been established.

A finite family of EP is the problem of identifying a point y' in intersection of nonempty closed convex subsets $Q = \bigcap_{j=1}^{N} Q_j$ of Hilbert space H such that

$$h_j(y', \bar{y}) \geq 0 \text{ for all } \bar{y} \in Q \text{ and } 1 \leq j \leq N, \tag{1.1}$$

where $h_j : Q_j \times Q_j \to \mathbb{R}$ are bifunctions. The problem of identifying a common element of the set of fixed points of nonlinear mappings and the set of equilibrium problem solutions has recently become a popular issue. The most popular methods to solve equilibrium problems is the proximal point method. Martinet [7] initially established the proximal point approach for problems involving monotone variational inequality and later, Rockafellar [8] extended the concept to monotone operators. Moudafi [9] has also suggested the proximal point approach for monotone equilibrium problems. Konnov [10] proposes an alternative variant of the proximal point method with weaker conditions. However, this technique cannot be used to solve a pseudomonotone equilibrium problem. To overcome this problem, Tran et al. [11] proposed an extragradient method for solving

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EP and proved a weak convergence theorem. Extragradient-like methods require an iterative sequence with prior knowledge of Lipschitz type constants. But all Lipschitz type constants are not easy to compute. As a result, various techniques for providing variable step sizes have been suggested as a means of overcoming this issue. For example, Rehman et al. [12] proposed Popov's explicit iterative extragradient method (PESM) with the subgradient method and proved a weak convergence theorem. To be more precise, they created a half-space to minimize the number of iteration steps and improve convergence. Since then, other authors have done extensive research on the PESM; see [13, 14]. But for some cases, it is difficult to compute that half space C_n; for C_n see [12].

On the other hand, The Armijo line-search rule is a widely accepted technology that can significantly improve the convergence process of the algorithm's iterative sequence. As a result, it can be used to solve equilibrium and related problems in Hilbert space. Many authors used linesearch method for solving equilibrium problem; see [15, 16, 17, 18] and the references therein. In 2021, Jolaoso et al. [17] proposed a parallel line search method for solving finite families of equilibrium and fixed point problems. They take condition on control parameter that $\sum_{j=0}^{M}\delta_{n,j}=1$; for $\delta_{n,j}$ see [17]. To remove this condition we proposed a viscosity type method for solving finite families of equilibrium and fixed point problems which does not require such condition.

Recently, various common problems namely common minimization problems, common variational inequalities, common equilibrium, and common variational inclusion problems have been studied by many authors. For example, In 2016, Hieu [19] proposed parallel hybrid Mann-type extragradient method to find common solutions to pseudomonotone equilibrium problems. They proved the stepsize meets some condition based on a prior estimate of the bifunctions Lipschitz-like constant. It should be noted that the Lipschitz-type constants is generally difficult to find. This difficulty may make it difficult for researchers to use these algorithms. In 2021, Ogwo et al. [20] proposed an iterative method to solve common solution of variational inequalities and fixed point problems for demicontractive mappings.

In this paper, influenced and inspired by the above results, we propose a new line search viscosity method and find a common solution of finite families of pseudomonotone equilibrium and fixed-point problems of demicontractive mappings in a real Hilbert space. We prove its strong convergence. The advantages of our new method are (1) it solves a single strongly convex problem for the only one iteration and uses an Armijo line searching rule to identify an optimal step size for the next step; (2) we do not require to compute the half space C_n; (3) we do not take any condition on control parameter that $\sum_{j=0}^{M}\delta_{n,j}=1$; (4) we do not require previous knowledge of Lipschitz constant; (5) we give strong convergence theorem under mild conditions. Moreover, we use this result for solving finite families of equilibrium problems and various problems like variational inclusion, equilibrium, null point, multiple-sets split feasibility and variational inequality problems. We also give numerical examples to show its relevance. We represent the solution set of finite families of equilibrium and fixed point problem for demicontractive operators by Γ.

2. Preliminaries

In this section, we provide some basic definitions and results that will be used to establish our main results. Suppose that H is a real Hilbert space and Q is a nonempty closed convex subset of H.

Definition 2.1. The bifunction $h: Q\times Q\to\mathbb{R}$ is called

(i) pseudomonotone if

$$h(p',y')\geq 0 \text{ implies } h(y',p')\leq 0 \text{ for all } p',y'\in Q.$$

(ii) monotone if

$$h(p',y')+h(y',p')\leq 0 \text{ for all } p',y'\in Q.$$

(iii) strongly pseudomonotone if there exists $\gamma>0$

$$h(p',y')\geq 0 \text{ implies } h(y',p')\leq -\gamma\|p'-y'\|^2 \text{ for all } p',y'\in Q.$$

Remark 2.2. Every strongly pseudomonotone mapping is pseudomonotone but converse is not true.

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Example 2.3. The bifunction $h : Q \times Q \to \mathbb{R}$ defined as $h(p',y') = \sum_{i=2}^{5}(y'_i - p'_i)\|p'\|$, $Q = \{x \in \mathbb{R}^5 : p'_1 \geq -1, p'_i \geq 0, i = 2,3,4,5\}$ is not strongly pseudomonotone but pseudomotone.

Proof. Clearly, the bifunction h is pseudomonotone on Q. Now, we will prove it is not strongly pseudomonotone on Q. For this we provide a counter example. Let $p' = (-1,0,0,0,0)$ and $y' = (0,0,0,0,0)$. Let $h(p',y') \geq 0$, which implies

$$\sum_{i=2}^{5}(y'_i - p'_i)\|p'\| \geq 0. \tag{2.1}$$

Now, $h(y',p') = \sum_{i=2}^{5}(p'_i - y'_i)\|y'\| = 0$ for $p' = (-1,0,0,0,0)$ and $y' = (0,0,0,0,0)$. Also,

$$\|p' - y'\|^2 = 1. \tag{2.2}$$

Clearly, following inequality is not satisfied for any $\gamma > 0$

$$h(y',p') \leq -\gamma\|p' - y'\|^2. \tag{2.3}$$

Therefore h is not strongly pseudomonotone on Q. □

Definition 2.4. A mapping P_Q is called metric projection from H to Q if

$$\|p' - P_Q p'\| = \inf\{\|p' - y'\|; y' \in Q\} \text{ for all } p' \in H. \tag{2.4}$$

Definition 2.5. An operator $S : H \to H$ is called

(i) nonexpansive if
$$\|Sp' - Sy'\| \leq \|p' - y'\| \text{ for all } p', y' \in H.$$
(ii) L-Lipschitz if there exists $L > 0$ such that
$$\|Sp' - Sy'\| \leq L\|p' - y'\| \text{ for all } p', y' \in H.$$
(iii) τ-demicontractive if $\tau \in [0,1)$ and $\text{Fix}(S) \neq \phi$ such that
$$\|Sp' - y'\|^2 \leq \|p' - y'\|^2 + \tau\|p' - Sp'\|^2 \text{ for all } p' \in H \text{ and } y' \in \text{Fix}(S).$$
(iv) contraction if there is a constant $\mu \in (0,1)$ such that
$$\|Sp' - Sy'\| \leq \mu\|p' - y'\| \text{ for all } p', y' \in H.$$

Definition 2.6. Assume that $j : H \to \mathbb{R}$ is lower semicontinuous, proper and convex function. Then subdifferential is defined as

$$\partial j(y') = \{p' \in H : \ j(q') - j(y') \geq \langle p', q' - y' \rangle\} \text{ for all } y', q' \in Dom(j). \tag{2.5}$$

Lemma 2.7. [18] *The metric projection P_Q satisfies:*

(i) For $p' \in H$ and $y' \in Q$, $y' = P_Q p'$ iff $\langle p' - y', y' - q' \rangle \leq 0$ for all $q' \in Q$.
(ii) $\|P_Q p' - y'\|^2 \leq \|p' - y'\|^2 - \|P_Q p' - p'\|^2$ for all $y' \in Q$ and $p' \in H$.
(iii) $\|P_Q p' - P_Q y'\|^2 \leq \langle p' - y', P_Q p' - P_Q y' \rangle$ for all $p' \in H$ and $y' \in Q$.

Lemma 2.8. [21] *Consider $\{l_n\}$ a sequence of real numbers and there exists a subsequence $\{n_t\}$ of $\{n\}$ satisfying $l_{n_t} < l_{n_t+1}$ for all $t \in \mathbb{N}$. Then there is a nondecreasing sequence $\{m_s\} \subset \mathbb{N}$ such that $m_s \to \infty$ and satisfies the following properties for every number $s \in \mathbb{N}$*

$$l_{m_s} \leq l_{m_s+1} \text{ and } l_s \leq l_{m_s+1}, \tag{2.6}$$

where $m_s = \max\{t \leq s : l_t < l_{t+1}\}$.

Lemma 2.9. [22] *Assume that l is real-valued function on H. Define $E = \{p' \in Q : l(p') \leq 0\}$. If E is nonempty and l is Lipschitz continuous with modulus $\theta > 0$, then*

$$d(p',E) \geq \frac{1}{\theta}\max\{l(p'),0\}, \tag{2.7}$$

where $d(p',E)$ is the distance function from p' onto E.

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Lemma 2.10. [23] *Assume that family of sequence* $\{\tau_k\}_{k=t}^{M} \subseteq \mathbb{R}$ *is countable , where* $t > 0$ *is a fixed integer and* $N_1 \in \mathbb{N}$ *is such that* $t+1 \leq N_1$. *Then*

$$\tau_t + \sum_{k=t+1}^{M} \tau_k \prod_{i=t}^{k-1}(1-\tau_i) + \prod_{i=t}^{M}(1-\tau_i) = 1. \tag{2.8}$$

Lemma 2.11. [23] *Assume that* $\{\tau_k\}_{k=t}^{M} \subseteq [0,1]$ *and* $\{r_k\}_{k=t}^{M-1} \subseteq H$ *are countable finite sets, where* $t > 0$ *is a fixed integer and* $N_1 \in \mathbb{N}$ *is such that* $t+1 \leq N_1$. *For* $p', q', y' \in H$, *define*

$$l' = \tau_t p' + \sum_{k=t+1}^{M} \tau_k \prod_{i=t}^{k-1}(1-\tau_i) r_{k-1} + \prod_{i=t}^{M}(1-\tau_i) y', \tag{2.9}$$

then

$$\begin{aligned} \|l' - q'\|^2 \leq \tau_t \|p' - q'\|^2 &+ \sum_{k=t+1}^{M} \tau_k \prod_{i=t}^{k-1}(1-\tau_i)\|r_{k-1} - q'\|^2 + \prod_{i=t}^{M}(1-\tau_i)\|y' - q'\|^2 \\ &- \tau_t \Big[\sum_{k=t+1}^{M} \tau_k \prod_{i=t}^{k-1}(1-\tau_i)\|p' - r_{k-1}\|^2 + \prod_{i=t}^{M}(1-\tau_i)\|p' - y'\|^2\Big]. \end{aligned} \tag{2.10}$$

Lemma 2.12. [24] *Let* $\{q_n\} \subset [0,\infty)$, $\{p_n\} \subset (0,1)$ *and* $\{b_n\}$ *are real sequences satisfying the inequality*

$$q_{n+1} \leq (1-p_n)q_n + p_n b_n \text{ for all } n \in \mathbb{N}. \tag{2.11}$$

Suppose $\sum_{n=0}^{\infty} p_n = \infty$ *and* $\limsup_{n\to\infty} b_n \leq 0$, *then* $\lim_{n\to\infty} q_n = 0$.

Assumption 1. *We need the following assumptions on the bifunction h to solve the equilibrium problem:*

1. h *is pseudomonotone on* Q.
2. $h(.,t)$ *is continuous on* Q *for every* $t \in Q$.
3. $h(x,.)$ *is convex and subdifferentiable on* Q *for each fixed* $x \in Q$.

3. Main Results

In this section, we introduce a new line search viscosity method and prove a strong convergence theorem for solving finite families of equilibrium and fixed point problems for demicontractive mappings.

Let Q_j be family of nonempty, closed and convex subsets of a real Hilbert space H such that $Q = \bigcap_{j=1}^{N} Q_j$, where $1 \leq j \leq N$ and $S_k : H \to H$, $k = 1,2,\ldots,M$ be a finite family of demicontractive mappings with constant η_k such that $I - S_k$ are demiclosed at zero with $\eta = \max\{\eta_k\}_{k=1}^{M}$. Let $h_j : Q_j \times Q_j \to \mathbb{R}$ be bifunctions satisfying Assumption 1. Assume that ϕ is a δ-contraction operator defined on H.

Algorithm 3.1. Consider $\phi_1, \rho \in (0,1)$, $\gamma \in (0,1)$, and $\{\kappa_n^j\} \subset [\kappa, \bar{\kappa}]$, where $0 < \kappa \leq \bar{\kappa}$, $1 \leq j \leq N$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. Choose $t_0, t_1 \in Q$ and γ_n such that $0 \leq \gamma_n \leq \bar{\gamma}_n$, where

$$\bar{\gamma}_n = \begin{cases} \min\left\{\frac{\tau_n}{\|t_n - t_{n-1}\|}, \gamma\right\} & \text{if } t_n \neq t_{n-1}, \\ \gamma & otherwise. \end{cases} \tag{3.1}$$

Moreover, τ_n is a positive sequence such that $\tau_n = o(\theta_n^1)$ satisfying $\lim_{n\to\infty} \frac{\tau_n}{\theta_n^1} = 0$.

Step 1: Compute

$$\begin{cases} x_n = t_n - \gamma_n(t_n - t_{n-1}), \\ u_n^j = argmin\left\{\dfrac{1}{2\kappa_n^j}\|y - x_n\|^2 + h_j(x_n, y) : y \in Q\right\}. \end{cases} \tag{3.2}$$

Step 2: (Armijo line-search rule) Find k_n as the smallest positive integer that satisfies

$$\begin{cases} v_{n,k}^j = (1-\rho^{k_n})x_n + \rho^{k_n} u_n^j, \\ -h_j(v_{n,k}^j, u_n^j) \geq \frac{\phi_1}{2\kappa_n^j} \|x_n - u_n^j\|^2. \end{cases} \tag{3.3}$$

Set $v_{n,k}^j = v_n^j$. If $0 \in \partial h_j(v_n^j, v_n^j)$, set $y_n = v_n^j$ and go to Step 4. Otherwise, do Step 3.
Step 3: Evaluate $l_n^j(t) = \left\langle x_n^j, t - v_n^j \right\rangle$, where $x_n^j \in \partial h_j(v_n^j, v_n^j)$ for $j = 1, 2, \ldots, N$. Construct the set

$$E_n = \{t \in H : \sum_{j=1}^{N} \alpha_n^j l_n^j(t) \leq 0\}, \tag{3.4}$$

where $\{\alpha_n^j\}_{j=1}^N \subset (0,1)$ such that $\sum_{j=1}^N \alpha_n^j = 1$ and E_n is nonempty set and l_n^j are Lipschitz continuous with modulus θ_n such that $\theta = \min\{\theta_n, ; n \in \mathbb{N}\} > 0$. Evaluate

$$y_n = P_{E_n}(x_n). \tag{3.5}$$

Step 4: Evaluate

$$\begin{aligned} t_{n+1} = \theta_n^1 \phi(t_n) + (1-\theta_n^1)[\theta_n^2 y_n + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1} (1-\theta_n^i) S_{k-2} y_n \\ + \prod_{i=2}^{M+1} (1-\theta_n^i) S_M y_n]. \end{aligned} \tag{3.6}$$

Remark 3.2. Clearly, from equation (3.1) and $\sum_{n=1}^{\infty} \tau_n < \infty$, we have

$$\sum_{n=1}^{\infty} \gamma_n \|t_{n-1} - t_n\| < \infty \text{ and } \frac{\gamma_n}{\theta_n^1} \|t_n - t_{n-1}\| \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Lemma 3.3. *Assume that $x_n \neq u_n^j$, then there is a nonnegative integer k_n satisfying (3.3).*

Proof. We prove the result by contradiction. Let for every $k_n > 0$, we estimate

$$v_{n,k}^j = (1-\rho^{k_n})x_n + \rho^{k_n} u_n^j \tag{3.8}$$

and

$$-h_j(v_{n,k}^j, u_n^j) < \frac{\phi_1}{2\kappa_n^j} \|x_n - u_n^j\|^2. \tag{3.9}$$

Taking limit $n \to \infty$ and using Assumption 1, we have

$$-h_j(x_n, u_n^j) < \frac{\phi_1}{2\kappa_n^j} \|x_n - u_n^j\|^2, \tag{3.10}$$

which implies

$$0 \leq h_j(x_n, u_n^j) + \frac{\phi_1}{2\kappa_n^j} \|x_n - u_n^j\|^2. \tag{3.11}$$

As u_n^j is the solution of strongly convex problem, therefore we have

$$\frac{1}{2\kappa_n^j} \|u_n^j - x_n\|^2 + h_j(x_n, u_n^j) \leq \frac{1}{2\kappa_n^j} \|y - x_n\|^2 + h_j(x_n, y), \ \forall y \in Q. \tag{3.12}$$

Put $y = x_n$ in equation (3.12), then we get

$$\frac{1}{2\kappa_n^j} \|u_n^j - x_n\|^2 + h_j(x_n, u_n^j) \leq 0. \tag{3.13}$$

From equations (3.11) and (3.13), we have

$$\frac{1}{2\kappa_n^j} \|u_n^j - x_n\|^2 \leq \frac{\phi_1}{2\kappa_n^j} \|x_n - u_n^j\|^2, \tag{3.14}$$

which implies either $x_n = u_n^j$ or $\phi_1 \geq 1$. If $x_n = u_n^j$, then this is contradiction to $x_n \neq u_n^j$. Also if $\phi_1 \geq 1$, then this is contradiction to $\phi_1 \in (0,1)$. Hence, we deduce that the Armijo line search rule in equation (3.3) is well defined. □

Lemma 3.4. *Assume that solution set is nonempty i.e., $y' \in \Gamma$. Let $l_n^j(y') = \left\langle x_n^j, y' - v_n^j \right\rangle$, where x_n^j and v_n^j are defined in Algorithm (3.1), then*

$$l_n^j(x_n) \geq \frac{\rho^{k_n}\phi_1}{2(1-\rho^{k_n})\kappa_n^j}\|x_n - u_n^j\|^2. \tag{3.15}$$

Additionally, $l_n^j(y') \leq 0$ and if $x_n \neq v_n^j$, then $l_n^j(y') > 0$.

Proof. Let $s^j(x_n) = x_n - v_n^j$. Using equation (3.3), we have

$$x_n = \frac{v_{n,k}^j - \rho^{k_n} u_n^j}{1-\rho^{k_n}}. \tag{3.16}$$

Subracting v_n^j on both sides of equation (3.16) and set $v_{n,k}^j = v_n^j$, we have

$$x_n - v_n^j = \frac{\rho^{k_n}}{1-\rho^{k_n}}(v_n^j - u_n^j). \tag{3.17}$$

Since $x_n^j \in \partial h_j(v_n^j, v_n^j)$, then using equations (3.3) and (3.17), we have

$$\begin{aligned} l_n^j(x_n) &= \left\langle x_n^j, x_n - v_n^j \right\rangle \\ &= \frac{\rho^{k_n}}{1-\rho^{k_n}}\left\langle x_n^j, v_n^j - u_n^j \right\rangle \\ &\geq \frac{\rho^{k_n}}{1-\rho^{k_n}}\left(h_j(v_n^j, v_n^j) - h_j(v_n^j, u_n^j)\right) \\ &\geq \frac{\rho^{k_n}}{2(1-\rho^{k_n})\kappa_n^j}\phi_1\|x_n - u_n^j\|^2. \end{aligned} \tag{3.18}$$

Now, if $x_n \neq v_n^j$, then $l_n^j(x_n) > 0$. As $y' \in \Gamma$, we have $h_j(y', z) \geq 0$ for all $z \in Q$, $j = 1, 2, \ldots N$. Also, each h_j is pseuedomonotone on Q, which impies $h_j(z, y') \leq 0$. Subsequently,

$$\begin{aligned} l_n^j(y') &= \left\langle x_n^j, y' - v_n^j \right\rangle \\ &\leq h_j(v_n^j, y') - h_j(v_n^j, v_n^j) \\ &\leq 0. \end{aligned} \tag{3.19}$$

□

Remark 3.5. Clearly $y' \in E_n$ and E_n is a closed convex subset of a real Hilbert space H.

Theorem 3.6. *Let the solution set $\bigcap_{j=1}^N EP(h_j, Q) \cap (\bigcap_{k=1}^M Fix(S_k))$ is nonempty and h_j for $j = 1, 2, \ldots, N$ satisfy Assumption 1. Assume that $\{\theta_n^k\}$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} \theta_n^1 = 0$, *and* $\sum_{n=0}^{\infty} \theta_n^1 = \infty$,

(ii) $\liminf_{n\to\infty} \theta_n^2 > \eta$ *where* $n \in \mathbb{N}$,

(iii) $\liminf_{n\to\infty} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i) > 0$ *for every* $k = 3, 4, \ldots M+1$ *and* $\liminf_{n\to\infty} \prod_{i=2}^{M+1}(1-\theta_n^i) > 0$.

Then $\{t_n\}$ converges strongly to $y' \in \bigcap_{j=1}^N EP(h_j, Q) \cap (\bigcap_{k=1}^M Fix(S_k))$.

Proof. Firstly, we show that the sequence $\{t_n\}$ is bounded. Assume that $y' \in \bigcap_{j=1}^N EP(h_j, Q) \cap (\bigcap_{k=1}^M Fix(S_k))$. From equation (3.5) and Lemma (2.7), we have

$$\begin{aligned} \|y_n - y'\|^2 &= \|P_{E_n}(x_n) - y'\|^2 \\ &\leq \|x_n - y'\|^2 - \|P_{E_n}(x_n) - x_n\|^2 \\ &\leq \|x_n - y'\|^2 - d(x_n, E_n) \qquad (3.20) \\ &\leq \|x_n - y'\|^2. \qquad (3.21) \end{aligned}$$

Using equation (3.2), we have

$$\|x_n - y'\| = \|t_n - \gamma_n(t_n - t_{n-1}) - y'\|$$
$$\leq \|t_n - y'\| + \gamma_n\|t_n - t_{n-1}\|. \tag{3.22}$$

Let

$$h_n = \theta_n^2 y_n + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)S_{k-2}y_n + \prod_{i=2}^{M+1}(1-\theta_n^i)S_M y_n. \tag{3.23}$$

Using Lemmas (2.10), (2.11), equation (3.21), and conditions (*ii*) and (*iii*), we have

$$\begin{aligned}
\|h_n - y'\|^2 &= \|\theta_n^2 y_n + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)S_{k-2}y_n + \prod_{i=2}^{M+1}(1-\theta_n^i)S_M y_n - y'\|^2 \\
&\leq \theta_n^2\|y_n - y'\|^2 + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)\|S_{k-2}y_n - y'\|^2 \\
&+ \prod_{i=2}^{M+1}(1-\theta_n^i)\|S_M y_n - y'\|^2 \\
&- \theta_n^2\Big[\sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)\|y_n - S_{k-2}y_n\|^2 + \prod_{i=2}^{M+1}(1-\theta_n^i)\|y_n - S_M y_n\|^2\Big] \\
&\leq \theta_n^2\|y_n - y'\|^2 + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)(\|y_n - y'\|^2 + \eta_{k-2}\|y_n - S_{k-2}y_n\|^2) \\
&+ \prod_{i=2}^{M+1}(1-\theta_n^i)(\|y_n - y'\|^2 + \eta_M\|y_n - S_M y_n\|^2) \\
&- \theta_n^2\Big[\sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)\|y_n - S_{k-2}y_n\|^2 + \prod_{i=2}^{M+1}(1-\theta_n^i)\|y_n - S_M y_n\|^2\Big] \\
&\leq \Big[\theta_n^2 + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i) + \prod_{i=2}^{M+1}(1-\theta_n^i)\Big]\|y_n - y'\|^2 \\
&- \Big[\sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)(\theta_n^2 - \eta_{k-2})\|y_n - S_{k-2}y_n\|^2 \\
&+ \prod_{i=2}^{M+1}(1-\theta_n^i)(\theta_n^2 - \eta_M)\|y_n - S_M y_n\|^2\Big] \\
&\leq \|y_n - y'\|^2 - \Big[\sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i)(\theta_n^2 - \eta)\|y_n - S_{k-2}y_n\|^2 \\
&+ \prod_{i=2}^{M+1}(1-\theta_n^i)(\theta_n^2 - \eta)\|y_n - U_N y_n\|^2\Big] &\quad (3.24) \\
&\leq \|y_n - y'\|^2 &\quad (3.25) \\
&\leq \|x_n - y'\|^2. &\quad (3.26)
\end{aligned}$$

From equations (3.6) and (3.26), we deduce that

$$\begin{aligned}
\|t_{n+1} - y'\| &= \|\theta_n^1\phi(t_n) + (1-\theta_n^1)h_n - y'\| \\
&= \|\theta_n^1(\phi(t_n) - y') + (1-\theta_n^1)(h_n - y')\| \\
&\leq \theta_n^1\|\phi(t_n) - \phi(y')\| + \theta_n^1\|\phi(y') - y'\| + (1-\theta_n^1)\|h_n - y'\| \\
&\leq \theta_n^1\delta\|t_n - y'\| + \theta_n^1\|\phi(y') - y'\| + (1-\theta_n^1)\|x_n - y'\| \\
&\leq \theta_n^1\delta\|t_n - y'\| + \theta_n^1\|\phi(y') - y'\| + (1-\theta_n^1)(\|t_n - y'\| + \gamma_n\|t_n - t_{n-1}\|) \\
&= (1-(1-\delta)\theta_n^1)\|t_n - y'\| + \theta_n^1\|\phi(y') - y'\| + \theta_n^1\frac{\gamma_n}{\theta_n^1}\|t_n - t_{n-1}\|. &\quad (3.27)
\end{aligned}$$

As $\lim_{n\to\infty} \frac{\gamma_n}{\theta_n^1}\|t_n - t_{n-1}\| = 0$, we have $\frac{\gamma_n}{\theta_n^1}\|t_n - t_{n-1}\| \le K$. By using equation (3.27) we find

$$\begin{aligned} \|t_{n+1} - y'\| &\le (1-(1-\delta)\theta_n^1)\|t_n - y'\| + \theta_n^1\Big(\|\phi(y') - y'\| + K\Big) \\ &\le \max\{\|t_n - y'\|, \frac{K + \|\phi(y') - y'\|}{1-\delta}\} \\ &\vdots \\ &\le \max\{\|t_1 - y'\|, \frac{K + \|\phi(y') - y'\|}{1-\delta}\}. \end{aligned} \tag{3.28}$$

□

Hence the sequence $\{t_n\}$ is bounded. Also, $\{\phi(t_n)\}$, $\{y_n\}$ and $\{x_n\}$ are bounded. Consider

$$\begin{aligned} \|x_n - y'\|^2 &= \|t_n - \gamma_n(t_n - t_{n-1}) - y'\|^2 \\ &= \|t_n - y'\|^2 + \gamma_n^2\|t_n - t_{n-1}\| - 2\gamma_n\left\langle t_n - t_{n-1}, t_n - y'\right\rangle \\ &\le \|t_n - y'\|^2 + \gamma_n^2\|t_n - t_{n-1}\| + 2\gamma_n\|t_n - t_{n-1}\|\|t_n - y'\| \\ &\le \|t_n - y'\|^2 + \gamma_n\|t_n - t_{n-1}\|[\theta_n^1 . \frac{\gamma_n\|t_n - t_{n-1}\|}{\theta_n^1} + 2\|t_n - y'\|] \\ &\le \|t_n - y'\|^2 + \gamma_n\|t_n - t_{n-1}\|K_0, \end{aligned} \tag{3.29}$$

where $K_0 = \sup\{\gamma_n\|t_n - t_{n-1}\| + 2\|t_n - y'\|;\ n \in \mathbb{N}\}$. Now, by equation (3.24), we get

$$\begin{aligned} \|t_{n+1} - y'\|^2 &= \|\theta_n^1\phi(t_n) + (1-\theta_n^1)h_n - y'\|^2 \\ &\le \theta_n^1\|\phi(t_n) - y'\|^2 + (1-\theta_n^1)\|h_n - y'\|^2 \\ &\le \theta_n^1\|\phi(t_n) - y'\|^2 + \|y_n - y'\|^2 - \Big[\sum_{k=3}^{M+1}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)(\theta_n^2 - \eta)\|y_n - S_{k-2}y_n\|^2 \\ &+ \prod_{i=2}^{M+1}(1-\theta_n^i)(\theta_n^2 - \eta)\|y_n - S_M y_n\|^2\Big]. \end{aligned} \tag{3.30}$$

Using equations (3.21) and (3.29), we get

$$\begin{aligned} &\sum_{k=3}^{M+1}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)(\theta_n^2 - \eta)\|y_n - S_{k-2}y_n\|^2 \\ &\quad + \prod_{i=2}^{M+1}(1-\theta_n^i)(\theta_n^2 - \eta)\|y_n - S_M y_n\|^2 \le \theta_n^1\|\phi(t_n) - y'\|^2 + \|t_n - y'\|^2 - \|t_{n+1} - y'\|^2 + \gamma_n\|t_n - t_{n-1}\|K_0. \end{aligned} \tag{3.31}$$

Consider,

$$\begin{aligned} \|t_{n+1} - y'\|^2 &= \|\theta_n^1\phi(t_n) + (1-\theta_n^1)h_n - y'\|^2 \\ &= \|\theta_n^1(\phi(t_n) - \phi(y')) + (1-\theta_n^1)(h_n - y') + \theta_n^1(\phi(y') - y')\|^2 \\ &\le \|\theta_n^1(\phi(t_n) - \phi(y')) + (1-\theta_n^1)(h_n - y')\|^2 + 2\theta_n^1\left\langle\phi(y') - y', t_{n+1} - y'\right\rangle \\ &\le \theta_n^1\|\phi(t_n) - \phi(y')\|^2 + (1-\theta_n^1)\|h_n - y'\|^2 + 2\theta_n^1\left\langle\phi(y') - y', t_{n+1} - y'\right\rangle \\ &\le \theta_n^1\delta\|t_n - y'\|^2 + (1-\theta_n^1)(\|t_n - y'\|^2 + \gamma_n\|t_n - t_{n-1}\|K_0) + 2\theta_n^1\left\langle\phi(y') - y', t_{n+1} - y'\right\rangle \\ &\le (1-(1-\delta)\theta_n^1)\|t_n - y'\|^2 + \gamma_n\|t_n - t_{n-1}\|K_0 + 2\theta_n^1\left\langle\phi(y') - y', t_{n+1} - y'\right\rangle \\ &\le (1-(1-\delta)\theta_n^1)\|t_n - y'\|^2 + (1-\delta)\theta_n^1\left[\frac{K_0}{1-\delta}\frac{\gamma_n\|t_n - t_{n-1}\|}{\theta_n^1} + \frac{2}{1-\delta}\left\langle\phi(y') - y', t_{n+1} - y'\right\rangle\right]. \end{aligned} \tag{3.32}$$

Hence, we get

$$q_{n+1} \le (1-p_n)q_n + p_n b_n, \tag{3.33}$$

where $q_n = \|t_n - y'\|^2$, $p_n = (1-\delta)\theta_n^1$ and $b_n = \frac{K_0}{1-\delta}\frac{\gamma_n\|t_n - t_{n-1}\|}{\theta_n^1} + \frac{2}{1-\delta}\langle \phi(y') - y', t_{n+1} - y'\rangle$.
Now, we show that $t_n \to y'$.

Case 1: Assume that there exists a $N \in \mathbb{N}$ such that $\{\|t_n - y'\|\}$ is decreasing for $n \geq N$. As $\{\|t_n - y'\|\}$ is bounded and monotonic and subsequently convergent. From equation (3.2), we get

$$\|x_n - t_n\| = \|-\gamma_n(t_n - t_{n-1})\| = \|\theta_n^1.\frac{\gamma_n(t_n - t_{n-1})}{\theta_n^1}\|. \tag{3.34}$$

Taking limit $n \to \infty$, we get

$$\lim_{n\to\infty}\|t_n - x_n\| = 0. \tag{3.35}$$

Consider

$$\begin{aligned}
\|t_{n+1} - y_n\|^2 &= \|\theta_n^1\phi(y_n) + (1-\theta_n^1)[\theta_n^2 y_n + \sum_{k=3}^{M+1}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)S_{k-2}y_n \\
&+ \prod_{i=2}^{M+1}(1-\theta_n^i)S_M y_n] - y_n\|^2 \\
&\leq \theta_n^1\|\phi(y_n) - y_n\|^2 + (1-\theta_n^1)[\sum_{k=3}^{M+1}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)\|S_{k-2}y_n - y_n\|^2 \\
&+ \prod_{i=2}^{M+1}(1-\theta_n^i)\|S_M y_n - y_n\|^2].
\end{aligned} \tag{3.36}$$

Taking limit $n \to \infty$ in equation (3.31) and using conditions (i)-(iii), we have

$$\lim_{n\to\infty}\|y_n - S_{k-2}y_n\| = 0 \text{ for } 3 \leq k \leq M+1 \text{ and } \lim_{n\to\infty}\|y_n - S_M y_n\| = 0. \tag{3.37}$$

Taking limit $n \to \infty$ in equation (3.36). From equation (3.37) and $\lim_{n\to\infty}\theta_n^1 = 0$, we have

$$\lim_{n\to\infty}\|t_{n+1} - y_n\| = 0. \tag{3.38}$$

Using equation (3.25), we have

$$\begin{aligned}
\|t_{n+1} - y'\|^2 &= \|\theta_n^1\phi(t_n) + (1-\theta_n^1)h_n - y'\|^2 \\
&\leq \theta_n^1\|\phi(t_n) - y'\|^2 + (1-\theta_n^1)\|h_n - y'\|^2 \\
&\leq \theta_n^1\|\phi(t_n) - y'\|^2 + (1-\theta_n^1)\|y_n - y'\|^2 \\
&\leq \theta_n^1\|\phi(t_n) - y'\|^2 + \|y_n - y'\|^2,
\end{aligned} \tag{3.39}$$

which implies

$$-\|y_n - y'\|^2 \leq \theta_n^1\|\phi(t_n) - y'\|^2 - \|t_{n+1} - y'\|^2. \tag{3.40}$$

As $y_n = P_{E_n}(x_n)$ and using Lemma (2.7), we have

$$\|y' - y_n\|^2 + \|x_n - y_n\|^2 \leq \|x_n - y'\|^2, \tag{3.41}$$

which implies

$$\|x_n - y_n\|^2 \leq \|x_n - y'\|^2 - \|y' - y_n\|^2. \tag{3.42}$$

Using equations (3.29), (3.40) and (3.42), we have

$$\begin{aligned}
\|x_n - y_n\|^2 &\leq \|x_n - y'\|^2 + \theta_n^1\|\phi(t_n) - y'\|^2 - \|t_{n+1} - y'\|^2 \\
&\leq \|t_n - y'\|^2 + \theta_n^1\frac{\gamma_n\|t_n - t_{n-1}\|K_0}{\theta_n^1} + \theta_n^1\|\phi(t_n) - y'\|^2 - \|t_{n+1} - y'\|^2.
\end{aligned} \tag{3.43}$$

Taking limit $n \to \infty$ in equation (3.43) and using and $\lim_{n\to\infty}\frac{\gamma_n\|t_n - t_{n-1}\|}{\theta_n^1} = 0$, $\lim_{n\to\infty}\theta_n^1 = 0$, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.44}$$

Using equations (3.35), (3.38) (3.44) and by triangular inequality, we estimate

$$\lim_{n\to\infty} \|t_{n+1} - t_n\| = 0. \tag{3.45}$$

Using Lemma (2.9) and equations (3.15), (3.20) we have

$$\begin{aligned}\|y_n - y'\|^2 &\le \|x_n - y'\|^2 - d(x_n, E_n) \\ &\le \|x_n - y'\|^2 - (\frac{1}{\theta} l_n^j(x_n)) \\ &\le \|x_n - y'\|^2 - (\frac{\rho^{k_n}\phi_1}{2\theta(1-\rho^{k_n})\kappa_n^j}\|x_n - u_n^j\|^2).\end{aligned} \tag{3.46}$$

Using equations (3.29), (3.39) and (3.46), we have

$$\begin{aligned}\|t_{n+1} - y'\|^2 &\le \theta_n^1\|\phi(t_n) - y'\|^2 + \|x_n - y'\|^2 - (\frac{\rho^{k_n}\phi_1}{2\theta(1-\rho^{k_n})\kappa_n^j}\|x_n - u_n^j\|^2) \\ &\le \theta_n^1\|\phi(t_n) - y'\|^2 - (\frac{\rho^{k_n}\phi_1}{2\theta(1-\rho^{k_n})\kappa_n^j}\|x_n - u_n^j\|^2) + \|t_n - y'\|^2 + \gamma_n\|t_n - t_{n-1}\|K_0,\end{aligned} \tag{3.47}$$

which implies

$$\frac{\rho^{k_n}\phi_1}{2\theta(1-\rho^{k_n})\kappa_n^j}\|x_n - u_n^j\|^2 \le \theta_n^1\|\phi(t_n) - y'\|^2 + \|t_n - y'\|^2 - \|t_{n+1} - y'\|^2 + \theta_n^1.\frac{\gamma_n\|t_n - t_{n-1}\|K_0}{\theta_n^1}. \tag{3.48}$$

Taking limit $n \to \infty$ and using $\lim_{n\to\infty} \frac{\gamma_n\|t_n - t_{n-1}\|}{\theta_n^1} = 0$ and $\lim_{n\to\infty} \theta_n^1 = 0$, we have

$$\lim_{n\to\infty} \frac{\rho^{k_n}\phi_1}{2\theta(1-\rho^{k_n})\kappa_n^j}\|x_n - u_n^j\|^2 = 0. \tag{3.49}$$

Hence

$$\lim_{n\to\infty} \rho^{k_n}\|x_n - u_n^j\| = 0 \text{ for all } 1 \le j \le N. \tag{3.50}$$

Let us take two different cases:

Case (i): If $\lim_{l\to\infty} \rho^{k_{n_l}} > 0$, then there is a $\bar{\rho} > 0$ and a subsequence $\{\rho^{k_{n_l}}\}$ of $\{\rho^{k_n}\}$ such that for some $l_0 > 0$, $\{\rho^{k_{n_l}}\} > \bar{\rho}$ for all $l > l_0$. Since $\{x_n\}$ and $\{u_n^j\}$ are bounded sequences, there exists subsequence $\{x_{n_l}\}$ and $\{u_{n_l}^j\}$ are subsequences of $\{x_n\}$ and $\{u_n^j\}$. Let $\{x_{n_l}\}$ converges weakly to some $p' \in Q$. This implies

$$\lim_{l\to\infty} \|x_{n_l} - u_{n_l}^j\| = 0 \text{ for all } 1 \le j \le N, \tag{3.51}$$

which implies $\{u_{n_l}^j\}$ also converges weakly to some p' as $l \to \infty$. Using the definition of $u_{n_l}^j$

$$u_{n_l}^j = argmin\left\{\frac{1}{2\kappa_{n_l}^j}\|y - x_{n_l}\|^2 + h_j(x_{n_l}, y) : y \in Q\right\}, \tag{3.52}$$

which gives

$$0 \in \partial h_j(x_{n_l}, u_{n_l}^j) + \frac{1}{\kappa_{n_l}^j}(u_{n_l}^j - x_{n_l}) + N_Q(u_{n_l}^j) \text{ for all } 1 \le j \le N. \tag{3.53}$$

There exists $g_{n_l}^j \in \partial h_j(x_{n_l}, u_{n_l}^j)$ for $1 \le j \le N$ such that

$$\left\langle g_{n_l}^j, y - u_{n_l}^j \right\rangle + \frac{1}{\kappa_{n_l}^j}\left\langle u_{n_l}^j - x_{n_l}, y - u_{n_l}^j \right\rangle \ge 0 \text{ for all } y \in Q. \tag{3.54}$$

As we know that

$$h_j(x_{n_l}, y) - h_j(x_{n_l}, u_{n_l}^j) \ge \left\langle g_{n_l}^j, y - u_{n_l}^j \right\rangle \text{ for all } y \in Q. \tag{3.55}$$

Also

$$\left\langle u_{n_l}^j - x_{n_l}, y - u_{n_l}^j \right\rangle \le \|u_{n_l}^j - x_{n_l}\|\|y - u_{n_l}^j\|. \tag{3.56}$$

Using equations (3.54), (3.55) and (3.56), we have

$$h_j(x_{n_l},y)-h_j(x_{n_l},u^j_{n_l})+\frac{1}{\kappa^j_{n_l}}\left\langle u^j_{n_l}-x_{n_l},y-u^j_{n_l}\right\rangle\geq 0 \text{ for all } y\in Q \tag{3.57}$$

and

$$h_j(x_{n_l},y)-h_j(x_{n_l},u^j_{n_l})+\frac{1}{\kappa^j_{n_l}}\|u^j_{n_l}-x_{n_l}\|\|y-u^j_{n_l}\|\geq 0 \text{ for all } y\in Q. \tag{3.58}$$

Letting $l\to\infty$ and using the weak continuity of h_j and equation (3.51), we have

$$h_j(p',y)-h_j(p',p')\geq 0 \text{ for all } y\in Q. \tag{3.59}$$

Therefore

$$h_j(p',y)\geq 0 \text{ for all } y\in Q. \tag{3.60}$$

Hence $p'\in\bigcap_{j=1}^N EP(h_j,Q)$.

Case (ii): Assume that $\lim_{n\to\infty}\rho^{k_{n_l}}=0$. Since $\{u^j_n\}$ is bounded sequence, there exists a subsequence $\{u^j_{n_l}\}$ of $\{u^j_n\}$. Let $\{u^j_{n_l}\}$ converges weakly to some $q'\in Q$. Put $y=x_{n_l}$ in equation (3.57), we have

$$h_j(x_{n_l},u^j_{n_l})\leq-\frac{1}{\kappa^j_{n_l}}\|u^j_{n_l}-x_{n_l}\|^2. \tag{3.61}$$

Putting $k=k_{n_l}$ and using equation (3.3), we have

$$-h_j(v^j_{n_l,k_{n_l}-1},u^j_{n_l})\geq\frac{\phi_1}{2\kappa^j_n}\|x_{n_l}-u^j_{n_l}\|^2. \tag{3.62}$$

From equations (3.61) and (3.62), we have

$$h_j(x_{n_l},u^j_{n_l})\leq-\frac{1}{\kappa^j_{n_l}}\|u^j_{n_l}-x_{n_l}\|^2\leq\frac{2}{\phi_1}(h_j(v^j_{n_l,k_{n_l}-1},u^j_{n_l})). \tag{3.63}$$

As $v^j_{n_l,k}=(1-\rho^{k_{n_l}})x_{n_l}+\rho^{k_n}u^j_{n_l}$ and $\rho^{k_{n_l}}\to 0$, $x_{n_l}\to p'$ and $u^j_{n_l}\to q'$ as $l\to\infty$, which implies $v^j_{n_l,k}\to p'$ as $l\to\infty$. Also, $\{\frac{1}{\kappa^j_{n_l}}\|u^j_{n_l}-x_{n_l}\|^2\}$ is bounded. So, we can assume that $\lim_{l\to\infty}\frac{1}{\kappa^j_{n_l}}\|u^j_{n_l}-x_{n_l}\|^2$ exists. Using weak continuity of h_j and taking limit $l\to\infty$ in equation (3.63), we have

$$h_j(p',q')\leq-\lim_{l\to\infty}\frac{1}{\kappa^j_{n_l}}\|u^j_{n_l}-x_{n_l}\|^2\leq\frac{2}{\phi_1}h_j(p',q'). \tag{3.64}$$

Hence, $h_j(p',q')=0$ and $\lim_{l\to\infty}\|u^j_{n_l}-x_{n_l}\|=0$. Using similar process as in Case (i), we have $p'\in\bigcap_{j=1}^N EP(h_j,Q)$. Taking limit $n\to\infty$ in equation (3.31) and using conditions (i)-(iii), we have

$$\lim_{n\to\infty}\|y_n-S_{k-2}y_n\|=0 \text{ for } 3\leq k\leq M+1 \text{ and } \lim_{n\to\infty}\|y_n-S_My_n\|=0. \tag{3.65}$$

As $\{x_{n_m}\}$ converges weakly to some $p'\in Q$ and from equation (3.44), we have $\lim_{n\to\infty}\|x_n-y_n\|=0$, which implies $\{y_{n_m}\}$ converges weakly to p'. Also, $I-S_k$ is demiclosed at zero for each $k=1,2,\ldots,M$, therefore $p'\in\bigcap_{k=1}^M Fix(S_k)$. Subsequently, $p'\in\bigcap_{k=1}^M Fix(S_k)\cap\bigcap_{j=1}^N EP(h_j,Q)$. Consider a subsequence $\{t_{n_l}\}$ of $\{t_n\}$ and clearly it converges weakly to p'. Also, by Lemma (2.7), we have

$$\begin{aligned}\limsup_{n\to\infty}\left\langle t_n-y',\phi(y')-y'\right\rangle&\leq\lim_{l\to\infty}\left\langle t_{n_l}-y',\phi(y')-y'\right\rangle\\&\leq\left\langle p'-y',\phi(y')-y'\right\rangle\\&\leq 0.\end{aligned} \tag{3.66}$$

As $\sum_{n=0}^\infty p_n=\infty$, from equations (3.33), (3.66) and Lemma (2.12) we get $\|t_n-w'\|\to 0$. This implies $t_n\to w'$.

Case 2: Suppose that there is a subsequence $\{q_{n_l}\}$ of $\{q_n\}$ such that

$$q_{n_l+1}\geq q_{n_l} \quad \text{for all} \quad l\in\mathbb{N}.$$

Hence, by Lemma 2.8, there is a nondecreasing sequence of natural numbers $\{n_s\} \subset \mathbb{N}$ such that $n_s \to \infty$ as $s \to \infty$, we have

$$\|t_{n_s} - y'\| \le \|t_{n_s+1} - y'\|, \tag{3.67}$$

and

$$\|t_s - y'\| \le \|t_{n_s+1} - y'\|. \tag{3.68}$$

Using equation (3.43), we have

$$\begin{aligned} \|x_{n_s} - a_{n_s}\|^2 &\le \|t_{n_s} - y'\|^2 + \gamma_{n_s}\|t_{n_s} - t_{n_s-1}\|K_0 + \theta^1_{n_s}\|\phi(t_{n_s}) - y'\|^2 - \|t_{n_s+1} - y'\|^2 \\ &\le \theta^1_{n_s}\frac{\gamma_{n_s}\|t_{n_s} - t_{n_s-1}\|K_0}{\theta^1_{n_s}} + \theta^1_{n_s}\|\phi(t_{n_s}) - y'\|^2. \end{aligned} \tag{3.69}$$

Taking limit as $s \to \infty$ and using $\lim_{s\to\infty} \frac{\gamma_{n_s}\|t_{n_s} - t_{n_s-1}\|}{\theta^1_{n_s}} = 0$ and $\lim\limits_{s\to\infty} \theta^1_{n_s} = 0$, we have

$$\lim_{s\to\infty} \|x_{n_s} - y_{n_s}\| = 0.$$

This with equation (3.31) gives

$$\begin{aligned} &\sum_{k=3}^{M+1} \theta^k_{n_s} \prod_{i=2}^{k-1} (1-\theta^i_{n_s})(\theta^2_{n_s} - \eta)\|y_{n_s} - S_{k-2}y_{n_s}\|^2 \\ &\quad + \prod_{i=2}^{M+1}(1-\theta^i_{n_s})(\theta^2_{n_s} - \eta)\|y_{n_s} - S_M y_{n_s}\|^2 \le \theta^1_{n_s}\|\phi(t_{n_s}) - y'\|^2 + \|t_{n_s} - y'\|^2 - \|t_{n_s+1} - y'\|^2 + \gamma_{n_s}\|t_{n_s} - t_{n_s-1}\|K_0 \\ &\qquad \le \theta^1_{n_s}\|\phi(t_{n_s}) - y'\|^2 + \theta^1_{n_s}\frac{\gamma_{n_s}\|t_{n_s} - t_{n_s-1}\|K_0}{\theta^1_{n_s}}. \end{aligned} \tag{3.70}$$

Taking limit as $s \to \infty$ and using $\lim\limits_{s\to\infty} \theta^1_{n_s} = 0$ and $\lim_{s\to\infty} \frac{\gamma_{n_s}\|t_{n_s} - t_{n_s-1}\|}{\theta^1_{n_s}} = 0$, we have

$$\lim_{s\to\infty} \|y_{n_s} - S_{k-2}y_{n_s}\| = 0, \text{ for } 3 \le k \le M+2. \tag{3.71}$$

Similarly, we can prove

$$\lim_{s\to\infty} \|t_{n_s} - x_{n_s}\| = 0 \text{ and } \lim_{s\to\infty} \|t_{n_s+1} - y_{n_s}\| = 0. \tag{3.72}$$

Also,

$$\lim_{s\to\infty} \|t_{n_s+1} - t_{n_s}\| = 0. \tag{3.73}$$

As in Case 1, we can prove that

$$\lim_{s\to\infty} \|x_{n_s} - y_{n_s}\| = 0. \tag{3.74}$$

Proceeding similarly as in Case 1, we have

$$\limsup_{s\to\infty} \left\langle \phi(y') - y', t_{n_s+1} - y' \right\rangle \le 0. \tag{3.75}$$

From equation (3.33), we have

$$q_{n_s+1} \le (1 - p_{n_s})q_{n_s} + p_{n_s}b_{n_s}. \tag{3.76}$$

As $q_{n_s+1} \ge q_{n_s}$, we get

$$\begin{aligned} p_{n_s}q_{n_s} &\le q_{n_s} - q_{n_s+1} + p_{n_s}b_{n_s} \\ &\le p_{n_s}b_{n_s} \end{aligned} \tag{3.77}$$

Using $p_{n_s} > 0$, we have $q_{n_s} \le b_{n_s}$. Subsequently,

$$\|t_{n_s} - y'\|^2 \le \frac{K_0}{1-\delta}\frac{\gamma_{n_s}\|t_{n_s} - t_{n_s-1}\|}{\theta^1_{n_s}} + \frac{2}{1-\delta}\left\langle \phi(y') - y', t_{n_s+1} - y' \right\rangle \tag{3.78}$$

As $\{t_{n_s}\}$ is bounded and $\frac{K_0}{1-\delta}\frac{\gamma_{n_s}\|t_{n_s}-t_{n_s-1}\|}{\theta^1_{n_s}} \to 0$ as $s\to\infty$. Hence, from equation (3.75), we obtain $\|t_{n_s}-y'\|\to 0$ as $s\to\infty$. This together with equation (3.77) implies that $\|t_{n_s+1}-y'\|\to 0$ as $s\to\infty$. Also from equation (3.68), we have $\|t_s-y'\|\le\|t_{n_s+1}-y'\|$ for all $s\in\mathbb{N}$, which gives $t_s\to y'$ as $s\to\infty$. The proof is complete.

Remark 3.7. (i) Hieu's Algorithm [19] requires to solve two strongly convex program at each iteration, whereas our algorithm requires to solve only one strongly convex program.

(ii) Theorem 3.6 generalizes the result of Isiogugu et al. [23] from Halpern method to the Line search extragradient method including viscosity approximation.

(iii) Theorem 3.6 generalizes and extends Ogbuisi and Isiogugu [25] (Theorem 3.1) from a pseudomonotone EP to a common solution of finite families of pseudomonotone EPs. Ogbuisi and Isiogugu [25] (Algorithm 3) use conditions on control parameters that $\beta_n>0$, $\sum_{k=1}^{\infty}\beta_n=\infty$, $\sum_{k=1}^{\infty}\beta_n^2<\infty$, and $\sum_{k=1}^{\infty}\beta_n\varepsilon_n=\infty$. We do not require such conditions.

(iv) Jolaoso et al. result [17] use the condition on control parameters that $\sum_{j=0}^{M}\delta_{n,j}=1$. We do not assume such condition.

(v) Theorem 3.6 generalizes and extends Hieu's [26] result from nonexpansive mappings to demicontractive mappings. Furthermore, our result is independent of prior knowledge of the Lipschitz constant and Hieu's Algorithm requires to solve two strongly convex program at each iteration.

4. Applications

In this section, we will study that how our result can be used to find the common solution of finite families of equilibrium and various nonlinear analysis problems like variational inequality problems, variational inclusion problems, null point problems, multiple-sets split feasibility problems and monotone equilibrium problems.

4.1. **Variational Inclusion Problem.** Suppose that H is a real Hilbert space. Let $S:H\to 2^H$ be set-valued maximal monotone operator. Let r be ν-ism operator in H. Take $\tau\in(0,2\nu)$. Then, monotone variational inclusion problem is to find $y'\in H$ such that

$$0\in r(y')+S(y'). \tag{4.1}$$

We know that $0\in r(y')+S(y')$ iff $y'\in Fix(J^S_\tau(I-\tau r))$, where J^S_τ is resolvent operator. Also, if $\tau\in(0,2\nu)$, then $J^S_\lambda(I-\tau r)$ is averaged mapping, see [27]. We now present a strong convergence theorem for approximating common solution of finite families of equilibrium and monotone variational inclusion problems.

Theorem 4.1. *Let $h_j:Q_j\times Q_j\to\mathbb{R}$ be bifunctions satisfying Assumption 1 and $S_k:H\to 2^H$ be set valued maximal monotone operator. Let r_k be ν_k-ism operator in H. Set $\nu=\max\{\nu_k\}$. Take $\tau\in(0,2\nu)$. Suppose $\bigcap_{j=1}^N EP(h_j,Q)\cap(\bigcap_{k=1}^M Fix(J^{S_k}_\tau(I-\tau r_k)))\neq\phi$ and for any $t_0,t_1\in H$, $\{t_n\}$ is defined by*

$$\begin{cases} x_n=t_n-\gamma_n(t_n-t_{n-1}),\\ u_n^j=argmin\left\{\dfrac{1}{2\kappa_n^j}\|y-x_n\|^2+h_j(x_n,y):y\in Q\right\},\\ v^j_{n,k}=(1-\rho^{k_n})x_n+\rho^{k_n}u_n^j,\\ h_j(v^j_{n,k},x_n)-h_j(v^j_{n,k},u^j_n)\ge\frac{\phi_1}{\kappa_n^j}\|x_n-u_n^j\|^2,\\ E_n=\{t\in H:\sum_{j=1}^N\alpha_n^j l_n^j(t)\le 0\},\\ y_n=P_{E_n}(x_n),\\ t_{n+1}=\theta_n^1\phi(t_n)+(1-\theta_n^1)[\theta_n^2y_n+\sum_{k=3}^{M+1}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)J^{S_{k-2}}_\tau(I-\tau r_{k-2})y_n\\ \qquad+\prod_{i=2}^{M+1}(1-\theta_n^i)J^{S_M}_\tau(I-\tau r_M)))y_n],\end{cases}$$

where $\gamma_n,\phi_1,\rho\in(0,1)$, $\{\kappa_n^j\}\subset[\kappa,\bar\kappa]$, where $0\le\kappa\le\bar\kappa$, $l_n^j(t)=\left\langle x_n^j,t-v_n^j\right\rangle$, where $x_n^j\in\partial h_j(v_n^j,v_n^j)$ for $j=1,2,...,N$. and ϕ is a δ- contraction operator defined on H. Also $\{\theta_n^k\}$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\theta_n^1=0$, $\lim_{n\to\infty}\frac{\gamma_n}{\theta_n^1}\|t_n-t_{n-1}\|=0$ *and* $\sum_{n=0}^{\infty}\theta_n^1=\infty$,
(ii) $\liminf_{n\to\infty}\theta_n^2>\nu$ *where* $n\in\mathbb{N}$,
(iii) $\liminf_{n\to\infty}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)>0$ *for every* $k=3,4,\ldots.M+1$ *and* $\liminf_{n\to\infty}\prod_{i=2}^{M+1}(1-\theta_n^i)>0$.

Then $\{t_n\}$ *converges strongly to* $y'\in\bigcap_{j=1}^{N}EP(h_j,Q)\cap(\bigcap_{k=1}^{M}Fix(J_\tau^{S_k}(I-\tau r_k)))$.

Proof: Let us take $S_k=J_\tau^{S_k}(I-\tau r_k)$ for all $k\in N$. Therefore, S_k is averaged mapping and thus nonexpansive. Additionally, $I-S_k$ is demiclosed at zero. Thus, Theorem 3.6 leads to the conclusion.

Remark 4.2. Cholamjiak [28] Algorithm requires the computation of C_{n+1} and the projection of x_0 onto C_{n+1}. If the feasible set is complex, it may be computationally expensive. Theorem 4.1 generalizes and extends Cholamjiak et al. result [28] from finite family of monotone inclusion problems to a common solution of finite families of monotone inclusion and equilibrium problems.

4.2. **Variational Inequality Problem.** Suppose that H is a real Hilbert space. Let $r:H\to H$ be an operator. Assume that Q is nonempty closed and convex subsets of H, then the variational inequality problem is to identify a point $y'\in Q$ such that

$$\langle r(y'),y-y'\rangle\geq 0\ \forall y\in Q. \tag{4.2}$$

We denote the solution set of variational inequality problem by VI(Q,r). If r is β-ism operator on H, it is clear that $P_Q(I-\tau r)$ is nonexpansive for every $\tau\in(0,2\beta)$. As $y'\in Fix(P_Q(I-\tau r))$ iff $y'\in$ VI(Q,r), see [27]. We now give a strong convergence theorem for approximating common solution of finite families of monotone variational inequality and equilibrium problems.

Theorem 4.3. *Let* $h_j:Q_j\times Q_j\to\mathbb{R}$ *be bifunctions satisfying Assumption 1 and* r_k *be* ν_k*-ism operator in* H. *Set* $\nu=\max\{\nu_k\}$. *Take* $\tau\in(0,2\nu)$. *Suppose* $\bigcap_{j=1}^{N}EP(h_j,Q)\cap(\bigcap_{k=1}^{M}VIP(r_k,Q_k)\neq\phi$ *and for any* t_0 *and* $t_1\in H,\{t_n\}$ *is defined by*

$$\begin{cases} x_n=t_n-\gamma_n(t_n-t_{n-1}),\\ u_n^j=argmin\left\{\dfrac{1}{2\kappa_n^j}\|y-x_n\|^2+h_j(x_n,y):y\in Q\right\},\\ v_{n,k}^j=(1-\rho^{k_n})x_n+\rho^{k_n}u_n^j,\\ h_j(v_{n,k}^j,x_n)-h_j(v_{n,k}^j,u_n^j)\geq\frac{\phi_1}{\kappa_n^j}\|x_n-u_n^j\|^2,\\ E_n=\{t\in H:\sum_{j=1}^{N}\alpha_n^jl_n^j(t)\leq 0\},\\ y_n=P_{E_n}(x_n),\\ t_{n+1}=\theta_n^1\phi(t_n)+(1-\theta_n^1)[\theta_n^2y_n+\sum_{k=3}^{M+1}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)P_{Q_{k-2}}(I-\tau r_{k-2})y_n\\ \qquad+\prod_{i=2}^{M+1}(1-\theta_n^i)P_{Q_M}(I-\tau r_M)y_n], \end{cases}$$

where $\gamma_n,\phi_1,\rho\in(0,1)$, $\{\kappa_n^j\}\subset[\kappa,\bar{\kappa}]$, *where* $0\leq\kappa\leq\bar{\kappa}$, $l_n^j(t)=\left\langle x_n^j,t-v_n^j\right\rangle$, *where* $x_n^j\in\partial h_j(v_n^j,v_n^j)$ *for* $j=1,2,\ldots,N$ *and* ϕ *is a* δ*-contraction operator defined on* H. *Also* $\{\theta_n^k\}$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\theta_n^1=0$, $\lim_{n\to\infty}\frac{\gamma_n}{\theta_n^1}\|t_n-t_{n-1}\|=0$ *and* $\sum_{n=0}^{\infty}\theta_n^1=\infty$,
(ii) $\liminf_{n\to\infty}\theta_n^2>\nu$ *where* $n\in\mathbb{N}$,
(iii) $\liminf_{n\to\infty}\theta_n^k\prod_{i=2}^{k-1}(1-\theta_n^i)>0$ *for every* $k=3,4,\ldots.M+1$ *and* $\liminf_{n\to\infty}\prod_{i=2}^{M+1}(1-\theta_n^i)>0$.

Then $\{t_n\}$ *converges strongly to* $y'\in\bigcap_{j=1}^{N}EP(h_j,Q)\cap(\bigcap_{k=1}^{M}VIP(r_k,Q_k))$.

Proof: Let us take $S_k=P_{Q_k}(I-\tau r_k)$ for all $k\in N$. Therefore, S_k is averaged mapping and thus nonexpansive. Additionally, $I-S_k$ is demiclosed at zero. Thus, Theorem 3.6 leads to the conclusion.

Remark 4.4. Theorem 4.3 generalizes and extends Hieu's result [19] from a common solution of pseudomonotone EPs to a common solution finite families of pseudomonotone EPs and monotone VIPs.

4.3. **Equilibrium Problem.** Suppose that Q is a non empty closed, convex subset of real Hilbert space H and r is a bifunction. Thus equillibrium problem for r is to find a point $y' \in Q$ and $r(y', p') \geq 0$ for all $p' \in Q$. EP(r) denotes the solution set of equillibrium problem.

Assumption 2: We assume that bifunction r satisfies the following conditions:

1. $r(y', y') \geq 0$.
2. r is monotone i.e. $r(p', y') + r(y', p') \leq 0$ for any $p', y' \in Q$.
3. For each $p', y', q' \in Q$, $\limsup_{s \to 0^+} r(sq' + (1-s)p', y') \leq r(p', y')$.
4. For each $p' \in Q, y' \to r(p', y')$ is lower semi-continuous and convex.

We next present an important lemma for solving the equilibrium problem.

Lemma 4.5 ([29]). *Assume that $r : Q \times Q \to \mathbb{R}$ is a bifunction satisfying Assumption 2. Let $s > 0$ and $p' \in H$, then there is $y' \in Q$ and*

$$r(q', y') + \frac{1}{s} \langle y' - q', q' - p' \rangle \geq 0 \text{ for all } y' \in Q.$$

Further if $T_s^r(p') = \{q' \in C : F(q', y') + \frac{1}{s} \langle y' - q', q' - p' \rangle \geq 0 \ \forall y \in C\}$, then following statements hold:

1. T_s^r is single valued.

2. T_s^r is firmly nonexpansive i.e. for any $p', y' \in H$

$$\|T_s^r(p') - T_s^r(y')\|^2 \leq \langle T_s^r(p') - T_s^r(y'), p' - y' \rangle.$$

3. Fix(T_s^r)=EP(r).

4. EP(r) is convex and closed.

The equilibrium problem can be solved as a fixed point problem, as shown by the above lemma. We now present a strong convergence theorem for estimating a common solution of finite families of equilibrium and monotone equilibrium problems.

Theorem 4.6. *Let $h_j : Q_j \times Q_j \to \mathbb{R}$ be bifunctions satisfying Assumption 1 and $r_k : Q_k \times Q_k \to \mathbb{R}$ be bifunctions satisfying Assumption 2. Suppose $\bigcap_{j=1}^N EP(h_j, Q) \cap (\bigcap_{k=1}^M EP(r_k, Q) \neq \phi$ and for any t_0 and $t_1 \in H, \{t_n\}$ is defined by*

$$\begin{cases} x_n = t_n - \gamma_n(t_n - t_{n-1}), \\ u_n^j = argmin\left\{ \frac{1}{2\kappa_n^j} \|y - x_n\|^2 + h_j(x_n, y) : y \in Q \right\}, \\ v_{n,k}^j = (1 - \rho^{k_n})x_n + \rho^{k_n} u_n^j, \\ h_j(v_{n,k}^j, x_n) - h_j(v_{n,k}^j, u_n^j) \geq \frac{\phi_1}{\kappa_n^j} \|x_n - u_n^j\|^2, \\ E_n = \{t \in H : \sum_{j=1}^N \alpha_n^j l_n^j(t) \leq 0\}, \\ y_n = P_{E_n}(x_n), \\ t_{n+1} = \theta_n^1 \phi(t_n) + (1 - \theta_n^1)[\theta_n^2 y_n + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1} (1 - \theta_n^i) T_s^{r_{k-2}} y_n \\ \qquad + \prod_{i=2}^{M+1} (1 - \theta_n^i) T_s^{r_M} y_n], \end{cases}$$

where $\gamma_n, \phi_1, \rho \in (0,1)$, $\{\kappa_n^j\} \subset [\kappa, \bar{\kappa}]$, where $0 \leq \kappa \leq \bar{\kappa}$, $l_n^j(t) = \left\langle x_n^j, t - v_n^j \right\rangle$, where $x_n^j \in \partial h_j(v_n^j, v_n^j)$ for $j = 1, 2, ..., N$. where ϕ is a δ-contraction operator defined on H. Also $\{\theta_n^k\}$ satisfying the following conditions:

(i) $\lim_{n \to \infty} \theta_n^1 = 0, \quad \lim_{n \to \infty} \frac{\gamma_n}{\theta_n^1} \|t_n - t_{n-1}\| = 0$ *and* $\sum_{n=0}^{\infty} \theta_n^1 = \infty,$

(ii) $\liminf_{n \to \infty} \theta_n^k \prod_{i=2}^{k-1} (1 - \theta_n^i) > 0$ *for every $k = 3, 4, ...M + 1$ and* $\liminf_{n \to \infty} \prod_{i=2}^{M+1} (1 - \theta_n^i) > 0.$

Then $\{t_n\}$ converges strongly to $y' \in \bigcap_{j=1}^N EP(h_j, Q) \cap (\bigcap_{k=1}^M EP(r_k, Q)$.

Proof: From Lemma 4.5 T_s^r is firmly nonexpansive and thus demicontractive. Hence, one can consider fixed point problem as equilibrium problem. Subsequently, the conclusion follows from Theorem 3.6 and Lemma 4.5.

4.4. **Multiple-Sets Split Feasiblity Problem.** Suppose that $D : H_1 \to H_2$ is a bounded linear operator. Let $C_i (1 \leq i \leq N)$ and $Q_j (1 \leq j \leq M)$ be two families of nonempty closed and convex subsets of Hilbert spaces H_1 and H_2 respectively. Then multiple-sets split feasibility problem (MSSFP) is to find a point $y' \in H_1$ and

$$y' \in \bigcap_{k=1}^{M} C_i \text{ and } Dy' \in \bigcap_{l=1}^{P} Q_j. \tag{4.3}$$

As we know that $y' \in Fix(P_{C_k}(I - \eta H^*(I - P_{Q_K})H))$ iff $y' \in C$ where C is convex, closed subset of Hilbert space H. The solution set of the multiple-sets split feasibility problem is denoted by Δ. Next, we prove the strong convergence theorem for estimating common solution of fixed point and multiple-sets split feasibility problems.

Theorem 4.7. *Let $h_j : Q_j \times Q_j \to \mathbb{R}$ be bifunctions satisfying Assumption 1. Suppose $\bigcap_{j=1}^{N} EP(h_j, Q) \cap \Delta \neq \phi$ and for any $t_0, t_1 \in H, \{t_n\}$ is defined by*

$$\begin{cases} x_n = t_n - \gamma_n(t_n - t_{n-1}), \\ u_n^j = argmin\left\{ \dfrac{1}{2\kappa_n^j} \|y - x_n\|^2 + h_j(x_n, y) : y \in Q \right\}, \\ v_{n,k}^j = (1 - \rho^{k_n})x_n + \rho^{k_n} u_n^j, \\ h_j(v_{n,k}^j, x_n) - h_j(v_{n,k}^j, u_n^j) \geq \frac{\phi_1}{\kappa_n^j} \|x_n - u_n^j\|^2, \\ E_n = \{t \in H : \sum_{j=1}^{N} \alpha_n^j l_n^j(t) \leq 0\}, \\ y_n = P_{E_n}(x_n), \\ t_{n+1} = \theta_n^1 \phi(t_n) + (1 - \theta_n^1)[\theta_n^2 y_n + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1} (1 - \theta_n^i) P_{C_{k-2}}(I - \eta H^*(I - P_{Q_{k-2}})H)y_n \\ \qquad + \prod_{i=2}^{M+1} (1 - \theta_n^i) P_{C_M}(I - \eta H^*(I - P_{Q_M})H)y_n], \end{cases}$$

where $\gamma_n, \phi_1, \rho \in (0,1)$, $\{\kappa_n^j\} \subset [\kappa, \bar{\kappa}]$, where $0 \leq \kappa \leq \bar{\kappa}$, $l_n^j(t) = \left\langle x_n^j, t - v_n^j \right\rangle$, where $x_n^j \in \partial h_j(v_n^j, v_n^j)$ for $j = 1, 2, ..., N$. and ϕ is a δ-contraction operator defined on H. Also $\{\theta_n^k\}$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \theta_n^1 = 0$, $\lim_{n\to\infty} \frac{\gamma_n}{\theta_n^1} \|t_n - t_{n-1}\| = 0$ *and* $\sum_{n=0}^{\infty} \theta_n^1 = \infty$,

(ii) $\liminf_{n\to\infty} \theta_n^2 > \nu$ *where* $n \in \mathbb{N}$,

(iii) $\liminf_{n\to\infty} \theta_n^k \prod_{i=2}^{k-1} (1 - \theta_n^i) > 0$ *for every* $k = 3, 4, ...M + 1$ *and* $\liminf_{n\to\infty} \prod_{i=2}^{M+1} (1 - \theta_n^i) > 0$.

Then $\{t_n\}$ converges strongly to $y' \in \bigcap_{j=1}^{N} EP(h_j, Q) \cap \Delta$.

Proof: Let us take $S_k = P_{C_k}(I - \eta H^*(I - P_{Q_K})H)$ for all $k \in N$. Therefore, S_k is averaged mapping and thus nonexpansive. Additionally, $I - S_k$ is demiclosed at zero. Thus, Theorem 3.6 leads to the conclusion.

Remark 4.8. Theorem 4.7 extends and generalizes Yao's result [30] from weak convergence to strong convergence.

4.5. **Multiple-Sets Split Common Null Point Problem.** Suppose that $L : H_1 \to H_2$ is a bounded linear operator. Let $N_k : H_1 \to 2^{H_1}$ and $K_l : H_2 \to 2^{H_2}$ be two set valued operators, where H_1 and H_2 are two Hilbert spaces and $1 \leq k \leq M$ and $1 \leq l \leq P$. The multiple-sets split common null point problem is to find $y' \in H_1$ and

$$0 \in \bigcap_{k=1}^{M} N_k(y') \text{ and } p' = Ly' \text{ solves } 0 \in \bigcap_{l=1}^{P} K_l(p'). \tag{4.4}$$

The solution set is denoted by Δ. We know that $0 \in \Delta$ iff $y' \in Fix(J_\eta^{N_k}(I - \eta L^*(I - J_\eta^{K_l})L))$, where $J_\eta^N = (I + \eta N)^{-1}$ is resolvent operator. Also, if $\eta > 0$, then $J_\eta^{N_k}(I - \eta L^*(I - J_\eta^{K_l})L)$ is nonexpansive mapping, see [31]. We now present a strong convergence theorem for approximating common solution of finite family of equilibrium, SCNPP for multiple sets problems.

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Theorem 4.9. *Let $h_j : Q_j \times Q_j \to \mathbb{R}$ be bifunctions satisfying Assumption 1 and $N_k : H \to 2^H$ be maximal monotone operator. Suppose $\bigcap_{j=1}^{N} EP(h_j, Q) \cap \Delta \neq \phi$ and for any $t_0, t_1 \in H$, $\{t_n\}$ is defined by*

$$\begin{cases} x_n = t_n - \gamma_n(t_n - t_{n-1}), \\ u_n^j = argmin\left\{ \dfrac{1}{2\kappa_n^j}\|y - x_n\|^2 + h_j(x_n, y) : y \in Q \right\}, \\ v_{n,k}^j = (1-\rho^{k_n})x_n + \rho^{k_n} u_n^j, \\ h_j(v_{n,k}^j, x_n) - h_j(v_{n,k}^j, u_n^j) \geq \frac{\phi_1}{\kappa_n^j}\|x_n - u_n^j\|^2, \\ E_n = \{t \in H : \sum_{j=1}^{N} \alpha_n^j l_n^j(t) \leq 0\}, \\ y_n = P_{E_n}(x_n), \\ t_{n+1} = \theta_n^1 \phi(t_n) + (1-\theta_n^1)[\theta_n^2 y_n + \sum_{k=3}^{M+1} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i) J_\eta^{N_{k-2}}(I - \eta L^*(I - J_\eta^{K_l})L)y_n \\ \qquad + \prod_{i=2}^{M+1}(1-\theta_n^i) J_\eta^{N_M}(I - \eta L^*(I - J_\eta^{K_l})L)y_n], \end{cases}$$

where $\gamma_n, \phi_1, \rho \in (0,1)$, $\{\kappa_n^j\} \subset [\kappa, \bar{\kappa}]$, where $0 \leq \kappa \leq \bar{\kappa}$, $l_n^j(t) = \left\langle x_n^j, t - v_n^j \right\rangle$, where $x_n^j \in \partial h_j(v_n^j, v_n^j)$ for $j = 1, 2, ..., N$ and ϕ is a δ-contraction operator defined on H. Also $\{\theta_n^k\}$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \theta_n^1 = 0$, $\lim\limits_{n\to\infty} \frac{\gamma_n}{\theta_n^1}\|t_n - t_{n-1}\| = 0$ *and* $\sum_{n=0}^{\infty} \theta_n^1 = \infty$,

(ii) $\liminf\limits_{n\to\infty} \theta_n^2 > \nu$ *where* $n \in \mathbb{N}$,

(iii) $\liminf\limits_{n\to\infty} \theta_n^k \prod_{i=2}^{k-1}(1-\theta_n^i) > 0$ *for every* $k = 3, 4, ...M+1$ *and* $\liminf\limits_{n\to\infty} \prod_{i=2}^{M+1}(1-\theta_n^i) > 0$.

Then $\{t_n\}$ converges strongly to $y' \in \bigcap_{j=1}^{N} EP(h_j, Q) \cap \Delta$.

Proof: Let us take $S_k = J_\eta^{N_k}(I - \eta L^*(I - J_\eta^{K_l})L)$ for all $k \in N$. Therefore, S_k is nonexpansive. Additionally, $I - S_k$ is demiclosed at zero. Thus, Theorem 3.6 leads to the conclusion.

Remark 4.10. Reich and Tuyen [32] method requires to compute three sets C_n, D_n, W_n, their intersection and projection of x_0 at each step, which is expensive and time consuming if the feasible set is complex. We do not require such computation. Theorem 4.9 generalizes and extends Reich and Tuyen result [32] from split null point problem to common solution of multiple-sets split null point problem and finite family of equilibrium problem.

5. Numerical Example

The numerical studies presented in this section show that Algorithm 3.1 is more effective than Wairojjana Alg in [33] and Jolaoso Alg in [17].

Example 5.1. Suppose that the bifunctions $h_j : Q \times Q \to \mathbb{R}$ is defined as $h_j(x,y) = \langle P_j x + Q_j y + q_j, y - x \rangle$ where $j = 1, 2$, $Q = \{x \in \mathbb{R}^5 : -4 \leq x_j \leq 4, j = 1,2,3,4,5\}$, $q_j \in \mathbb{R}^5$ and $P_j, Q_j \in \mathbb{R}^{5\times 5}$ are two matrices of order 5 taken same as in [34] and $H = \mathbb{R}^5$. The bifunction h_j is Lipschitz-type continuous with constants $L_1 = L_2 = \frac{1}{2}\|Q_j - P_j\|$, see [11, 34] and satisfy the Assumption 1. Take $P_1 = P_2, Q_1 = Q_2, q_1 = q_2$, $\phi_1 = 0.001$, $\alpha_n^j = 1/2$, $\gamma = 0.1$, $\tau_n = \frac{1}{n^2}$ for $j = 1,2$ and $\rho = 0.00005$. Take $M = N = 2$, we can choose a contraction mapping $p : Q \to Q$ as $p(x) = \frac{x}{4}$ for all $x \in Q$ and two demicontractive mappings $U_k : Q \to Q$ as $U_k(x) = \frac{-2kx}{k+1}$ for all $x \in Q, k = 1,2$ where $x = (x_1, x_2, x_3, x_4, x_5)^{\mathsf{T}}$. Clearly, the solution set is non empty and $0 \in \bigcap_{k=1}^{2} \mathrm{Fix}(U_k) \bigcap \bigcap_{j=1}^{2} \mathrm{EP}(h_j, Q)$.

Let $\theta_n^k = \frac{49(k^2-1)n^2 + 13n}{51[(nk)^2+1]}$ and take $\|t_{n-1} - t_n\| < 10^{-4}$ as stopping criterion. We also provide numerical findings for the following t_0 and t_1 values.

Case 1: $t_0 = (2,2,2,2,2)^T$, $t_1 = (5,5,5,5,5)^T$;

Case 2: $t_0 = (20,20,20,20,20)$, $t_1 = (50,50,50,50,50)^T$;

Case 3: $t_0 = (200,200,200,200,200)^T$, $t_1 = (500,500,500,500,500)^T$;

Case 4: $t_0 = (2000,2000,2000,2000,2000)^T$, $t_1 = (5000,5000,5000,5000,5000)^T$.

We also show that Algorithm (Alg) 3.1 is more effective than Wairojjana Alg in [33] and Jolaoso Alg in [17]. We plot the graphs of number of iterations n with errors $\|t_{n-1} - t_n\|$. Tables 1-2 and Figures 1-2 show the numerical results.

Algorithm	Iterations Number	Time (Seconds)
Alg 3.1	13	0.04083
Wairojjana Alg	31	0.2606
Lateff Alg	88	0.7235

TABLE 1. Comparison of Alg 3.1 with Wairojjana Alg [33] and Jolaoso Alg [17] when $t_0 = (2,2,2,2,2)^T$, $t_1 = (5,5,5,5,5)^T$

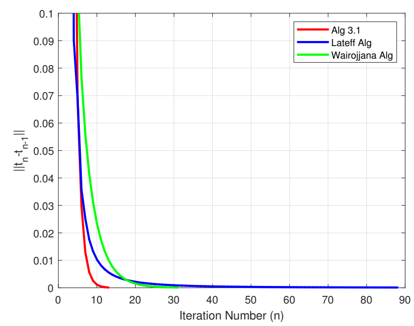

FIGURE 1. Comparison of Alg 3.1 with Wairojjana Alg [33] and Jolaoso Alg [17] when $t_0 = (2,2,2,2,2)^T$, $t_1 = (5,5,5,5,5)^T$

Cases	Iteration Number	Time (Seconds)
1	13	0.04083
2	16	0.4975
3	18	0.5708
4	21	0.7905

TABLE 2. Example 5.3: Numerical study of Alg 3.1 for different values of t_0 and t_1.

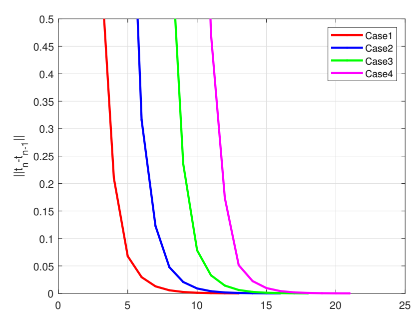

FIGURE 2. Example 5.3: Numerical study of Alg 3.1 for different values of t_0 and t_1

Example 5.2. Suppose that the bifunctions $h_j : Q \times Q \to \mathbb{R}$ is defined as $h_j(x,y) = (3 - \|x\|)\langle x, y-x\rangle$ where $j = 1,2$, $H = l_2$ be a real Hilbert space with the inner product $\|.\| : l_2 \to \mathbb{R}$ and $\langle .,. \rangle : l_2 \times l_2 \to \mathbb{R}$ are defined as $\|x\| = (\sum_{k=1}^{\infty} |x_k|^2)^{\frac{1}{2}}$ and $\langle x,y\rangle = \sum_{k=1}^{\infty} x_k y_k$, where $x = \{x_k\}_{k=1}^{\infty}$, $y = \{y_k\}_{k=1}^{\infty}$ and $Q = \{x \in H : \|x\| \leq 1\}$. Take $M = 2$. Let a contraction mapping $p : Q \to Q$ be defined as $p(x) = \frac{x}{4}$ for all $x \in Q$ and two demicontractive mappings $U_k : K \to K$ be defined as $U_k(x) = \frac{-2kx}{k+1}$ for all $x \in Q, k = 1,2$ where $x = (x_1, x_2, ..., x_k, ...), x_k \in \mathbb{R}$. Clearly bifunctions h_j for $j = 1,2$ satisfy Assumption 1 and the solution set is non empty. Also, $0 \in \bigcap_{k=1}^{2} \mathrm{Fix}(U_k) \bigcap \bigcap_{j=1}^{2} \mathrm{EP}(h_j, Q)$. Let $\theta_n^k = \frac{49(k^2-1)n^2+13n}{51[(nk)^2+1]}$ and use $\|t_{n-1} - t_n\| < 10^{-4}$ as stopping criterion. Take $\gamma = 0.1$, $\tau_n = \frac{1}{n^2}$, $\phi_1 = 0.001$, $\alpha_n^j = 1/2$ for $j = 1,2$ and $\rho = 0.00005$. We compare the numerical result of Alg 3.1 with Wairojjana Alg [33] and Jolaoso Alg [17]. We also provide numerical findings for the following t_0 and t_1 values.

Case 1: $t_0 = (1,1,1,10,10,0,0,...,...),\ t_1 = (4,4,4,40,40,0,0,...,...)$;

Case 2: $t_0 = (10,10,10,100,100,0,0,...,...),\ t_1 = (40,40,40,400,400,0,0,...,...)$;

Case 3: $t_0 = (1,1,0,0,0,0,0,...,...),\ t_1 = (4,4,400,7000,4000,0,0,...,...)$;

Case 4: $t_0 = (1,1,0,0,0,0,0,...,...),\ t_1 = (4,4,400,40000,4000,0,0,...,...)$.

We also show that Algorithm (Alg) 3.1 is more effective than Wairojjana Alg in [33] and Jolaoso Alg in [17]. We plot the graphs of number of iterations n with errors $\|t_{n-1} - t_n\|$. Tables 3-4 and Figures 3-4 show the numerical results.

Algorithm	number of iterations	Time (Seconds)
Alg 3.1	16	0.1596
Lateef Alg	73	0.9678
Wairojjana Alg	459	1.5156

TABLE 3. Comparison of Alg 3.1 with Wairojjana Alg [33] and Jolaoso Alg [17] when $t_0 = (1,1,1,10,10,0,0,...,...),\ t_1 = (4,4,4,40,40,0,0,...,...)$

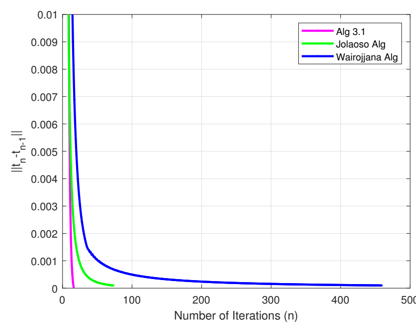

FIGURE 3. Comparison of Alg 3.1 with Wairojjana Alg [33] and Jolaoso Alg [17] when $t_0 = (1,1,1,10,10,0,0,...,...),\ t_1 = (4,4,4,40,40,0,0,...,...)$

Cases	number of iterations	Execution Time in Seconds
1	16	0.1596
2	17	0.1981
3	19	0.8073
4	21	1.1999

TABLE 4. Example 5.3: Numerical study of Alg 3.1 for different values of t_0 and t_1.

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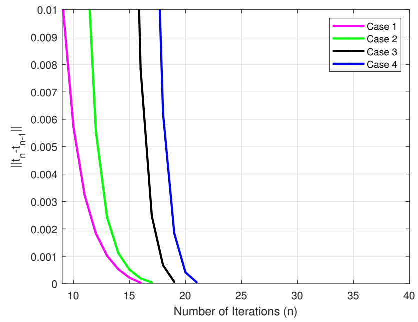

FIGURE 4. Numerical study of Alg 3.1 for different values of t_0 and t_1.

Example 5.3. Suppose that the bifunctions $h_j : Q \times Q \to \mathbb{R}$ is defined as $h_j(x,y) = \sum_{i=2}^{5}(y_i - x_i)\|x\|$ where $j = 1,2$, $Q = \{x \in \mathbb{R}^5 : x_1 \geq -1, x_i \geq 0, i = 2,3,4,5\}$. The bifunctions h_j is Lipschitz continuous with $L_1 = L_2 = 2$; see [35]. Take $\phi_1 = 0.001$, $\alpha_n^j = 1/2$, $\gamma = 0.1$, $\tau_n = \frac{1}{n^2}$ for $j = 1,2$ and $\rho = 0.00005$. Take $M = N = 2$, we can choose a contraction mapping $p : Q \to Q$ as $p(x) = \frac{x}{4}$ for all $x \in Q$ and two demicontractive mappings $U_k : Q \to Q$ as $U_k(x) = \frac{-2kx}{k+1}$ for all $x \in Q, k = 1,2$ where $x = (x_1,x_2,x_3,x_4,x_5)^{\mathsf{T}}$. Clearly, the solution set is non empty and $0 \in (\bigcap_{k=1}^{2} \mathrm{Fix}(U_k)) \bigcap (\bigcap_{j=1}^{2} \mathrm{EP}(h_j,Q))$.

Let $\theta_n^k = \frac{49(k^2-1)n^2+13n}{51[(nk)^2+1]}$ and take $\|t_{n-1} - t_n\| < 10^{-4}$ as stopping criterion. We also provide numerical findings for the following t_0 and t_1 values.

Case 1: $t_0 = (1,1,1,1,1)^T$, $t_1 = (5,5,5,5,5)^T$;

Case 2: $t_0 = (10,10,10,10,10)$, $t_1 = (50,50,50,50,50)^T$;

Case 3: $t_0 = (100,100,100,100,100)^T$, $t_1 = (500,500,500,500,500)^T$;

Case 4: $t_0 = (1000,1000,1000,1000,1000)^T$, $t_1 = (5000,5000,5000,5000,5000)^T$.

We also show that Algorithm (Alg) 3.1 is more effective than Wairojjana Alg in [33] and Jolaoso Alg in [17]. We plot the graphs of number of iterations n with errors $\|t_{n-1} - t_n\|$. Tables 5-6 and Figures 5-6 show the numerical results.

Algorithm	number of iterations	Time (Seconds)
Alg 3.1	20	0.9995
Lateef Alg	80	1.5690
Wairojjana Alg	183	2.1586

TABLE 5. Comparison of Alg 3.1 with Wairojjana Alg [33] and Jolaoso Alg [17] when $t_0 = (1000,1000,1000,1000,1000)^T$, $t_1 = (5000,5000,5000,5000,5000)^T$

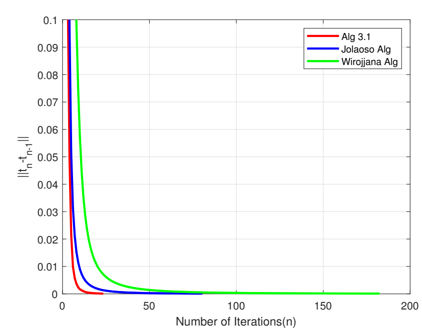

FIGURE 5. Comparison of Alg 3.1 with Wairojjana Alg [33] and Jolaoso Alg [17] when $t_0 = (1000,1000,1000,1000,1000)^T$, $t_1 = (5000,5000,5000,5000,5000)^T$

Cases	number of iterations	Execution Time in Seconds
1	6	0.0706
2	7	0.7990
3	13	0.9976
4	20	0.9995

TABLE 6. Example 5.3: Numerical study of Alg 3.1 for different values of t_0 and t_1.

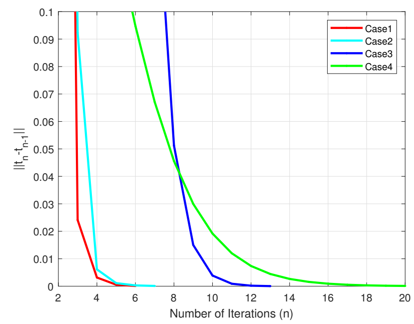

FIGURE 6. Numerical study of Alg 3.1 for different values of t_0 and t_1.

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Conflict of Interest

The authors declare that there is no conflict of interest.

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