This is a resubmission of the paper RMJ 240225. A resubmission was permitted by the editor in chief, Professor Goodrich.
In the last two pages, we respond to the reviewer comments as a cover letter.

RESUBMISSION COVER LETTER FOR THE PAPER FURTHER NUMERICAL RADIUS INEQUALITIES SUBMITTED TO RMJM

MOHAMMAD SABABHEH, CRISTIAN CONDE, AND HAMID REZA MORADI

Dear editor in chief of RMJM and reviewer

We, the authors, would like first to thank you for considering our paper and giving us the opportunity to resubmit our paper.

We read the reviewer comments carefully, and we believe that we have revised our paper accordingly.

In this short letter, we mention the major changes we did.

First: The reviewer mentions that our presented new estimates are sophisticated, and that it is not clear how close they are to the original approximated quantities.

We do agree with the reviewer with this major comments. In fact yes, our obtained bounds seem more complicated than previous bounds, and it was not clear how close they are to the estimated quantities in the previous version.

Before mentioning what we did in response to this comment, we remark that the calculations of the exact value of the numerical radius are not easy. It is indeed harder than the operator norm. So, it is expected that to get better estimates, more complicated calculations are needed! Moreover, the obtained bounds, although seem complicated, they provide concrete relations that could help other researchers in the field.

Furthermore, nowadays computers and Math algebras can calculate complicated terms in no time. So, these complicated terms are not really complicated for computer calculations.

In response to the reviewer major comment, we have added many remarks and comments (in red) explaining the advantage of almost all obtained results. More precisely, for most of the results we have given numerical examples that show that our new approximation can be better than previously obtained ones. For other results, we showed how the obtained bounds are always sharper than previous ones.

In fact, our numerical calculations indicate that our results are almost always better than previous ones, as we mention in Remark 2.3.

2010 *Mathematics Subject Classification.* Primary 47A12, 47A30, Secondary 47B15, 15A60.

Key words and phrases. Numerical radius, operator norm, inequality.

5 Aug 2024 10:43:02 PDT
240805-Sababheh Version 1 - Submitted to Rocky Mountain J. Math.

We like to mention that numerical examples are not proofs of anything, but they provide an indication of what is going on. Having ”too many” examples where our new bounds are better than previous ones indicate that the obtained results are not trivial.

Further, we have added more particular cases from the general one to make some comparisons with the literature.

To make it clear for the reviewer, we colored all comparisons with the previous results by red.

Second: The reviewer suggested some minor suggestions.

In response to this, we added the spectral radius relation with the numerical radius, as suggested, and we added the definition of the absolute value of an operator.

Sincerely yours

The authors

FURTHER NUMERICAL RADIUS INEQUALITIES

MOHAMMAD SABABHEH, CRISTIAN CONDE, AND HAMID REZA MORADI

Abstract. In this article, we present some new inequalities for the numerical radius of products of Hilbert space operators and the generalized Aluthge transform. In particular, we show some upper bounds for $\omega(ABC+DEF)$ using the celebrated Buzano inequality, and then some consequences that generalize some results from the literature are discussed. After that, inequalities that involve the generalized Aluthge transform are shown using some known bounds for the numerical radius of the product of two operators.

1. Introduction

Given a complex Hilbert space $\mathcal{H}$, endowed with the inner product $\langle\cdot,\cdot\rangle$, we use $\mathcal{B}(\mathcal{H})$ to denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, the numerical radius and the operator norm of A are defined respectively by

$$\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle| \text{ and } \|A\| = \sup_{\|x\|=1} \|Ax\|.$$

A related quantity is the spectral radius, which is defined for $A \in \mathcal{B}(\mathcal{H})$ by $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A.

It is well known that if A is normal, in the sense that $A^*A = AA^*$, then $\|A\| = \omega(A) = r(A)$. However, for non-normal operators, this equality fails. In general, the following hold for any $A \in \mathcal{B}(\mathcal{H})$:

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|, \tag{1.1}$$

and

$$r(A) \le \omega(A) \le \|A\|.$$

These inequalities are important because they approximate $\omega(A)$ in terms of $\|A\|$ and $r(A)$, which are easier quantities to compute than $\omega(A)$. Sharpening (1.1) and other inequalities for the numerical radius has been the interest of numerous researchers in the past few years; see [6, 7, 10, 17, 18, 19, 20, 21, 22] for example.

2010 *Mathematics Subject Classification.* Primary 47A12, 47A30, Secondary 47B15, 15A60.

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Among the most interesting bounds, we have the following lemma, in which the notation $|\cdot|$ refers to the absolute value operator, defined for $A\in\mathcal{B}(\mathcal{H})$ by $|A|=(A^*A)^{\frac{1}{2}}$.

Lemma 1.1. [14, 15] *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then*

$$\omega\left(AB\right)\le\frac{1}{2}\left\||A^*|^2+|B|^2\right\|, \tag{1.2}$$

$$\omega^2(A)\le\frac{1}{2}\|\,|A|^2+|A^*|^2\,\|,$$

$$\omega(A)\le\frac{1}{2}\|\,|A|+|A^*|\,\|,$$

and

$$\omega(A)\le\frac{1}{2}\left(\|A\|+\|A^2\|^{\frac{1}{2}}\right). \tag{1.3}$$

In studying the notion of the numerical radius, the Aluthge transform has played an interesting role. Recall that this transform is defined for $A\in\mathcal{B}(\mathcal{H})$ by $\widetilde{A}=|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ where U is the partial isometry that appears in the polar decomposition $A=U|A|$ of A. In fact, for $0\le t\le 1$, the generalized Aluthge transform of A is defined by $\widetilde{A_t}=|A|^{1-t}U|A|^t$. In [23], it is shown that

$$\omega(A)\le\frac{1}{2}\left(\|A\|+\omega(\widetilde{A})\right). \tag{1.4}$$

Using the inequalities [23]

$$\omega(\widetilde{A})\le\|\widetilde{A}\|\le\|A^2\|^{\frac{1}{2}}\le\|A\|, \tag{1.5}$$

it follows that (1.4) refines the second inequality in (1.1) and the inequality (1.3).

Although the numerical radius and the operator norm are equivalent, the numerical radius differs from the operator norm, being not submultiplicative. That is, if $A,B\in\mathcal{B}(\mathcal{H})$, then $\|AB\|\le\|A\|\,\|B\|$, but $\omega(AB)\not\le\omega(A)\omega(B)$. In fact, it has been a standing question about the best possible constant in $\omega(AB)\le k\omega(A)\omega(B)$. We refer the reader to [2] for discussion of submultilicativity of $\omega(\cdot)$.

It was shown in [8] that if $A,B\in\mathcal{B}\left(\mathcal{H}\right)$, then

$$\omega\left(AB+BA\right)\le 2\sqrt{2}\omega\left(A\right)\|B\| \tag{1.6}$$

and

$$\omega(AB+B^*A)\le 2\omega(A)\|B\|. \tag{1.7}$$

We refer the reader to [3] for different results regarding the numerical radius of the product and commutators of Hilbert space operators.

Some critical tools in obtaining numerical radius inequalities are listed next.

Lemma 1.2. [13] *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators. Then*

$$\|A+B\| \leq \max\{\|A\|, \|B\|\} + \left\| |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \right\|.$$

Lemma 1.3. [12] *Let $T \in \mathcal{B}(\mathcal{H})$ and $0 \leq t \leq 1$. Then for any $x, y \in \mathcal{H}$,*

$$|\langle Tx, y\rangle| \leq \sqrt{\left\langle |T|^{2t} x, x\right\rangle \left\langle |T^*|^{2(1-t)} y, y\right\rangle}.$$

Lemma 1.4. [5] *Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle \langle y, e\rangle| \leq \frac{1}{2}\left(|\langle x, y\rangle| + \|x\| \|y\|\right).$$

Lemma 1.5. [2, Theorem 3.4] *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega(A+B) \leq \frac{1}{2}\left((\omega(A) + \omega(B)) + \sqrt{(\omega(A) - \omega(B))^2 + 4\sup_{\theta \in \mathbb{R}} \left\|\mathfrak{R}\left(e^{\mathrm{i}\theta} A\right) \mathfrak{R}\left(e^{\mathrm{i}\theta} B\right)\right\|}\right).$$

This inequality is clearly a refinement of the triangle inequality $\omega(A+B) \leq \omega(A) + \omega(B)$, due to the fact that [23]

$$\omega(A) = \sup_{\theta \in \mathbb{R}} \|\mathfrak{R}(e^{i\theta} A)\|.$$

Lemma 1.6. [11, Lemma 2.1] *Let $X, Y \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega\begin{pmatrix} O & X \\ Y & O \end{pmatrix} = \omega\begin{pmatrix} O & Y \\ X & O \end{pmatrix} = \omega\begin{pmatrix} O & X \\ -Y & O \end{pmatrix}.$$

Our target in this article is to show some inequalities for the numerical radius of products of Hilbert space operators, such as finding bounds for $\omega(ABC + DEF)$ and some inequalities for the Aluthge transform that refine some of the known results in the literature. Many consequences and other inequalities for $\omega(AB), \omega(A), \omega(A+B)$ and $\omega(\widetilde{T}_t)$ will be presented. In this context, $\widetilde{T}_t$ denotes the generalized Aluthge transform, defined for $T \in \mathcal{B}(\mathcal{H})$ and $0 \leq t \leq 1$ by

$$\widetilde{T}_t = |T|^{1-t} U |T|^t,$$

where U is the partial isometry in the polar decomposition $T = U|T|$ of T.

2. Main Result

In this section, we present our main results. The first part deals mainly with inequalities of the numerical radius of the product of operators; then, we discuss some inequalities that involve the Aluthge transform.

Block operators have played a significant role in obtaining numerical radius inequalities for certain Hilbert space operators. We present the following excellent application of Lemma 1.5, where more straightforward block operators can be used as a bound for $\omega(AB + CD)$.

Theorem 2.1. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega\left(AB+CD\right) \le \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\|\left(AB\right)^2\| + \|\left(CD\right)^2\| + \|ABD^*C^* + D^*C^*AB\|}.$$

In particular,

$$\omega\left(A \pm C\right) \le \omega\left(\begin{bmatrix} O & A \\ C & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\|A^2\| + \|C^2\| + \|AC^* + C^*A\|}. \tag{2.1}$$

Proof. Applying Lemma 1.5 in what follows, we notice that

$$\begin{aligned}
&\omega\left(AB+CD\right) \\
&= \omega\left(\begin{bmatrix} O & AB+CD \\ AB+CD & O \end{bmatrix}\right) \\
&= \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix} + \begin{bmatrix} O & CD \\ AB & O \end{bmatrix}\right) \\
&\le \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{4\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{R}\begin{bmatrix} O & e^{\mathrm{i}\theta}AB \\ e^{\mathrm{i}\theta}CD & O \end{bmatrix}\mathfrak{R}\begin{bmatrix} O & e^{\mathrm{i}\theta}CD \\ e^{\mathrm{i}\theta}AB & O \end{bmatrix}\right\|} \\
&= \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} \left(e^{\mathrm{i}\theta}AB + e^{-\mathrm{i}\theta}D^*C^*\right)^2 & O \\ O & \left(e^{-\mathrm{i}\theta}B^*A^* + e^{\mathrm{i}\theta}CD\right)^2 \end{bmatrix}\right\|} \\
&= \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} \left(e^{\mathrm{i}\theta}AB + e^{-\mathrm{i}\theta}D^*C^*\right)^2 & O \\ O & \left(\left(e^{\mathrm{i}\theta}AB + e^{-\mathrm{i}\theta}D^*C^*\right)^*\right)^2 \end{bmatrix}\right\|} \\
&= \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} \left(e^{\mathrm{i}\theta}AB + e^{-\mathrm{i}\theta}D^*C^*\right)^2 & O \\ O & \left(\left(e^{\mathrm{i}\theta}AB + e^{-\mathrm{i}\theta}D^*C^*\right)^2\right)^* \end{bmatrix}\right\|} \\
&= \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\sup_{\theta\in\mathbb{R}}\left\|\left(e^{\mathrm{i}\theta}AB + e^{-\mathrm{i}\theta}D^*C^*\right)^2\right\|}.
\end{aligned}$$

Thus,

$$\omega\left(AB+CD\right) \le \omega\left(\begin{bmatrix} O & AB \\ CD & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\sup_{\theta\in\mathbb{R}}\left\|\left(e^{\mathrm{i}\theta}AB + e^{-\mathrm{i}\theta}D^*C^*\right)^2\right\|}.$$

On the other hand,

$$\begin{aligned}
\left\|\left(e^{i\theta}AB + e^{-i\theta}D^*C^*\right)^2\right\| &= \left\|e^{2\mathrm{i}\theta}\left(AB\right)^2 + e^{-2\mathrm{i}\theta}\left(D^*C^*\right)^2 + ABD^*C^* + D^*C^*AB\right\| \\
&\le \|\left(AB\right)^2\| + \|\left(D^*C^*\right)^2\| + \|ABD^*C^* + D^*C^*AB\|
\end{aligned}$$

where we have used triangle inequality for the usual operator norm to obtain the last line. This completes the proof. □

Remark 2.1. *In this remark, we give a numerical example that supports the advantage of Theorem 2.1. More precisely, we know that*

$$\omega(A+C) \le \omega(A)+\omega(C) \text{ and } \omega(A+C) \le \|A+C\|. \tag{2.2}$$

On the other hand, Theorem 2.1 presented a new upper bound for $\omega(A+C)$, which is stated in (2.1). If we let $A=\begin{bmatrix} -2 & -2 \\ -1 & -2 \end{bmatrix}$ and $C=\begin{bmatrix} -3 & -1 \\ 3 & 0 \end{bmatrix}$, numerical calculations show that

$$\omega(A+C) \approx 5.08114, \omega(A)+\omega(C) \approx 6.80278, \|A+C\| \approx 5.8823,$$

while

$$\omega\left(\begin{bmatrix} O & A \\ C & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\|A^2\|+\|C^2\|+\|AC^*+C^*A\|} \approx 5.8439.$$

This shows that, in this example, the bound Theorem 2.1 presented can be better than other known bounds. We emphasize that other examples show that the other bounds are better than that in Theorem 2.1. This means that neither bound is uniformly better than the other, but it also means that Theorem 2.1 introduces a new valuable bound independent from the other known bounds.

Another importance of the bound we found in (2.1) is stated in the following remark.

Remark 2.2. *It is known that* [11] *if $A, C \in \mathcal{B}(\mathcal{H})$, then*

$$\omega\left(\begin{bmatrix} O & A \\ C & O \end{bmatrix}\right) \ge \frac{1}{2}\max\{\omega(A+C), \omega(A-C)\}. \tag{2.3}$$

Notice that (2.1) provides another lower bound for $\omega\left(\begin{bmatrix} O & A \\ C & O \end{bmatrix}\right)$ as follows

$$\omega\left(\begin{bmatrix} O & A \\ C & O \end{bmatrix}\right) \ge \omega(A \pm C) - \frac{1}{2}\sqrt{\|A^2\|+\|C^2\|+\|AC^*+C^*A\|}. \tag{2.4}$$

If we let $A=\begin{bmatrix} 0 & -2 \\ -10 & -5 \end{bmatrix}$ and $C=\begin{bmatrix} -3 & 5 \\ -3 & 3 \end{bmatrix}$, then numerical calculations show that

$$\omega(A \pm C) - \frac{1}{2}\sqrt{\|A^2\|+\|C^2\|+\|AC^*+C^*A\|} \approx 6.08511,$$

and

$$\frac{1}{2}\max\{\omega(A+C), \omega(A-C)\} \approx 5.70112,$$

which shows the advantage of (2.4) over (2.3) in this example. We point out that this is not always the case, as there are other examples with opposite conclusions.

Another interesting consequence of Theorem 2.1 is the following bound related to (1.6) and (1.7), which results from Theorem 2.1 by letting $C = B$ and $D = A$ first, then $C = B^*$ and $D = A$.

Corollary 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

(2.5)
$$\omega(AB + BA) \le \omega\left(\begin{bmatrix} O & AB \\ BA & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\|(AB)^2\| + \|(BA)^2\| + \|ABA^*B^* + A^*B^*AB\|},$$

and

(2.6)
$$\omega(AB + B^*A) \le \omega\left(\begin{bmatrix} O & AB \\ B^*A & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\|(AB)^2\| + \|(B^*A)^2\| + \|ABA^*B + A^*BAB\|}.$$

Remark 2.3. *In this remark, we first give an example that supports (2.5) over (1.6). Notice first that a stronger version of (1.6) can be stated in the following form*

$$\omega(AB + BA) \le 2\sqrt{2}\min\{\omega(A)\|B\|, \omega(B)\|A\|\}. \tag{2.7}$$

If we let $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, then numerical calculations show that

$$\omega(AB + BA) \approx 5.76783, 2\sqrt{2}\min\{\omega(A)\|B\|, \omega(B)\|A\|\} \approx 31.0381,$$

and

$$\omega\left(\begin{bmatrix} O & AB \\ BA & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\|(AB)^2\| + \|(BA)^2\| + \|ABA^*B^* + A^*B^*AB\|} \approx 7.08945.$$

This example shows the advantage of (2.5) over (2.7). We like to mention here that we could not find any example where (2.7) is better than (2.5).

On the other hand, if we let $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ 2 & -3 \end{bmatrix}$, we find that

$$\omega(AB + B^*A) \approx 11.6033, 2\omega(A)\|B\| \approx 20.1388,$$

and

$$\omega\left(\begin{bmatrix} O & AB \\ B^*A & O \end{bmatrix}\right) + \frac{1}{2}\sqrt{\|(AB)^2\| + \|(B^*A)^2\| + \|ABA^*B + A^*BAB\|} \approx 11.8136,$$

which shows the advantage of (2.6) over (1.7). We point out that there are examples where (1.7) is better than (2.6), as one can check with $A = \begin{bmatrix} 2 & -6 \\ 10 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 10 \\ 10 & 3 \end{bmatrix}$.

We continue with the following general inequality for the product of operators, which implies several consequences for more uncomplicated cases.

Theorem 2.2. *Let $A, B, C, D, E, F \in \mathcal{B}(\mathcal{H})$. Then for any $0 \le t \le 1$,*

$$\omega^2(ABC+DEF) \le \frac{1}{2}\omega\left(\left(A^*|B^*|^{2(1-t)}A + D^*|E^*|^{2(1-t)}D\right)\left(C^*|B|^{2t}C + F^*|E|^{2t}F\right)\right) + \frac{1}{2}\left\|C^*|B|^{2t}C + F^*|E|^{2t}F\right\|\left\|A|B^*|^{2(1-t)}A^* + D|E^*|^{2(1-t)}D^*\right\|.$$

In particular,

$$\omega^2(ABC+DEF) \le \frac{1}{2}\min\{\alpha, \beta\},$$

where

$$\alpha = \omega\left(\left(|A|^2+|D|^2\right)\left(C^*|B|^2C + F^*|E|^2F\right)\right) + \left\|C^*|B|^2C + F^*|E|^2F\right\|\left\||A^*|^2+|D^*|^2\right\|,$$

and

$$\beta = \omega\left(\left(A^*|B^*|^2A + D^*|E^*|^2D\right)\left(|C|^2+|F|^2\right)\right) + \left\||C|^2+|F|^2\right\|\left\|A|B^*|^2A^* + D|E^*|^2D^*\right\|.$$

Proof. Let $T = \begin{bmatrix} ABC+DEF & O \\ O & O \end{bmatrix} \in \mathcal{B}(\mathcal{H}\oplus\mathcal{H})$, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a unit vector in $\mathcal{H}\oplus\mathcal{H}$. Indeed, $\|x_1\|^2 + \|x_2\|^2 = 1$. Then

$$\begin{aligned}
&\left|\left\langle \begin{bmatrix} ABC+DEF & O \\ O & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right|^2 \\
&= \left|\left\langle \begin{bmatrix} A & D \\ O & O \end{bmatrix}\begin{bmatrix} B & O \\ O & E \end{bmatrix}\begin{bmatrix} C & O \\ F & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right|^2 \\
&= \left|\left\langle \begin{bmatrix} B & O \\ O & E \end{bmatrix}\begin{bmatrix} C & O \\ F & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} A^* & O \\ D^* & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right|^2 \\
&\le \left\langle \begin{bmatrix} C^*|B|^{2t}C + F^*|E|^{2t}F & O \\ O & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle \left\langle \begin{bmatrix} A|B^*|^{2(1-t)}A^* + D|E^*|^{2(1-t)}D^* & O \\ O & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle \\
&\qquad \text{(by Lemma 1.3)} \\
&\le \frac{1}{2}\left|\left\langle \begin{bmatrix} \left(A^*|B^*|^{2(1-t)}A + D^*|E^*|^{2(1-t)}D\right)\left(C^*|B|^{2t}C + F^*|E|^{2t}F\right) & O \\ O & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\rangle\right| \\
&\quad + \frac{1}{2}\left\|\begin{bmatrix} C^*|B|^{2t}C + F^*|E|^{2t}F & O \\ O & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\|\left\|\begin{bmatrix} A|B^*|^{2(1-t)}A^* + D|E^*|^{2(1-t)}D^* & O \\ O & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\| \\
&\qquad \text{(by Lemma 1.4)} \\
&\le \frac{1}{2}\omega\left(\left(A^*|B^*|^{2(1-t)}A + D^*|E^*|^{2(1-t)}D\right)\left(C^*|B|^{2t}C + F^*|E|^{2t}F\right)\right) \\
&\quad + \frac{1}{2}\left\|C^*|B|^{2t}C + F^*|E|^{2t}F\right\|\left\|A|B^*|^{2(1-t)}A^* + D|E^*|^{2(1-t)}D^*\right\|.
\end{aligned}$$

Therefore,

$$\omega^2(ABC+DEF) \le \frac{1}{2}\omega\left(\left(A^*|B^*|^{2(1-t)}A + D^*|E^*|^{2(1-t)}D\right)\left(C^*|B|^{2t}C + F^*|E|^{2t}F\right)\right) + \frac{1}{2}\left\|C^*|B|^{2t}C + F^*|E|^{2t}F\right\|\left\|A|B^*|^{2(1-t)}A^* + D|E^*|^{2(1-t)}D^*\right\|,$$

as desired. □

In the following remark and corollaries, we attempt to present some simple consequences from Theorem 2.2.

Remark 2.4. *If we let $B = C = E = F = I$, Theorem 2.2 reduces to*

$$\omega^2(A+D) \le \|A^*A + D^*D\| + \|AA^* + DD^*\|. \tag{2.8}$$

A simple upper bound for $\omega(A+D)$ follows from the triangle inequality as follows $\omega(A+D) \le \omega(A) + \omega(D)$. In this remark, we give numerical examples that show the advantage of (2.8) over this latter bound. Indeed, if we let $A = \begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$, then numerical calculations show that

$$\omega^2(A+D) \approx 6.25, (\omega(A)+\omega(D))^2 \approx 32.4806,$$

and

$$\|A^*A + D^*D\| + \|AA^* + DD^*\| \approx 26.9188.$$

This shows that our bound in (2.8) can be better than the upper bound resulting from the triangle inequality for the numerical radius. On the other hand, if we let

Remark 2.5.

(I) *The case $B = E = I$, in Theorem 2.2, reduces to*

$$\omega^2(AC+DF) \le \frac{1}{2}\left(\omega\left(\left(|A|^2+|D|^2\right)\left(|C|^2+|F|^2\right)\right) + \left\||C|^2+|F|^2\right\|\left\||A^*|^2+|D^*|^2\right\|\right).$$

(II) *The case $D = E = F = O$, in Theorem 2.2, reduces to*

$$\omega^2(ABC) \le \frac{1}{2}\min\{\alpha', \beta'\}$$

where

$$\alpha' = \omega\left(|A|^2\left(C^*|B|^2C\right)\right) + \|A\|^2\left\|C^*|B|^2C\right\|,$$

and

$$\beta' = \omega\left(\left(A^*|B^*|^2A\right)|C|^2\right) + \|C\|^2\left\|A|B^*|^2A^*\right\|.$$

From Lemma 1.1, we know that if $A, B \in \mathcal{B}(\mathcal{H})$, then
$$\omega(AB) \le \frac{1}{2}\left\| |A^*|^2 + |B|^2 \right\|.$$
As a consequence of Theorem 2.2, we deduce the following refinement of this latter inequality.

Corollary 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*
$$\omega(AB) \le \frac{1}{2}\min\{\alpha_0, \beta_0\},$$
where
$$\alpha_0 = \sqrt{2}\left\| B^*|A|^2 B + A|B^*|^2 A^* \right\|^{\frac{1}{2}},$$
and
$$\beta_0 = \left\| |A^*|^2 + |B|^2 \right\|.$$

Proof. Letting $A = e^{\mathrm{i}\theta}$, $B = A$, $C = B$, $D = e^{-\mathrm{i}\theta}$, $E = B^*$, and $F = A^*$ in Theorem 2.2, and then taking supremum over $\theta \in \mathbb{R}$, we get the desired result. □

The above corollary implies two inequalities, as follows:
$$\omega^2(AB) \le \frac{1}{2}\left\| B^*|A|^2 B + A|B^*|^2 A^* \right\| \text{ and } \omega(AB) \le \frac{1}{2}\left\| |A^*|^2 + |B|^2 \right\|.$$
Notice that if $B = I$, in the first inequality, we get the second inequality in Lemma 1.1.

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then for any $0 \le t \le 1$*
$$\omega(T) \le \frac{1}{2}\min\{\alpha_1, \beta_1\},$$
where
$$\alpha_1 = \sqrt{2}\left\| |T|^2 + |T^*|^2 \right\|^{\frac{1}{2}},$$
and
$$\beta_1 = \left\| |T^*|^{2(1-t)} + |T|^{2t} \right\|.$$

We notice that the above corollary implies two particular inequalities as follows:
$$\omega^2(T) \le \frac{1}{2}\| \, |T|^2 + |T^*|^2 \| \text{ and } \omega(T) \le \frac{1}{2}\| \, |T| + |T^*| \, \|,$$
which are the two inequalities from Lemma 1.1. This shows how Theorem 2.2 can be used to obtain some inequalities from the literature.

We also have the following result for the product of two operators. This result will be used to obtain the main result of the Aluthge transform.

Theorem 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*
$$\begin{aligned}&\omega(AB)\\ &\le \frac{1}{2}\sqrt{\frac{1}{4}\left\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right\|^2 + \omega(BA)^2 + \frac{1}{2}\omega\left(\left(\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right)BA + BA\left(\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right)\right)}.\end{aligned}$$

Proof. By the Polarization identity, we can write, for any unit vector $x \in \mathcal{H}$,

$$\begin{aligned}
&\left\langle \Re\left(e^{i\theta}AB\right)x, x\right\rangle \\
&= \Re\left\langle e^{i\theta}ABx, x\right\rangle \\
&= \Re\left\langle e^{i\theta}Bx, A^*x\right\rangle \\
&= \frac{1}{4}\left\|\left(e^{i\theta}B + A^*\right)x\right\|^2 - \frac{1}{4}\left\|\left(e^{i\theta}B - A^*\right)x\right\|^2 \\
&\le \frac{1}{4}\left\|\left(e^{i\theta}B + A^*\right)x\right\|^2 \\
&\le \frac{1}{4}\left\|e^{i\theta}B + A^*\right\|^2 \\
&= \frac{1}{4}\left\|\left(e^{i\theta}B + A^*\right)\left(e^{i\theta}B + A^*\right)^*\right\| \\
&= \frac{1}{2}\left\|\frac{|A|^2 + |B^*|^2}{2} + \Re e^{i\theta}BA\right\| \\
&= \frac{1}{2}\left\|\left(\frac{|A|^2 + |B^*|^2}{2}\right)^2 + \left(\Re e^{i\theta}BA\right)^2 + \left(\frac{|A|^2 + |B^*|^2}{2}\right)\Re e^{i\theta}BA + \Re e^{i\theta}BA\left(\frac{|A|^2 + |B^*|^2}{2}\right)\right\|^{\frac{1}{2}} \\
&= \frac{1}{2}\left\|\left(\frac{|A|^2 + |B^*|^2}{2}\right)^2 + \left(\Re e^{i\theta}BA\right)^2 + \frac{1}{2}\left(\Re e^{i\theta}\left(\left(|A|^2 + |B^*|^2\right)BA + BA\left(|A|^2 + |B^*|^2\right)\right)\right)\right\|^{\frac{1}{2}} \\
&\le \frac{1}{2}\sqrt{\frac{1}{4}\left\||A|^2 + |B^*|^2\right\|^2 + \left\|\Re e^{i\theta}BA\right\|^2 + \frac{1}{2}\left\|\Re e^{i\theta}\left(\left(|A|^2 + |B^*|^2\right)BA + BA\left(|A|^2 + |B^*|^2\right)\right)\right\|} \\
&\le \frac{1}{2}\sqrt{\frac{1}{4}\left\||A|^2 + |B^*|^2\right\|^2 + \omega(BA)^2 + \frac{1}{2}\omega\left(\left(|A|^2 + |B^*|^2\right)BA + BA\left(|A|^2 + |B^*|^2\right)\right)}.
\end{aligned}$$

Thus,

$$\omega(AB) \le \frac{1}{2}\sqrt{\frac{1}{4}\left\||A|^2 + |B^*|^2\right\|^2 + \omega(BA)^2 + \frac{1}{2}\omega\left(\left(|A|^2 + |B^*|^2\right)BA + BA\left(|A|^2 + |B^*|^2\right)\right)}. \tag{2.9}$$

Replacing A by $\sqrt{\frac{\|B\|}{\|A\|}}A$ and B by $\sqrt{\frac{\|A\|}{\|B\|}}B$ in (2.9), we get

$$\begin{aligned}
&\omega(AB) \\
&\le \frac{1}{2}\sqrt{\frac{1}{4}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|^2 + \omega(BA)^2 + \frac{1}{2}\omega\left(\left(\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right)BA + BA\left(\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right)\right)},
\end{aligned}$$

as desired. □

It is worth mentioning here that, Theorem 2.3 improves [3, Theorem 3.2] when $X = I$.

Now, we discuss inequalities that involve the Aluthge transform. The following numerical radius inequality in terms of the Aluthge transform improves the upper bound obtained in [1, Theorem 3.2].

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and $T = U|T|$ be the polar decomposition of T, and let $\widetilde{T}_t = |T|^t U |T|^{1-t}$ $(0 \le t \le 1)$ be the generalized Aluthge transformation of T. Then*

$$\begin{aligned}&\omega(T)\\&\le \frac{1}{2}\sqrt{\|T\|^2 + \omega\left(\widetilde{T}_t\right)^2 + \frac{1}{2}\omega\left(\left(\|T\|^{2t-1}|T|^{2(1-t)} + \|T\|^{1-2t}|T|^{2t}\right)\widetilde{T}_t + \widetilde{T}_t\left(\|T\|^{2t-1}|T|^{2(1-t)} + \|T\|^{1-2t}|T|^{2t}\right)\right)}.\end{aligned}$$

Proof. Replacing $A = U|T|^{1-t}$ and $B = |T|^t$, in Theorem 2.3, and noting $|B^*|^2 = |B|^2 = |T|^{2t}$, $|A^*|^2 = U|T|^{1-t}|T|^{1-t}U^* = U|T|^{2(1-t)}U^* = |T^*|^{2(1-t)}$ (see [9, (1)]), $|A|^2 = |T|^{1-t}U^*U|T|^{1-t} = |T|^{2(1-t)}$, $AB = U|T| = T$, $BA = |T|^t U |T|^{1-t} = \widetilde{T}_t$.
Here, we also used the following identity

$$\left\| \|T\|^{2t-1}|T|^{2(1-t)} + \|T\|^{1-2t}|T|^{2t} \right\| = 2\|T\|. \tag{2.10}$$

In fact,

$$\begin{aligned} 2\|T\| &= 2\|U|T|\| \\ &= 2\left\|U|T|^{1-t}|T|^t\right\| \\ &= 2\left\|\left(\frac{\|T\|^t}{\|T\|^{1-t}}\right)^{\frac{1}{2}} U|T|^{1-t}\left(\frac{\|T\|^{1-t}}{\|T\|^t}\right)^{\frac{1}{2}}|T|^t\right\|. \end{aligned} \tag{2.11}$$

From [16], we also know that

$$2\|AXB\| \le \|A^*AX + XBB^*\|, \tag{2.12}$$

where $A, B, X \in \mathcal{B}(\mathcal{H})$. Replacing $A = \left(\frac{\|T\|^t}{\|T\|^{1-t}}\right)^{\frac{1}{2}} U|T|^{1-t}$, $X = I$, and $B = \left(\frac{\|T\|^{1-t}}{\|T\|^t}\right)^{\frac{1}{2}}|T|^t$ in (2.12), we get

$$\begin{aligned} 2\|T\| &= 2\left\|\left(\frac{\|T\|^t}{\|T\|^{1-t}}\right)^{\frac{1}{2}} U|T|^{1-t}\left(\frac{\|T\|^{1-t}}{\|T\|^t}\right)^{\frac{1}{2}}|T|^t\right\| \quad \text{(by (2.11))} \\ &\le \left\|\|T\|^{2t-1}|T|^{2(1-t)} + \|T\|^{1-2t}|T|^{2t}\right\|. \end{aligned}$$

On the other hand, by employing the triangle inequality for the usual operator norm, we have

$$\left\| \|T\|^{2t-1}|T|^{2(1-t)} + \|T\|^{1-2t}|T|^{2t} \right\| \le 2\|T\|. \tag{2.13}$$

Combining inequalities (2) and (2.13), we conclude (2.10). ∎

Remark 2.6. *By* (1.7)*, we infer that*

$$\begin{aligned}
&\omega\left(T\right)\\
&\le \frac{1}{2}\sqrt{\left\|T\right\|^2+\omega^2\left(\widetilde{T}_t\right)+\frac{1}{2}\omega\left(\left(\left\|T\right\|^{2t-1}\left|T\right|^{2(1-t)}+\left\|T\right\|^{1-2t}\left|T\right|^{2t}\right)\widetilde{T}_t+\widetilde{T}_t\left(\left\|T\right\|^{2t-1}\left|T\right|^{2(1-t)}+\left\|T\right\|^{1-2t}\left|T\right|^{2t}\right)\right)}\\
&\le \frac{1}{2}\sqrt{\left\|T\right\|^2+\omega^2\left(\widetilde{T}_t\right)+\omega\left(\widetilde{T}_t\right)\left\|\left\|T\right\|^{2t-1}\left|T\right|^{2(1-t)}+\left\|T\right\|^{1-2t}\left|T\right|^{2t}\right\|}\\
&= \frac{1}{2}\sqrt{\left\|T\right\|^2+\omega^2\left(\widetilde{T}_t\right)+2\omega\left(\widetilde{T}_t\right)\left\|T\right\|}\\
&= \frac{1}{2}\left(\left\|T\right\|+\omega\left(\widetilde{T}_t\right)\right).
\end{aligned}$$

This shows that Theorem 2.4 is an improvement of the inequality

$$\omega\left(T\right)\le\frac{1}{2}\left(\omega\left(\widetilde{T}_t\right)+\left\|T\right\|\right)$$

obtained by Abu Omar and Kittaneh [1, Theorem 3.2]*. Of course, the case $t=1/2$ also improves to the main result of* [23]*.*

In the following remark, we further emphasize the significance of Theorem 2.3.

Remark 2.7. *It follows from the inequality* (2.9) *that*

$$\begin{aligned}
&\omega\left(T\right)\\
&\le \frac{1}{2}\sqrt{\frac{1}{4}\left\|\left|T\right|^{2(1-t)}+\left|T\right|^{2t}\right\|^2+\omega^2\left(\widetilde{T}_t\right)+\frac{1}{2}\omega\left(\left(\left|T\right|^{2(1-t)}+\left|T\right|^{2t}\right)\widetilde{T}_t+\widetilde{T}_t\left(\left|T\right|^{2(1-t)}+\left|T\right|^{2t}\right)\right)}\\
&\le \frac{1}{2}\sqrt{\frac{1}{4}\left\|\left|T\right|^{2(1-t)}+\left|T\right|^{2t}\right\|^2+\omega^2\left(\widetilde{T}_t\right)+\omega\left(\widetilde{T}_t\right)\left\|\left|T\right|^{2(1-t)}+\left|T\right|^{2t}\right\|}\\
&= \frac{1}{4}\left\|\left|T\right|^{2(1-t)}+\left|T\right|^{2t}\right\|+\frac{1}{2}\omega\left(\widetilde{T}_t\right).
\end{aligned}$$

This indicates that our result improves [20, Corollary 2.11]*.*

Remark 2.8. *Combining the first inequality in* (1.1) *and Theorem 2.4, we conclude that if $\widetilde{T}_t=0$, then $1/2\ \left\|T\right\|=\omega\left(T\right)$.*

Data availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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