Construction and determination of univalent biharmonic mappings

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Abstract: In this paper, a family of biharmonic mappings $W_{\alpha,\beta}$, which arises from analytic functions and has two parameters, is constructed. Some sufficient conditions for $W_{\alpha,\beta}$ to be sense-preserving and univalent are explored. The radii of full convexity and starlikeness of $W_{\alpha,\beta}$ are determined. Some sufficient conditions for biharmonic mappings to be fully starlike and fully convex are obtained. Many related previous results are generalized.

Keywords: biharmonic mappings; univalent; fully starlike; fully convex; radius

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1 Introduction

Let $\mathcal{A}$ denote the class of normalized functions $G$ of the form

$$G(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in unit disk $\mathbb{D} = \{z : |z| < 1\}$. Set $\mathcal{S} = \{G \in \mathcal{A} : G \text{ is univalent in } \mathbb{D}\}$, $\mathcal{S}^* = \{G \in \mathcal{S} : G(\mathbb{D}) \text{ is a starlike domain with respect to the origin}\}$ and $\mathcal{K} = \{G \in \mathcal{S} : G(\mathbb{D}) \text{ is a convex domain}\}$ (see [10]).

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The convexity and starlikeness are hereditary properties for conformal mappings. That is to say, if an analytic function maps \( D \) onto a convex domain or starlike domain, then it also maps each concentric subdisk onto a convex domain or starlike domain respectively. However, these hereditary properties do not generalize to harmonic mappings (see [8, 14]). A function \( F \) is harmonic in \( D \), if \( F \) satisfies the Laplace’s equation \( \Delta F = 0 \), where

\[
\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}
\]

is the Laplacian. The failure of the hereditary properties of starlike and convex harmonic mapping leads to the notation of fully starlike and fully convex functions, which was discussed in [8].

**Definition 1.** A function \( F \) on \( D \) with \( F(0) = 0 \) is said to be fully starlike if it is sense-preserving, and maps every circle \( |z| = r < 1 \) in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin satisfying

\[
\frac{\partial}{\partial \theta} (\arg F(re^{i\theta})) > 0, \quad 0 \leq \theta < 2\pi, 0 < r < 1.
\] (1.1)

**Definition 2.** A function \( F \) on \( D \) is said to be fully convex if it is sense-preserving, and maps every circle \( |z| = r < 1 \) in a one-to-one manner onto a convex curve satisfying

\[
\frac{\partial}{\partial \theta} (\arg \frac{\partial}{\partial \theta} F(re^{i\theta})) > 0, \quad 0 \leq \theta < 2\pi, 0 < r < 1.
\] (1.2)

It is important to point out that, by the degree principle [9], both of the fully starlike mapping and fully convex mapping defined as above are univalent. Indeed, the fully starlike mapping and fully convex mapping defined as above are starlike and convex, respectively. Furthermore, it is obvious that if \( F \) is fully convex in \( D \), then it is full starlike in \( D \).

Let

\[
DF = zF_z - \bar{z}F_{\bar{z}} \quad \text{and} \quad D^2F = D(DF).
\]

[18, Theorem 1] says that if the function \( F \) belongs to the class \( C^1(\mathbb{D}) \) and satisfies the following conditions: (i) \( F(0) = 0, F(z) \neq 0 \) for all \( z \in \mathbb{D} \setminus \{0\} \); (ii) \( J_F(z) > 0 \) for all \( z \in \mathbb{D} \); (iii)

\[
\Re \left( \frac{DF(z)}{F(z)} \right) > 0
\] (1.3)

for \( z \in \mathbb{D} \setminus \{0\} \), then \( F \) is fully starlike in \( \mathbb{D} \). [18, Theorem 3] says that if the function \( F \) belongs to the class \( C^1(\mathbb{D}) \) and satisfies the following conditions: (i) \( F(0) = 0, F(z)DF(z) \neq 0 \) for all \( z \in \mathbb{D} \setminus \{0\} \); \( DF \in C^1(\mathbb{D}) \); (ii) \( J_F(z) > 0 \) for all \( z \in \mathbb{D} \); (iii)

\[
\Re \left( \frac{D^2F}{DF} \right) > 0
\] (1.4)

for \( z \in \mathbb{D} \setminus \{0\} \), then \( F \) is fully convex in \( \mathbb{D} \). It is easy to verify that condition (1.1) is equivalent to (1.3) and (1.2) is equivalent to (1.4).

As to the radius of full convexity, full starlikeness, or univalency of harmonic mappings, we refer to [5, 12-14, 19, 20, 25].

A complex-valued function \( F = u + iv \) which is four times continuously differentiable in \( \mathbb{D} \) is biharmonic
if $F$ satisfies the biharmonic equation $\Delta^2 F = \Delta(\Delta F) = 0$. Every biharmonic function $F$ on $\mathbb{D}$ has the form

$$F(z) = |z|^2 G(z) + H(z)$$

where $G$ and $H$ are harmonic in $\mathbb{D}$, see [1-3]. A biharmonic function $F$ on $\mathbb{D}$ is said to be sense-preserving if

$$J_F(z) := |F_z(z)|^2 - |F_{\bar{z}}(z)|^2 > 0$$

for $z \in \mathbb{D} \setminus \{0\}$.

It has been proved that, if $G \in \mathcal{S}$, then $F = |z|^2 G$ is not necessarily univalent in $\mathbb{D}$ (see [2] and [3]). On the other hand, it is easy to see that, $F = |z|^2 G$ is univalent in $\mathbb{D}$ whenever $G$ is harmonic and starlike in $\mathbb{D}$, see [1] and [15].

For a given analytic function $G$, let

$$W_\beta(z) = |z|^2 G(z) - G(z) + \beta \int_0^z \frac{G(\xi)}{\xi} d\xi.$$  \hfill (1.5)

In [3], Muhanna proved two theorems concerning the sense-preserving property and univalency of biharmonic function $W_2(z)$, i.e. $\beta = 2$ in (1.5). Indeed for $G \in \mathcal{S}$, the sharp inequality $\left(1 - |z|^2\right) |G'(z)| \leq 4$ holds for $z \in \mathbb{D}$. The number 4 in this inequality cannot be replaced by a smaller number. As pointed out in [22] a closer examination of the proof of [3, Theorem 3 and Proposition 3] shows that these results are valid for $W_4(z)$ rather than for $W_2(z)$. Correct formulation of these results as follows.

**Theorem A.** If $G \in \mathcal{S}$, then $W_4(z)$ defined by (1.5) has Jacobian $J_{W_4(z)} > 0$ except at $z = 0$. Moreover, if $G$ is starlike, then the biharmonic function $W_4(z)$ is univalent in $\mathbb{D}$.

For $\lambda > 0$, let

$$\mathcal{R}(\lambda) = \{G \in \mathcal{A} : |G'(z)| - 1 < \lambda, z \in \mathbb{D}\}$$

and $\mathcal{R}(1) = \mathcal{R}$. Functions in $\mathcal{R}$ are known to be univalent in $\mathbb{D}$, but functions in $\mathcal{R}(\lambda)$ are not necessarily univalent in $\mathbb{D}$ if $\lambda > 1$. Also, it is important to remark that functions in the class $\mathcal{R}$ are not necessarily starlike in $\mathbb{D}$. However, functions in $\mathcal{R}(\lambda)$ are known to be starlike in $\mathbb{D}$ provided $0 < \lambda < 2/\sqrt{5}$ (see [21, 26]) and $2/\sqrt{5}$ cannot be replaced by a larger one (see [11]). On the converse part, even a normalized convex function $G$ does not necessarily satisfy the condition $\Re G'(z) > 0$ in $\mathbb{D}$ and hence need not belong to $\mathcal{R}$. In [4], the authors identify a family $\mathcal{R}(\lambda)$ which contains also non-univalent functions $G \in \mathcal{R}(\lambda)$ such that $W_\beta(z)$ is sense-preserving and univalent in $\mathbb{D}$.

More research results on biharmonic mappings in recent years, see [6, 7, 16, 23].

For a given analytic function $G$, let

$$W(z) = W_{\alpha, \beta}(z) = |z|^2 G(z) + \alpha G(z) + \beta \int_0^z \frac{G(\xi)}{\xi} d\xi, \quad \alpha \in \mathbb{R} \quad \beta \in \mathbb{C}.$$  \hfill (1.6)

It is obvious that the family $W_{\alpha, \beta}(z)$ defined by (1.6) is biharmonic.

In section 2 of this paper, we explore some sufficient conditions for $W_{\alpha, \beta}(z)$ to be sense-preserving and univalent. In section 3, let $G \in \mathcal{R}(\lambda)$, we determine some sufficient conditions for $W_{\alpha, \beta}(z)$ to be sense-
preserving and univalent. In section 4, we present the radii of full starlikeness and full convexity of $W_{\alpha,\beta}(z)$. In section 5, we give some sufficient conditions for some more general biharmonic mappings to be fully convex or fully starlike. The results of this paper generalize some previous related results.

2 Univalency of $W_{\alpha,\beta}(z)$ with $G \in S$

**Lemma 2.1.** Let $G \in S$. Then for $z \in \mathbb{D}$, we have

$$\frac{1 - r}{1 + r} \leq \frac{zG'(z)}{G(z)} \leq \frac{1 + r}{1 - r},$$

(2.1)

where $r = |z|$. Furthermore, if $G \in S^*$, then

$$\Re \frac{zG'(z)}{G(z)} \in \left[\frac{1 - r}{1 + r}, \frac{1 + r}{1 - r}\right].$$

(2.2)

**Proof.** The double inequality (2.1) just comes from [10, Theorem 2.7]. Furthermore, if $G \in S^*$, then $\Re \frac{zG'(z)}{G(z)} > 0$. Therefore, (2.1) implies (2.2) holds.

**Theorem 2.2.** Let $G(z) \in S$ and

$$W(z) = |z|^2 G(z) - G(z) + \beta \int_0^z \frac{G(\xi)}{\xi} d\xi.$$  

(2.3)

If $\Re \beta \in (-\infty, -6] \cup [4, +\infty)$, then $J_W(z) > 0$ except at $z = 0$.

**Proof.** Let $z = re^{i\theta}$. Then it follows from (2.3) that

$$Wz = zG, \quad \bar{W}z = \bar{z}G + r^2 Gz - Gz + \beta G/z.$$  

If $\mu$ denotes the dilatation of $W$, then

$$|\mu(z)| = \left|\frac{Wz}{W_\bar{z}}\right| = \left|\frac{1}{e^{-2i\theta} + \frac{zG'(z)}{G(z)} - \frac{G'(z)}{zG(z)} + \frac{\beta}{z^2}}\right| = \frac{1}{\left|1 + \frac{1}{r^2} [\beta - (1 - r^2) \frac{zG'(z)}{G(z)}]\right|}.$$  

If $\Re \beta \geq 4$, then Lemma 2.1 leads to

$$\Re \left(\frac{1}{r^2} (\beta - (1 - r^2) \frac{zG'(z)}{G(z)})\right) \geq \frac{1}{r^2} \left(4 - (1 - r^2) \frac{1 + r}{1 - r}\right) = \frac{4 - (1 + r)^2}{r^2} > 0$$

for $r \in (0, 1)$. This implies that $|\mu(z)| < 1$ for $z \in \mathbb{D} \setminus \{0\}$. 

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If $\Re\beta \leq -6$, then by Lemma 2.1, one has
\[
\Re \left( \frac{1}{r^2} (\beta - (1 - r^2) \frac{z G'(z)}{G(z)}) \right) \leq \frac{1}{r^2} \left( -6 + (1 - r^2) \frac{1 + r}{1 - r} \right) = \frac{-6 + (1 + r)^2}{r^2} < -2
\]
for $r \in (0, 1)$. This implies that $|\mu(z)| < 1$ for $z \in \mathbb{D} \setminus \{0\}$, too.

\[\square\]

**Theorem 2.3.** Let $G(z) \in S^*$ and
\[
W(z) = |z|^2 G(z) + \alpha G(z) + \beta \int_0^z \frac{G(\xi)}{\xi} d\xi. \quad (2.4)
\]
For each case, if the stated conditions are satisfied, then $J_W(z) > 0$ except at $z = 0$.

- **Case (1):** $\alpha \geq 0$ and $\Re\beta \geq 0$.
- **Case (2):** $\alpha \in (-1, 0)$ and $\Re\beta \geq 4$.
- **Case (3):** $\alpha = -1$ and $\Re\beta \in (-\infty, -2] \cup [4, \infty)$.
- **Case (4):** $\alpha < -1$ and $\Re\beta \leq -2$.

**Proof.** Let $z = re^{i\theta}$. Then it follows from (2.4) that
\[
W(z) = |z|^2 G(z) + \alpha G(z) + \beta \int_0^z \frac{G(\xi)}{\xi} d\xi.
\]
If $\mu$ denotes the dilatation of $W$, then
\[
|\mu(z)| = \left| \frac{W_z}{W_z} \right| = \frac{1}{\left| e^{-2i\theta} + (r^2 + \alpha) \frac{G'(z)}{G(z)} + \frac{\beta}{r^2} \right|} = \frac{1}{\left| 1 + (1 + \alpha \frac{r^2}{r^2}) \frac{G'(z)}{G(z)} + \frac{\beta}{r^2} \right|}.
\]
Let
\[
\Delta = (1 + \alpha \frac{r^2}{r^2}) \frac{G'(z)}{G(z)} + \frac{\beta}{r^2}. \quad (2.5)
\]
Then it is sufficient to get $|\mu(z)| < 1$ if
\[
\Re\Delta > 0 \quad \text{or} \quad \Re\Delta < -2. \quad (2.6)
\]
We divide the proof into four subcases.

**Subcase (1).** $\alpha \geq 0$ and $\Re\beta \geq 0$. It follows from (2.2) that
\[
\Re\Delta \geq \Re \left( 1 + \alpha \frac{r^2}{r^2} \frac{1 - r}{1 + r} + \frac{\beta}{r^2} \right) \geq \frac{1 - r}{1 + r} > 0.
\]

**Subcase (2).** $\alpha \in (-1, 0)$ and $\Re\beta \geq 4$. 
If $\alpha \in (-1, 0)$ and $\Re \beta \geq 4$, then it follows from (2.2) that
\[
\Re \Delta = \Re \left( \frac{\beta}{r^2} - \frac{1 - r^2 z G'(z)}{r^2 G(z)} + \frac{1 + \alpha z G'(z)}{r^2 G(z)} \right)
\geq \frac{4}{r^2} \left( 1 - r^2 \right) \left( \frac{1 + \alpha}{1 + r} \right)
\geq \frac{4 - (1 + r)^2}{r^2} > 0.
\]

**Subcase (3).** $\alpha = -1$ and $\Re \beta \in (-\infty, -2] \cup [4, \infty)$.

Since $G(z) \in S^*$ and $S^* \subset S$, the part of “$\alpha = -1$ and $\Re \beta \geq 4$” comes from Theorem 2.2. The part of “$\alpha = -1$ and $\Re \beta \leq -2$” will come from the following part of Subcase (4).

**Subcase (4).** $\alpha \leq -1$ and $\Re \beta \leq -2$. It follows from (2.2) that
\[
\Re \Delta \leq \Re \left( 1 + \frac{\alpha}{r^2} \right) \left( \frac{1 - r}{1 + r} \right) \left( \frac{\beta}{r^2} \right)
\leq \left( 1 - \frac{1}{r^2} \right) \left( 1 - r \right) - \frac{2}{r^2} = \frac{-2(1 - r)^2 - 2}{r^2} < -2.
\]

\[\square\]

**Theorem 2.4.** Under the hypotheses of Theorem 2.3, $W(z)$ defined by (2.4) is univalent in $D$.

**Proof.** Fix $\rho \in (0, 1)$. Consider the function
\[
V(\varphi) = \rho^2 G(\rho \varphi) + \alpha G(\rho \varphi) + \beta \int_0^{\rho \varphi} \frac{G(\xi)}{\xi} d\xi,
\]
where $|\varphi| \leq 1$. Then a computation gives
\[
V'(\varphi) = \rho X(\rho \varphi),
\]
where
\[
X(\rho \varphi) = (\alpha + \rho^2) G'(\rho \varphi) + \beta \frac{G(\rho \varphi)}{\rho \varphi}.
\]

Let $U(z) = \int_0^z \frac{G(\xi)}{\xi} d\xi$. It is obvious that $zU'(z) = G(z)$. Hence $U(z)$ and $-U(z)$ are convex as $G(z)$ is starlike. It follows that $U(\rho z)$ and $-U(\rho z)$ are convex for fixed $\rho \in (0, 1)$.

Direct computation leads to
\[
\frac{dV(\varphi)/d\varphi}{dU(\rho \varphi)/d\varphi} = \frac{\rho \varphi X(\rho \varphi)}{G(\rho \varphi)} = (\alpha + \rho^2) \frac{\rho \varphi G'(\rho \varphi)}{G(\rho \varphi)} + \beta
\]
and
\[
\frac{dV(\varphi)/d\varphi}{d(-U(\rho \varphi))/d\varphi} = -(\alpha + \rho^2) \frac{\rho \varphi G'(\rho \varphi)}{G(\rho \varphi)} - \beta.
\]
Subcase (1). $\alpha \geq 0$ and $\Re \beta \geq 0$. By (2.2), we have

$$\Re \left( \frac{dV(\varphi)}{dU(\rho \varphi)} \right) > 0.$$

Subcase (2). $\alpha \in (-1, 0)$ and $\Re \beta \geq 4$. By (2.2), we get

$$\Re \left( \frac{dV(\varphi)}{dU(\rho \varphi)} \right) > 4 + (-1 + \rho^2) \Re \left( \frac{\rho \varphi G'(\rho \varphi)}{G(\rho \varphi)} \right) > 4 + (-1 + \rho^2) \frac{1 + \rho |\varphi|}{1 - \rho} > 0.$$

Subcase (3). $\alpha = -1$ and $\Re \beta \in (-\infty, -2] \cup [4, \infty)$.

If $\alpha = -1$ and $\Re \beta \geq 4$, then

$$\Re \left( \frac{dV(\varphi)}{dU(\rho \varphi)} \right) \geq 4 + (-1 + \rho^2) \Re \left( \frac{\rho \varphi G'(\rho \varphi)}{G(\rho \varphi)} \right) > 0.$$

If $\alpha = -1$ and $\Re \beta \leq -2$, then

$$\Re \left( \frac{dV(\varphi)}{d(-U(\rho \varphi))} \right) \geq -(-1 + \rho^2) \Re \left( \frac{\rho \varphi G'(\rho \varphi)}{G(\rho \varphi)} \right) + 2 \geq 2 + (1 - \rho^2) \frac{1 - \rho |\varphi|}{1 + \rho |\varphi|} > 0.$$

Subcase (4). $\alpha < -1$ and $\Re \beta \leq -2$.

$$\Re \left( \frac{dV(\varphi)}{d(-U(\rho \varphi))} \right) > -(-1 + \rho^2) \Re \left( \frac{\rho \varphi G'(\rho \varphi)}{G(\rho \varphi)} \right) + 2 > 0.$$

Thus, in each case, we have

$$\Re \left( \frac{dV(\varphi)}{dU(\rho \varphi)} \right) > 0 \quad \text{or} \quad \Re \left( \frac{dV(\varphi)}{d(-U(\rho \varphi))} \right) > 0.$$

It follows, by [10, p.46], that $V(\varphi)$ is close-to-convex and in particular, univalent in $D$. This implies, as $V(\varphi) = W(\rho \varphi)$ for $|\varphi| = 1$, $W(z)$ is univalent on $|z| = \rho$.

Since $J_W(z) > 0$ except at $z = 0$ as shown by Theorem 2.3, the degree principle [9] implies that $W(z)$ is univalent on $|z| \leq \rho$.

As $\rho$ is arbitrary, we conclude that $W(z)$ is univalent in $D$.

Remark. If $\alpha = -1$ and $\beta = 4$, then Theorem 2.2 and Theorem 2.4 reduce to Theorem A.

3 Univalency of $W_{\alpha,\beta}(z)$ with $G \in \mathcal{R}(\lambda)$
Theorem 3.1. Let $G \in \mathcal{R}(\lambda)$ for some $2/\sqrt{5} < \lambda < 2$ and the biharmonic function $W$ defined by (2.4). For each case, if the stated conditions are satisfied, then $J_W(z) > 0$ in $\mathbb{D}\{0\}$.

Case (1): $\alpha > 0$ and $\Re \beta \geq \frac{2(1+\alpha)(1+\lambda)}{2-\lambda}$.
Case (2): $\alpha > 0$ and $\Re \beta \leq -2\left(\frac{1+\alpha(1+\lambda)}{2-\lambda}\right) + 1$.
Case (3): $\alpha \leq -1$ and $\Re \beta \leq -2 + \frac{2\alpha(1+\lambda)}{2-\lambda}$.
Case (4): $\alpha \leq -1$ and $\Re \beta \geq -\frac{2\alpha(1+\lambda)}{2-\lambda}$.

Proof. It is sufficient to get $J_W(z) > 0$ in $\mathbb{D}\{0\}$ if $\Re \Delta > 0$ or $\Re \Delta < -2$, where $\Delta$ defined by (2.5). Now we let $G \in \mathcal{R}(\lambda)$ for some $\lambda \in (2/\sqrt{5}, 2)$. Then, there exists a function $\omega(z)$ analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$, such that

$$G'(z) = 1 + \lambda \omega(z).$$

(3.1)

It follows that

$$G(z) = z + \lambda z \int_0^1 \omega(tz)dt$$

and

$$\frac{G(z)}{z} = 1 + \lambda \int_0^1 \omega(tz)dt. \quad (3.2)$$

Thus,

$$\frac{zG'(z)}{G(z)} = \frac{1 + \lambda \omega(z)}{1 + \lambda \int_0^1 \omega(tz)dt}.$$

By the Schwarz lemma, we obtain

$$|\omega(z)| \leq |z| \quad (3.3)$$

in $\mathbb{D}$. So,

$$\left| \frac{zG'(z)}{G(z)} \right| \leq \frac{1 + \lambda |z|}{1 - \frac{1}{2}|z|} = \frac{2(1 + \lambda r)}{2 - \lambda r}.$$

Let $b = \Re \beta$. We divide the proof into four cases as follows.

Case (1). If $\alpha > 0$ and $b \geq \frac{2(1+\alpha)(1+\lambda)}{2-\lambda}$, then

$$\Re \Delta \geq -2\left(\frac{1+\alpha}{r^2}\right)(1+\lambda r) + \frac{b}{r^2} \geq -\frac{2(r^2 + \alpha)(1 + \lambda r)}{r^2(2 - \lambda r)} + \frac{2(1 + \alpha)(1 + \lambda)}{r^2(2 - \lambda)}$$

$$> -\frac{2(r^2 + \alpha)(1 + \lambda r)}{r^2(2 - \lambda r)} + \frac{2(r^2 + \alpha)(1 + \lambda)}{r^2(2 - \lambda r)} = 0.$$
Case (2). If $\alpha > 0$ and $b \leq -2 \frac{(1 + \alpha)(1 + \lambda)}{2 - \lambda} + 1$, then
\[
\Re \Delta \leq 2 \left(1 + \frac{\alpha}{r^2} \right) \left(1 + \frac{\alpha}{r^2} \right) + b \frac{\alpha}{r^2} \\
\leq \frac{2(r^2 + \alpha)(1 + \lambda r)}{r^2(2 - \lambda r)} - \frac{2}{r^2} \left(\frac{1 + \alpha}{2 - \lambda} \right) + 1] \\
< \frac{2(r^2 + \alpha)(1 + \lambda r)}{r^2(2 - \lambda r)} - \frac{2}{r^2} \left[\frac{(r^2 + \alpha)(1 + \lambda r)}{2 - \lambda r} + 1\right] \\
= -\frac{2}{r^2} < -2.
\]

Case (3). If $\alpha \leq -1$ and $b \leq -2 + \frac{2\alpha(1 + \lambda)}{2 - \lambda}$, then
\[
\Re \Delta \leq -2 \left(1 + \frac{\alpha}{r^2} \right) \left(1 + \frac{\alpha}{r^2} \right) + \frac{1}{r^2} (-2 + \frac{2\alpha(1 + \lambda)}{2 - \lambda}) \\
= -\frac{2}{r^2} \left(1 - \frac{\alpha(1 + \lambda)}{2 - \lambda} + \frac{\alpha(1 + \lambda)}{2 - \lambda} + \frac{r^2(1 + \lambda r)}{2 - \lambda r}\right) \\
< -\frac{2}{r^2} \left(1 - \frac{\alpha(1 + \lambda)}{2 - \lambda} + \frac{\alpha(1 + \lambda)}{2 - \lambda}\right) \\
= -\frac{2}{r^2} < -2.
\]

Case (4). If $\alpha \leq -1$ and $b \geq -\frac{2\alpha(1 + \lambda)}{2 - \lambda}$, then
\[
\Re \Delta \geq 2 \left(1 + \frac{\alpha}{r^2} \right) \left(1 + \frac{\alpha}{r^2} \right) - \frac{1}{r^2} \frac{2\alpha(1 + \lambda)}{2 - \lambda} \\
= \frac{2}{r^2} \left(\frac{r^2(1 + \lambda r)}{2 - \lambda r} + \frac{\alpha(1 + \lambda)}{2 - \lambda} - \frac{\alpha(1 + \lambda)}{2 - \lambda}\right) \\
> \frac{2}{r^2} \left(\frac{r^2(1 + \lambda r)}{2 - \lambda r} + \frac{\alpha(1 + \lambda)}{2 - \lambda} - \frac{\alpha(1 + \lambda)}{2 - \lambda}\right) > 0.
\]

\[\square\]

**Theorem 3.2.** Under the hypotheses of Theorem 3.1, $W(z)$ is univalent in $\mathbb{D}$.

**Proof.** Let $V(\varphi)$ and $X(\rho \varphi)$ defined as in (2.7) and (2.8), respectively. Then (3.1) and (3.2) lead to that
\[
X(\rho \varphi) = (\alpha + \rho^2)G'(\rho \varphi) + \beta \frac{G'(\rho \varphi)}{\rho \varphi} \\
= (\alpha + \rho^2)(1 + \lambda \omega(\rho \varphi)) + \beta(1 + \lambda) \int_0^1 \omega(t \rho \varphi) dt). \tag{3.4}
\]

Now we divide the proof into four cases as follows.
Case (1). If $\alpha > 0$ and $\Re\beta > \frac{2(1+\alpha)(1+\lambda)}{2-\lambda}$, then

$$\Re X(\rho \varphi) \geq (\alpha + \rho^2)(1 - \lambda \rho |\varphi|) + (1 - \lambda \rho |\varphi|) \Re\beta$$
$$\geq (\alpha + \rho^2)(1 - \lambda \rho) + \frac{2(1 + \alpha)(1 + \lambda)}{2 - \lambda} (1 - \lambda \rho)$$
$$\geq (\alpha + \rho^2)(1 - \lambda \rho) + \frac{(\alpha + \rho^2)(1 + \lambda)}{2 - \lambda} (2 - \lambda)$$
$$\geq (\alpha + \rho^2)(1 - \lambda \rho) + (\alpha + \rho^2)(1 + \lambda) (2 - \lambda)$$
$$\geq 2(\alpha + \rho^2) > 0.$$ 

Case (2). If $\alpha > 0$ and $\Re\beta \leq -2\left(\frac{1+\alpha(1+\lambda)}{2-\lambda} + 1\right)$, then

$$\Re X(\rho \varphi) \leq (\alpha + \rho^2)(1 + \lambda \rho |\varphi|) - 2\left(\frac{1 + \alpha(1 + \lambda)}{2 - \lambda} + 1\right)(1 - \lambda \rho |\varphi|)$$
$$\leq (\alpha + \rho^2)(1 + \lambda \rho) - \left[\frac{1 + \alpha(1 + \lambda)}{2 - \lambda} + 1\right](2 - \lambda \rho)$$
$$< (\alpha + 1)(1 + \lambda) - \left[\frac{1 + \alpha(1 + \lambda)}{2 - \lambda} + 1\right](2 - \lambda)$$
$$\leq \lambda - 2 < 0.$$ 

Case (3). If $\alpha \leq -1$ and $\Re\beta \leq -2 + \frac{2\alpha(1+\lambda)}{2-\lambda}$, then

$$\Re X(\rho \varphi) \leq (\alpha + \rho^2)(1 - \lambda \rho |\varphi|) - 2\left[1 - \frac{\alpha(1 + \lambda)}{2 - \lambda}\right](1 - \lambda \rho |\varphi|) < 0.$$ 

Case (4). If $\alpha \leq -1$ and $\Re\beta \geq -\frac{2\alpha(1+\lambda)}{2-\lambda}$, then

$$\Re X(\rho \varphi) \geq (\alpha + \rho^2)(1 + \lambda \rho |\varphi|) - \frac{2\alpha(1 + \lambda)}{2 - \lambda} (1 - \lambda \rho |\varphi|)$$
$$\geq (\alpha + \rho^2)(1 + \lambda \rho) - \frac{\alpha(1 + \lambda)}{2 - \lambda} (2 - \lambda \rho)$$
$$\geq (\alpha + \rho^2)(1 + \lambda) - \frac{\alpha(1 + \lambda)}{2 - \lambda} (2 - \lambda)$$
$$= \rho^2(1 + \lambda) > 0.$$ 

Thus, by [10 p.46], it follows that in each case $V(\varphi)$ is close-to-convex and in particular, univalent in $\mathbb{D}$. This implies, as $V(\varphi) = W(\rho \varphi)$ for $|\varphi| = 1$, $W(z)$ is univalent on $|z| = \rho$.

Since, $J_W(z) > 0$ except at $z = 0$ as shown by Theorem 3.1, the degree principle implies that $W(z)$ is univalent on $|z| \leq \rho$.

As $\rho$ is arbitrary, we conclude that $W(z)$ is univalent in $\mathbb{D}$. 

\[\square\]
4 Fully convex and fully starlike radii of biharmonic mappings

Theorem 4.1. Let \( F(z) = |z|^2 G(z) + H(z) \) be a biharmonic mapping in \( \mathbb{D} \), where

\[
G(z) = \sum_{k=1}^{\infty} a_k^{(2)} z^k + \sum_{k=1}^{\infty} \overline{b_k^{(2)}} z^k \quad \text{and} \quad H(z) = z + \sum_{k=2}^{\infty} a_k^{(1)} z^k + \sum_{k=1}^{\infty} \overline{b_k^{(1)}} z^k
\] (4.1)

are sense-preserving harmonic in \( \mathbb{D} \), and satisfy the condition

\[
\sum_{k=2}^{\infty} k(|a_k^{(1)}| + |b_k^{(1)}|) + \sum_{k=1}^{\infty} (k+2)(|a_k^{(2)}| + |b_k^{(2)}|) + |b_1^{(1)}| \leq 1.
\] (4.2)

Then \( F \) is sense-preserving and fully starlike in \( \mathbb{D} \).

Proof. A computation gives

\[
F_z(z) = \overline{z}(\sum_{k=1}^{\infty} a_k^{(2)} z^k + \sum_{k=1}^{\infty} \overline{b_k^{(2)}} z^k) + |z|^2 \sum_{k=1}^{\infty} k a_k^{(2)} z^{k-1} + \sum_{k=2}^{\infty} k a_k^{(1)} z^{k-1} + 1,
\]

\[
F_{\overline{z}}(z) = z(\sum_{k=1}^{\infty} a_k^{(2)} z^k + \sum_{k=1}^{\infty} \overline{b_k^{(2)}} z^k) + |z|^2 \sum_{k=1}^{\infty} k \overline{b_k^{(2)}} z^{k-1} + \sum_{k=1}^{\infty} k \overline{b_k^{(1)}} z^{k-1}.
\]

Then \( J_F(0) = |F_z(0)|^2 - |F_{\overline{z}}(0)|^2 = 1 - |b_1^{(1)}|^2 > 0 \) and for \( z \neq 0, J_F(z) = (|F_z(z)| + |F_{\overline{z}}(z)|)(|F_z(z)| - |F_{\overline{z}}(z)|) > 0 \) because for \( z \neq 0, \)

\[
|F_z(z)| - |F_{\overline{z}}(z)| \geq 1 - \sum_{k=1}^{\infty} |a_k^{(2)}| - \sum_{k=1}^{\infty} |b_k^{(2)}| - \sum_{k=2}^{\infty} k |a_k^{(2)}| - \sum_{k=2}^{\infty} k |a_k^{(1)}| - \sum_{k=1}^{\infty} k |b_k^{(1)}| - \sum_{k=1}^{\infty} k |b_k^{(2)}| - (k+2) \sum_{k=1}^{\infty} |a_k^{(2)}| + |b_k^{(2)}| - |b_1^{(1)}| \geq 0.
\]

Thus, \( F \) is sense-preserving in \( \mathbb{D} \).

Moreover, for each fixed \( r_0 \in (0, 1) \), let

\[
F_{r_0}(z) := r_0^2 G(z) + H(z).
\] (4.3)
Then
\[
F_{r_0}(z) = r_0^2 \left( \sum_{k=1}^{\infty} a_k^{(2)} z^k + \sum_{k=1}^{\infty} b_k^{(2)} z^k \right) + z + \sum_{k=2}^{\infty} a_k^{(1)} z^k + \sum_{k=1}^{\infty} b_k^{(1)} z^k
\]
\[
= (r_0^2 a_1^{(2)} + 1)z + \sum_{k=2}^{\infty} (r_0^2 a_k^{(2)} + a_k^{(1)}) z^k + \sum_{k=1}^{\infty} (r_0^2 b_k^{(2)} + b_k^{(1)}) z^k,
\]
which is harmonic in \( \mathbb{D} \). By condition (4.2), we have
\[
\sum_{k=2}^{\infty} k |r_0^2 a_k^{(2)} + a_k^{(1)}| + \sum_{k=1}^{\infty} k |r_0^2 b_k^{(2)} + b_k^{(1)}| \\
\leq \sum_{k=2}^{\infty} k (|a_k^{(1)}| + |b_k^{(1)}|) + \sum_{k=1}^{\infty} k (|a_k^{(2)}| + |b_k^{(2)}|) - |a_1^{(2)}| - |b_1^{(1)}| \\
\leq \sum_{k=2}^{\infty} k (|a_k^{(1)}| + |b_k^{(1)}|) + \sum_{k=1}^{\infty} (k+2) (|a_k^{(2)}| + |b_k^{(2)}|) - |a_1^{(2)}| - |b_1^{(1)}| \\
\leq 1 - |a_1^{(2)}|.
\]
So
\[
\sum_{k=2}^{\infty} k |r_0^2 a_k^{(2)} + a_k^{(1)}| + \sum_{k=1}^{\infty} k |r_0^2 b_k^{(2)} + b_k^{(1)}| \leq 1 - |a_1^{(2)}| \\
\leq 1 - |r_0^2 a_1^{(2)}| \leq |r_0^2 a_1^{(2)} + 1|.
\]

It follows from [17, Lemma 1.2] that \( F_{r_0}(z) \) is univalent and fully starlike in \( \mathbb{D} \), which implies that for each \( z = re^{i\theta} \neq 0 \),
\[
\frac{\partial}{\partial \theta} (\text{arg } F_{r_0}(re^{i\theta})) > 0.
\]
Letting \( r_0 = r \) yields that
\[
\frac{\partial}{\partial \theta} (\text{arg } F(re^{i\theta})) = \frac{\partial}{\partial \theta} (\text{arg } F_r(re^{i\theta})) > 0
\]
for each \( z = re^{i\theta} \neq 0 \), and also \( F \) is univalent on each circle \( |z| = r \).

From the sensen-preserving property of \( F \) and the univalence of \( F \) on the each circle \( |z| = r \), it follows that \( F \) is univalent in \( \mathbb{D} \). By using (4.4), we have that the biharmonic function \( F \) is indeed fully starlike in \( \mathbb{D} \). \( \square \)

**Remark.** If \( b_k^{(1)} = 0 \) and \( b_k^{(2)} = 0 \) for \( k = 1, 2, \ldots \), then Theorem 4.1 reduces to [22, Lemma 6].

Theorem 4.1 can also be obtained from the Theorem 3.1 and Theorem 3.2 of [24]. But for the convenience of readers, a detailed proof is still given here.
Theorem 4.2. Let $F(z) = |z|^2 G(z) + H(z)$ be a biharmonic mapping in $\mathbb{D}$, where $G(z)$ and $H(z)$ are sense-preserving harmonic mappings defined by (4.1). If

$$
\sum_{k=2}^{\infty} k^2 \left( |a_k^{(1)}| + |b_k^{(1)}| \right) + \sum_{k=2}^{\infty} k^2 \left( |a_k^{(2)}| + |b_k^{(2)}| \right) + 3(|a_1^{(2)}| + |b_1^{(2)}|) + |b_1^{(1)}| \leq 1. \tag{4.5}
$$

Then $F$ is sense-preserving and fully convex in $\mathbb{D}$.

Proof. A computation gives $J_F(0) = 1 - |b_1^{(1)}|^2 > 0$ and for $z \neq 0$, $J_F(z) = (|F_z(z)| + |F\bar{z}(z)|)(|F_z(z)| - |F\bar{z}(z)|) > 0$ because for $z \neq 0$,

$$
|F_z(z)| - |F\bar{z}(z)| \geq 1 - \sum_{k=2}^{\infty} k(|a_k^{(1)}| + |b_k^{(1)}|) - \sum_{k=2}^{\infty} (k + 2)(|a_k^{(2)}| + |b_k^{(2)}|) - |b_1^{(1)}|
$$

$$
\geq 1 - \sum_{k=2}^{\infty} k^2(|a_k^{(1)}| + |b_k^{(1)}|) - \sum_{k=2}^{\infty} k^2(|a_k^{(2)}| + |b_k^{(2)}|) - 3(|a_1^{(2)}| + |b_1^{(2)}|) - |b_1^{(1)}|
$$

$$
\geq 0.
$$

Thus, $F$ is sense-preserving in $\mathbb{D}$.

Moreover, for each fixed $r_0 \in (0, 1)$, let $F_{r_0}(z)$ be defined as (4.3). Then

$$
F_{r_0}(z) = (r_0^2 a_1^{(2)} + 1) z + \sum_{k=2}^{\infty} \left( r_0^2 a_k^{(2)} + a_k^{(1)} \right) z^k + \sum_{k=1}^{\infty} \left( r_0^2 b_k^{(2)} + b_k^{(1)} \right) z^k
$$

is harmonic in $\mathbb{D}$. Condition (4.5) implies that

$$
\sum_{k=2}^{\infty} k^2 |r_0^2 a_k^{(2)} + a_k^{(1)}| + \sum_{k=1}^{\infty} k^2 |r_0^2 b_k^{(2)} + b_k^{(1)}|
$$

$$
\leq \sum_{k=2}^{\infty} k^2 |a_k^{(2)} + a_k^{(1)}| + \sum_{k=1}^{\infty} k^2 |b_k^{(2)} + b_k^{(1)}|
$$

$$
\leq \sum_{k=2}^{\infty} k^2 (|a_k^{(1)}| + |b_k^{(1)}|) + \sum_{k=1}^{\infty} k^2 (|a_k^{(2)}| + |b_k^{(2)}|) - |a_1^{(2)}| + |b_1^{(1)}|
$$

$$
\leq \sum_{k=2}^{\infty} k^2 (|a_k^{(1)}| + |b_k^{(1)}|) + \sum_{k=2}^{\infty} k^2 (|a_k^{(2)}| + |b_k^{(2)}|) + 3(|a_1^{(2)}| + |b_1^{(2)}|) - |a_1^{(2)}| + |b_1^{(1)}|
$$

$$
\leq 1 - |a_1^{(2)}|.
$$

So

$$
\sum_{k=2}^{\infty} k^2 |r_0^2 a_k^{(2)} + a_k^{(1)}| + \sum_{k=1}^{\infty} k^2 |r_0^2 b_k^{(2)} + b_k^{(1)}| \leq 1 - |a_1^{(2)}|
$$

$$
\leq 1 - |r_0^2 a_1^{(2)}| \leq |1 + r_0^2 a_1^{(2)}|.
$$

It follows from [17. Lemma 1.1] that $F_{r_0}(z)$ is univalent and fully convex in $\mathbb{D}$, which implies that for each
\( z = re^{i\theta} \neq 0, \)

\[
\frac{\partial}{\partial \theta} \left( \arg \frac{\partial}{\partial \theta} F(re^{i\theta}) \right) > 0.
\]

Letting \( r_0 = r \) yields that

\[
\frac{\partial}{\partial \theta} \left( \arg \frac{\partial}{\partial \theta} F(re^{i\theta}) \right) = \frac{\partial}{\partial \theta} \left( \arg \frac{\partial}{\partial \theta} F_r(re^{i\theta}) \right) > 0 \tag{4.6}
\]

for each \( z = re^{i\theta} \neq 0 \), and also \( F \) is univalent on each circle \( |z| = r \).

From the sense-preserving property of \( F \) and the univalence of \( F \) on the each circle \( |z| = r \), it follows that \( F \) is univalent in \( \mathbb{D} \). By using (4.6), we have that the biharmonic function \( F \) is indeed fully convex in \( \mathbb{D} \).

**Theorem 4.3.** Let \( G(z) = z + \sum_{k=2}^{\infty} a_k z^k \) be analytic in \( \mathbb{D} \) such that \( |a_k| \leq k \) for \( k \geq 2 \). Then the biharmonic function \( W(z) \) defined by (2.4) with \( \alpha > 0 \) and \( \beta > 0 \) has Jacobian \( J_W(z) > 0 \) except at \( z = 0 \) and univalent in \( \{ z : |z| < r_S \} \), where \( r_S \) is the root of the equation

\[
(1 - 2\alpha - 2\beta)r^3 + 3(2\alpha + 2\beta - 1)r^2 - (7\alpha + 5\beta)r + \alpha + \beta = 0 \tag{4.7}
\]

in the interval \((0, 1)\). Moreover, \( W \) is fully starlike for \( \{ z : |z| < r_S \} \).

**Proof.** Let

\[
W_r(z) = \frac{1}{r} W(rz) \tag{4.8}
\]

for \( 0 < r < 1 \), where \( W \) is defined by (2.4), and \( G(z) = z + \sum_{k=2}^{\infty} a_k z^k \) such that \( |a_k| \leq k \) for \( k \geq 2 \). Then a computation gives that

\[
W_r(z) = |z|^2 \sum_{k=1}^{\infty} a_k r^{k+1} z^k + \sum_{k=1}^{\infty} \frac{\beta}{k} a_k r^{k-1} z^k,
\]

so that

\[
\frac{W_r(z)}{\alpha + \beta} = |z|^2 \sum_{k=1}^{\infty} B_k z^k + \sum_{k=2}^{\infty} A_k z^k,
\]

where

\[
B_k = \frac{1}{\alpha + \beta} a_k r^{k+1} \quad \text{and} \quad A_k = \frac{\alpha k + \beta}{k(\alpha + \beta)} a_k r^{k-1}. \tag{4.9}
\]

Note that \( W(z) \) is univalent and fully starlike in \( |z| < r \) if and only if \( W_r(z) \) is univalent and fully starlike.
in the unit disk $|z| < 1$. Thus, by Theorem 4.1, it suffices to show that

$$S(r) := \sum_{k=1}^{\infty} (k+2)|B_k| + \sum_{k=2}^{\infty} k|A_k| \leq 1$$

for $0 < r \leq r_S$. Now, since $|a_k| \leq k$ for $k \geq 2$, it follows that $S(r) \leq \frac{1}{\alpha + \beta} T(r)$, where

$$T(r) = \sum_{k=1}^{\infty} k(k+2)r^{k+1} + \sum_{k=2}^{\infty} k|a_k + \beta| r^{k-1}.$$  

By the last inequality, $S(r) \leq 1$ holds if $T(r) \leq \alpha + \beta$. Again, since

$$\frac{r}{(1-r)^2} = \sum_{k=1}^{\infty} kr^k \quad \text{and} \quad \frac{r(1+r)}{(1-r)^3} = \sum_{k=1}^{\infty} k^2 r^k,$$

we find that

$$T(r) = r \sum_{k=1}^{\infty} k^2 r^k + 2r \sum_{k=1}^{\infty} kr^k + \alpha \left(\frac{1}{r} \sum_{k=1}^{\infty} k^2 r^k - 1\right) + \beta \left(\frac{1}{r} \sum_{k=1}^{\infty} kr^k - 1\right) = \frac{r^2(1+r)}{(1-r)^3} + \frac{2r^2}{(1-r)^2} + \alpha \left(\frac{1+r}{(1-r)^3} - 1\right) + \beta \left(\frac{1}{(1-r)^2} - 1\right)$$

and therefore, by a computation, we see that $T(r) \leq \alpha + \beta$ is equivalent to the inequality $\phi(r) \geq 0$, where

$$\phi(r) = (1-2\alpha - 2\beta)r^3 + 3(2\alpha + 2\beta - 1)r^2 - (7\alpha + 5\beta)r + \alpha + \beta.$$  

The inequality $\phi(r) \geq 0$ holds if $0 < r \leq r_S$, where $r_S$ is the root of the equation $\phi(r) = 0$ in the interval $(0,1)$. Thus, $S(r) \leq 1$ for $0 < r \leq r_S$ and so, by Theorem 4.1, $W_r(z)$ is univalent sense-preserving and fully starlike in $\mathbb{D}$ with $0 < r \leq r_S$. The proof is completed.

\begin{theorem}
Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq 1$ for $k \geq 2$. Then the biharmonic function $W(z)$ defined by (2.4) with $\alpha > 0$ and $\beta > 0$ has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_K\}$, where $r_K$ is the root of the equation

$$2r^5 - 6r^4 + [5 - 2(\alpha + \beta)]r^3 + [6(\alpha + \beta) - 3]r^2 - (7\alpha + 5\beta)r + \alpha + \beta = 0 \quad (4.10)$$

in the interval $(0,1)$. Moreover, $W(z)$ is fully convex for $\{z : |z| < r_K\}$.

\end{theorem}

\begin{proof}
Let $W_r(z)$ be defined as (4.8). $W(z)$ is univalent and fully convex in $|z| < r$ if and only if $W_r(z)$ is univalent and fully convex in the unit disk $|z| < 1$. Thus, by Theorem 4.2, it suffices to show that

$$S(r) := \sum_{k=2}^{\infty} k^2|B_k| + \sum_{k=2}^{\infty} k^2|A_k| + \frac{3r^2}{\alpha + \beta} \leq 1.$$ 


for $0 < r \leq r_K$, where $A_k$ and $B_k$ are defined as in (4.9). Now, since $|a_k| \leq 1$ for $k \geq 2$, it follows that

$$S(r) \leq \frac{1}{\alpha + \beta} T(r),$$

where

$$T(r) = \sum_{k=2}^{\infty} k^2 r^{k+1} + \sum_{k=2}^{\infty} k|a_k + \beta r^{k-1} + 3r^2|.$$ 

By the last inequality, $S(r) \leq 1$ holds if $T(r) \leq \alpha + \beta$. A direct computation leads to that

$$T(r) = r \sum_{k=1}^{\infty} k^2 r^k - r^2 + \alpha \left( \frac{1}{r} \sum_{k=1}^{\infty} k^2 r^k - 1 \right) + \beta \left( \frac{1}{r} \sum_{k=1}^{\infty} k r^k - 1 \right) + 3r^2$$

$$= \frac{r^2(1 + r)}{(1 - r)^3} + \alpha \left( \frac{1 + r}{(1 - r)^2} - 1 \right) + \beta \left( \frac{1}{(1 - r)^2} - 1 \right) + 2r^2.$$ 

Therefore, $T(r) \leq \alpha + \beta$ is equivalent to the inequality

$$\phi(r) \geq 0,$$

where

$$\phi(r) = 2r^5 - 6r^4 + [5 - 2(\alpha + \beta)]r^3 + [6(\alpha + \beta) - 3]r^2 - (7\alpha + 5\beta)r + \alpha + \beta.$$ 

The inequality $\phi(r) \geq 0$ holds if $0 < r \leq r_K$, where $r_K$ is the root of the equation $\phi(r) = 0$ in the interval $(0, 1)$. Thus, $S(r) \leq 1$ for $0 < r \leq r_K$ and so, by Theorem 4.2, $W_r(z)$ is univalent sense-preserving and fully convex in $\mathbb{D}$ with $0 < r \leq r_K$. The proof is completed.

Remark. If we substitute the parts of assumption “$\alpha > 0$ and $\beta > 0$” by “$\alpha < 0$ and $\beta < 0$” in Theorem 4.3 and Theorem 4.4, and substitute $\alpha$ and $\beta$ by $-\alpha$ and $-\beta$ respectively in equations (4.7) and (4.10), then Theorem 4.3 and Theorem 4.4 still hold.

## 5 Criteria for fully starlikeness or fully convexity of biharmonic mappings

**Theorem 5.1.** If $F$ is a biharmonic mapping in $\mathbb{D}$ with the form $F(z) = |z|^2 G(z) + \alpha G(z)$, where $\alpha \geq 0$ and $G$ is a fully starlike harmonic mapping in $\mathbb{D}$, then $F$ is fully starlike univalent in $\mathbb{D}$.

**Proof.** Firstly, $G$ is a fully starlike harmonic mapping, then $G(0) = 0, G(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$. Since $F(z) = |z|^2 G(z) + \alpha G(z) = (|z|^2 + \alpha) G(z)$ with $\alpha \geq 0$, we have $F(0) = 0, F(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$.

Secondly, Let $|z| = r$. Then $F(z) = |z|^2 G(z) + \alpha G(z)$ deduces that

$$F_z = \bar{z} G + (r^2 + \alpha) G_\bar{z}, \quad F_\bar{z} = z G + (r^2 + \alpha) G_z.$$
Thus the Jacobian of $F$ is given by

$$J_F(z) = |F_z|^2 - |F_{\bar z}|^2$$

$$= (r^2 + \alpha)^2(|G_z|^2 - |G_{\bar z}|^2) + 2(r^2 + \alpha)\Re[zG_G z - z\bar{G_G} \bar{z}]$$

$$= (r^2 + \alpha)^2 J_G(z) + 2(r^2 + \alpha)|G|^2 \Re \frac{DG_G}{G}.$$

Since $G$ is a fully starlike harmonic mapping, $J_G(z) > 0$ and $\Re \frac{DG_G}{G} > 0$. It follows that

$$J_F(z) > 0$$

for all $z \in \mathbb{D}$. Hence $F$ is orientation-preserving and locally univalent.

Thirdly, a direct computation leads to

$$\frac{D F}{F} = \frac{z[zG + (r^2 + \alpha)G_z] - \bar{z}[zG + (r^2 + \alpha)G_{\bar z}]}{(r^2 + \alpha)G} = \frac{DG_G}{G}.$$

Hence, by [18, Theorem 1], $F$ is fully starlike in $\mathbb{D}$ follows from the full starlikeness of $G$. □

**Remark.** If $\alpha = 0$, then Theorem 5.1 reduces to [1, Theorem 2.6].

**Proposition 5.2.** Let $G$ be a fully starlike harmonic mapping in $\mathbb{D}$ and $F$ be a biharmonic mapping with the form $F = k|z|^2G_G(z) + G(z)$, where $k \in [-1, 1]$. Then $\Re \frac{DF}{F} > 0$ for $z \neq 0$.

**Proof.** Let $F_s = \frac{DF}{F}$ and $G_s = \frac{DG_G}{G}$. Direct computations yield

$$F_s = \frac{z(kzG_G + kr^2G_{\bar z} + G_z) - \bar{z}(kzG_G + kr^2G_{\bar z} + G_{\bar z})}{(r^2 + \alpha)G}$$

$$= \frac{kr^2(zG_G - \bar{z}G_{\bar z}) + zG_z - \bar{z}G_{\bar z}}{(r^2 + \alpha)G}$$

$$= \frac{kr^2G_G G_s + GG_s}{F}.$$

It is easy to verify that $G_s = -\bar{G_s}$. Therefore, for $k \in [-1, 1]$, $z \neq 0$ and fully starlike harmonic mapping $G$, it holds that

$$\Re F_s = \Re \frac{GG_s - kr^2G_s\bar{G_s}}{F}$$

$$= \Re \frac{(GG_s - kr^2G_s\bar{G_s})(kr^2G + G)}{|F|^2}$$

$$= \frac{(1 - k^2r^4)|G|^2}{|F|^2} \Re G_s > 0.$$

□

**Theorem 5.3.** If $F$ is a biharmonic mapping in $\mathbb{D}$ with the form $F(z) = |z|^2G(z) + \alpha G(z)$, where $\alpha \geq 0$ and $G$ is a fully convex harmonic mapping in $\mathbb{D}$, then $F$ is fully convex univalent in $\mathbb{D}$.

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Proof. Firstly, we have $G(0) = 0, G(z)DG(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$. Since $F(z) = \vert z \vert^2 G(z) + \alpha G(z) = (\vert z \vert^2 + \alpha)G(z)$ with $\alpha \geq 0$ and

$$DF = zF_z - \bar{z}F_{\bar{z}} = (r^2 + \alpha)(zG_z - \bar{z}G_{\bar{z}}) = (r^2 + \alpha)DG,$$

it follows that $F(0) = 0, F(z)DF(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$.

Secondly, since $G$ is fully convex, it is fully starlike. So, by Theorem 5.1, (5.1) still holds.

Thirdly, direct computations yield

$$D^2 F = D(DF) = z[(r^2 + \alpha)(zG_z - \bar{z}G_{\bar{z}})] - \bar{z}[(r^2 + \alpha)(zG_z - \bar{z}G_{\bar{z}})]\bar{z}$$

$$= (r^2 + \alpha)(zG_z + \bar{z}G_{\bar{z}} + z^2G_{zz} + \bar{z}G_{\bar{z}z})$$

$$= (r^2 + \alpha)D^2 G.$$

It follows that

$$\frac{D^2 F}{DF} = \frac{D^2 G}{DG}.$$

Hence, by [18, Theorem 3], $F$ is fully convex follows from the full convexity of $G$. 

Proposition 5.4. Let $G$ be a fully convex harmonic mapping in $\mathbb{D}$ and $F$ be a biharmonic mapping with the form $F(z) = k\vert z \vert^2 G(z) + G(z)$, where $k \in [-1, 1]$. Then $\Re \frac{D^2 F}{DF} > 0$ for $z \neq 0$.

Proof. Since $F(z) = k\vert z \vert^2 G(z) + G(z)$, direct computations lead to

$$DF = zF_z - \bar{z}F_{\bar{z}} = z(k\bar{z}G + kr^2\bar{G}_z + G_z) - \bar{z}(kzG + kr^2G_{\bar{z}} + G_{\bar{z}})$$

$$= kr^2(z\bar{G}_z - \bar{z}G_z) + zG_z - \bar{z}G_{\bar{z}}$$

$$= kr^2DG + DG$$

and

$$D^2 F = D(DF)$$

$$= z[kr^2(z\bar{G}_z - \bar{z}G_z) + zG_z - \bar{z}G_{\bar{z}}] - \bar{z}[kr^2(z\bar{G}_z - \bar{z}G_z) + zG_z - \bar{z}G_{\bar{z}}]$$

$$= kr^2(z\bar{G}_z + z^2G_{zz} + \bar{z}G_{\bar{z}} + \bar{z}^2G_{\bar{z}z}) + zG_z + z^2G_{zz} + \bar{z}G_{\bar{z}} + \bar{z}^2G_{\bar{z}z}$$

$$= kr^2D^2G + D^2 G.$$

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It is easy to verify that $\nabla F = -\nabla F$, $\nabla^2 F = \nabla^2 F$. It follows that
\[
\Re \frac{\nabla^2 F}{\nabla F} = \Re \frac{kr^2 \nabla^2 G + \nabla^2 G}{kr^2 \nabla G + \nabla G} = \frac{1 - k^2 r^4}{|kr^2 \nabla G + \nabla G|^2} \Re (\nabla^2 G \nabla G)
\]
\[
= \frac{(1 - k^2 r^4)|\nabla G|^2}{|kr^2 \nabla G + \nabla G|^2} \Re \nabla^2 G \nabla G.
\]

Therefore, $\Re \frac{\nabla^2 F}{\nabla F} > 0$ for $k \in [-1, 1]$ as $\Re \frac{\nabla^2 G}{\nabla G} > 0$. 

References


