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ON THE SOLUTIONS AND STABILITY OF A CERTAIN FAMILY OF EQUATIONS ARISING FROM ENGINEERING MODELS

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ABSTRACT. A certain interesting family of functional equations (FEs) arose recently from many interesting engineering models like e.g. Networks and fog computing. We investigate the solutions of such family of equations given by

$$\begin{aligned}\zeta_1(\eta_1,\eta_2)\Pi(\eta_1,\eta_2) &= \zeta_2(\eta_1,\eta_2)\Pi(\eta_1,0)+\zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2)\\ &\quad+\zeta_4(\eta_1,\eta_2)\Pi(0,0)+\zeta_5(\eta_1,\eta_2),\end{aligned}$$

for some known functions (generally polynomials) $\zeta_i(\eta_1,\eta_2), i=1,\ldots,5$ of two complex variables η_1 and η_2. We use an example to illustrate our findings. Moreover, we give remark on the stability of this interesting family of FEs. In this way, we generalize recent interesting results.

1. Introduction

Functional equations (FEs) are almost everywhere in applications. They currently constitute a modern branch of mathematics with interesting applications in many fields. They continue to grow rapidly over the last few decades and have been proven to be a powerful tool in many disciplines see e.g. [9, 5, 13]. They have numerous applications; for instance, in communication and network models see [16, 20, 10], in information theory [17, 2], in decision theory [1, 23], and in digital filtering [22].

We use the current article to investigate the solution of the family of FEs (see e.g. [9]). As far as we know there is no general solution available for such interesting family of equations. Therefore, this article can be considered as a step forward towards a general solution theory.

The problem of interest can be stated as follows: Find a general solution of the family of FEs given by

$$\begin{aligned}\zeta_1(\eta_1,\eta_2)\Pi(\eta_1,\eta_2) &= \zeta_2(\eta_1,\eta_2)\Pi(\eta_1,0)+\zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2)\\ &\quad+\zeta_4(\eta_1,\eta_2)\Pi(0,0)+\zeta_5(\eta_1,\eta_2),\end{aligned}\tag{1}$$

where the functions $\zeta_i(\eta_1,\eta_2), i=1,\ldots,5$ are some given functions of two complex variables η_1, η_2. The unknowns of (1) are defined as follows

$$\Pi(\eta_1,\eta_2):=\sum_{m,n=0}^{\infty} p_{m,n}\eta_1^m\eta_2^n, \qquad |\eta_1|\le 1,\ |\eta_2|\le 1$$

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$$\Pi(\eta_1,0) := \sum_{m=0}^{\infty} p_{m,0}\eta_1^m : D \to \mathbb{C}$$

$$\Pi(0,\eta_2) := \sum_{n=0}^{\infty} p_{0,n}\eta_2^n : D \to \mathbb{C}$$

and

$$\Pi(0,0) = \Pi(\eta_1,0)|_{\eta_1=0} = \Pi(0,\eta_2)|_{\eta_2=0} = p_{0,0}.$$

It is easy to see that the unknown $\Pi(0,0)$ is just some probability value ($p_{0,0}$). The problem of interest is originally the problem of extracting the sequence $p_{m,n}$ which represents many interesting applications see e.g. [19].

The family of FEs (1) has been investigated by the authors in [9]. A solution of a special case of (1) is introduced in [20]. In [10] a special case of (1) has been studied. A solution in terms of a conformal mapping has been introduced for a special case of (1) in [3].

Equation (1) is interesting because of the following:

- A special case of (1) arose from a network model (see [25]) where

$$\zeta_1(\eta_1,\eta_2) = \eta_1\eta_2 - (\eta_1 r + \tilde{r})(\eta_2 r + \tilde{r})((\eta_1+\eta_2)p\tilde{p} + \eta_1\eta_2(p^2+\tilde{p}^2)),$$
$$\zeta_2(\eta_1,\eta_2) = p(\eta_1 r + \tilde{r})(\eta_2 r + \tilde{r})(\eta_1(\eta_2-1) - p(2\eta_1\eta_2 - \eta_1 - \eta_2)),$$
$$\zeta_3(\eta_1,\eta_2) = p(\eta_1 r + \tilde{r})(\eta_2 r + \tilde{r})(\eta_2(\eta_1-1) - p(2\eta_1\eta_2 - \eta_1 - \eta_2)),$$

  and

$$\zeta_4(\eta_1,\eta_2) = p^2(\eta_1 r + \tilde{r})(\eta_2 r + \tilde{r})(2\eta_1\eta_2 - \eta_1 - \eta_2).$$

  for some parameters $p, r, \tilde{r}, \tilde{p}$.
- A special case of (1) arose from a model of database systems (see [24]) where

$$\zeta_1(\eta_1,\eta_2) = (1+\alpha+\beta)\eta_1\eta_2 - \alpha\eta_2 - \beta\eta_1 - \eta_1^2\eta_2^2,$$
$$\zeta_2(\eta_1,\eta_2) = \beta\eta_1(\eta_2-1),$$

  and

$$\zeta_3(\eta_1,\eta_2) = \alpha\eta_2(\eta_1-1),$$

  and $\zeta_4(\eta_1,\eta_2) = 0$.
- A special case of (1) arose from a model of Asymmetric clocked buffered switch (see [26]) where

$$\zeta_1(\eta_1,\eta_2) = \eta_1\eta_2 - [1-a_1+a_1(r_{11}\eta_1 + r_{12}\eta_2)][1-a_2+a_2(r_{21}\eta_1 + r_{22}\eta_2)],$$
$$\zeta_2(\eta_1,\eta_2) = (\eta_2-1)[1-a_1+a_1(r_{11}\eta_1 + r_{12}\eta_2)][1-a_2+a_2(r_{21}\eta_1 + r_{22}\eta_2)],$$
$$\zeta_3(\eta_1,\eta_2) = (\eta_1-1)[1-a_1+a_1(r_{11}\eta_1 + r_{12}\eta_2)][1-a_2+a_2(r_{21}\eta_1 + r_{22}\eta_2)],$$

  and

$$\zeta_4(\eta_1,\eta_2) = (\eta_1-1)(\eta_2-1)[1-a_1+a_1(r_{11}\eta_1 + r_{12}\eta_2)][1-a_2+a_2(r_{21}\eta_1 + r_{22}\eta_2)],$$

  for some parameters $a_i, r_{ij}, i, j = 1, 2$.

- A special case of (1) arose from a model of Non-work conserving generalized processor sharing queue (see [27]) where

$$\zeta_1(\eta_1,\eta_2) = -\lambda_1\eta_1^2\eta_2 - \lambda_2\eta_1\eta_2^2 + (\lambda_1+\lambda_2+\phi_1\mu_1+\phi_2\mu_2)\eta_1\eta_2 - \phi_1\mu_1\eta_2 - \phi_2\mu_2\eta_1,$$

$$\zeta_2(\eta_1,\eta_2) = \phi_2\mu_2\eta_1(\eta_2-1) - (1-\phi_1)\mu_1\eta_2(\eta_1-1),$$

$$\zeta_3(\eta_1,\eta_2) = \phi_1\mu_1\eta_2(\eta_1-1) - (1-\phi_2)\mu_2\eta_1(\eta_2-1),$$

and

$$\zeta_4(\eta_1,\eta_2) = (1-\phi_1)\mu_1\eta_2(\eta_1-1) + (1-\phi_2)\mu_2\eta_1(\eta_2-1).$$

for some system parameters $\lambda_i,\phi_i,\mu_i, i=,2$.

The article is organized as follows. In Section 2 we introduce some preliminaries, in Section 3 we present the main results, in Section 4 we investigate the stability of the family of equations, and in Section 5 we conclude our work.

2. Preliminaries

Throughout the article, we use $\mathbb{N}$ to denote the set of positive integers, $\mathbb{C}$ the set of complex numbers, D the unit disk in the complex plane, $\overline{D}$ denotes the closure of D, $\mathbb{N}_0$ to denote $\mathbb{N}\cup\{0\}$, and $\deg_{\eta_1} H$ to denote the degree of the polynomial H with respect to η_1. In [19] the authors end up with a particular case of (1). Since till the moment of writing this article and according to the best of our knowledge such interesting equations don't have a general solution theory then it worth investigation.

It should be remarked that the authors investigated in [9] the solution of (1) as follows. They used the following sets

$$\mathscr{K} := \{(\eta_1,\eta_2)\in\overline{D}^2 : \zeta_1(\eta_1,\eta_2)=0\},$$

$$\mathscr{K}_0 := \{\eta_1\in\overline{D} : (\eta_1,0)\in\mathscr{K}\}, \qquad \mathscr{K}^0 := \{\eta_1\in\overline{D} : (0,\eta_1)\in\mathscr{K}\},$$

and used the following two hypotheses.

($\mathscr{A}$) $\zeta_1(\eta_1,0)=\zeta_2(\eta_1,0)$, $\zeta_3(\eta_1,0)=-\zeta_4(\eta_1,0)$, $\zeta_5(\eta_1,0)=0$ for $\eta_1\in\overline{D}\setminus\mathscr{K}_0$,

($\mathscr{B}$) $\zeta_1(0,\eta_2)=\zeta_3(0,\eta_2)$, $\zeta_2(0,\eta_2)=-\zeta_4(0,\eta_2)$, $\zeta_5(0,\eta_2)=0$ for $\eta_2\in\overline{D}\setminus\mathscr{K}^0$,

to introduce the theorem.

Theorem 1. *Given a function $P:\overline{D}^2\to\mathbb{C}$ satisfies equation (1) for aa $\eta_1,\eta_2\in\overline{D}$, then functions $g,h:\overline{D}\to\mathbb{C}$ exist with $h(0)=g(0)$,*

$$\zeta_2(\eta_1,\eta_2)h(\eta_1)+\zeta_3(\eta_1,\eta_2)g(\eta_2)+\zeta_4(\eta_1,\eta_2)g(0)+\zeta_5(\eta_1,\eta_2)=0, \qquad (\eta_1,\eta_2)\in\mathscr{K}, \tag{2}$$

and

$$\Pi(\eta_1,\eta_2) = \frac{\zeta_2(\eta_1,\eta_2)h(\eta_1)+\zeta_3(\eta_1,\eta_2)g(\eta_2)+\zeta_4(\eta_1,\eta_2)g(0)+\zeta_5(\eta_1,\eta_2)}{\zeta_1(\eta_1,\eta_2)}, \qquad (\eta_1,\eta_2)\in\overline{D}^2\setminus\mathscr{K}. \tag{3}$$

In addition, if $(\mathscr{A})$ *and* $(\mathscr{B})$ *hold and* $\mathscr{K}_0$ *and* $\mathscr{K}^0$ *are at most countable, then every continuous function* $f : \overline{D}^2 \to \mathbb{C}$*, given by* (3)*, with some continuous functions* $g, h : \overline{D} \to \mathbb{C}$ *with* $h(0) = g(0)$ *and* (2) *is valid, satisfies equation* (1) *for all* $\eta_1, \eta_2 \in \overline{D}$*; in particular,*

$$\Pi(\eta_1, 0) = h(\eta_1), \qquad \Pi(0, \eta_1) = g(\eta_1), \qquad \eta_1 \in \overline{D}. \tag{4}$$

The proof is introduced in [9]. Also in [9] they assumed a somewhat different case, where

$(\mathscr{C})$ $\zeta_2(\eta_1, \eta_2) = 0$ for $\eta_1, \eta_2 \in \overline{D}$,

and used the following hypothesis.

$(\mathscr{D})$ $\zeta_4(0, \eta_2) = \zeta_5(0, \eta_2) = 0$, $\zeta_3(0, \eta_2) = \zeta_1(0, \eta_2)$ for $\eta_2 \in \overline{D} \setminus \mathscr{K}^0$.

to introduce the theorem.

Theorem 2. *Assume that* $(\mathscr{C})$ *is valid. Let* $f : \overline{D}^2 \to \mathbb{C}$ *satisfies* (1) *for all* $(\eta_1, \eta_2) \in \overline{D}$*. Then a function* $g : \overline{D} \to \mathbb{C}$ *exists with*

$$\zeta_3(\eta_1, \eta_2) g(\eta_2) + \zeta_4(\eta_1, \eta_2) g(0) + \zeta_5(\eta_1, \eta_2) = 0, \qquad (\eta_1, \eta_2) \in \mathscr{K}, \tag{5}$$

and

$$\Pi(\eta_1, \eta_2) = \frac{\zeta_3(\eta_1, \eta_2) g(\eta_2) + \zeta_4(\eta_1, \eta_2) g(0) + \zeta_5(\eta_1, \eta_2)}{\zeta_1(\eta_1, \eta_2)}, \qquad (\eta_1, \eta_2) \in \overline{D}^2 \setminus \mathscr{K}. \tag{6}$$

Moreover, if $(\mathscr{D})$ *holds and* $\mathscr{K}^0$ *is at most countable, then every continuous function* $P : \overline{D}^2 \to \mathbb{C}$*, given by* (6) *with some continuous function* $g : \overline{D} \to \mathbb{C}$ *fulfilling* (5)*, satisfies equation* (1) *for all* $\eta_1, \eta_2 \in \overline{D}$*; in particular,*

$$g(\eta_2) = \Pi(0, \eta_2), \qquad \eta_2 \in \overline{D}. \tag{7}$$

It is easy to see that by plugging $\eta_1 = 0$ and/or $\eta_2 = 0$ in the main equation (1) we could get some expressions for the unknowns $\Pi(\eta_1, 0)$ and $\Pi(0, \eta_2)$ based on some conditions as follows. If $\eta_2 = 0$ in (1) then

$$\begin{aligned} \zeta_1(\eta_1, 0)\Pi(\eta_1, 0) &= \zeta_2(\eta_1, 0)\Pi(\eta_1, 0) + \zeta_3(\eta_1, 0)\Pi(0, 0) \\ &\quad + \zeta_4(\eta_1, 0)\Pi(0, 0) + \zeta_5(\eta_1, 0), \end{aligned} \tag{8}$$

from (8) if $\zeta_1(\eta_1, 0) \neq \zeta_2(\eta_1, 0)$ for some η_1 then in this case it is possible to get an expression for $\Pi(\eta_1, 0)$ as follows

$$\Pi(\eta_1, 0) = \frac{(\zeta_3(\eta_1, 0) + \zeta_4(\eta_1, 0))\Pi(0, 0) + \zeta_5(\eta_1, 0)}{\zeta_1(\eta_1, 0) - \zeta_2(\eta_1, 0)}. \tag{9}$$

Similarly, when plugging $\eta_1 = 0$ in (1) to get

$$\begin{aligned} \zeta_1(0, \eta_2)\Pi(0, \eta_2) &= \zeta_2(0, \eta_2)\Pi(0, 0) + \zeta_3(0, \eta_2)\Pi(0, \eta_2) \\ &\quad + \zeta_4(0, \eta_2)\Pi(0, 0) + \zeta_5(0, \eta_2), \end{aligned} \tag{10}$$

from (10) if $\zeta_1(0, \eta_2) \neq \zeta_3(0, \eta_2)$ for some η_2 then in this case it is possible to get an expression for $\Pi(0, \eta_2)$ as follows

$$\Pi(0, \eta_2) = \frac{(\zeta_2(0, \eta_2) + \zeta_4(0, \eta_2))\Pi(0, 0) + \zeta_5(0, \eta_2)}{\zeta_1(0, \eta_2) - \zeta_3(0, \eta_2)}. \tag{11}$$

In the following, we will validate the obtained expressions (11) and (9) as follows: plugging (9) and (11) in (1) to get

$$\begin{aligned}\Pi(\eta_1,\eta_2) = \frac{1}{\zeta_1(\eta_1,\eta_2)}\Big(&\zeta_2(\eta_1,\eta_2)\frac{\zeta_3(\eta_1,0)\Pi(0,0)+\zeta_4(\eta_1,0)\Pi(0,0)+\zeta_5(\eta_1,0)}{\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)}\\ &+\zeta_3(\eta_1,\eta_2)\frac{\zeta_2(0,\eta_2)\Pi(0,0)+\zeta_4(0,\eta_2)\Pi(0,0)+\zeta_5(0,\eta_2)}{\zeta_1(0,\eta_2)-\zeta_3(0,\eta_2)}\\ &+\zeta_4(\eta_1,\eta_2)\Pi(0,0)+\zeta_5(\eta_1,\eta_2)\Big).\end{aligned} \tag{12}$$

In order to say that (12) represents a potential solution of (1) we should get (9) when plugging $\eta_2=0$ in (12). So plugging $\eta_2=0$ in (12) to get

$$\begin{aligned}\Pi(\eta_1,0) &= \frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)\frac{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)}{\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)}\\ &+\zeta_3(\eta_1,0)\frac{(\zeta_2(0,0)+\zeta_4(0,0))\Pi(0,0)+\zeta_5(0,0)}{\zeta_1(0,0)-\zeta_3(0,0)}\\ &+\zeta_4(\eta_1,0)\Pi(0,0)+\zeta_5(\eta_1,0)\Big).\end{aligned} \tag{13}$$

Now its time to check whether (13) is equivalent to (9) or not so

$$\begin{aligned}&\frac{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)}{\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)}\\ &=\frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)\frac{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)}{\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)}\\ &+\zeta_3(\eta_1,0)\frac{(\zeta_2(0,0)+\zeta_4(0,0))\Pi(0,0)+\zeta_5(0,0)}{\zeta_1(0,0)-\zeta_3(0,0)}\\ &+\zeta_4(\eta_1,0)\Pi(0,0)+\zeta_5(\eta_1,0)\Big).\end{aligned}$$

which is satisfied if and only if

$$\begin{aligned}&(\zeta_1(0,0)-\zeta_3(0,0))\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)\}\\ &=\frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)(\zeta_1(0,0)-\zeta_3(0,0))\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)\}\\ &+\zeta_3(\eta_1,0)\{(\zeta_2(0,0)+\zeta_4(0,0))\Pi(0,0)+\zeta_5(0,0)\}(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0))\\ &+\{\zeta_4(\eta_1,0)\Pi(0,0)+\zeta_5(\eta_1,0)\}(\zeta_1(0,0)-\zeta_3(0,0))(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0))\Big).\end{aligned} \tag{14}$$

Since (14) is somehow a complicated expression. For simplicity, we assume that

$$\zeta_1(0,0)-\zeta_3(0,0)=B_1,\ \zeta_2(0,0)+\zeta_4(0,0)=B_2, \tag{15}$$

where $B_k, k = 1,2$ are some nonzero constants. Using this assumption to rewrite (14) in the form

$$
\begin{aligned}
&B_1\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)\} \\
&\qquad = \frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)B_1\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)\} \\
&\qquad + \zeta_3(\eta_1,0)\{B_2\Pi(0,0)+\zeta_5(0,0)\}(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)) \\
&\qquad + \{\zeta_4(\eta_1,0)\Pi(0,0)+\zeta_5(\eta_1,0)\}B_1(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0))\Big). \qquad (16)
\end{aligned}
$$

Since (16) is still complicated then for simplicity, we can consider some particular cases as follows.

(1) $\Pi(0,0) = 0$. In this case (16) is

$$
\begin{aligned}
B_1\zeta_5(\eta_1,0) &= \frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)B_1\zeta_5(\eta_1,0)+\zeta_3(\eta_1,0)\zeta_5(0,0)(\zeta_1(\eta_1,0) \\
&\quad -\zeta_2(\eta_1,0))+\zeta_5(\eta_1,0)B_1(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0))\Big).
\end{aligned}
$$

(2) $\zeta_5(0,0) = 0$. In this case (16) is

$$
\begin{aligned}
&B_1\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)\} \\
&\quad = \frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)B_1\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)\} \\
&\qquad + \zeta_3(\eta_1,0)B_2\Pi(0,0)(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)) \\
&\qquad + \{\zeta_4(\eta_1,0)\Pi(0,0)+\zeta_5(\eta_1,0)\}B_1(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0))\Big).
\end{aligned}
$$

(3) $\zeta_5(\eta_1,\eta_2) = 0$. In this case (16) is

$$
\begin{aligned}
&B_1\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)\} \\
&\qquad = \frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)B_1\{(\zeta_3(\eta_1,0)+\zeta_4(\eta_1,0))\Pi(0,0)\} \\
&\qquad + \zeta_3(\eta_1,0)B_2\Pi(0,0)(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)) \\
&\qquad + \zeta_4(\eta_1,0)\Pi(0,0)B_1(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0))\Big).
\end{aligned}
$$

(4) $\zeta_4(\eta_1,\eta_2) = 0$. In this case (16) is

$$
\begin{aligned}
&B_1\{(\zeta_3(\eta_1,0))\,\Pi(0,0)+\zeta_5(\eta_1,0)\} \\
&\qquad = \frac{1}{\zeta_1(\eta_1,0)}\Big(\zeta_2(\eta_1,0)B_1\{(\zeta_3(\eta_1,0))\Pi(0,0)+\zeta_5(\eta_1,0)\} \\
&\qquad + \zeta_3(\eta_1,0)\{B_2\Pi(0,0)+\zeta_5(0,0)\}(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0)) \\
&\qquad + \zeta_5(\eta_1,0)B_1(\zeta_1(\eta_1,0)-\zeta_2(\eta_1,0))\Big).
\end{aligned}
$$

It should be remarked that from the above four cases that, most of them depend on the unknown $\Pi(0,0)$. This means that plugging arbitrary values of $\Pi(0,0)$ will give us more insights.

3. Main Results

In this section, we investigate the solution of the family of FEs introduced in section (2). This approach is based on assuming some particular cases of the polynomial coefficients of (1) namely, if $\zeta_4(\eta_1,\eta_2) = \zeta_5(\eta_1,\eta_2) = 0$ in (1) we get

$$\zeta_1(\eta_1,\eta_2)\Pi(\eta_1,\eta_2) = \zeta_2(\eta_1,\eta_2)\Pi(\eta_1,0) + \zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2). \tag{17}$$

From (17) it is easy to see that (by plugging $\eta_1 = \eta_2 = 0$)

$$(\zeta_1(0,0) - \zeta_2(0,0) - \zeta_3(0,0))\Pi(0,0) = 0,$$

which is fulfilled if the following cases holds:

Case 1

$$\zeta_1(0,0) = \zeta_2(0,0) + \zeta_3(0,0),$$

in this case $\Pi(0,0)$ is just arbitrary.

Case 2

$$\zeta_1(0,0) \neq \zeta_2(0,0) + \zeta_3(0,0),$$

in this case we must have

$$\Pi(0,0) = 0.$$

Now, if we plug $\eta_2 = 0$ in (17) we get

$$\zeta_1(\eta_1,0)\Pi(\eta_1,0) = \zeta_2(\eta_1,0)\Pi(\eta_1,0) + \zeta_3(\eta_1,0)\Pi(0,0), \tag{18}$$

which is equivalent to

$$(\zeta_1(\eta_1,0) - \zeta_2(\eta_1,0))\Pi(\eta_1,0) = \zeta_3(\eta_1,0)\Pi(0,0),$$

from which we could have two cases namely:

Case A $\zeta_1(\eta_1,0) = \zeta_2(\eta_1,0)$ (for all η_1) in this case

$$\zeta_3(\eta_1,0)\Pi(0,0) = 0, \quad \text{for all } \eta_1.$$

Case B $\zeta_1(\eta_1,0) \neq \zeta_2(\eta_1,0)$ (for all η_1 except finitely many) in this case

$$\Pi(\eta_1,0) = \frac{\zeta_3(\eta_1,0)\Pi(0,0)}{\zeta_1(\eta_1,0) - \zeta_2(\eta_1,0)}.$$

Similarly, when plugging $\eta_1 = 0$ in (17) we get

$$\zeta_1(0,\eta_2)\Pi(0,\eta_2) = \zeta_2(0,\eta_2)\Pi(0,0) + \zeta_3(0,\eta_2)\Pi(0,\eta_2),$$

which is equivalent to

$$(\zeta_1(0,\eta_2) - \zeta_3(0,\eta_2))\Pi(0,\eta_2) = \zeta_2(0,\eta_2)\Pi(0,0)$$

from which we could have two cases namely:

Case α $\zeta_1(0,\eta_2) = \zeta_3(0,\eta_2)$ (for all η_2) in this case

$$\zeta_2(0,\eta_2)\Pi(0,0) = 0, \quad \text{for all } \eta_2.$$

Case β $\zeta_1(0,\eta_2) \neq \zeta_3(0,\eta_2)$ (for all η_2 except finitely many) in this case

$$\Pi(0,\eta_2) = \frac{\zeta_2(0,\eta_2)\Pi(0,0)}{\zeta_1(0,\eta_2) - \zeta_3(0,\eta_2)}.$$

So in general we have a combination of different cases namely:

$$1A\alpha, 1A\beta, 1B\alpha, 1B\beta, 2A\alpha, 2A\beta, 2B\alpha, 2B\beta$$

with the assumption that the functions $\zeta_1(\eta_1,\eta_2), \zeta_2(\eta_1,\eta_2), \zeta_3(\eta_1,\eta_2)$ are nonzero polynomials with degrees

$$\deg_{\eta_1} \zeta_i, \deg_{\eta_2} \zeta_i \leq k \in \mathbb{N}, i = 1,2,3.$$

We can consider some special cases as follows. In the case $2B\alpha$ we have

Lemma 3. *If* $\Pi(0,0) = 0$, *then the main unknown function is given by*

$$\Pi(\eta_1,\eta_2) = \frac{\zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2)}{\zeta_1(\eta_1,\eta_2)},$$

Proof. If

$$\Pi(0,0) = 0, \qquad \zeta_1(0,0) \neq \zeta_2(0,0)$$

and

$$\zeta_1(0,\eta_2) = \zeta_3(0,\eta_2), \quad \zeta_1(\eta_1,0) \neq \zeta_2(\eta_1,0)$$

in this case we have

$$\Pi(\eta_1,0) = \frac{\zeta_3(\eta_1,0)\Pi(0,0)}{\zeta_1(\eta_1,0) - \zeta_2(\eta_1,0)} = 0, \text{ for all } \eta_1 \text{ by continuity,}$$

therefore

$$\zeta_1(\eta_1,\eta_2)\Pi(\eta_1,\eta_2) = \zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2),$$

giving

$$\Pi(\eta_1,\eta_2) = \frac{\zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2)}{\zeta_1(\eta_1,\eta_2)},$$

for any $\Pi(0,\eta_2)$ for which $\zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2)$ is divisible by $\zeta_1(\eta_1,\eta_2)$.

□

In the case $2B\beta$ we have the following:

Lemma 4. *If*

$$\Pi(0,0) = 0, \qquad \Pi(\eta_1,0) = 0 \textit{ (except finitely many points),}$$

then

$$\Pi(\eta_1,\eta_2) = 0.$$

Proof. If

$$\Pi(0,0)=0, \quad \Pi(\eta_1,0)=0 \text{ (except finitely many points)},$$

$$\Pi(0,\eta_2)=0 \text{ (except finitely many points)},$$

then by continuity

$$\Pi(\eta_1,0)=0, \quad \Pi(0,\eta_2)=0.$$

therefore

$$\zeta_1(\eta_1,\eta_2)\Pi(\eta_1,\eta_2)=0,$$

giving

$$\Pi(\eta_1,\eta_2)=0.$$

□

Now, we use a different perspective to solve the family of FEs. We will assume some special values of the polynomial coefficients in (1). In fact its like the previous section but using a different perspective. For $\zeta_4(\eta_1,\eta_2)=\zeta_5(\eta_1,\eta_2)=0$ we have (1) in the form

$$\zeta_1(\eta_1,\eta_2)\Pi(\eta_1,\eta_2)=\zeta_2(\eta_1,\eta_2)\Pi(\eta_1,0)+\zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2),$$

if

$$\zeta_1(\eta_1,0)=\zeta_2(\eta_1,0),\ \zeta_1(0,\eta_2)=\zeta_3(0,\eta_2),\ \text{ and } \Pi(0,0)=0$$

then

$$\Pi(\eta_1,\eta_2)=\frac{\zeta_2(\eta_1,\eta_2)\Pi(\eta_1,0)+\zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2)}{\zeta_1(\eta_1,\eta_2)}.$$

Let

$$\zeta_1(\eta_1,0)=\zeta_2(\eta_1,0)=P(\eta_1),$$

then in this case the functions $\zeta_1(\eta_1,\eta_2),\zeta_2(\eta_1,\eta_2)$ can be of the form

$$\zeta_1(\eta_1,\eta_2)=P(\eta_1)+\sum_{j=1}^{k}\eta_2^j q_j(\eta_1),$$

$$\zeta_2(\eta_1,\eta_2)=P(\eta_1)+\sum_{j=1}^{k}\eta_2^j r_j(\eta_1).$$

Similarly, we can assume that

$$\zeta_1(0,\eta_2)=\zeta_3(0,\eta_2)=S(\eta_2),$$

then in this case the functions $\zeta_1(\eta_1,\eta_2),\zeta_2(\eta_1,\eta_2)$ can be of the form

$$\zeta_1(\eta_1,\eta_2)=S(\eta_2)+\sum_{j=1}^{k}\eta_1^j t_j(\eta_2),$$

and

$$\zeta_3(\eta_1,\eta_2)=S(\eta_2)+\sum_{j=1}^{k}\eta_1^j u_j(\eta_2).$$

Example 5. Let's assume that the functions $\zeta_1(\eta_1,\eta_2),\zeta_2(\eta_1,\eta_2),\zeta_3(\eta_1,\eta_2)$ are of degree one:

$$\zeta_1(\eta_1,\eta_2)=\alpha+\beta\eta_1+\gamma\eta_2,$$

$$\zeta_2(\eta_1,\eta_2)=\alpha+\beta\eta_1+\delta\eta_2,$$

$$\zeta_3(\eta_1,\eta_2)=\alpha+\varepsilon\eta_1+\gamma\eta_2.$$

So it is clear that

$$\alpha+\beta\eta_1=\zeta_1(\eta_1,0)=\zeta_2(\eta_1,0),$$

and

$$\alpha+\gamma\eta_2=\zeta_1(0,\eta_2)=\zeta_3(0,\eta_2).$$

In this case the functional equation can be rewritten as

$$\begin{aligned}\Pi(\eta_1,\eta_2) &= \frac{(\alpha+\beta\eta_1+\delta\eta_2)\Pi(\eta_1,0)+(\alpha+\varepsilon\eta_1+\gamma\eta_2)\Pi(0,\eta_2)}{\alpha+\beta\eta_1+\gamma\eta_2},\\ &= \frac{(\alpha+\delta\eta_2+\beta\eta_1)\Pi(\eta_1,0)+(\alpha+\gamma\eta_2+\varepsilon\eta_1)\Pi(0,\eta_2)}{\alpha+\gamma\eta_2+\beta\eta_1}. \end{aligned} \tag{19}$$

The denominator of (19) is zero if and only if

$$(\eta_1)_{\eta_2}=-\frac{\alpha+\gamma\eta_2}{\beta}, \tag{20}$$

and the numerator of (19) should be zero at the value of η_1 given by (20) . So that

$$\begin{aligned} 0 &= (\alpha+\delta\eta_2+\beta\frac{\alpha+\gamma\eta_2}{-\beta})\Pi(\frac{\alpha+\gamma\eta_2}{-\beta},0)+(\alpha+\gamma\eta_2+\varepsilon\frac{\alpha+\gamma\eta_2}{-\beta})\Pi(0,\eta_2)\\ &= (\delta-\gamma)\eta_2\Pi(\frac{\alpha+\gamma\eta_2}{-\beta},0)+(\alpha+\gamma\eta_2+\varepsilon\frac{\alpha+\gamma\eta_2}{-\beta})\Pi(0,\eta_2), \end{aligned}$$

which is equivalent to

$$\Pi(0,\eta_2)=\frac{(\delta-\gamma)\eta_2}{\alpha+\gamma\eta_2+\varepsilon\frac{\alpha+\gamma\eta_2}{-\beta}}\Pi(\frac{\alpha+\gamma\eta_2}{-\beta},0).$$

Now $\Pi(\eta_1,0)$ can be arbitrary chosen (as a continuous function), $\Pi(0,0)=0$, and

$$\Pi(0,\eta_2)=\begin{cases} \frac{(\delta-\gamma)\eta_2}{\alpha+\gamma\eta_2+\varepsilon\frac{\alpha+\gamma\eta_2}{-\beta}}\Pi(\frac{\alpha+\gamma\eta_2}{-\beta},0), & \text{for } \eta_2\neq-\frac{\alpha}{\gamma}\\ \\ \lim_{\eta_2\to-\frac{\alpha}{\gamma}}\{\frac{(\delta-\gamma)\eta_2}{\alpha+\gamma\eta_2+\varepsilon\frac{\alpha+\gamma\eta_2}{-\beta}}\Pi(\frac{\alpha+\gamma\eta_2}{-\beta},0)\}, & \text{for } \eta_2=-\frac{\alpha}{\gamma} \end{cases}$$

then

$$\Pi(\eta_1,\eta_2)=\frac{(\alpha+\beta\eta_1+\delta\eta_2)\Pi(\eta_1,0)+(\alpha+\varepsilon\eta_1+\gamma\eta_2)\frac{(\delta-\gamma)\eta_2}{\alpha+\gamma\eta_2+\varepsilon\frac{\alpha+\gamma\eta_2}{-\beta}}\Pi(\frac{\alpha+\gamma\eta_2}{-\beta},0)}{\alpha+\gamma\eta_2+\beta\eta_1}.$$

4. Remarks on the Stability of (1)

Stability issue of FEs is nowadays a hot topic with many recent interesting applications see e.g. [28, 29, 30, 31, 28] and hundreds of mathematicians are interested in such issue. It has applications in optimization theory see e.g. [32], see also [33, 34, 30] for more applications.

There are numerous tools illustrated in the literature see e.g. [31] to investigate stability (the direct, invariant means, weighted space, the fixed point, shadowing) method, and the method based on the sandwich theorems. The concept of stability of FEs arises when one replace the FE by the corresponding functional inequality. The stability issue now is:

How do the solutions of the functional inequality differ from those of the given FE?

In this section, we give remarks on the stability of the family (1). It should be noted that the general solution of equation (1) is a function defined as follows

$$\Pi(\cdot,\cdot) : D \subset \mathbb{C}^2 \to \mathbb{C} \tag{21}$$

where D is the unit disk. Equation (1) is called stable in the sense of Ulam-Hyers if there is a $r > 0$: for any $\varepsilon > 0$,

$$\Pi : D \subset \mathbb{C}^2 \to \mathbb{C}$$

with

$$d(\zeta_1(\eta_1,\eta_2)\Pi(\eta_1,\eta_2)\,, \zeta_2(\eta_1,\eta_2)\Pi(\eta_1,0) + \zeta_3(\eta_1,\eta_2)\Pi(0,\eta_2) \\ + \zeta_4(\eta_1,\eta_2)\Pi(0,0) + \zeta_5(\eta_1,\eta_2)) \leq \varepsilon$$

there exists a function Π^* with

$$d(\Pi,\Pi^*) \leq r\varepsilon.$$

5. Conclusion

In this paper, we investigate the solution of a family of FEs. Potential future work could be to introduce particular exact-form solutions for some particular cases. Moreover, one can investigate the hyperstability of such interesting FEs.

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