# Homological dimension based on a class of Gorenstein modules and recollement situations

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Abstract

In this paper, we study the Gorenstein $(\mathcal{L},\mathcal{A})$-projective resolution dimension of modules and the left global Gorenstein $(\mathcal{L},\mathcal{A})$-projective dimension of rings in detail, where $(\mathcal{L},\mathcal{A})$ represents a complete duality pair and Gorenstein $(\mathcal{L},\mathcal{A})$-projective modules were introduced by Gillespie in [17]. We give some characterizations of the (finite) Gorenstein $(\mathcal{L},\mathcal{A})$-projective resolution dimension of modules and reveal some connections between this dimension and other Gorenstein homological dimensions. We find some lower and upper bounds of the (finite) left global Gorenstein $(\mathcal{L},\mathcal{A})$-projective dimension of rings by some classical homological invariants and recollements of abelian categories.
*Keywords:* Gorenstein $(\mathcal{L},\mathcal{A})$-projective resolution dimension; left global Gorenstein $(\mathcal{L},\mathcal{A})$-projective dimension; invariant; recollement of abelian categories.
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## 1. Introduction

The concept of G-dimension for commutative noetherian rings was introduced by Auslander and Bridger in [1]. The G-dimension generalizes the projective dimension and is a refinement of it. This concept has been extended to modules over any ring $R$ by Enochs and Jenda in [12] through the notion of Gorenstein projective modules. A left $R$-module $M$ is called Gorenstein projective if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ of projective left $R$-modules which remains exact when applying the functor $\mathrm{Hom}_R(-,P)$ for any projective left $R$-module $P$, such that $M \cong \mathrm{Ker}(P_{-1} \to P_{-2})$. Then modules of finite Gorenstein projective dimension are defined in the standard way, using resolutions by Gorenstein projective modules. The corresponding classes of Gorenstein injective and flat modules were defined in [12] and [14] respectively. They are key elements of Gorenstein homological algebra. Modules of finite Gorenstein flat dimension are also defined in the standard way, using resolutions by Gorenstein flat modules. Holm systematically studied these Gorenstein homological dimensions in [19]. Gorenstein homological algebra has developed rapidly during the past years. The Gorenstein methods have proved to be very useful in commutative and noncommutative algebra, as well as in representation theory and



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model category theory. This is one reason why the existence of Gorenstein resolutions and the properties of Gorenstein modules have been studied intensively. There have been some interesting applications of the theory of modules of finite Gorenstein dimensions.

The notion of a duality pair of $R$-modules was introduced by Holm and Jørgensen in [20]. A duality pair over the ring $R$ is a pair $(\mathcal{L}, \mathcal{A})$, where $\mathcal{L}$ is a class of left $R$-modules and $\mathcal{A}$ is a class of right $R$-modules, satisfying the conditions: $L \in \mathcal{L}$ if and only if $L^+ \in \mathcal{A}$ and $\mathcal{A}$ is closed under direct summands and finite direct sums, where $L^+ = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z})$ denotes the character module of $L$. Duality pairs are related to purity and to the existence of covers and envelopes, which implies that duality pairs are very useful in relative homological algebra. Gillespie investigated Gorenstein homological algebra with respect to a complete duality pair $(\mathcal{L}, \mathcal{A})$ in [17]. Several kinds of Gorenstein modules are similarly defined by means of exact complexes of projective, injective and flat modules staying exact when applying some functors at most. A left $R$-module $M$ is called Gorenstein $(\mathcal{L}, \mathcal{A})$-projective if there exists a $\mathrm{Hom}_R(-, \mathcal{L})$-exact exact sequence $\mathbb{P} : \cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ of projective left $R$-modules such that $M \cong \mathrm{Ker}(P_{-1} \to P_{-2})$. The corresponding classes of Gorenstein $(\mathcal{L}, \mathcal{A})$-injective and flat modules were also defined in [17]. We are interested in Gorenstein homological algebra with respect to a complete duality pair. In fact, Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules are Gorenstein projective and Gorenstein $(\mathcal{L}, \mathcal{A})$-flat modules are Gorenstein flat. In this paper, we show that a flat module which is not projective is not Gorenstein $(\mathcal{L}, \mathcal{A})$-projective (see Remark 3.6). That is, the class of all Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules can be strictly included in the class of all Gorenstein $(\mathcal{L}, \mathcal{A})$-flat modules. When Gorenstein projective modules are Gorenstein flat is still open. But Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules are always Gorenstein $(\mathcal{L}, \mathcal{A})$-flat, and of course, are Gorenstein flat (see [6, Remark 3.13]).

Gorenstein homological algebra is a relative version of homological algebra. A guiding principle in Gorenstein homological algebra is to seek analogous of results about absolute homological dimensions. Motivated by this, we study the Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension of modules and the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension of rings in detail. Modules of finite Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension are defined in the standard way, using resolutions by Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules. The Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension is a refinement of the projective dimension. We found that it shares many properties analogous to those of the projective dimension, as well as the Gorenstein projective dimension.

We give some characterizations of the (finite) Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension of modules and reveal some connections between this dimension and other Gorenstein homological dimensions. Projectively coresolved Gorenstein flat modules were introduced and investigated by Saroch and Stovicek in [22]. A left $R$-module $M$ is called projectively coresolved Gorenstein flat if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ of projective left $R$-modules which remains exact after tensoring by every injective right $R$-module such that $M \cong \mathrm{Ker}(P_{-1} \to P_{-2})$. We use $\mathcal{PGF}$ to denote the class of all projectively coresolved Gorenstein flat modules. The relative homological dimension based on the class of all projectively coresolved Gorenstein flat modules was studied in [9]. Since Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules are both Gorenstein projective and Gorenstein flat modules, the theory of Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension is closely related to the study of the classical homological invariants $\mathrm{spli}R$, $\mathrm{silp}R$ and $\mathrm{sfli}R$ that were defined by Gedrich and Gruenberg in [15], where $\mathrm{spli}R$ is defined as the supremum of the projective dimension of injective modules, $\mathrm{silp}R$ is defined as the supremum of the injective dimension of projective modules, and $\mathrm{sfli}R$ is defined as the supremum of the flat dimension of injective modules. If the class of all Gorenstein $(\mathcal{L}, \mathcal{A})$-flat modules is closed under extensions, then for any $R$-module $M$, we obtain that

$$\mathrm{resdim}_{_R\mathcal{GF}}M \leqslant \mathrm{resdim}_{_R\mathcal{GP}}M \leqslant \mathrm{resdim}_{_R\mathcal{GF}}M + \mathrm{splf}R,$$

where $\mathrm{resdim}_{_R\mathcal{GP}}M$ denotes the Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension of $M$, $\mathrm{resdim}_{_R\mathcal{GF}}M$ denotes the Gorenstein $(\mathcal{L}, \mathcal{A})$-flat resolution dimension of $M$, and $\mathrm{splf}R$ denotes the supremum of the projective dimension of flat modules. That is, we give a lower bound and upper bound of the Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension of $M$. Furthermore, we give a corresponding version of bounds of the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension of rings (see Theorem 4.4). If a ring is of finite left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension, then this dimension equals both $\mathrm{spli}R$ and $\mathrm{silp}R$, as well as the left global projectively coresolved Gorenstein flat dimension of rings, and every Gorenstein projective module is Gorenstein flat over the ring (see Propositions 4.7 and 4.9).

Recollements of abelian categories provide a very useful framework for investigating homological connections among categories. Psaroudakis studied how various homological invariants and dimensions of categories involved in a recollement of abelian categories are related in [21]. He gave a series of bounds among the global, finitistic and representation dimensions of the categories linked by recollements of abelian categories. Zhang and Zhu investigated the Gorenstein global dimension with respect to recollements of abelian categories and gave some upper bounds of Gorenstein global dimensions of categories involved in a recollement of abelian categories by the classical homological invariants in [23]. In this paper, we discuss the finiteness of the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimensions and the left global projectively coresolved Gorenstein flat dimensions of rings by recollement situations and the classical homological invariants. Associated to some results in [23], we give some upper bounds of these global dimensions of rings from categories involved in a recollement of abelian categories by the classical homological invariants.

This paper is organized as follows. In section 2, we give some basic notions and facts. In section 3, we deeply study the Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension of modules. We give some characterizations of the finiteness of the Gorenstein $(\mathcal{L}, \mathcal{A})$-projective resolution dimension of modules and show some connections between this dimension and other Gorenstein homological dimensions. In sections 4 and 5, we investigate the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension of rings and find some lower and upper bounds of the finiteness of this dimension.



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## 2. Preliminaries

We recall some notions and basic facts which we need in the later sections.

**$\mathcal{X}$-resolution dimension.** Let $\mathcal{H}$ be an abelian category, $\mathcal{X}$ a class of objects of $\mathcal{H}$ which can be also regarded as a full subcategory of $\mathcal{H}$, and $M$ an object in $\mathcal{H}$. The class $\mathcal{X}$ is thick if it is closed under extensions, kernels of epimorphisms, cokernels of monomorphisms and direct summands. We denote $\mathcal{X}^{\perp} = \{H \in \mathcal{H} \,|\, \mathrm{Ext}^1_{\mathcal{H}}(X, H) = 0,\ \forall\ X \in \mathcal{X}\}$. Recall that a morphism $\varphi : X \to M$ is an $\mathcal{X}$-precover (or a right $\mathcal{X}$-approximation) of $M$ if $X \in \mathcal{X}$ and if for every morphism $\alpha : X_0 \to M$ with $X_0 \in \mathcal{X}$ there exists $\beta : X_0 \to X$ such that $\alpha = \varphi\beta$. If, in addition, $\beta$ is an automorphism of $X$ in the case where $X_0 = X$ and $\alpha = \varphi$, then $\varphi$ is called an $\mathcal{X}$-cover. Recall that an $\mathcal{X}$-precover $\varphi$ of $M$ is special if $\varphi$ is surjective and $\mathrm{Ker}\varphi \in \mathcal{X}^{\perp}$. $\mathcal{X}$ is precovering if every object of $\mathcal{H}$ has an $\mathcal{X}$-precover. $\mathcal{X}$ is covering if every object of $\mathcal{H}$ has an $\mathcal{X}$-cover. $\mathcal{X}$ is special precovering if every object of $\mathcal{H}$ has an special $\mathcal{X}$-precover.

In addition, let $\mathcal{H}$ have enough projective objects. Assume that $\mathcal{X}$ is precovering and contains the class of all projective objects of $\mathcal{H}$. The $\mathcal{X}$-resolution dimension of $M$ is defined as the smallest non-negative integer $n$ such that there is an exact sequence $0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$ with each $X_i \in \mathcal{X}$. If such $n$ does not exist, we say that the $\mathcal{X}$-resolution dimension of $M$ is infinite. We denote by $\mathcal{X}^{\wedge}$ the class of objects in $\mathcal{H}$ having finite $\mathcal{X}$-resolution dimension.

**Recollement of abelian categories.** Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A recollement between $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ is a diagram (for example, see [21, Definition 2.1] or an earlier reference)

$$\begin{tikzcd} \mathcal{B} \arrow[r, "i"] & \mathcal{C} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & \mathcal{D} \arrow[l, bend right, "l"'] \arrow[l, bend left, "r"] \end{tikzcd}$$

denoted by $(\mathcal{B}, \mathcal{C}, \mathcal{D})$, satisfying the following conditions:

(R1) $(q, i, p)$ is an adjoint triple,

(R2) $(l, e, r)$ is an adjoint triple,

(R3) the functors $i$, $l$ and $r$ are fully faithful,

(R4) $\mathrm{Im}i = \mathrm{Ker}e$.

Unless stated otherwise, we assume in the following that $R$ is an associative ring with an identity, and all modules are left $R$-modules. We use $R$-Mod to denote the category of all left $R$-modules and $R^{op}$ to denote the opposite ring of $R$. We treat right $R$-modules as left modules over the opposite ring $R^{op}$. $\mathcal{P}$ denotes the class of all projective left $R$-modules and $\mathcal{F}$ denotes the class of all flat left $R$-modules. We use $\mathrm{pd}_R$, $\mathrm{id}_R$, and $\mathrm{fd}_R$ to stand for projective, injective, and flat dimension respectively. For other notions and results not specified in this paper, we refer readers to [7, 13].

**Duality pair.** A duality pair $(\mathcal{L}, \mathcal{A})$ is called perfect (see [20]) if $\mathcal{L}$ contains the $R$-module ${}_RR$, and is closed under coproducts and extensions. $\{\mathcal{L}, \mathcal{A}\}$ is a symmetric duality pair over the ring $R$ (see [17]) if $(\mathcal{L}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{L})$ are duality pairs. A duality pair $(\mathcal{L}, \mathcal{A})$ is complete (see [17]) if $\{\mathcal{L}, \mathcal{A}\}$ is a symmetric duality pair and $(\mathcal{L}, \mathcal{A})$ is a perfect duality pair over $R$.

Throughout this paper, $(\mathcal{L}, \mathcal{A})$ stands for a complete duality pair.

**Gorenstein $(\mathcal{L}, \mathcal{A})$-projective and $(\mathcal{L}, \mathcal{A})$-flat modules.** We use $\mathcal{GP}$ to denote the class of all Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules (see [17]). It is clear that all kernels, images and cokernels of the aforementioned $\mathbb{P}$ are in $\mathcal{GP}$. Also, it is clear that $\mathcal{P} \subseteq \mathcal{GP}$ and $\mathrm{Ext}^i_R(N, L) = 0$ for any $N \in \mathcal{GP}$, any $L \in \mathcal{L}$ and any $i \geqslant 1$. It follows from [17] that $\mathcal{GP}$ is closed under direct sums, extensions, direct summands and kernels of epimorphisms. Since the existence of the perfect duality pair $(\mathcal{L}, \mathcal{A})$ implies that $\mathcal{P} \subseteq \mathcal{F} \subseteq \mathcal{L}$ by [17, Proposition 2.3], any Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module is Ding projective, and of course, it is Gorenstein projective (see [10, 12]). We use $\mathcal{DP}$ and $\mathcal{GP}(R)$ to denote the classes of all Ding projective and all Gorenstein projective $R$-modules, respectively.

An $R$-module $M$ is called Gorenstein $(\mathcal{L}, \mathcal{A})$-flat if there exists an $\mathcal{A}\otimes_R-$-exact exact sequence

$$\mathbb{F} : \cdots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots$$

with each $F_i \in \mathcal{F}$ such that $M \cong \mathrm{Ker}(F_{-1} \to F_{-2})$. $\mathcal{GF}$ denotes the class of all Gorenstein $(\mathcal{L}, \mathcal{A})$-flat modules (see [17]). It is clear that all kernels, images and cokernels of $\mathbb{F}$ are in $\mathcal{GF}$, $\mathcal{F} \subseteq \mathcal{GF}$, and $\mathrm{Tor}^R_i(A, M) = 0$ for any $A \in \mathcal{A}$ and any $i \geqslant 1$. By [17, Proposition 2.3 and Remark], one knows that $\mathcal{A}$ contains the classes of all injective $R^{op}$-modules and FP-injective $R^{op}$-modules. So any Gorenstein $(\mathcal{L}, \mathcal{A})$-flat module is Ding flat, and of course, it is Gorenstein flat (see [16, 14]).

**Remark 2.1.** It is known that $\mathcal{P}$ and $\mathcal{F}$ are special precovering. Since $(\mathcal{L}, \mathcal{A})$ is a perfect duality pair, it follows from [20, Theorem 3.1] that $\mathcal{L}$ is covering. By [22, Theorem 4.9], $\mathcal{PGF}$ is special precovering. By [17, Theorem 4.9], $\mathcal{GP}$ is special precovering. By [17, Proposition 5.2], if $\mathcal{GF}$ is closed under extensions, then $\mathcal{GF}$ is special precovering. According to these facts, we can study the corresponding resolution dimensions.

## 3. The $\mathcal{GP}$-resolution dimension of modules

The goal of this section is to investigate the $\mathcal{GP}$-resolution dimension of modules. We give some homological characterizations of the finiteness of the $\mathcal{GP}$-resolution dimension and discuss connections between it and some known dimensions.

The canonical example of a complete duality pair is the level duality pair $(\mathcal{L}, \mathcal{A})$ over any ring, given in [5], where $\mathcal{L}$ represents the class of level modules and $\mathcal{A}$ represents the class of absolutely clean $R^{op}$-modules. To describe it, we recall that an $R^{op}$-module $F$ is said to be of type $\mathrm{FP}_\infty$ if it has a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to F \to 0$$

with each $P_i$ finitely generated. An $R^{op}$-module $A$ is called absolutely clean if $\mathrm{Ext}^1_R(F, A) = 0$ for any $R^{op}$-module $F$ of type $\mathrm{FP}_\infty$. An $R$-module $L$ is called level if $\mathrm{Tor}^R_1(F, L) = 0$ for any $R^{op}$-module $F$ of type $\mathrm{FP}_\infty$. In this case, $\mathcal{GP}$ is exactly the class of Gorenstein AC-projective modules in [5], denoted by $\mathcal{G}_{ac}\mathcal{P}$, and $\mathcal{GF}$ is exactly the class of Gorenstein AC-flat modules in [4]. The results in the paper are valid in particular for the level duality pair.

At first, we give some characterizations of the finiteness of the $\mathcal{GP}$-resolution dimension. The following proposition is an extension of [6, Proposition 3.6].

**Proposition 3.1.** *The following conditions are equivalent for an $R$-module $M$ and a non-negative integer $n$.*

(1) $resdim_{_R\mathcal{GP}} M \leqslant n$.

(2) *There exists an exact sequence of $R$-modules* $0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ *with each* $G_i \in \mathcal{GP}$.

(3) *For any exact sequence of $R$-modules* $0 \to K_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ *with each* $G_i \in \mathcal{GP}$, *one gets* $K_n \in \mathcal{GP}$.

(4) *For each exact sequence of $R$-modules* $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *with each* $P_i \in \mathcal{P}$, *one gets* $K_n \in \mathcal{GP}$.

(5) *There exists an exact sequence of $R$-modules* $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *with each* $P_i \in \mathcal{P}$ *and* $K_n \in \mathcal{GP}$.

(6) *There exists an exact sequence of $R$-modules* $0 \to K \to P \to M \to 0$ *with* $P \in \mathcal{P}$ *and* $resdim_{_R\mathcal{GP}} K \leqslant n-1$.

*Proof.* (1) ⇒ (2) and (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (1) are straightforward.

(2) ⇒ (3) It follows using the arguments appearing in [6, proof of Proposition 3.6]. □

The following two propositions concerning some properties of $\mathcal{GP}$ can be proved as for $\mathcal{GP}(R)$.

**Proposition 3.2.** *Let $(M_i)_{i\in I}$ be a family of $R$-modules, $I$ a set and $M = \oplus_{i\in I} M_i$. Then* $resdim_{_R\mathcal{GP}} M = sup\{resdim_{_R\mathcal{GP}} M_i \,|\, i \in I\}$. *In particular, $\mathcal{GP}^\wedge$ is closed under finite direct sums and direct summands.*

**Proposition 3.3.** *Let $0 \to A \to B \to C \to 0$ be an exact sequence of $R$-modules. Then the following conditions hold.*

(1) $resdim_{_R\mathcal{GP}} B \leqslant max\{resdim_{_R\mathcal{GP}} A,\ resdim_{_R\mathcal{GP}} C\}$.

(2) $resdim_{_R\mathcal{GP}} A \leqslant max\{resdim_{_R\mathcal{GP}} B,\ resdim_{_R\mathcal{GP}} C\}$.

(3) $resdim_{_R\mathcal{GP}} C \leqslant 1 + max\{resdim_{_R\mathcal{GP}} A,\ resdim_{_R\mathcal{GP}} B\}$.

*Moreover, $\mathcal{GP}^\wedge$ is a thick class.*

**Proposition 3.4.** *Let $M$ be an $R$-module and $n$ a non-negative integer. Consider the following conditions.*

(1) $resdim_{_R\mathcal{GP}} M \leqslant n$.

(2) $\mathrm{Ext}_R^{n+i}(M, L) = 0$ *for any* $i \geqslant 1$ *and any* $L \in \mathcal{L}$.

(3) $\mathrm{Ext}_R^{n+i}(M, F) = 0$ *for any* $i \geqslant 1$ *and any* $F \in \mathcal{F}$.

(4) $\mathrm{Ext}_R^{n+i}(M, P) = 0$ *for any* $i \geqslant 1$ *and any* $P \in \mathcal{P}$.

(5) $\mathrm{Ext}_R^{n+i}(M, H) = 0$ *for any* $i \geqslant 1$ *and any* $H \in \mathcal{L}^\wedge$.

(6) $\mathrm{Ext}_R^{n+i}(M, J) = 0$ *for any* $i \geqslant 1$ *and any* $J \in \mathcal{F}^\wedge$.

(7) $\mathrm{Ext}_R^{n+i}(M,Q)=0$ *for any* $i \geqslant 1$ *and any* $Q \in \mathcal{P}^{\wedge}$.

(8) *There exists an exact sequence of* $R$*-modules* $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *with each* $P_i \in \mathcal{P}$ *and* $K_n \in \mathcal{GP}$.

*Then* (1) $\Leftrightarrow$ (8), (8) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *and* (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (4) *hold. Assume that* $M$ *is of finite* $\mathcal{GP}$*-resolution dimension. One gets that* (4) $\Rightarrow$ (8).

*Proof.* (1)$\Leftrightarrow$ (8) It immediately follows from Proposition 3.1.

(8) $\Rightarrow$ (2) and (2) $\Rightarrow$ (5) follow by dimension shifting.

(7) $\Rightarrow$ (4) is obvious.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are clear since $\mathcal{P} \subseteq \mathcal{F} \subseteq \mathcal{L}$ by [17, Proposition 2.3].

(4) $\Rightarrow$ (8) Pick a partial projective resolution of $M$,

$$0 \to K \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with each $P_i \in \mathcal{P}$. By dimension shifting and by assumption, one gets that $\mathrm{Ext}_R^i(K,P)=0$ for any $i \geqslant 1$ and any $P \in \mathcal{P}$. Since $M$ is of finite $\mathcal{GP}$-resolution dimension and $\mathcal{P} \subseteq \mathcal{GP}$, one obtains that $K \in \mathcal{GP}^{\wedge}$ by Proposition 3.3. Assume that $\mathrm{resdim}_{_R\mathcal{GP}} K = m$. By Proposition 3.1, there exists an exact sequence of $R$-modules $0 \to G_m \to G_{m-1} \overset{f_{m-1}}{\to} G_{m-2} \to \cdots \to G_1 \overset{f_1}{\to} G_0 \overset{f_0}{\to} K \to 0$ with each $G_i \in \mathcal{GP}$. Let $K_j = \mathrm{Ker} f_j$ for $0 \leqslant j \leqslant m-1$. Obviously, $K_{m-1} = G_m$. Since $\mathrm{Ext}_R^k(G_i,P)=0$ for any $k \geqslant 1$ and any $P \in \mathcal{P}$, by dimension shifting, one gets that $\mathrm{Ext}_R^1(K_j,P)=0$ for $0 \leqslant j \leqslant m-1$. Consider the exact sequence

$$0 \to G_m \to G_{m-1} \to K_{m-2} \to 0.$$

The following proof follows the arguments in [6, proof of (4) $\Rightarrow$ (1) in Proposition 3.1]. Since $G_m \in \mathcal{GP}$, there exists an exact sequence

$$0 \to G_m \to P \to G_m' \to 0$$

with $P \in \mathcal{P}$ and $G_m' \in \mathcal{GP}$. Consider the following pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G_m \arrow[r] \arrow[d] & G_{m-1} \arrow[r] \arrow[d] & K_{m-2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & K_{m-2} \arrow[r] & 0 \\
 & G_m' \arrow[r, equal] \arrow[d] & G_m' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $G_{m-1}, G_m' \in \mathcal{GP}$, $X \in \mathcal{GP}$. Because $\mathrm{Ext}_R^1(K_{m-2},P)=0$, the exact sequence $0 \to P \to X \to K_{m-2} \to 0$ splits. That is, $X \cong P \oplus K_{m-2}$. Since $\mathcal{GP}$ is closed under direct summands,

$K_{m-2} \in \mathcal{GP}$. By continuing this process, one obtains $K \in \mathcal{GP}$, and this completes the proof. □

The following proposition is attributed to [6, Proposition 3.1], since it is basically an extension of the equivalence (1) $\Leftrightarrow$ (4) of the cited result.

**Proposition 3.5.** *Let $M$ be an $R$-module. Consider the following conditions.*

(1) *$M \in \mathcal{GP}$.*

(2) *$\mathrm{Ext}^1_R(M, L) = 0$ for any $L \in \mathcal{L}$.*

(3) *$\mathrm{Ext}^1_R(M, F) = 0$ for any $F \in \mathcal{F}$.*

(4) *$\mathrm{Ext}^1_R(M, P) = 0$ for any $P \in \mathcal{P}$.*

*Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) holds. Assume that $resdim_{{}_R\mathcal{GP}}M \leqslant 1$. One gets that (4) $\Rightarrow$ (1) holds.*

By [6, Proposition 3.4], we know that

$$\mathcal{GP} \cap \mathcal{P}^\wedge = \mathcal{GP} \cap \mathcal{F}^\wedge = \mathcal{GP} \cap \mathcal{L}^\wedge = \mathcal{GP} \cap \mathcal{L} = \mathcal{GP} \cap \mathcal{F} = \mathcal{P},$$

which is useful in the sequel. In fact, for a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective module $M$, either $\mathrm{resdim}_{{}_R\mathcal{L}}M = \mathrm{fd}_R M = \mathrm{pd}_R M = \infty$ or $M \in \mathcal{P}$.

**Remark 3.6.** (1) If $M \in \mathcal{F}$, then $\mathrm{pd}_R M = \mathrm{resdim}_{{}_R\mathcal{GP}}M$. In particular, if $M \in \mathcal{F}$, then $M \in \mathcal{P}$ if and only if $M \in \mathcal{GP}$.

(2) If $M$ is flat but not projective, then $\mathrm{resdim}_{{}_R\mathcal{GP}}M > 0$. By [6, Remark 3.13], we know that Gorenstein $(\mathcal{L}, \mathcal{A})$-projective modules are Gorenstein $(\mathcal{L}, \mathcal{A})$-flat over any ring. Note that a flat $R$-module which is not projective is not Gorenstein $(\mathcal{L}, \mathcal{A})$-projective. That is, $\mathcal{GP}$ can be strictly included in $\mathcal{GF}$.

(3) $\mathcal{GP}^\wedge \cap \mathcal{F}^\wedge = \mathcal{P}^\wedge$.

(4) Let $\mathcal{L}$ be closed under kernels of epimorphisms. If $M \in \mathcal{L}$, then $\mathrm{pd}_R M = \mathrm{resdim}_{{}_R\mathcal{GP}}M$. In particular, if $M \in \mathcal{L}$, then $M \in \mathcal{P}$ if and only if $M \in \mathcal{GP}$. By [5, Proposition 2.10], one gets that the class of all level $R$-modules is closed under kernels of epimorphisms. Let $M$ be a level $R$-module. Then $\mathrm{pd}_R M = \mathrm{resdim}_{{}_R\mathcal{G}_{ac}\mathcal{P}}M$. Further, $M \in \mathcal{P}$ if and only if $M \in \mathcal{G}_{ac}\mathcal{P}$.

The following result is inspired by [9, Theorem 10]. It further gives some homological characterizations of the finiteness of the $\mathcal{GP}$-resolution dimension.

**Theorem 3.7.** *The following conditions are equivalent for an $R$-module $M$ and a non-negative integer $n$.*

(1) *$resdim_{{}_R\mathcal{GP}}M = n$.*

(2) *There exists an exact sequence $0 \to K \to G \xrightarrow{\pi} M \to 0$, where $G \in \mathcal{GP}$, $pd_R K = n-1$ and $\pi$ is a special $\mathcal{GP}$-precover of $M$. If $n = 0$, this is understood to mean $K = 0$. If $n = 1$, we also require that the exact sequence be non-split.*

(3) *There exists an exact sequence $0 \to M \to K \to G \to 0$ with $G \in \mathcal{GP}$ and $pd_R K = n$.*

(4) *There exists a projective $R$-module $P$, such that the $R$-module $M' = M \oplus P$ fits into an exact sequence $0 \to G \to M' \to K \to 0$, where $G \in \mathcal{GP}$ and $pd_R K = n$, which remains exact after applying the functor $\mathrm{Hom}_R(-, H)$ for any $R$-module $H \in \mathcal{GP}^\perp$.*



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(5) *There exists a projective $R$-module $P$, such that the $R$-module $M^{'} = M \oplus P$ fits into an exact sequence $0 \to G \to M^{'} \to K \to 0$, where $G \in \mathcal{GP}$ and $\mathrm{pd}_R K = n$, which remains exact after applying the functor* $\mathrm{Hom}_R(-, L)$ *for any $R$-module $L \in \mathcal{L}^{\wedge}$.*

(6) *There exists a projective $R$-module $P$, such that the $R$-module $M^{'} = M \oplus P$ fits into an exact sequence $0 \to G \to M^{'} \to K \to 0$, where $G \in \mathcal{GP}$ and $\mathrm{pd}_R K = n$, which remains exact after applying the functor* $\mathrm{Hom}_R(-, F)$ *for any $R$-module $F \in \mathcal{F}^{\wedge}$.*

(7) *There exists a projective $R$-module $P$, such that the $R$-module $M^{'} = M \oplus P$ fits into an exact sequence $0 \to G \to M^{'} \to K \to 0$, where $G \in \mathcal{GP}$ and $\mathrm{pd}_R K = n$, which remains exact after applying the functor* $\mathrm{Hom}_R(-, Q)$ *for any $R$-module $Q \in \mathcal{P}^{\wedge}$.*

(8) *There exists a Gorenstein $(\mathcal{L}, \mathcal{A})$-projective $R$-module $T$, such that the $R$-module $M^{'} = M \oplus T$ fits into an exact sequence $0 \to G \to M^{'} \to K \to 0$, where $G \in \mathcal{GP}$ and $\mathrm{pd}_R K = n$. If $n = 1$, we also require that the exact sequence remain exact after applying the functor* $\mathrm{Hom}_R(-, P)$ *for any $R$-module $P \in \mathcal{P}$.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It can be deduced from Auslander-Buchweitz approximation theory in [2].

(3) $\Rightarrow$ (4) By (3), there exists an exact sequence $0 \to M \to K \to G \to 0$ with $G \in \mathcal{GP}$ and $\mathrm{pd}_R K = n$. We also have an exact sequence $0 \to G^{'} \to P \to G \to 0$ with $P \in \mathcal{P}$ and $G^{'} \in \mathcal{GP}$. Consider the following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G^{'} \arrow[r, equal] \arrow[d] & G^{'} \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & M^{'} \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & K \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $P \in \mathcal{P}$, the exact sequence $0 \to M \to M^{'} \to P \to 0$ splits. That is, $M^{'} \cong M \oplus P$. Thus we obtain an exact sequence $0 \to G^{'} \to M \oplus P \to K \to 0$ with $G^{'} \in \mathcal{GP}$ and $\mathrm{pd}_R K = n$. For any $R$-module $H \in \mathcal{GP}^{\perp}$, apply the functor $\mathrm{Hom}_R(-, H)$ to the above pullback diagram.

Then we have the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Hom}_R(G,H) \arrow[r] \arrow[d] & \mathrm{Hom}_R(K,H) \arrow[r] \arrow[d] & \mathrm{Hom}_R(M,H) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(P,H) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \mathrm{Hom}_R(M^{'},H) \arrow[r] \arrow[d, "\beta"] & \mathrm{Hom}_R(M,H) \arrow[r] & 0 \\
 & \mathrm{Hom}_R(G^{'},H) \arrow[r, equal] \arrow[d] & \mathrm{Hom}_R(G^{'},H) & & \\
 & 0 & & &
\end{tikzcd}$$

Since $\beta\alpha = \gamma$ is an epimorphism, $\beta$ is also an epimorphism.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) It follows from $\mathcal{P}^{\wedge} \subseteq \mathcal{F}^{\wedge} \subseteq \mathcal{L}^{\wedge} \subseteq \mathcal{GP}^{\perp}$ and $\mathcal{P} \subseteq \mathcal{GP}$.

(8) $\Rightarrow$ (1) Firstly, assume that $n = 1$. Then $\mathrm{pd}_R K = 1$. Note that $\mathrm{resdim}_{_R\mathcal{GP}} K \leqslant \mathrm{pd}_R K = 1$. By Proposition 3.3 and the exact sequence $0 \to G \to M^{'} \to K \to 0$, we can get $\mathrm{resdim}_{_R\mathcal{GP}} M^{'} \leqslant 1$. By Proposition 3.2, we can get $\mathrm{resdim}_{_R\mathcal{GP}} M = \mathrm{resdim}_{_R\mathcal{GP}} M^{'} \leqslant 1$. Assume that $\mathrm{resdim}_{_R\mathcal{GP}} M = 0$, namely, $M \in \mathcal{GP}$. Then $M^{'} \in \mathcal{GP}$. Since the exact sequence $0 \to G \to M^{'} \to K \to 0$ remains exact after applying the functor $\mathrm{Hom}_R(-, P)$ for any $R$-module $P \in \mathcal{P}$, we can get that $\mathrm{Ext}^1_R(K, P) = 0$. By Proposition 3.5, $K \in \mathcal{GP}$. By Remark 3.6, $K \in \mathcal{P}$. Then $\mathrm{pd}_R K = 0$. This is a contradiction. Thus $\mathrm{resdim}_{_R\mathcal{GP}} M = 1$.

Then assume that $n > 1$. Similarly, we get that $\mathrm{resdim}_{_R\mathcal{GP}} M = \mathrm{resdim}_{_R\mathcal{GP}} M^{'} \leqslant n$. Next, assume that $\mathrm{resdim}_{_R\mathcal{GP}} M \leqslant n-1$. Then $\mathrm{resdim}_{_R\mathcal{GP}} M^{'} \leqslant n-1$. By Proposition 3.4, $\mathrm{Ext}^n_R(M^{'}, P) = 0$ for any $P \in \mathcal{P}$. By the exact sequence $0 \to G \to M^{'} \to K \to 0$ and dimension shifting, we have $\mathrm{Ext}^n_R(K, P) = 0$. Since $\mathrm{pd}_R K = n$, there exists an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to K \to 0$ with each $P_i \in \mathcal{P}$. Let $N = \mathrm{Im}(P_{n-1} \to P_{n-2})$. By dimension shifting, we obtain that $\mathrm{Ext}^1_R(N, P) \cong \mathrm{Ext}^n_R(K, P) = 0$. Consider the exact sequence $0 \to P_n \to P_{n-1} \to N \to 0$. Then the exact sequence splits. Thus $N \in \mathcal{P}$. This implies that $\mathrm{pd}_R K \leqslant n-1$. This is a contradiction. Therefore, $\mathrm{resdim}_{_R\mathcal{GP}} M = n$. This completes the proof. $\square$

**Remark 3.8.** By Remark 3.6, we know that $\mathcal{GP}$ can be strictly included in $\mathcal{GF}$. Assume that $\mathcal{GF}$ is closed under extensions. By [17, Proposition 5.2], we get that $\mathcal{GF}$ is closed under kernels of epimorphisms. On the one hand, any Gorenstein $(\mathcal{L}, \mathcal{A})$-flat $R$-module is Gorenstein flat. On the other hand, we know that any Gorenstein flat $R$-module with finite projective dimension is flat by [13, Corollary 10.3.4]. So any Gorenstein $(\mathcal{L}, \mathcal{A})$-flat $R$-module with finite projective dimension is flat. Based on these facts, the conditions in Theorem 3.7 are equivalent for any Gorenstein $(\mathcal{L}, \mathcal{A})$-flat $R$-module $M$, with the additional fact that the $R$-module $K$ in (2), (3), (4), (5), (6), (7) and (8) is flat.



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**Proposition 3.9.** *Let $M$ be an $R$-module. Consider the following conditions.*

(1) $M \in \mathcal{GP}$.

(2) $\mathrm{Ext}_R^1(M, H) = 0$ *for any* $H \in \mathcal{L}^\wedge$.

(3) $\mathrm{Ext}_R^1(M, J) = 0$ *for any* $J \in \mathcal{F}^\wedge$.

(4) $\mathrm{Ext}_R^1(M, Q) = 0$ *for any* $Q \in \mathcal{P}^\wedge$.

*Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *holds. Assume that* $M \in \mathcal{GP}^\wedge$. *One gets that* (4) $\Rightarrow$ (1).

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) It follows from Proposition 3.4.

(4) $\Rightarrow$ (1) Since $M \in \mathcal{GP}^\wedge$, by Theorem 3.7, there exists an exact sequence $0 \to K \to G \to M \to 0$ with $G \in \mathcal{GP}$ and $K \in \mathcal{P}^\wedge$. By assumption, $\mathrm{Ext}_R^1(M, K) = 0$. Then the sequence splits. Hence $M$ is a direct summand of $G$. So $M \in \mathcal{GP}$. $\square$

**Proposition 3.10.** *Let $M$ be an $R$-module of finite $\mathcal{GP}$-resolution dimension. Then the following conditions are equivalent.*

(1) $M \in \mathcal{P}^\wedge$.

(2) $\mathrm{Ext}_R^1(G, M) = 0$ *for any* $G \in \mathcal{GP}$.

(3) $\mathrm{Ext}_R^1(Q, M) = 0$ *for any* $Q \in \mathcal{PGF}$.

*Proof.* (1) $\Rightarrow$ (2) It follows from Proposition 3.4.

(2) $\Rightarrow$ (1) Since $M$ is of finite $\mathcal{GP}$-resolution dimension, by Theorem 3.7, there exists an exact sequence $0 \to M \to K \to G \to 0$ with $G \in \mathcal{GP}$ and $K \in \mathcal{P}^\wedge$. Since $\mathrm{Ext}_R^1(G, M) = 0$, the exact sequence splits. Hence $M$ is a direct summand of $K$. Then $M \in \mathcal{P}^\wedge$.

(1) $\Leftrightarrow$ (3) Note that $\mathcal{GP} \subseteq \mathcal{PGF}$ by [5, Theorem A.6]. Then $M$ is of finite $\mathcal{PGF}$-resolution dimension. It is obtained from [9, Proposition 14]. $\square$

We use $\mathrm{Gpd}_R M$, $\mathrm{Dpd}_R M$ and PGF-$\mathrm{dim}_R M$ to denote the Gorenstein projective, Ding projective and projectively coresolved Gorenstein flat dimensions of $M$, respectively.

**Proposition 3.11.** *Let $M$ be an $R$-module.*

(1) *If* $\mathrm{resdim}_{_R\mathcal{GP}} M < \infty$, *then* $\mathrm{Gpd}_R M = \mathrm{Dpd}_R M = \mathrm{resdim}_{_R\mathcal{GP}} M = \text{PGF-dim}_R M$.

(2) *If* $\mathrm{pd}_R M < \infty$, *then* $\mathrm{Gpd}_R M = \mathrm{Dpd}_R M = \mathrm{resdim}_{_R\mathcal{GP}} M = \text{PGF-dim}_R M = \mathrm{pd}_R M$.

*Proof.* (1) Assume that $\mathrm{resdim}_{_R\mathcal{GP}} M = n < \infty$. Since $\mathcal{GP} \subseteq \mathcal{DP} \subseteq \mathcal{GP}(R)$, we get that $\mathrm{Gpd}_R M \leqslant \mathrm{Dpd}_R M \leqslant \mathrm{resdim}_{_R\mathcal{GP}} M = n$. Suppose that $\mathrm{Gpd}_R M \leqslant n-1$. By [19, Theorem 2.20], $\mathrm{Ext}_R^i(M, P) = 0$ for any $i \geqslant n$ and any $P \in \mathcal{P}$. By Proposition 3.4, $\mathrm{resdim}_{_R\mathcal{GP}} M \leqslant n-1$. This is a contradiction. So $\mathrm{Gpd}_R M = n$. Thus $\mathrm{Gpd}_R M = \mathrm{Dpd}_R M = \mathrm{resdim}_{_R\mathcal{GP}} M$. We know that $\mathcal{GP} \subseteq \mathcal{PGF}$. Then PGF-$\mathrm{dim}_R M \leqslant \mathrm{resdim}_{_R\mathcal{GP}} M = n$. By [9, Corollary 13], $\mathrm{Gpd}_R M = \text{PGF-dim}_R M$. So $\mathrm{Gpd}_R M = \mathrm{Dpd}_R M = \mathrm{resdim}_{_R\mathcal{GP}} M = \text{PGF-dim}_R M$.

(2) Assume that $\mathrm{pd}_R M = n < \infty$. Note that $\mathrm{resdim}_{_R\mathcal{GP}} M \leqslant \mathrm{pd}_R M = n$. Suppose that $\mathrm{resdim}_{_R\mathcal{GP}} M \leqslant n-1$. It is clear from Proposition 3.4 and Thoerem 3.7 that $\mathrm{pd}_R M \leqslant n-1$. This is a contradiction. So $\mathrm{resdim}_{_R\mathcal{GP}} M = n$. By (1), we obtain that $\mathrm{Gpd}_R M = \mathrm{Dpd}_R M = \mathrm{resdim}_{_R\mathcal{GP}} M = \text{PGF-dim}_R M = \mathrm{pd}_R M$. $\square$

We give the following result, which follows the definitions of the classes $\mathcal{GP}(R)$, $\mathcal{DP}$, $\mathcal{GP}$ and $\mathcal{PGF}$.

**Corollary 3.12.** $\mathcal{P}^\wedge \cap \mathcal{GP}(R) = \mathcal{P}^\wedge \cap \mathcal{DP} = \mathcal{P}^\wedge \cap \mathcal{GP} = \mathcal{P}^\wedge \cap \mathcal{PGF} = \mathcal{P}$.

## 4. The left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension of rings

In this section, we use classical homological invariants to characterize the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension of rings and give a lower and upper bound of it, in terms of the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-flat dimension and the invariant $\mathrm{splf}R$.

At first, we compare the $\mathcal{GP}$-resolution dimension of Gorenstein $(\mathcal{L}, \mathcal{A})$-flat modules with the projective dimension of flat modules. We use $\mathrm{splf}R$ to denote the supremum of the projective dimension of flat modules. We have the following equality.

**Proposition 4.1.** *For any ring $R$, $sup\{resdim_{{}_R\mathcal{GP}} M \,|\, M \in \mathcal{GF}\} = splfR$. Moreover, $\mathcal{F} \subseteq \mathcal{P}^\wedge$ if and only if $\mathcal{GF} \subseteq \mathcal{GP}^\wedge$.*

*Proof.* By Remark 3.6, the projective dimension of a flat $R$-module equals its $\mathcal{GP}$-resolution dimension. Since $\mathcal{F} \subseteq \mathcal{GF}$, $\mathrm{splf}R \leqslant \sup\{\mathrm{resdim}_{{}_R\mathcal{GP}} M \,|\, M \in \mathcal{GF}\}$. If $\mathrm{splf}R = \infty$, then $\sup\{\mathrm{resdim}_{{}_R\mathcal{GP}} M \,|\, M \in \mathcal{GF}\} = \infty$. Now, let $\mathrm{splf}R = n < \infty$. By [17, Proposition 2.3], $\mathcal{A}$ contains the class of all injective $R^{op}$-modules. For a Gorenstein $(\mathcal{L}, \mathcal{A})$-flat $R$-module $M$, by [5, Theorem A.6] and [18, Remark 38], there exists an exact sequence $0 \to M \to F \to G \to 0$ with $F \in \mathcal{F}$ and $G \in \mathcal{GP}$. Then $\mathrm{pd}_R F \leqslant n$. Naturally, $\mathrm{resdim}_{{}_R\mathcal{GP}} F \leqslant n$. By Proposition 3.3, one gets that $\mathrm{resdim}_{{}_R\mathcal{GP}} M \leqslant n$. Thus $\sup\{\mathrm{resdim}_{{}_R\mathcal{GP}} M \,|\, M \in \mathcal{GF}\} \leqslant \mathrm{splf}R = n$. Therefore, $\sup\{\mathrm{resdim}_{{}_R\mathcal{GP}} M \,|\, M \in \mathcal{GF}\} = \mathrm{splf}R$.

We let $\mathcal{F} \subseteq \mathcal{P}^\wedge$, and $M \in \mathcal{GF}$. By [18, Remark 38] again, there exists an exact sequence $0 \to F \to G \to M \to 0$ with $F \in \mathcal{F}$ and $G \in \mathcal{GP}$. Note that $\mathcal{F} \subseteq \mathcal{P}^\wedge \subseteq \mathcal{GP}^\wedge$. By Proposition 3.3, $M \in \mathcal{GP}^\wedge$. That is, $\mathcal{F} \subseteq \mathcal{P}^\wedge$ implies $\mathcal{GF} \subseteq \mathcal{GP}^\wedge$. Next, we let $\mathcal{GF} \subseteq \mathcal{GP}^\wedge$. Since $\mathcal{F} \subseteq \mathcal{GF}$, $\mathcal{F} \subseteq \mathcal{GP}^\wedge$. By Remark 3.6, $\mathcal{F} \subseteq \mathcal{P}^\wedge$. Then $\mathcal{GF} \subseteq \mathcal{GP}^\wedge$ implies $\mathcal{F} \subseteq \mathcal{P}^\wedge$. This completes the proof. □

**Proposition 4.2.** *The following conditions are equivalent for a non-negative integer $n$.*

(1) *$splfR \leqslant n$.*

(2) *Every Gorenstein $(\mathcal{L}, \mathcal{A})$-flat $R$-module has $\mathcal{GP}$-resolution dimension at most $n$.*

(3) *Every flat $R$-module has $\mathcal{GP}$-resolution dimension at most $n$.*

*If any of the previous conditions holds,*

$$sup\{resdim_{{}_R\mathcal{GP}} M \,|\, M \in \mathcal{F}\} = sup\{resdim_{{}_R\mathcal{GP}} N \,|\, N \in \mathcal{GF}\} = splfR \leqslant n.$$

*Proof.* (1) ⇒ (2) It immediately follows from Proposition 4.1.

(2) ⇒ (3) It is clear.

(3) ⇒ (1) Let $M \in \mathcal{F}$ and $\mathrm{resdim}_{{}_R\mathcal{GP}} M \leqslant n$. By Remark 3.6, $\mathrm{pd}_R M \leqslant n$. Thus $\mathrm{splf}R \leqslant n$.

In the case, one easily gets that $\sup\{\mathrm{resdim}_{{}_R\mathcal{GP}} M \,|\, M \in \mathcal{F}\} = \sup\{\mathrm{resdim}_{{}_R\mathcal{GP}} N \,|\, N \in \mathcal{GF}\} = \mathrm{splf}R \leqslant n$. □

**Proposition 4.3.** *Assume that $\mathcal{GF}$ is closed under extensions, and let $M$ be an $R$-module. One gets that*

$$resdim_{{}_R\mathcal{GF}} M \leqslant resdim_{{}_R\mathcal{GP}} M \leqslant resdim_{{}_R\mathcal{GF}} M + splfR.$$

*Proof.* Since $\mathcal{GP} \subseteq \mathcal{GF}$, $\mathrm{resdim}_{R\mathcal{GF}}M \leqslant \mathrm{resdim}_{R\mathcal{GP}}M$. Next, we show that $\mathrm{resdim}_{R\mathcal{GP}}M \leqslant \mathrm{resdim}_{R\mathcal{GF}}M + \mathrm{splf}R$. If $\mathrm{splf}R = \infty$ or $\mathrm{resdim}_{R\mathcal{GF}}M = \infty$, then the result obviously holds. Now, set $\mathrm{splf}R = n < \infty$ and $\mathrm{resdim}_{R\mathcal{GF}}M = m < \infty$. Then there exists an exact sequence $0 \to G_m \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ with each $G_i \in \mathcal{GF}$. Pick a partial projective resolution of $M$, $0 \to K \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each $P_i \in \mathcal{P}$. Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_m \arrow[r] & G_{m-1} \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

By taking the mapping cone, we obtain an exact sequence

$$0 \to K \to G_m \oplus P_{m-1} \to \cdots \to G_1 \oplus P_0 \to G_0 \oplus M \to M \to 0.$$

Therefore, we have the other exact sequence $0 \to K \to G_m \oplus P_{m-1} \to \cdots \to G_1 \oplus P_0 \to G_0 \to 0$. Since $P_i \in \mathcal{P}$, $G_i \in \mathcal{GF}$ and $\mathcal{P} \subseteq \mathcal{GF}$, $G_{i+1} \oplus P_i \in \mathcal{GF}$. By assumption and [17, Proposition 5.2], we know that $\mathcal{GF}$ is closed under kernels of epimorphisms. Thus $K \in \mathcal{GF}$. By Proposition 4.2, $\mathrm{resdim}_{R\mathcal{GP}}K \leqslant n$. Thus there exists an exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to K \to 0$ with each $C_i \in \mathcal{GP}$. Thus we have an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to P_{m-1} \to P_{m-2} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

By Proposition 3.1, $\mathrm{resdim}_{R\mathcal{GP}}M \leqslant m + n$. Namely, $\mathrm{resdim}_{R\mathcal{GP}}M \leqslant \mathrm{resdim}_{R\mathcal{GF}}M + \mathrm{splf}R$, as desired. $\square$

In the following, we define the left global $\mathcal{X}$-resolution dimension, denoted by $\mathcal{X}$-lgl.dim$R$, of the ring $R$, by letting

$$\mathcal{X}\text{-lgl.dim}R = \sup\{\mathrm{resdim}_{R\mathcal{X}}M \,|\, M \text{ an } R\text{-module}\}.$$

The left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension, the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-flat dimension, and the left global projectively coresolved Gorenstein flat dimension of $R$ are defined by setting $\mathcal{X} = \mathcal{GP}$, $\mathcal{X} = \mathcal{GF}$ and $\mathcal{X} = \mathcal{PGF}$, respectively.

By Proposition 4.3, we can give a lower and upper bound of the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension of $R$, in terms of the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-flat dimension of $R$ and the invariant splf$R$.

**Theorem 4.4.** *Suppose $\mathcal{GF}$ is closed under extensions. Then*

$$\mathcal{GF}\text{-}lgl.dimR \leqslant \mathcal{GP}\text{-}lgl.dimR \leqslant \mathcal{GF}\text{-}lgl.dimR + splfR.$$

We know that $\mathcal{GP} \subseteq \mathcal{PGF}$. In the following, in order to use classical homological invariants to characterize those rings over which all modules are of finite $\mathcal{GP}$-resolution dimension, we give some conditions such that $\mathcal{GP} = \mathcal{PGF}$.



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**Lemma 4.5.** *Suppose that every $R^{op}$-module in $\mathcal{A}$ is of finite injective or finite flat dimension. Then $\mathcal{GP} = \mathcal{PGF}$.*

*Proof.* The containment $\mathcal{GP} \subseteq \mathcal{PGF}$ is known. On the other hand, it can be proved s-traightforwardly from the assumption and [5, Theorem A.6] that the converse containment holds. $\square$

In the following, we characterize the finiteness of $\mathcal{GP}$-lgl.dim$R$ by the invariants spli$R$, silp$R$ and sfli$R$.

**Proposition 4.6.** *Consider the following conditions.*

(1) $\mathcal{GP}$*-lgl.dim*$R < \infty$.

(2) *resdim*$_{_R\mathcal{GP}}M < \infty$ *for any* $R$*-module* $M$.

(3) *resdim*$_{_R\mathcal{PGF}}M < \infty$ *for any* $R$*-module* $M$.

(4) *spli*$R =$ *silp*$R < \infty$ *and sfli*$R =$ *sfli*$R^{op} < \infty$.

(5) *spli*$R < \infty$ *and sfli*$R^{op} < \infty$.

*Then* (1) $\Leftrightarrow$ (2), (2) $\Rightarrow$ (3) *and* (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) *hold. Assume that every* $R^{op}$*-module in* $\mathcal{A}$ *is of finite injective or finite flat dimension. Then* (3) $\Rightarrow$ (2). *In this case,*

$$\mathcal{GP}\text{-}lgl.dimR = \mathcal{PGF}\text{-}lgl.dimR < \infty.$$

*Proof.* (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Assume that $\mathcal{GP}$-lgl.dim$R = \infty$. Then for any non-negative integer $n$, there is an $R$-module $M_n$ such that $\mathrm{resdim}_{_R\mathcal{GP}}M_n > n$. Let $M = \oplus M_n$. By Proposition 3.2,

$$\mathrm{resdim}_{_R\mathcal{GP}}M = \sup\{\mathrm{resdim}_{_R\mathcal{GP}}M_n\} = \infty.$$

This is a contradiction. So $\mathcal{GP}$-lgl.dim$R < \infty$.

(2) $\Rightarrow$ (3) Since $\mathcal{GP} \subseteq \mathcal{PGF}$, $\mathrm{resdim}_{_R\mathcal{PGF}}M \leqslant \mathrm{resdim}_{_R\mathcal{GP}}M < \infty$ for any $R$-module $M$.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) It immediately follows from [9, Theorem 21].

(3) $\Rightarrow$ (2) It is immediate from Lemma 4.5.

In this case, we get that $\mathcal{GP}$-lgl.dim$R = \mathcal{PGF}$-lgl.dim$R < \infty$. $\square$

**Proposition 4.7.** *If* $\mathcal{GP}$*-lgl.dim*$R < \infty$, *then* $\mathcal{GP}$*-lgl.dim*$R = \mathcal{PGF}$*-lgl.dim*$R =$ *spli*$R =$ *silp*$R$.

*Proof.* Since $\mathcal{GP}$-lgl.dim$R < \infty$, one gets that $\mathrm{resdim}_{_R\mathcal{GP}}M < \infty$ for any $R$-module $M$. By Proposition 3.11, one obtains that $\mathrm{Gpd}_R M = \text{PGF-dim}_R M = \mathrm{resdim}_{_R\mathcal{GP}}M < \infty$. Then $\sup\{\mathrm{Gpd}_R M \mid M \text{ a left } R\text{-module}\} = \mathcal{PGF}$-lgl.dim$R = \mathcal{GP}$-lgl.dim$R < \infty$. Since $\mathcal{GP}$-lgl.dim$R < \infty$ again, one has that spli$R =$ silp$R < \infty$ by Proposition 4.6. It follows from [3, Theorem 3.3] that $\sup\{\mathrm{Gpd}_R M \mid M \text{ a left } R\text{-module}\} = \max\{\mathrm{spli}R, \mathrm{silp}R\}$. Thus $\mathcal{GP}$-lgl.dim$R = \mathcal{PGF}$-lgl.dim$R =$ spli$R =$ silp$R$. $\square$

**Corollary 4.8.** *If* $\mathcal{GF}$ *is closed under extensions and* $\mathcal{GP}$*-gl.dim*$R < \infty$, *then spli*$R =$ *silp*$R \leqslant$ $\mathcal{GF}$*-gl.dim*$R+$*splf*$R$.

**Proposition 4.9.** *If* $\mathcal{GP}$*-gl.dim*$R < \infty$, *then every Gorenstein projective module is Gorenstein flat.*

*Proof.* Since $\mathcal{GP}$-gl.dim$R < \infty$, by Proposition 4.6, sfli$R$ = sfli$R^{op} < \infty$. So it immediately follows from [11, Theorem 5.3 and Corollary 5.5] that every Gorenstein projective module is Gorenstein flat. $\square$

## 5. Recollement situation of the finite left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension

By [22, Theorem 4.4], we know that $\mathcal{PGF} \subseteq \mathcal{GP}(R)$, that is, $\mathcal{PGF}$ is a special class of Gorenstein projective modules. Of course, so is $\mathcal{GP}$. Indeed, we know that recollements of abelian categories provide a very useful framework for investigating homological connections among categories. In this section, we will discuss the finiteness of the left global Gorenstein $(\mathcal{L}, \mathcal{A})$-projective dimension and the finiteness of the left global projectively coresolved Gorenstein flat dimension by recollement situations and invariants.

Throughout this section, we assume that $S$ and $T$ are associative rings with an identity.

${}_R\mathcal{P}$ denotes the class of all projective left $R$-modules.

${}_S\mathcal{P}$ denotes the class of all projective left $S$-modules.

${}_R\mathcal{I}$ denotes the class of all injective left $R$-modules.

${}_S\mathcal{I}$ denotes the class of all injective left $S$-modules.

Gwgldim$T$ denotes the Gorenstein weak global dimension of a ring $T$.

By [8], we know that the Gorenstein weak global dimension of a ring is a left-right symmetric invariant. Let ($R$-Mod, $S$-Mod, $T$-Mod) be a recollement of categories of corresponding modules.

$$\begin{tikzcd} R\text{-Mod} \arrow[r, "i"] & S\text{-Mod} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & T\text{-Mod} \arrow[l, bend right, "l"'] \arrow[l, bend left, "r"] \end{tikzcd}$$

We also need the following notions.

pgl.dim$_S R = \sup\{\text{pd}_S i(M) \mid M$ a left $R$-module$\}$

igl.dim$_S R = \sup\{\text{id}_S i(M) \mid M$ a left $R$-module$\}$

pgl.dim$_{S\,R}\mathcal{P} = \sup\{\text{pd}_S i(P) \mid P \in {}_R\mathcal{P}\}$

igl.dim$_{S\,R}\mathcal{I} = \sup\{\text{id}_S i(I) \mid I \in {}_R\mathcal{I}\}$

Motivated by [23, Theorem 3.5], we give the following result about the left global $\mathcal{GP}$-resolution dimension of rings by recollement situations.

**Proposition 5.1.** *Let ($R$-Mod, $S$-Mod, $T$-Mod) be a recollement of categories of corresponding modules.*

$$\begin{tikzcd} R\text{-Mod} \arrow[r, "i"] & S\text{-Mod} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & T\text{-Mod} \arrow[l, bend right, "l"'] \arrow[l, bend left, "r"] \end{tikzcd}$$

(1) *If $\mathcal{GP}$-lgl.dim$S < \infty$ and $\mathcal{GP}$-lgl.dim$T < \infty$, then*

$$\begin{aligned}\mathcal{GP}\text{-}lgl.dimT &\leqslant min\{sup\{id_T e(I) \,|\, I \in {}_S\mathcal{I}\}, sup\{pd_T e(P) \,|\, P \in {}_S\mathcal{P}\}\} + \mathcal{GP}\text{-}lgl.dimS \\ &\leqslant min\{sup\{id_T e(I) \,|\, I \in {}_S\mathcal{I}\}, sup\{pd_T e(P) \,|\, P \in {}_S\mathcal{P}\}\} + GwgldimS + splfS,\end{aligned}$$

$$\begin{aligned}\mathcal{PGF}\text{-}lgl.dimT &\leqslant min\{sup\{id_T e(I) \,|\, I \in {}_S\mathcal{I}\}, sup\{pd_T e(P) \,|\, P \in {}_S\mathcal{P}\}\} + \mathcal{PGF}\text{-}lgl.dimS \\ &\leqslant min\{sup\{id_T e(I) \,|\, I \in {}_S\mathcal{I}\}, sup\{pd_T e(P) \,|\, P \in {}_S\mathcal{P}\}\} + GwgldimS + splfS.\end{aligned}$$

(2) *If the functor $e$ preserves both projective and injective objects, $\mathcal{GP}$-lgl.dim$S < \infty$ and $\mathcal{GP}$-lgl.dim$T < \infty$, then*

$$\begin{aligned}\mathcal{GP}\text{-}lgl.dimS &\leqslant min\{pgl.dim_S R, igl.dim_S R\} + \mathcal{GP}\text{-}lgl.dimT + 1 \\ &\leqslant min\{pgl.dim_S R, igl.dim_S R\} + GwgldimT + splfT + 1,\end{aligned}$$

$$\begin{aligned}\mathcal{PGF}\text{-}lgl.dimS &\leqslant min\{pgl.dim_S R, igl.dim_S R\} + \mathcal{PGF}\text{-}lgl.dimT + 1 \\ &\leqslant min\{pgl.dim_S R, igl.dim_S R\} + GwgldimT + splfT + 1.\end{aligned}$$

(3) *If $\mathcal{GP}$-lgl.dim$R < \infty$, $\mathcal{GP}$-lgl.dim$S < \infty$, and one of the following conditions holds,*

(*a*) *$pgl.dim_{_SR}\mathcal{P} \leqslant 1$ and $igl.dim_{_SR}\mathcal{I} < \infty$;*

(*b*) *$pgl.dim_{_SR}\mathcal{P} < \infty$ and $igl.dim_{_SR}\mathcal{I} \leqslant 1$,*

*then*

$$\mathcal{GP}\text{-}lgl.dimR \leqslant \mathcal{GP}\text{-}lgl.dimS \text{ and } \mathcal{PGF}\text{-}lgl.dimR \leqslant \mathcal{PGF}\text{-}lgl.dimS.$$

*Proof.* (1) By [23, Proposition 3.4], we get that

$$\text{spli}T \leqslant \sup\{\text{pd}_T e(P) \,|\, P \in {}_S\mathcal{P}\} + \text{spli}S \text{ and } \text{silp}T \leqslant \sup\{\text{id}_T e(I) \,|\, I \in {}_S\mathcal{I}\} + \text{silp}S.$$

By assumption and Proposition 4.7, we have that

$$\mathcal{GP}\text{-lgl.dim}S = \mathcal{PGF}\text{-lgl.dim}S = \text{spli}S = \text{silp}S,$$

$$\mathcal{GP}\text{-lgl.dim}T = \mathcal{PGF}\text{-lgl.dim}T = \text{spli}T = \text{silp}T.$$

Then

$$\text{spli}T \leqslant \min\{\sup\{\text{id}_T e(I) \,|\, I \in {}_S\mathcal{I}\}, \sup\{\text{pd}_T e(P) \,|\, P \in {}_S\mathcal{P}\}\} + \text{spli}S.$$

By Proposition 3.11 and [8, Theorem 3.3], we know that

$$\mathcal{GP}\text{-lgl.dim}S \leqslant \text{Gwgldim}S + \text{splf}S.$$

So we obtain inequalities of (1).

(2) Since the functor $e$ preserves both projective and injective objects, by [23, Proposition 3.4], we get that

$$\text{spli}S \leqslant \text{pgl.dim}_S R + \text{spli}T + 1 \text{ and } \text{silp}S \leqslant \text{igl.dim}_S R + \text{silp}T + 1.$$

Since $\mathcal{GP}$-lgl.dim$S < \infty$ and $\mathcal{GP}$-lgl.dim$T < \infty$, we have that

$$\mathcal{GP}\text{-lgl.dim}S = \mathcal{PGF}\text{-lgl.dim}S = \text{spli}S = \text{silp}S,$$

$$\mathcal{GP}\text{-lgl.dim}T = \mathcal{PGF}\text{-lgl.dim}T = \text{spli}T = \text{silp}T$$



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by Proposition 4.7. Then

$$\mathrm{spli}S \leqslant \min\{\mathrm{pgl.dim}_S R, \mathrm{igl.dim}_S R\} + \mathrm{spli}T + 1.$$

By Proposition 3.11 and [8, Theorem 3.3], we know that

$$\mathcal{GP}\text{-lgl.dim}T \leqslant \mathrm{Gwgldim}T + \mathrm{splf}T.$$

So we obtain inequalities of (2).

(3) By assumption, it follows from Proposition 3.11 and [23, Theorem 3.5]. □

In the following, we apply Proposition 5.1 to some rings and Morita rings.

**Example 5.2.** Let $R$ be a ring and $e$ an idempotent element of $R$. By [21, Example 2.7], we have a recollement

$$\begin{tikzcd}[column sep=large] R/ReR\text{-Mod} \arrow[r, "\mathrm{inc}"] & R\text{-Mod} \arrow[l, bend right, "R/ReR\otimes_R -"'] \arrow[l, bend left, "\mathrm{Hom}_R(R/ReR,-)"] \arrow[r, "e(-)"] & eRe\text{-Mod} \arrow[l, bend right, "Re\otimes_{eRe}-"'] \arrow[l, bend left, "\mathrm{Hom}_{eRe}(eR,-)"] \end{tikzcd}$$

where the category $R/ReR$-Mod of modules over $R/ReR$ is the kernel of the functor $e(-) : R$-Mod$\to$ $eRe$-Mod. Based on this recollement, the inequalities from Proposition 5.1 can be obtained under the corresponding assumptions of finiteness for the relative global dimensions of the rings.

**Example 5.3.** Let $R$, $S$ be rings, $M$ an $S$-$R$-bimodule and $N$ an $R$-$S$-bimodule. Let $\phi : M \otimes_R N \to S$ be an $S$-$S$-bimodule homomorphism and let $\psi : N \otimes_S M \to R$ be an $R$-$R$-bimodule homomorphism. Then the above data allow one to define the Morita ring (see [21, Example 2.8]):

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} R & {}_RN_S \\ {}_SM_R & S \end{pmatrix}$$

where the addition of elements of $\Lambda_{(\phi,\psi)}$ is componentwise and multiplication is given by

$$\begin{pmatrix} r & n \\ m & s \end{pmatrix}\begin{pmatrix} r' & n' \\ m' & s' \end{pmatrix} = \begin{pmatrix} rr' + \psi(n\otimes m') & rn' + ns' \\ mr' + sm' & ss' + \phi(m\otimes n') \end{pmatrix}.$$

Then $\Lambda_{(\phi,\psi)}$ is a ring with an identity. We assume that $\phi(m\otimes n)m' = m\psi(n\otimes m')$ and $n\phi(m\otimes n') = \psi(n\otimes m)n'$ for all $m,\ m' \in M$ and $n,\ n' \in N$. This condition ensures that $\Lambda_{(\phi,\psi)}$ is an associative ring. One knows that $e_1 = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1_S \end{pmatrix}$ are idempotent elements of $\Lambda_{(\phi,\psi)}$. So one obtains tow recollements of categories of modules

$$\begin{tikzcd}[column sep=large] \Lambda/\Lambda e_1\Lambda\text{-Mod} \arrow[r, "\mathrm{inc}"] & \Lambda_{(\phi,\psi)}\text{-Mod} \arrow[l, bend right, "\Lambda/\Lambda e_1\Lambda\otimes_\Lambda -"'] \arrow[l, bend left, "\mathrm{Hom}_\Lambda(\Lambda/\Lambda e_1\Lambda,-)"] \arrow[r, "e_1(-)"] & e_1\Lambda e_1\text{-Mod} \arrow[l, bend right, "\Lambda e_1\otimes_{e_1\Lambda e_1}-"'] \arrow[l, bend left, "\mathrm{Hom}_{e_1\Lambda e_1}(e_1\Lambda,-)"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
\Lambda/\Lambda e_2\Lambda\text{-Mod} \arrow[r, "\mathrm{inc}"] & \Lambda_{(\phi,\psi)}\text{-Mod} \arrow[l, bend right, "\Lambda/\Lambda e_2\Lambda\otimes_\Lambda -"'] \arrow[l, bend left, "\mathrm{Hom}_\Lambda(\Lambda/\Lambda e_2\Lambda,-)"] \arrow[r, "e_2(-)"] & e_2\Lambda e_2\text{-Mod} \arrow[l, bend right, "\Lambda e_2\otimes_{e_2\Lambda e_2} -"'] \arrow[l, bend left, "\mathrm{Hom}_{e_2\Lambda e_2}(e_2\Lambda,-)"]
\end{tikzcd}$$

where $\alpha : \Lambda/\Lambda e_1\Lambda\text{-Mod} \to S/\mathrm{Im}\phi\text{-Mod}$, $\beta : e_1\Lambda e_1\text{-Mod} \to R\text{-Mod}$, $\gamma : \Lambda/\Lambda e_2\Lambda\text{-Mod} \to R/\mathrm{Im}\psi\text{-Mod}$, and $\eta : e_2\Lambda e_2\text{-Mod} \to S\text{-Mod}$ are equivalences of categories. Naturally, one can get the following two recollements of categories of modules by above equivalences

$$\begin{tikzcd}[column sep=large]
S/\mathrm{Im}\phi\text{-Mod} \arrow[r, "\mathrm{inc}\circ\alpha^{-1}"] & \Lambda_{(\phi,\psi)}\text{-Mod} \arrow[l, bend right, "\alpha\circ\Lambda/\Lambda e_1\Lambda\otimes_\Lambda -"'] \arrow[l, bend left, "\alpha\circ\mathrm{Hom}_\Lambda(\Lambda/\Lambda e_1\Lambda,-)"] \arrow[r, "\beta\circ e_1(-)"] & R\text{-Mod} \arrow[l, bend right, "\Lambda e_1\otimes_{e_1\Lambda e_1} -\circ\beta^{-1}"'] \arrow[l, bend left, "\mathrm{Hom}_{e_1\Lambda e_1}(e_1\Lambda,-)\circ\beta^{-1}"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
R/\mathrm{Im}\psi\text{-Mod} \arrow[r, "\mathrm{inc}\circ\gamma^{-1}"] & \Lambda_{(\phi,\psi)}\text{-Mod} \arrow[l, bend right, "\gamma\circ\Lambda/\Lambda e_2\Lambda\otimes_\Lambda -"'] \arrow[l, bend left, "\gamma\circ\mathrm{Hom}_\Lambda(\Lambda/\Lambda e_2\Lambda,-)"] \arrow[r, "\eta\circ e_2(-)"] & S\text{-Mod} \arrow[l, bend right, "\Lambda e_2\otimes_{e_2\Lambda e_2} -\circ\eta^{-1}"'] \arrow[l, bend left, "\mathrm{Hom}_{e_2\Lambda e_2}(e_2\Lambda,-)\circ\eta^{-1}"]
\end{tikzcd}$$

The inequalities from Proposition 5.1 can be obtained within these particular contexts under the corresponding assumptions of finiteness for the relative global dimensions of the rings.


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