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GRADED PRÜFER DOMAINS HAVING NOETHERIAN HOMOGENEOUS SPECTRUM

DONG KYU KIM

ABSTRACT. In this paper, we investigate some equivalent conditions for a graded Prüfer domain to have Noetherian homogeneous spectrum. More precisely, when R is a graded Prüfer domain, we show that R has Noetherian homogeneous spectrum if and only if R satisfies graded property (##) and R satisfies the ascending chain condition on homogeneous prime ideals, if and only if each finitely generated homogeneous ideal of R has only finitely many minimal homogeneous prime ideals and R satisfies the ascending chain condition on homogeneous prime ideals, if and only if R is a graded RTP domain and R satisfies the ascending chain condition on homogeneous prime ideals.

1. Introduction

1.1. ***Graded rings.*** In this paper, we always assume that all monoids are torsion-free cancellative commutative monoid written additively. Hence Γ admits a total order compatible with its monoid operation [5, Corollary 3.4]. Let R be a commutative ring with identity and let Γ be a torsion-free cancellative monoid. Then R is said to be a Γ-*graded ring* if there exists a family $\{R_\alpha \mid \alpha \in \Gamma\}$ of additive abelian groups such that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ and $R_\alpha \cdot R_\beta \subseteq R_{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded ring. We define $\bigcup_{\alpha\in\Gamma} R_\alpha$ as the set of homogeneous elements of R and we denote by H the set of nonzero homogeneous elements of R. Then H is a multiplicative subset of R if R is an integral domain. In this case, the quotient ring R_H is called the *homogeneous quotient field* of R.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded ring and let I be an ideal of R. We say that I is a *homogeneous ideal* of R if $I = \bigoplus_{\alpha\in\Gamma}(I \cap R_\alpha)$ (or equivalently, I has a set of homogeneous generators). It is well known that an arbitrary sum, an arbitrary intersection and a finite product of homogeneous ideals of R are also homogeneous. We say that I is a *homogeneous prime ideal* of R if it is both homogeneous and prime; and I is a *maximal homogeneous ideal* of R if it is maximal among proper homogeneous ideals of R. We denote by h-Spec(R) the set of homogeneous prime ideals of R and h-Max(R) the set of maximal homogeneous ideals of R. It is well known that h-Max(R) is a nonempty subset of h-Spec(R).

Let Γ be a torsion-free cancellative monoid with quotient group $\langle\Gamma\rangle$ and let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Let H be the set of nonzero homogeneous elements of R and let R_H be the homogeneous quotient field of R. Then $R_H = \bigoplus_{\alpha\in\langle\Gamma\rangle}(R_H)_\alpha$, where $(R_H)_\alpha = \left\{\frac{f}{g} \,\middle|\, f \in R_\beta \text{ and } g \in R_\gamma \text{ with } \beta - \gamma = \alpha\right\}$ for each $\alpha \in \langle\Gamma\rangle$. Hence R_H is a graded integral domain. Let T be an overring of

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R contained in R_H. Then T is said to be a *homogeneous overring* of R if $T = \bigoplus_{\alpha \in \langle \Gamma \rangle}(T \cap (R_H)_\alpha)$ [1, page 198]. A fractional ideal I of R is said to be a *homogeneous fractional ideal* of R if there exists an element $h \in H$ such that hI is a homogeneous ideal of R [1, page 198]. We denote by $\mathbf{HF}(R)$ the set of nonzero homogeneous fractional ideals of R. Then $R = R_H$ if and only if $\mathbf{HF}(R) = \{R\}$. To avoid this case, we assume that $R \neq R_H$ unless otherwise mentioned in this paper. In this case, R has a nonzero nonunit homogeneous element.

1.2. ***Star-operations.*** Let R be an integral domain with quotient field K and let $\mathbf{F}(R)$ be the set of nonzero fractional ideals of R. A *star-operation* on R is a mapping $I \mapsto I_*$ of $\mathbf{F}(R)$ into $\mathbf{F}(R)$ satisfying the following three conditions for all $0 \neq x \in K$ and $I, J \in \mathbf{F}(R)$:

(1) $(x)_* = (x)$ and $(xI)_* = xI_*$.
(2) $I \subseteq I_*$; and if $I \subseteq J$, then $I_* \subseteq J_*$.
(3) $(I_*)_* = I_*$.

The map $v : \mathbf{F}(R) \to \mathbf{F}(R)$ given by $I \mapsto I_v := (I^{-1})^{-1}$, where $I^{-1} = (R : I) = \{x \in K \,|\, xI \subseteq R\}$, is a star-operation on R and we call it the *v-operation* on R.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded integral domain. Let H be the set of nonzero homogeneous elements of R and let R_H be the homogeneous quotient field of R. Then for each $I, J \in \mathbf{HF}(R)$, $(I :_{R_H} J) = (I : J) \in \mathbf{HF}(R)$ [1, Proposition 2.5]. Hence $I^{-1}, I_v \in \mathbf{HF}(R)$ for all $I \in \mathbf{HF}(R)$.

1.3. ***Main results.*** In [18], the authors studied the rings having Noetherian spectrum. Let R be a commutative ring with identity and let $\mathrm{Spec}(R)$ be the set of prime ideals of R. We say that R has *Noetherian spectrum* (or $\mathrm{Spec}(R)$ is Noetherian) if $\mathrm{Spec}(R)$ with the Zariski topology satisfies the descending chain condition on closed subsets (or equivalently, R satisfies the ascending chain condition on radical ideals). Hence each Noetherian ring has Noetherian spectrum. (Recall that R is said to be a *Noetherian ring* if it satisfies the ascending chain condition on ideals.) It is well known that R has Noetherian spectrum if and only if for each ideal I of R, there exists a finitely generated ideal J of R such that $I \subseteq \sqrt{J} \subseteq \sqrt{I}$ [18, Proposition 2.1], if and only if every (prime) ideal of R is the radical of a finitely generated ideal [18, Corollary 2.4], if and only if R satisfies the ascending chain condition on prime ideals and each ideal of R has only finitely many minimal prime ideals [11, Theorem 88 and Exercise 25, page 65].

In [9], the authors defined the concept of Noetherian homogeneous spectrum, which is the concept corresponding to graded rings of that of Noetherian spectrum, and studied whether Noetherian homogeneous spectrum in graded rings has Noetherian spectrum. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded ring. We say that R has *Noetherian homogeneous spectrum* (or h-$\mathrm{Spec}(R)$ is Noetherian) if h-$\mathrm{Spec}(R)$ with the Zariski topology satisfies the descending chain condition on closed subsets. Then R has Noetherian homogeneous spectrum if and only if for each homogeneous ideal I of R, there exists a finitely generated (homogeneous) ideal J of R such that $I \subseteq \sqrt{J} \subseteq \sqrt{I}$ [9, page 1575]. Hence each graded Noetherian ring has Noetherian homogeneous spectrum. (Recall that R is called a *graded Noetherian ring* if each homogeneous ideal of R is finitely generated.)

An integral domain R is said to be a *Prüfer domain* if every nonzero finitely generated ideal of R is invertible. In [8], the authors defined the concept of the radical trace property and studied some properties of Prüfer domains having Noetherian spectrum. Recall that R is said to satisfy the *radical trace property* (RTP domain) if for each nonzero ideal I of R, II^{-1} is a radical ideal of R. More precisely, when R is a Prüfer domain having Noetherian spectrum and I is an ideal of R such that I^{-1} is a ring, $I^{-1} = (I : I)$ if and only if $I = \sqrt{I}$, if and only if I is contained in only maximal ideals of $(I : I)$ [8, Theorem 2.5]. Also, when R is a Prüfer domain satisfying the ascending chain condition on prime ideals, R is an RTP domain if and only if R has Noetherian spectrum, if and only if R satisfies property (##) [8, Theorem 2.7]. (Recall from [7] that R *satisfies property* (#) if for any two distinct subsets Δ_1 and Δ_2 of $\mathrm{Max}(R)$, where $\mathrm{Max}(R)$ is the set of maximal ideals of R, $\bigcap_{M\in\Delta_1} R_M \neq \bigcap_{M\in\Delta_2} R_M$; and R satisfies *property* (##) if each overring of R satisfies property (#).)

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded integral domain. Let H be the set of nonzero homogeneous elements of R and let R_H be the set of nonzero homogeneous elements of R. We say that R is a *graded valuation domain* if for any nonzero homogeneous element $x \in R_H$, $x \in R$ or $x^{-1} \in R$; and R is a *graded Prüfer domain* if every nonzero finitely generated homogeneous ideal of R is invertible. In [2, Theorem 2.2], the authors investigated some equivalent conditions for a graded integral domain to be a graded valuation domain. More precisely, R is a graded valuation domain if and only if the set of (principal) homogeneous ideals of R is linearly ordered under the inclusion. In [3], the authors studied some properties of graded Prüfer domains. In detail, R is a graded Prüfer domain if and only if $R_{H\setminus P}$ is a graded valuation domain for all $P \in \text{h-Spec}(R)$, if and only if $R_{H\setminus M}$ is a graded valuation domain for all $M \in \text{h-Max}(R)$ [3, Theorem 3.1]. In [3, Theorem 3.5], the authors also examined some properties of homogeneous overrings of graded Prüfer domains. These results are useful in this paper. We will continue to use [2, Theorem 2.2] and [3, Theorem 3.1] in this paper without mentioning from now on.

In this paper, we examine some equivalent conditions for a graded Prüfer domain to have Noetherian homogeneous spectrum. To do this, we apply the results in [8] and [18], including the results mentioned above, to graded rings.

This paper consists of three sections including the introduction. In Section 2, we investigate when the equality $I^{-1} = (I : I)$ holds if R is a graded Prüfer domain and I is a nonzero proper homogeneous ideal of R. To do this, we investigate some equivalent conditions for a graded ring to have Noetherian homogeneous spectrum. More precisely, we show that R has Noetherian homogeneous spectrum if and only if every homogeneous prime ideal of R is the radical of a finitely generated homogeneous ideal, if and only if every homogeneous ideal of R is the radical of a finitely generated homogeneous ideal, if and only if R satisfies the ascending chain condition on homogeneous radical ideals, if and only if R satisfies the ascending chain condition on homogeneous prime ideals and each homogeneous ideal of R has only finitely many minimal homogeneous prime ideals (Theorem 2.5). As the main result of this section, when R is a graded Prüfer domain having Noetherian homogeneous spectrum and I is a nonzero proper homogeneous ideal of R such that I^{-1} is a ring, we show that $I^{-1} = (I : I)$ if and only if $I = \sqrt{I}$, if and only if I is contained in only maximal homogeneous ideals of $(I : I)$ (Theorem 2.9). In Section 3, we investigate some equivalent conditions for a graded Prüfer domain to have Noetherian homogeneous spectrum. As the main result of this paper, when R is a graded Prüfer domain, we show that R has Noetherian homogeneous spectrum if and only if R satisfies graded property (##) and R satisfies the ascending chain condition on homogeneous prime ideals, if and only if each finitely generated homogeneous ideal of R has only finitely many minimal homogeneous prime ideals and R satisfies the ascending chain condition on homogeneous prime ideals, if and only if R is a graded RTP domain and R satisfies the ascending chain condition on homogeneous prime ideals (Theorem 3.5).

2. When the equality $I^{-1} = (I : I)$ holds?

From now on, we always assume that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a Γ-graded ring, H is the set of nonzero homogeneous elements of R and R_H is the homogeneous quotient field of R.

In this section, we investigate when the equality $I^{-1} = (I : I)$ holds if R is a graded Prüfer domain and I is a nonzero proper homogeneous ideal of R such that I^{-1} is a ring. We first show that this equality does not hold in general but this is true in some cases. To do this, we require the following concept.

In [12], the author generalized the concept of primary ideals to graded rings and studied some properties on it. Let R be a graded ring and let I be a proper homogeneous ideal of R. We say that I is a *graded primary ideal* of R if for any $a, b \in H$ with $ab \in I$, $a \in I$ or $b \in \sqrt{I}$; and I is a *graded P-primary ideal* of R if I is a graded primary ideal of R such that $\bigoplus_{\alpha\in\Gamma} \left(\sqrt{I} \cap R_\alpha\right) = P$. Note that $\sqrt{I}$ is the intersection of the set of minimal prime ideals of I. Then $\sqrt{I}$ is the intersection of the set of minimal homogeneous prime ideals of R. Hence $\sqrt{I}$ is a homogeneous ideal of R. Note that each graded primary ideal of R is primary [17, page 125, Lemma 14].

Remark 2.1. (1) In [8, Example 2.6], the authors constructed a Prüfer domain D and an ideal I of D such that I^{-1} is a ring but $I^{-1} \neq (I : I)$. Let G be a torsion-free abelian group and let $R = D[G]$ be the group ring of G over D. Then R is a graded Prüfer domain [16, Example 2.16]. Note that $J[G]^{-1} = J^{-1}[G]$ and $(J[G] : J[G]) = (J : J)[G]$ for each nonzero ideal J of D [4, Lemma 2.3]. Thus $I[G]$ is a homogeneous ideal of R such that $I[G]^{-1}$ is a ring but $I[G]^{-1} \neq (I[G] : I[G])$.

(2) Let R be a graded Prüfer domain and let Q be a graded primary ideal of R. Then Q^{-1} is a ring if and only if $Q^{-1} = (Q : Q)$ [14, Lemma 3.4].

We first show that the equality holds if $I = \sqrt{I}$. To do this, we review the concept of minimal homogeneous prime ideals.

Let R be a graded ring. Let I be a proper homogeneous ideal of R and let P be a homogeneous prime ideal of R containing I. If there does not exist a homogeneous prime ideal of R properly between I and P, then P is called a *minimal homogeneous prime ideal* of I. It is well known that if Q is a minimal prime ideal of I, then Q is a homogeneous (prime) ideal of R. Hence Q is a minimal prime ideal of I if and only if Q is a minimal homogeneous prime ideal of I.

Lemma 2.2. *Let R be a graded Prüfer domain and let I be a homogeneous radical ideal of R. Then the following assertions are equivalent.*

(1) $I^{-1} = (I : I)$.
(2) I^{-1} *is a ring.*

Proof. We may assume that I is a nonzero proper homogeneous ideal of R.

(1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Suppose that I^{-1} is a ring. Let $\{P_\delta \,|\, \delta \in \Delta\}$ be the set of minimal homogeneous prime ideals of I and let $\{M_\beta \,|\, \beta \in \mathscr{B}\}$ be the set of maximal homogeneous ideals of R not containing I. Then $I = \bigcap_{\delta\in\Delta} P_\delta$ and $I^{-1} = \left(\bigcap_{\delta\in\Delta} R_{H\setminus P_\delta}\right) \cap \left(\bigcap_{\beta\in\mathscr{B}} R_{H\setminus M_\beta}\right)$ [14, Theorem 3.2]. Let $x \in I^{-1}$ and $y \in I$. Then $xy \in P_\delta R_{H\setminus P_\delta} \cap R = P_\delta$ for all $\delta \in \Delta$. Hence $xy \in I$ and $x \in (I : I)$. Thus $I^{-1} = (I : I)$. □

The following example shows that there exists a homogeneous radical ideal I of R such that I^{-1} is not a ring.

Example 2.3. Let R be a graded Dedekind domain and let $M \in$ h-Max(R). (Recall that R is said to be a *graded Dedekind domain* if each nonzero homogeneous ideal of R is invertible.) Then R is a graded Prüfer domain and M is invertible. Hence $M^{-1} \neq (M : M)$. Thus by Remark 2.1(2), M^{-1} is not a ring.

The goal of this section is to show that $I^{-1} = (I : I)$ if and only if $I = \sqrt{I}$ when R has Noetherian homogenous spectrum and I is a nonzero homogeneous ideal of R such that I^{-1} is a ring. We first investigate some equivalent conditions for a graded ring to have Noetherian homogeneous spectrum. To do this, we require the following lemma.

Lemma 2.4. *Let R be a graded ring. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of homogeneous radical ideals of R and let $\{P_1, \ldots, P_k\}$ be a finite subset of* h-Spec(R) *such that $I_1 = \bigcap_{i=1}^k P_i$. If the chain $\sqrt{I_1 + P_i} \subseteq \sqrt{I_2 + P_i} \subseteq \cdots$ is stationary for all $i \in \{1, \ldots, k\}$, then the chain $I_1 \subseteq I_2 \subseteq \cdots$ is stationary.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of homogeneous radical ideals of R and let $\{P_1, \ldots, P_k\}$ be a finite subset of h-Spec(R) such that $I_1 = \bigcap_{i=1}^k P_i$. By the hypothesis, there exists an integer $n \geq 1$ such that $\sqrt{I_n + P_i} = \sqrt{I_m + P_i}$ for all $m \geq n$ and $i \in \{1, \ldots, k\}$. Let $m \geq n$ be an integer and let $x \in I_m$. Then $x \in \sqrt{I_n + P_i}$ for all $i \in \{1, \ldots, k\}$. Hence there exists an integer $\ell \geq 1$ such that $x^\ell \in I_n + P_i$ for all $i \in \{1, \ldots, k\}$. For each $i \in \{1, \ldots, k\}$, there exist $a_i \in I_n$ and $b_i \in P_i$ such that $x^\ell = a_i + b_i$. Then we have

$$\prod_{i=1}^k (x^\ell - a_i) = \prod_{i=1}^k b_i \in \bigcap_{i=1}^k P_i = I_1 \subseteq I_n.$$

Since $a_i \in I_n$ for all $i = 1, \ldots, k$, $x \in \sqrt{I_n} = I_n$. Hence $I_m = I_n$ for all $m \geq n$. Thus the chain $I_1 \subseteq I_2 \subseteq \cdots$ is stationary. □

Theorem 2.5. *Let R be a graded ring. Then the following statements are equivalent.*

(1) *R has Noetherian homogeneous spectrum.*
(2) *Every homogeneous prime ideal of R is the radical of a finitely generated homogeneous ideal.*
(3) *Every homogeneous ideal of R is the radical of a finitely generated homogeneous ideal.*
(4) *R satisfies the ascending chain condition on homogeneous radical ideals.*
(5) *R satisfies the ascending chain condition on homogeneous prime ideals and each homogeneous ideal of R has only finitely many minimal homogeneous prime ideals.*

Proof. (1) $\Rightarrow$ (2) It follows from that $\sqrt{P} = P$ for all $P \in$ h-Spec(R).

(2) $\Rightarrow$ (3) Suppose to the contrary that there exists a homogeneous ideal of R which is not the radical of a finitely generated homogeneous ideal. Let $\mathscr{A}$ be the set of homogeneous ideals of R which are

not the radical of a finitely generated homogeneous ideal. Then $\mathscr{A}$ is a nonempty set. Hence $\mathscr{A}$ has a maximal element P and $P \in \text{h-Spec}(R)$ [18, Proposition 2.3]. This contradicts the hypothesis. Thus every homogeneous ideal of R is the radical of a finitely generated homogeneous ideal.

(3) $\Rightarrow$ (1) This is obvious.

(3) $\Rightarrow$ (4) Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of homogeneous radical ideals of R and let $I = \bigcup_{n=1}^{\infty} I_n$. By the hypothesis, there exists a finitely generated homogeneous ideal J of R such that $I = \sqrt{J}$. Then there exists an integer $m \geq 1$ such that $J \subseteq I_m$. Since I_m is a radical ideal of R, $I = I_m$. Hence $I_k = I_m$ for all $k \geq m$. Thus the chain $I_1 \subseteq I_2 \subseteq \cdots$ is stationary.

(4) $\Rightarrow$ (2) Suppose to the contrary that there exists a homogeneous prime ideal P of R which is not the radical of a finitely generated homogeneous ideal. Then $\sqrt{(a)} \subsetneq P$ for all $a \in P \cap H$. For $n \geq 2$, assume that there exist $a_1, \dots, a_{n-1} \in P \cap H$ such that $\sqrt{(a_1)} \subsetneq \cdots \subsetneq \sqrt{(a_1, \dots, a_{n-1})} \subsetneq P$. Since $\sqrt{(a_1, \dots, a_{n-1})}$ is a homogeneous ideal of R, there exists an element $a_n \in P \cap H$ such that $a_n \notin \sqrt{(a_1, \dots, a_{n-1})}$. Then $\sqrt{(a_1)} \subsetneq \cdots \subsetneq \sqrt{(a_1, \dots, a_n)} \subsetneq P$ is a chain of homogeneous radical ideals of R. By the induction, there exists a chain $\left\{ \sqrt{(a_1, \dots, a_n)} \,\middle|\, n \in \mathbb{N} \right\}$ of homogeneous radical ideals of R. This contradicts the hypothesis. Thus every homogeneous prime ideal of R is the radical of a finitely generated homogeneous ideal.

(4) $\Rightarrow$ (5) This direction comes from [19, Corollary 1.2].

(5) $\Rightarrow$ (4) Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of homogeneous radical ideals of R. Then by the hypothesis, there exist $P_1, \dots, P_k \in \text{h-Spec}(R)$ such that $I_1 = \bigcap_{i=1}^{k} P_i$. Hence by Lemma 2.4, it suffices to show that the chain $P_i = \sqrt{I_1 + P_i} \subseteq \sqrt{I_2 + P_i} \subseteq \cdots$ is stationary for all $i \in \{1, \dots, k\}$. Suppose to the contrary that there exists an integer $i \in \{1, \dots, k\}$ such that the chain $P_i = \sqrt{I_1 + P_i} \subseteq \sqrt{I_2 + P_i} \subseteq \cdots$ is not stationary. By choosing an infinite subsequence, we may assume that $\sqrt{I_n + P_i} \subsetneq \sqrt{I_{n+1} + P_i}$ for all $n \in \mathbb{N}$. Let $M_1 = P_i$. For $n \geq 1$, assume that there exist $M_1, \dots, M_n \in \text{h-Spec}(R)$ such that $M_1 \subsetneq \cdots \subsetneq M_n \subsetneq \sqrt{I_{n+1} + M_n} \subsetneq \cdots$. By the hypothesis, $I_{n+1} + M_n$ has only finitely many minimal homogeneous prime ideals. Then by Lemma 2.4, there exists an element $M_{n+1} \in \text{h-Spec}(R)$ with $M_n \subsetneq M_{n+1}$ such that the chain $M_{n+1} = \sqrt{I_{n+1} + M_{n+1}} \subseteq \sqrt{I_{n+2} + M_{n+1}} \subseteq \cdots$ is not stationary. By the induction, there exists a chain $\{M_n \,|\, n \in \mathbb{N}\}$ of homogeneous prime ideals of R. This contradicts the hypothesis. Thus the chain $P_i = \sqrt{I_1 + P_i} \subseteq \sqrt{I_2 + P_i} \subseteq \cdots$ is stationary for all $i \in \{1, \dots, k\}$. ☐

Lemma 2.6. *Let R be a graded Prüfer domain. Let I be a nonzero proper homogeneous ideal of R and let P be a minimal homogeneous prime ideal of I. If there exists a finitely generated homogeneous ideal J of R such that $I \subseteq J \subseteq P$, then I^{-1} is not a ring.*

Proof. Suppose to the contrary that I^{-1} is a ring. Then $I^{-1} \subseteq R_{H \setminus P}$ [14, Theorem 3.2]. Since R is a graded Prüfer domain, J is invertible. Hence $1 \in JJ^{-1} \subseteq PI^{-1} \subseteq PR_{H \setminus P}$, which is absurd. Thus I^{-1} is not a ring. ☐

Let R be a graded ring and let I be a homogeneous ideal of R. Then $R/I = \bigoplus_{\alpha \in \Gamma} (R/I)_\alpha$, where $(R/I)_\alpha = \{r_\alpha + I \,|\, r_\alpha \in R_\alpha\}$ for each $\alpha \in \Gamma$. Hence R/I can be regarded as a Γ-graded ring. Let $Z(R/I)$ be the set of zero-divisors of R/I and let $\{P_\delta / I \,|\, \delta \in \Delta\}$ be the set of homogeneous prime ideals of R/I not meeting $(R/I) \setminus Z(R/I)$. Then $(R/I) \setminus Z(R/I)$ is a saturated multiplicative subset of R/I.

Hence $\bigcup_{\delta\in\Delta} P_\delta/I \subseteq Z(R/I)$. Suppose that R is a graded valuation domain and let $P = \bigcup_{\delta\in\Delta} P_\delta$. Then $P \in \text{h-Spec}(R)$ such that $x+I \notin Z(R/I)$ for all $x \in H \setminus P$.

Lemma 2.7. *Let R be a graded valuation domain and let I be a nonzero proper homogeneous ideal of R. Let $\{P_\delta/I \mid \delta \in \Delta\}$ be the set of homogeneous prime ideals of R/I not meeting $(R/I)\setminus Z(R/I)$ and let $P = \bigcup_{\delta\in\Delta} P_\delta$. Then $(I:I) = R_{H\setminus P}$.*

Proof. Let $x \in (I:I)$ be nonzero homogeneous. Since R is a graded valuation domain, we may assume that $x^{-1} \in R$. Then $I = x^{-1}I$. Suppose to the contrary that $x^{-1} \in P$. Since $P/I \subseteq Z(R/I)$, there exists an element $y \in R\setminus I$ such that $x^{-1}y \in I = x^{-1}I$, which is absurd. Then $x^{-1} \in H\setminus P$ and $x \in R_{H\setminus P}$. Hence $(I:I) \subseteq R_{H\setminus P}$. Conversely, let $a \in R_{H\setminus P}$ be nonzero homogeneous. Since $R \subseteq (I:I)$, we may assume that $a^{-1} \in H\setminus P$. Then $P \subsetneq (a^{-1})$. Hence there exists a homogeneous ideal J of R such that $I = a^{-1}J$. Since $a^{-1} \in H\setminus P$, $a^{-1}+I$ is not a zero-divisor of R/I. Then $I = J$ and $I = a^{-1}I$. Hence $a \in (I:I)$. Thus $(I:I) = R_{H\setminus P}$. □

Let R be a graded integral domain and let I be a homogeneous ideal of R. Then $I = \bigcap_{M\in\text{h-Max}(R)} IR_{H\setminus M}$ [3, Corollary 2.5]. Since this result is often used in this paper, we will use this fact without mentioning from now on. The following lemma plays an important role in proving the main result of this section.

Lemma 2.8. *Let R be a graded Prüfer domain having Noetherian homogeneous spectrum and let I be a homogeneous ideal of R such that I^{-1} is a ring. If $I \subsetneq \sqrt{I}$, then there exist homogeneous prime ideals P and Q of R such that $I \subseteq P \subsetneq Q$, $I^{-1} \not\subseteq R_{H\setminus Q}$ and $(I:I) \subseteq R_{H\setminus Q}$. Hence $I^{-1} \neq (I:I)$.*

Proof. Suppose that $I \subsetneq \sqrt{I}$. Then there exists an element $M \in \text{h-Max}(R)$ such that $IR_{H\setminus M}$ is not a radical ideal of $R_{H\setminus M}$. Since $R_{H\setminus M}$ is a graded valuation domain, there exists an element $P \in \text{h-Spec}(R)$ such that $\sqrt{IR_{H\setminus M}} = PR_{H\setminus M}$. Hence P is a minimal homogeneous prime ideal of I.

Suppose to the contrary that $IR_{H\setminus P} \subsetneq PR_{H\setminus P}$. Since $R_{H\setminus P}$ is a graded valuation domain, there exists an element $p \in P\cap H$ such that $IR_{H\setminus P} \subsetneq (p)R_{H\setminus P} \subseteq PR_{H\setminus P}$. By Theorem 2.5, there exists a finitely generated homogeneous ideal A of R such that $P = \sqrt{A}$. Let $J = A+(p)$ and take an element $N \in \text{h-Max}(R)$. Then $JR_{H\setminus N} = R_{H\setminus N}$ if $P \not\subseteq N$. Assume that $P \subseteq N$. Since $IR_{H\setminus P} \subsetneq (p)R_{H\setminus P}$, $Ip^{-1} \subseteq PR_{H\setminus P} = PR_{H\setminus N}$ [2, Theorem 2.3(6)]. Then $IR_{H\setminus N} \subseteq (p)R_{H\setminus N} \subseteq JR_{H\setminus N}$ and $I \subseteq J \subseteq P$. This contradicts Lemma 2.6. Hence $IR_{H\setminus P} = PR_{H\setminus P}$.

Let $\{Q_\delta R_{H\setminus M}/IR_{H\setminus M} \mid \delta \in \Delta\}$ be the set of homogeneous prime ideals of $R_{H\setminus M}/IR_{H\setminus M}$ not meeting $(R_{H\setminus M}/IR_{H\setminus M})\setminus Z(R_{H\setminus M}/IR_{H\setminus M})$. Since $R_{H\setminus M}$ is a graded valuation domain, there exists an element $Q \in \text{h-Spec}(R)$ such that $\bigcup_{\delta\in\Delta} Q_\delta R_{H\setminus M} = QR_{H\setminus M}$. Then $PR_{H\setminus M} = \sqrt{IR_{H\setminus M}} \subseteq QR_{H\setminus M}$. Let $x \in PR_{H\setminus M}$ be homogeneous such that $x \notin IR_{H\setminus M}$ and let $H(R_{H\setminus M})$ be the set of nonzero homogeneous elements of $R_{H\setminus M}$. Then we have

$$PR_{H\setminus M} = PR_{H\setminus P} = IR_{H\setminus P} = (IR_{H\setminus M})_{H(R_{H\setminus M})\setminus PR_{H\setminus M}},$$

where the first equality follows from [2, Theorem 2.3(6)]. Hence there exists an element $y \in H(R_{H\setminus M})\setminus PR_{H\setminus M}$ such that $xy \in IR_{H\setminus M}$. Since $y+IR_{H\setminus M}$ is a zero-divisor of $R_{H\setminus M}/IR_{H\setminus M}$, $y \in QR_{H\setminus M}$. Then $PR_{H\setminus M} \subsetneq QR_{H\setminus M}$. Hence $P \subsetneq Q$. By Theorem 2.5, there exists a finitely generated homogeneous ideal

B of R such that $Q = \sqrt{B}$. Then $P \subseteq B$. Since B is invertible, $1 \in BB^{-1} \subseteq QI^{-1}$. Hence $I^{-1} \nsubseteq R_{H \backslash Q}$. By Lemma 2.7, we have

$$(I : I) \subseteq (IR_{H\backslash M} : IR_{H\backslash M}) = (R_{H\backslash M})_{H(R_{H\backslash M})\backslash QR_{H\backslash M}} = R_{H\backslash Q}.$$

Thus there exist homogeneous prime ideals P and Q of R such that $I \subseteq P \subsetneq Q$, $I^{-1} \nsubseteq R_{H\backslash Q}$ and $(I : I) \subseteq R_{H\backslash Q}$. □

We are now ready to prove the main result of this section.

Theorem 2.9. *Let R be a graded Prüfer domain having Noetherian homogeneous spectrum and let I be a nonzero proper homogeneous ideal of R such that I^{-1} is a ring. Then the following conditions are equivalent.*

(1) $I^{-1} = (I : I)$.
(2) $I = \sqrt{I}$.
(3) *I is contained in only maximal homogeneous ideals of $(I : I)$.*

Proof. (1) ⇒ (3) Suppose that $I^{-1} = (I : I)$. Let M be a homogeneous prime ideal of I^{-1} containing I. Then $M \cap R$ is a homogeneous prime ideal of R containing I such that $M = (M \cap R)I^{-1}$ [3, Theorem 3.5(4)]. Let P be a minimal homogeneous prime ideal of I contained in $M \cap R$. If PI^{-1} is not a maximal homogeneous ideal of I^{-1}, then there exists a homogeneous prime ideal Q of R with $P \subsetneq Q$ such that $PI^{-1} \subsetneq QI^{-1} \in \text{h-Spec}(I^{-1})$ [3, Theorem 3.5(5)]. By Theorem 2.5, there exists a finitely generated homogeneous ideal A of R such that $Q = \sqrt{A}$. Then $P \subseteq A$. Since A is invertible, $1 \in AA^{-1} \subseteq QI^{-1}$, which is absurd. Hence PI^{-1} is a maximal homogeneous ideal of I^{-1}. Since $PI^{-1} \subseteq M$, $M = PI^{-1} \in \text{h-Max}(I^{-1})$. Thus I is contained in only maximal homogeneous ideals of I^{-1}.

(3) ⇒ (2) Suppose to the contrary that $I \subsetneq \sqrt{I}$. Then by Lemma 2.8, there exist homogeneous prime ideals P and Q of R such that $I \subseteq P \subsetneq Q$ and $(I : I) \subseteq R_{H\backslash Q}$. Hence $PR_{H\backslash Q} \cap (I : I)$ is a homogeneous prime ideal of $(I : I)$ containing I and properly contained in $QR_{H\backslash Q} \cap (I : I)$. This contradicts the hypothesis. Thus $I = \sqrt{I}$.

(2) ⇒ (1) It follows immediately from Lemma 2.2. □

Remark 2.10. (1) Let R be a graded valuation domain. Then I^{-1} is a ring if and only if I is a noninvertible homogeneous prime ideal of R [14, Lemma 3.5]. Hence by Theorem 2.9, I^{-1} is a ring if and only if $I^{-1} = (I : I)$.

(2) Let R be a graded Prüfer domain and let Q be a graded P-primary ideal of R. Then by Remark 2.1(2), Q^{-1} is a ring if and only if $Q^{-1} = (Q : Q)$. Hence by Theorem 2.9, Q^{-1} is a ring if and only if $Q = P$ when R has Noetherian homogeneous spectrum.

(3) Let R be a graded Prüfer domain and let P be a homogeneous prime ideal of R such that $P^{-1} = R$ and $P \neq P^2$. Then $P^{-2} = R = (P^2 : P^2)$ and $P^2 \neq \sqrt{P^2}$.

The following example shows that the condition that R has Noetherian homogeneous spectrum in Theorem 2.9 is essential.

Example 2.11. Let D be the ring of entire function. Then D is a Bézout domain and there exists a maximal ideal M of D such that $P = \bigcap_{n=1}^{\infty} M^n \subsetneq M$ and $P^{-1} = D$ [10, Example 3.12]. Hence $M^{-1} = D$ and $M \neq M^2$. Let G be a torsion-free abelian group and let $R = D[G]$ be the group ring of G over D. Then R is a graded Prüfer domain [16, Example 2.16]. Let $Q = M[G]$. Then Q is a maximal homogeneous ideal of R such that $Q^{-1} = M^{-1}[G] = R$ and $Q \neq Q^2$. Hence by Remark 2.10(3), $Q^{-2} = R = (Q^2 : Q^2)$ and $Q^2 \neq \sqrt{Q^2}$. Thus the condition that R has Noetherian homogeneous spectrum in Theorem 2.9 is essential.

3. Graded Prüfer domains having Noetherian homogeneous spectrum

In this section, we investigate some equivalent conditions for a graded Prüfer domain to have Noetherian homogeneous spectrum. To do this, we require several lemmas.

Lemma 3.1. *Let R be a graded Prüfer domain and let A be a finitely generated homogeneous ideal of R. If A has only finitely many minimal homogeneous prime ideals $P_1, \dots, P_k$, then there exists an element $x_i \in P_i \cap H$ such that P_i is the radical of $A + (x_i)$ for each $i \in \{1, \dots, k\}$.*

Proof. Take an index $i \in \{1, \dots, k\}$. Without loss of generality, we may assume that $i = 1$. Since R is a graded Prüfer domain, $P_1 + P_j = R$ for all $j \in \{2, \dots, k\}$. Then $P_1 + \prod_{j=2}^{n} P_j = R$. Hence there exist homogeneous $x_1 \in P_1$ and $y_1 \in \prod_{j=2}^{n} P_j$ such that $x_1 + y_1 = 1$. Since $P_j + (x_1) = R$ for all $j \in \{2, \dots, k\}$, P_1 is the minimal homogeneous prime ideal of $A + (x_1)$. Thus $P_1 = \sqrt{A + (x_1)}$. □

Lemma 3.2. *Let R be a graded Prüfer domain and let P be a nonzero homogeneous prime ideal of R. Suppose that P is both the radical of a finitely generated homogeneous ideal and contained in only finitely many maximal homogeneous ideals. Then there exists an element $p \in P \cap H$ such that P is a minimal homogeneous prime ideal of (p). In this case, there exists an element $x \in P \cap H$ such that $1 - x$ belongs to each maximal homogeneous ideal of R containing (p) and not containing P.*

Proof. Let $A = (a_1, \dots, a_n)$ be a finitely generated homogeneous ideal of R such that $P = \sqrt{A}$. Then there exists an index $i \in \{1, \dots, n\}$ such that P is a minimal homogeneous prime ideal of (a_i) [7, Lemma 4]. Without loss of generality, we may assume that $i = 1$. Let $\{M_1, \dots, M_k\}$ be the set of maximal homogeneous ideals of R containing P. Take an index $i \in \{1, \dots, k\}$ and let $H(R_{H \setminus M_i})$ be the set of nonzero homogeneous elements of $R_{H \setminus M_i}$. Then $R_{H \setminus M_i}$ is a graded valuation domain such that $(R_{H \setminus M_i})_{H(R_{H \setminus M_i}) \setminus PR_{H \setminus M_i}} = R_{H \setminus P}$. Suppose to the contrary that there exists an index $j \in \{2, \dots, n\}$ such that a_j^m divides a_1 in $R_{H \setminus M_i}$ for all $m \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} (a_j^n) R_{H \setminus P} \in$ h-Spec$(R_{H \setminus P})$ such that $(a_1) R_{H \setminus P} \subseteq \bigcap_{n=1}^{\infty} (a_j^n) R_{H \setminus P} \subsetneq PR_{H \setminus P}$ [14, Lemma 2.1(1)]. This contradicts that P is a minimal homogeneous prime ideal of (a_1). Note that $P = \sqrt{(a_1, a_2^m, \dots, a_n^m)}$ for all $m \in \mathbb{N}$. Hence we may assume that a_1 divides a_i in $R_{H \setminus M_j}$ for all $i \in \{2, \dots, n\}$ and $j \in \{1, \dots, k\}$. Since A is invertible, there exists a homogeneous ideal B of R such that $(a_1) = AB$. For each $i \in \{1, \dots, k\}$, we have

$$(a_1) R_{H \setminus M_i} = AR_{H \setminus M_i} BR_{H \setminus M_i} = (a_1) R_{H \setminus M_i} BR_{H \setminus M_i}.$$

Then $BR_{H \setminus M_i} = R_{H \setminus M_i}$ for all $i = 1, \dots, k$. Hence $B \not\subseteq M_i$ for all $i = 1, \dots, k$. Since $\{M_1, \dots, M_k\}$ is the set of maximal homogeneous ideals of R containing P, $A + B = R$. Then there exist homogeneous

$x \in A$ and $y \in B$ such that $x+y=1$. Let M be a maximal homogeneous ideal of R containing (a_1) and not containing P. Then $A \not\subseteq M$ and $AB = (a_1) \subseteq M$. Hence $B \subseteq M$. Thus $1-x = y \in B \subseteq M$. □

Although we obtain the following result in a similar way to [6, Lemma 8], we insert the proof for the sake of completeness.

Lemma 3.3. *Let R be a graded ring and let $\{M_\delta \mid \delta \in \Delta\}$ be the set of maximal homogeneous ideals of R. If there exists an element $m_\delta \in M_\delta \cap H$ such that $1 - m_\delta \in \bigcap_{\lambda \neq \delta} M_\lambda$ for each $\delta \in \Delta$, then $\{M_\delta \mid \delta \in \Delta\}$ is a finite set.*

Proof. Suppose to the contrary that $\{M_\delta \mid \delta \in \Delta\}$ is an infinite set. Then there exists a well-ordering $\leq$ of Δ such that Δ has no largest element. Let $I_\delta = \bigcap_{\lambda > \delta} M_\lambda$ for each $\delta \in \Delta$ and let $I = \bigcup_{\delta \in \Delta} I_\delta$. Then $\{I_\delta \mid \delta \in \Delta\}$ is a chain of proper homogeneous ideals of R. Hence I is a proper homogeneous ideal of R. By the hypothesis, $1 - m_\delta \in I_\delta \setminus M_\delta$ for each $\delta \in \Delta$. Then $I \not\subseteq M_\delta$ for all $\delta \in \Delta$. Hence $I = R$, which is absurd. Thus $\{M_\delta \mid \delta \in \Delta\}$ is a finite set. □

In [14], the author generalized the concepts property (#) and property (##) to graded integral domains and studied some properties of graded Prüfer domains. Let R be a graded integral domain. We say that R *satisfies graded property* (#) if for any two distinct subsets Δ_1 and Δ_2 of h-Max(R), $\bigcap_{M \in \Delta_1} R_{H\setminus M} \neq \bigcap_{M \in \Delta_2} R_{H\setminus M}$; and R satisfies *graded property* (##) if each homogeneous overring of R satisfies graded property (#).

When R is a graded Prüfer domain, the author showed that R satisfies graded property (#) if and only if R is uniquely expressed as an intersection of a set $\{V_\beta \mid \beta \in \mathscr{B}\}$ of graded valuation overrings of R such that there are no containment among the V_β's [14, Theorem 4.6]; and R satisfies graded property (##) if and only if for each homogeneous prime ideal P of R, there exists a finitely generated homogeneous ideal A of R such that $A \subseteq P$ and each maximal homogeneous ideal of R containing A contains P [14, Theorem 5.2].

Lemma 3.4. *Let R be a graded Prüfer domain satisfying graded property* (##). *Then each finitely generated homogeneous ideal of R has only finitely many minimal homogeneous prime ideals.*

Proof. It suffices to show that each principal homogeneous ideal of R has only finitely many minimal homogeneous prime ideals [7, Lemma 4]. Let $x \in H$ and let $\{P_\delta \mid \delta \in \Delta\}$ be the set of minimal homogeneous prime ideals of (x). Then $\{R_{H\setminus P_\delta} \mid \delta \in \Delta\}$ is a set of pairwise order-incomparable graded valuation overrings of R. Let $T = \bigcap_{\delta \in \Delta} R_{H\setminus P_\delta}$. Then T is a graded Prüfer domain [3, Theorem 3.5(2)]. Since R satisfies graded property (##), T satisfies graded property (##). Hence $T = \bigcap_{\delta \in \Delta} R_{H\setminus P_\delta}$ is the unique representation of T as an intersection of graded valuation overrings [14, Theorem 4.6]. Let $\{M_\beta \mid \beta \in \mathscr{B}\}$ be the set of maximal homogeneous ideals of T and let $H(T)$ be the set of nonzero homogeneous elements of T. Then $\{R_{H\setminus P_\delta} \mid \delta \in \Delta\} = \{T_{H(T)\setminus M_\beta} \mid \beta \in \mathscr{B}\}$. Note that $T_{H(T)\setminus P_\delta T} = R_{H\setminus P_\delta}$ for each $\delta \in \Delta$ [3, Theorem 3.5(1)]. Hence $\{P_\delta T \mid \delta \in \Delta\}$ is the set of maximal homogeneous ideals of T [3, Theorem 3.5(5)]. Take an index $\delta \in \Delta$. Then $P_\delta T$ is a minimal homogeneous prime ideal of xT [3, Theorem 3.5]. Since T satisfies graded property (##), there exists a finitely generated homogeneous ideal A of T such that $P_\delta T$ is the unique maximal homogeneous ideal of T containing A [14, Theorem

5.2]. Since $P_\delta T$ is a minimal homogeneous prime ideal of xT, $\sqrt{A+xT} = P_\delta T$. Then by Lemma 3.2, there exists an element $y_\delta \in P_\delta T \cap H(T)$ such that $1 - y_\delta \in P_\lambda$ for all $\lambda \neq \delta$. Hence by Lemma 3.3, $\{P_\delta T \mid \delta \in \Delta\}$ is a finite set. Thus each finitely generated homogeneous ideal of R has only finitely many minimal homogeneous prime ideals. □

In [15], the author defined a graded integral domain R to satisfy the *graded radical trace property* (graded RTP domain) if for each nonzero homogeneous ideal I of R, $II^{-1} = \sqrt{II^{-1}}$.

Recall from [14] that $P \in$ h-Spec(R) is said to be *homogeneous branched* if there exists a graded P-primary ideal Q of R with $Q \neq P$. We are now ready to prove the main result of this paper.

Theorem 3.5. *Let R be a graded Prüfer domain. Then the following statements are equivalent.*

(1) *R has Noetherian homogeneous spectrum.*
(2) *R satisfies both graded property (##) and the ascending chain condition on homogeneous prime ideals.*
(3) *Each finitely generated homogeneous ideal of R has only finitely many minimal homogeneous prime ideals and R satisfies the ascending chain condition on homogeneous prime ideals.*
(4) *R is a graded RTP domain and R satisfies the ascending chain condition on homogeneous prime ideals.*

Proof. (1) ⇒ (2) This direction comes from Theorem 2.5 and [14, Theorem 5.2].

(2) ⇒ (3) It follows immediately from Lemma 3.4.

(3) ⇒ (1) Suppose that R satisfies the ascending chain condition on homogeneous prime ideals and let P be a nonzero homogeneous prime ideal of R. Then there exists a homogeneous prime ideal Q of R properly contained in P such that there are no homogeneous prime ideals of R properly between Q and P. Hence there exists an element $p \in (P \cap H) \setminus Q$. Since $R_{H \setminus P}$ is a graded valuation domain, $QR_{H\setminus P} \subsetneq (p)R_{H\setminus P}$. Then P is a minimal homogeneous prime ideal of (p). By the hypothesis, (p) has only finitely many minimal homogeneous prime ideals. Hence by Lemma 3.1, P is the radical of a finitely generated homogeneous ideal of R. Thus by Theorem 2.5, R has Noetherian homogeneous spectrum.

(1) ⇒ (4) Suppose that R has Noetherian homogeneous spectrum. Then by Theorem 2.5, R satisfies the ascending chain condition on homogeneous prime ideals. Let I be a nonzero homogeneous ideal of R. Then $(II^{-1})^{-1} = (II^{-1} : II^{-1})$ [13, Corollary 2.4]. Hence by Theorem 2.9, $II^{-1} = \sqrt{II^{-1}}$. Thus R is a graded RTP domain.

(4) ⇒ (2) Suppose to the contrary that R does not satisfy graded property (##). Then there exists a homogeneous prime ideal P of R such that for each finitely generated homogeneous ideal A of R such that $A \subseteq P$, there exists a maximal homogeneous ideal of R containing A not containing P [14, Theorem 5.2]. Hence $\bigcap_{\beta\in\mathscr{B}} R_{H\setminus M_\beta} \subseteq R_{H\setminus P}$, where $\{M_\beta \mid \beta \in \mathscr{B}\}$ is the set of maximal homogeneous ideals of R not containing P [14, Lemma 4.4]. Since P is not invertible, P^{-1} is a ring [14, Theorem 3.6]. Then $P^{-1} = \bigcap_{\beta\in\mathscr{B}} R_{H\setminus M_\beta} \cap R_{H\setminus P} = \bigcap_{\beta\in\mathscr{B}} R_{H\setminus M_\beta}$ [14, Theorem 3.2]. Since R satisfies the ascending chain condition on homogeneous prime ideals, there exists a homogeneous prime ideal M of R with $M \subsetneq P$ such that there are no homogeneous prime ideals of R properly between M and

P. Then P is homogeneous branched [14, Theorem 2.7]. Hence there exists a graded P-primary ideal Q of R with $Q \neq P$. Let $T(Q)$ be the ideal transform of Q and let $P_0 = \bigcap_{n=1}^{\infty} Q^n$. Then $T(Q) = R_{H \backslash P_0} \cap \left(\bigcap_{\beta \in \mathscr{B}} R_{H \backslash M_\beta} \right)$ [14, Corollary 3.7(2)]. Hence we obtain

$$\begin{aligned} \bigcap_{\beta \in \mathscr{B}} R_{H \backslash M_\beta} &= P^{-1} \\ &= R_{H \backslash P} \cap \left(\bigcap_{\beta \in \mathscr{B}} R_{H \backslash M_\beta} \right) \\ &\subseteq R_{H \backslash P_0} \cap \left(\bigcap_{\beta \in \mathscr{B}} R_{H \backslash M_\beta} \right) \\ &= T(Q) \\ &\subseteq \bigcap_{\beta \in \mathscr{B}} R_{H \backslash M_\beta}. \end{aligned}$$

Since $Q^{-1} \subseteq T(Q) = P^{-1}$, $P^{-1} = T(Q) = Q^{-1}$. Then by Remark 2.1(2), $QQ^{-1} = Q$. Since R is a graded RTP domain, $Q = \sqrt{Q} = P$, which is absurd. Thus R satisfies graded property (##). □

Remark 3.6. Suppose that R is a graded Prüfer domain satisfying the ascending condition on homogeneous prime ideals and not satisfying graded property (##). By the proof of (4) ⇒ (2) in Theorem 3.5, there exists a homogeneous ideal Q of R such that $Q^{-1} = (Q : Q)$ but $Q \subsetneq \sqrt{Q}$. Thus the condition that R has Noetherian homogeneous spectrum in Theorem 2.9 is not replaced by that R satisfies the ascending condition on homogeneous prime ideals.

In Remark 2.1(1), we constructed a graded Prüfer domain R and a homogeneous ideal I of R such that I^{-1} is a ring but $I^{-1} \neq (I : I)$. We examine in which cases this does not happen. To do this, we require the following lemma.

Lemma 3.7. *Let R be a graded Prüfer domain. Let P be a homogeneous prime ideal of R such that $PR_{H \backslash P}$ is not principal and let I be a homogeneous ideal of R such that $\sqrt{I} = P$ and $IR_{H \backslash P} = PR_{H \backslash P}$. Then $I = P$.*

Proof. Take an element $M \in$ h-Max(R). Then $I \nsubseteq M$ if $P \nsubseteq M$. Hence $IR_{H \backslash M} = R_{H \backslash M}$. Assume that $P \subseteq M$ and take an element $x \in P \cap H$. Since $PR_{H \backslash P}$ is not principal, $(x)R_{H \backslash P} \subsetneq PR_{H \backslash P} = IR_{H \backslash P}$. Then there exists an element $y \in I \cap H$ such that $(x)R_{H \backslash P} \subsetneq (y)R_{H \backslash P}$. Note that $PR_{H \backslash P} = PR_{H \backslash M}$ [2, Theorem 2.3(6)]. Then $\frac{x}{y} \in PR_{H \backslash M}$ and $x \in yR_{H \backslash M} \subseteq IR_{H \backslash M}$. Hence $PR_{H \backslash M} \subseteq IR_{H \backslash M}$. Thus $I = P$. □

Theorem 3.8. *Let R be a graded Prüfer domain having Noetherian homogeneous spectrum. Suppose that R satisfies the descending chain condition on homogeneous prime ideals. Then the following conditions are equivalent.*

(1) *For any homogeneous ideal I of R such that I^{-1} is a ring, $I^{-1} = (I : I)$.*
(2) *For any homogeneous ideal I of R such that I^{-1} is a ring, $I = \sqrt{I}$.*

(3) *For each nonzero homogeneous prime ideal P of R which is not maximal homogeneous such that $PR_{H\backslash P}$ is principal, there exists a homogeneous prime ideal Q of R such that $P \subsetneq Q$ and each maximal homogeneous ideal of R containing Q also contains P.*

Proof. (1) $\Leftrightarrow$ (2) The equivalence comes from Theorem 2.9.

(1) $\Rightarrow$ (3) Suppose to the contrary that there exists a nonzero homogeneous prime ideal P of R which is not maximal homogeneous such that $PR_{H\backslash P}$ is principal and for each homogeneous prime ideal Q of R with $P \subsetneq Q$, there exists a maximal homogeneous ideal of R containing P and not containing Q. Since R satisfies the descending chain condition on homogeneous prime ideals, there exist homogeneous prime ideals Q_1 and Q_2 of R properly containing P such that there are no homogeneous prime ideals of R properly between P and Q_i for $i = 1,2$. Let $p \in P \cap H$ be such that $PR_{H\backslash P} = (p)R_{H\backslash P}$ and let $A = PR_{H\backslash Q_1} \cap (p)R_{H\backslash Q_2} \cap R$. By Theorem 2.5, there exists a finitely generated homogeneous ideal B of R such that $P = \sqrt{B}$.

Let $I = A + B$. We claim that I^{-1} ring such that $I^{-1} \neq (I : I)$. Since $I \subseteq P$, $P^{-1} \subseteq I^{-1}$. Let $\{M_\beta \,|\, \beta \in \mathscr{B}\}$ be the set of maximal homogeneous ideals of R not containing P. Since $P \notin$ h-Max(R), P^{-1} is a ring [14, Theorem 3.6]. Then $P^{-1} = R_{H\backslash P} \cap \left(\bigcap_{\beta\in\mathscr{B}} R_{H\backslash M_\beta}\right)$ [14, Theorem 3.2]. Let $a \in I^{-1}$ and take an index $\beta \in \mathscr{B}$. Since $\sqrt{I} = P$, $I \nsubseteq M_\beta$. Then there exists an element $b_\beta \in (I \cap H) \setminus M_\beta$. Since $ab_\beta \in R$, $a \in R_{H\backslash M_\beta}$. Hence $I^{-1} \subseteq R_{H\backslash M_\beta}$. Let $x \in I^{-1}$. Then we have

$$x(PR_{H\backslash Q_1} \cap (p)R_{H\backslash Q_2} \cap R + B) = xI \subseteq R.$$

Since $R_{H\backslash Q_1} \nsubseteq R_{H\backslash Q_2}$, $R_{H\backslash Q_2} \subsetneq (R_{H\backslash Q_2})_{H\backslash Q_1} \subseteq R_{H\backslash P}$. Then $(R_{H\backslash Q_2})_{H\backslash Q_1} = R_{H\backslash P}$ [2, Theorem 2.3(4)]. Hence we get

$$x(PR_{H\backslash Q_1} \cap (p)R_{H\backslash P} \cap R_{H\backslash Q_1} + BR_{H\backslash Q_1}) \subseteq R_{H\backslash Q_1}.$$

Note that $(p)R_{H\backslash P} = PR_{H\backslash P} = PR_{H\backslash Q_1}$ [2, Theorem 2.3(6)]. Then $xPR_{H\backslash Q_1} \subseteq R_{H\backslash Q_1}$. Hence $x \in (PR_{H\backslash Q_1})^{-1}$. Note that $(PR_{H\backslash Q_1})^{-1}$ is a ring [14, Theorem 3.6]. Then $(PR_{H\backslash Q_1})^{-1} = (R_{H\backslash Q_1})_{H(R_{H\backslash Q_1})\backslash PR_{H\backslash Q_1}} = R_{H\backslash P}$, where $H(R_{H\backslash Q_1})$ is the set of nonzero homogeneous elements of $R_{H\backslash Q_1}$ [14, Theorem 3.2]. Hence $I^{-1} \subseteq R_{H\backslash P} \cap \left(\bigcap_{\beta\in\mathscr{B}} R_{H\backslash M_\beta}\right)$. Thus $I^{-1} = P^{-1}$ and I^{-1} is a ring.

By Theorem 3.5, R satisfies graded property (##). Then there exists a finitely generated homogeneous ideal J of R such that $P \subsetneq J \subseteq Q_2$ [14, Lemma 5.6]. Hence $Q_2P^{-1} = P^{-1}$ [14, Lemma 5.5]. Since $I^{-1} = P^{-1}$, $I^{-1} \nsubseteq R_{H\backslash Q_2}$. Since $R_{H\backslash Q_2} \nsubseteq R_{H\backslash Q_1}$, $R_{H\backslash Q_1} \subsetneq (R_{H\backslash Q_1})_{H\backslash Q_2} \subseteq R_{H\backslash P}$. Then $(R_{H\backslash Q_1})_{H\backslash Q_2} = R_{H\backslash P}$ [2, Theorem 2.3(4)]. Note that $(p)R_{H\backslash P} = PR_{H\backslash P} = PR_{H\backslash Q_1}$ [2, Theorem 2.3(6)]. Hence $AR_{H\backslash Q_2} = (p)R_{H\backslash Q_2}$. Since R is a graded Prüfer domain, $(p) + B$ is invertible. Then we obtain

$$(I : I) \subseteq (IR_{H\backslash Q_2} : IR_{H\backslash Q_2}) = (((p) + B)R_{H\backslash Q_2} : ((p) + B)R_{H\backslash Q_2}) = R_{H\backslash Q_2}.$$

Hence I^{-1} is a ring such that $I^{-1} \neq (I : I)$. This contradicts the hypothesis. Thus this implication holds.

(3) $\Rightarrow$ (2) Let I be a homogeneous ideal of R such that I^{-1} is a ring. Suppose to the contrary that $I \subsetneq \sqrt{I}$. Then there exists an element $M \in$ h-Max(R) such that $IR_{H\backslash M}$ is not a radical ideal of $R_{H\backslash M}$. Since $R_{H\backslash M}$ is a graded valuation domain, there exists an element $P \in$ h-Spec(R) such that $\sqrt{IR_{H\backslash M}} = PR_{H\backslash M}$. By the proof of Lemma 2.8, $IR_{H\backslash P} = PR_{H\backslash P}$. Hence $P \subsetneq M$. Let $H(R_{H\backslash M})$ be

the set of nonzero homogeneous elements of $R_{H\setminus M}$. Then $(R_{H\setminus M})_{H(R_{H\setminus M})\setminus PR_{H\setminus M}} = R_{H\setminus P}$. Hence by Lemma 3.7, $PR_{H\setminus P}$ is principal.

By the hypothesis, there exists a homogeneous prime ideal Q of R such that $P \subsetneq Q$ and each maximal homogeneous ideal of R containing Q also contains P. Suppose to the contrary that $IR_{H\setminus Q} = PR_{H\setminus Q}$. Let $x \in P \cap H$. Note that $PR_{H\setminus Q}$ is not principal [14, Lemma 2.2(1)]. Then $(x)R_{H\setminus Q} \subsetneq PR_{H\setminus Q} = IR_{H\setminus Q}$. Hence there exists an element $y \in I \cap H$ such that $(x)R_{H\setminus Q} \subsetneq (y)R_{H\setminus Q}$. Since $P \subseteq M$, $Q \subseteq M$. Then $QR_{H\setminus Q} = QR_{H\setminus M}$ [2, Theorem 2.3(6)]. Since $\frac{x}{y} \in QR_{H\setminus Q}$, $x \in (y)R_{H\setminus M} \subseteq IR_{H\setminus M}$. Then $IR_{H\setminus M} = PR_{H\setminus M}$, which is impossible. Hence $IR_{H\setminus Q} \subsetneq PR_{H\setminus Q}$.

Since $R_{H\setminus Q}$ is a graded valuation domain, there exists an element $p \in P \cap H$ such that $IR_{H\setminus Q} \subsetneq (p)R_{H\setminus Q}$. By Theorem 2.5, there exists a finitely generated homogeneous ideal A of R such that $P = \sqrt{A}$. Let $J = A + (p)$ and take an element $N \in$ h-Max(R). Then $J \not\subseteq N$ if $P \not\subseteq N$. Hence $JR_{H\setminus N} = R_{H\setminus N}$. Assume that $P \subseteq N$. Then $Q \subseteq N$. Hence $p^{-1}I \subseteq QR_{H\setminus Q} = QR_{H\setminus N}$ [2, Theorem 2.3(6)]. It follows that $IR_{H\setminus N} \subseteq (p)R_{H\setminus N} \subseteq JR_{H\setminus N}$. Then $I \subseteq J \subseteq P$. Hence by Lemma 2.6, I^{-1} is not a ring, which is absurd. Thus $I = \sqrt{I}$. □

Remark 3.9. Let R be a graded integral domain and let I be a nonzero homogeneous ideal of R. Then I^{-1} is a ring if and only if $I^{-1} = (I_v : I_v)$, if and only if $I^{-1} = (II^{-1} : II^{-1})$ [10, Proposition 2.2]. Hence we obtain additional equivalent conditions in Theorem 3.8 as follows:

(4) For any nonzero homogeneous ideal I of R such that I^{-1} is a ring, $(I : I) = (I_v : I_v)$.
(5) For any nonzero homogeneous ideal I of R such that I^{-1} is a ring, $(I : I) = (II^{-1} : II^{-1})$.

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(KIM) DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, KYUNGPOOK NATIONAL UNIVERSITY, DAEGU 41566, REPUBLIC OF KOREA

E-mail address: dongkyu0397@gmail.com