On some eigenvalue properties related to fractional Sturm-Liouville problems

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Dedicated to Professor Chun-Kong Law on the occasion of his 65th birthday

Abstract

In this paper, we consider two variant types of fractional Sturm-Liouville problems (FSLP). For a regular FSLP, we represent some elementary properties of eigenvalues and eigenfunctions. We also use a fixed point theorem to give a sufficient condition on eigenvalues for the existence and uniqueness of the associated eigenfunctions. Next, we consider a non-self-adjoint two-term FSLP. Employing a recent and significant result related to the analysis of Mittag-Leffler functions, we investigate the existence and asymptotic behaviour of the real eigenvalues for this problem.

1 Introduction

Fractional calculus is the emerging mathematical field devoted to study convoluting-type pseudo-differential operators, specifically integrals and derivatives of any arbitrary real or complex order. The major difference between fractional and ordinary derivatives lies in the global nature of the former and to the local nature of the latter. Thus to get information on the fractional derivative of a function at a given point one needs to have a knowledge of the original function on a semi-interval. The ordinary circular trigonometric functions that occur in the classical theory are replaced by Mittag-Leffler functions. In most of the fractional Sturm-Liouville formulations presented recently, the ordinary derivatives are replaced with fractional derivatives, and the resulting problems are solved by using some numerical schemes [1, 2, 3]. Indeed, the basic case with zero potential function has been dealt rarely. Of note here is the work of Klimek *et al.* [9, 10]. The authors

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considered the self-adjoint fractional Sturm-Liouville eigenvalue problems. They showed the existence of real eigenvalues and the orthogonal property of eigenfunctions as in the classical case. They also derived many general properties by using the variational principle. More recently, Mingarelli *et al.* [6, 7] provided existence and uniqueness results for the initial value problems associated with mixed Riemann-Liouville/Caputo differential equations in the real domain. The authors also investigated the spectral and oscillation theory for a class of fractional differential equations subject to specific boundary conditions.

Next, the following results are more relevant to our work. Dehghan and Mingarelli [4] considered two-term fractional Sturm-Liouville eigenvalue problems (FSLP). The authors investigated the operator is a composition of a left Riemann-Liouville fractional derivative with a left Caputo fractional derivative coupled with a Dirichlet-type boundary conditions. For the fractional order $1/2 < \alpha < 1$, the authors showed that there is a finite number of real eigenvalues, an infinite number of non-real eigenvalues, and that the number of such real eigenvalues grows without bound as $\alpha \to 1^-$. By an analysis of the generalized Mittag-Leffer function $E_{2\alpha,2}(-\lambda)$, the authors also derived the existence and asymptotic distribution of the real eigenvalues. In 2022, the authors [5] continued the study of similar issues on a non-self-adjoint fractional three-term Sturm-Loiuville boundary value problem. Next, we also mention a work of Klimek, Odzijewicz and Malinowska [10]. The authors considered a different form of fractional differential equations (cf. (1.1) below). The authors presented an interesting application of fractional variational calculus, namely, using the fact that the fractional Sturm–Liouville eigenvalue problem can be remodeled as a fractional isoperimetric variational problem. They showed that an increasing sequence of eigenvalues and a corresponding sequence of eigenfunctions exist, for which the fractional Sturm–Liouville equation is satisfied. Motivated by the above results, we plan to deal with two variant types of fractional Sturm-Liouville operators as in [4, 5, 10]. Specifically, we first consider the following regular fractional Sturm-Loiuville equation

$$[D^{c,\alpha}_{b-}D^{c,\alpha}_{a+} + q(x)]y(x) = \lambda\omega(x)y(x) \quad on\ (a, b), \tag{1.1}$$

subject to the Dirichlet-Neumann type boundary conditions

$$y(a) = D^{c,\alpha}_{a+}y(x)|_{x=b} = 0, \tag{1.2}$$

where the right and left Caputo fractional derivatives are denoted by $D^{c,\alpha}_{b-}$ and $D^{c,\alpha}_{a+}$, respectively. Here we also assume $q,\ \omega \in C[a, b]$ with $\omega > 0$. Observe that in the case $\alpha = 1$ we have $D^{c,1}_{a+}y = y'$ and $D^{c,1}_{b-}y = -y'$, hence (1.1) is consistent with the classical Sturm-Liouville equation. Here we mention a work as in [11]. The author considered a so-called q-fractional Sturm– Liouville problem and applied a fixed point theory to give a sufficient condition on the parameter λ to guarantee the existence and uniqueness of solutions. For the FSLP (1.1)-(1.2), we plan to deduce the exact form of solutions (cf. Lemma 4.1) and employ a fixed point theory to develop the similar result as in [11] (cf. Theorem 4.3). Also, a fractional version of Wronskian associated with this problem is defined (cf. Theorem 4.4). This part about the Wronskian also parallels the results in [11, Section 4]. Next, we turn to consider the following non-self-adjoint FSLP

$$-D_{0+}^{\alpha} D_{0+}^{c,\alpha} y(x) = \lambda y(x), \qquad 1/2 < \alpha < 1, \quad 0 < x < 1, \tag{1.3}$$

with boundary conditions

$$y(0) = 0 \quad \text{and} \quad I_{0+}^{2-2\alpha} y(x)|_{x=1} = 0, \tag{1.4}$$

where the left Riemann-Liouville fractional derivative and the left Riemann-Liouville fractional integral are denoted by D_{0+}^{α} and $I_{0+}^{2-2\alpha}$, respectively. Employing a recent result in [4, 5], we have the following result related to the existence and asymptotic behaviour of the real eigenvalues.

Theorem 1.1. *The following distribution of real eigenvalues of the FSLP* (1.3)-(1.4) *is valid.*

(i) For fixed $1/2 < \alpha < 1$*, there exists* $N^* \in \mathbb{N}$ *such that for each* $n = 0, 1, 2, 3, \cdots, N^* - 1$ *the interval*

$$I_n(\alpha) := \left(\left(\frac{(2n + \frac{1}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha}, \left(\frac{(2n + \frac{3}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha} \right), \tag{1.5}$$

contains two real eigenvalues.

(ii) Furthermore, if the first of two real eigenvalues in $I_n(\alpha)$ *is denoted by* $\lambda_n(\alpha)$*,* (1.5) *also gives the a-priori estimate*

$$\left(\frac{(2n + \frac{1}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha} \leq \lambda_n(\alpha) \leq \left(\frac{(2n + \frac{3}{2} + \frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})} \right)^{2\alpha}. \tag{1.6}$$

Our plan of this paper is as follows. In Section 2, we recall some definitions and known properties of fractional calculus and deduce the general solutions of several two-term fractional differential equations. Then, the proof of Theorem 1.1 will be given in Section 3. Next, the issues related to the FSLP (1.1)-(1.2) will be discussed in Section 4. Finally, we give a short section to conclude the result of this paper.

2 Preliminary and general solutions of fractional differential equations

We start with some definitions and preliminary properties of fractional calculus and refer the reader to some known results [4, 8, 13] for the details.

Definition 2.1. *Let* $0 < \alpha < 1$*.*

(i) The left and right Riemann-Liouville fractional integrals I^α_{a+} and I^α_{b-} of order α are defined by

$$I^\alpha_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \qquad x \in (a,b],$$

and

$$I^\alpha_{b-}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}, \qquad x \in [a,b),$$

respectively. Here $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ denotes Euler's gamma function.

(ii) The left and right Riemann-Liouville fractional derivatives D^α_{a+} and D^α_{b-} are defined by

$$D^\alpha_{a+}f(x) := DI^{1-\alpha}_{a+}f(x), \qquad x > a,$$

and

$$D^\alpha_{b-}f(x) := -DI^{1-\alpha}_{b-}f(x), \qquad x < b,$$

respectively, where $D = \frac{d}{dx}$ and f is sufficiently differentiable.

(iii) The left and right Caputo fractional derivatives $D^{c,\alpha}_{a+}$ and $D^{c,\alpha}_{b-}$ are defined by

$$D^{c,\alpha}_{a+}f(x) = D^\alpha_{a+}[f(x) - f(a)] = I^{1-\alpha}_{a+}Df(x), \qquad x > a,$$

and

$$D^{c,\alpha}_{b-}f(x) = D^\alpha_{b-}[f(x) - f(b)] = -I^{1-\alpha}_{b-}Df(x), \qquad x < b,$$

respectively, where f is sufficiently differentiable.

Property 2.2. *Let $0 < \alpha < 1$.*

(i) If $f \in L^p(a,b)$, then

$$D^\alpha_{a+}I^\alpha_{a+}f(x) = f(x),$$
$$D^\alpha_{b-}I^\alpha_{b-}f(x) = f(x)$$

for almost all $x \in [a,b]$.

(ii) If $f \in L^1(a,b)$ and $I^{1-\alpha}_{a+}f,\ I^{1-\alpha}_{b-}f \in AC[a,b]$. Then,

$$I^\alpha_{a+}(D^\alpha_{a+}f(x)) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\alpha}_{a+}f(x)|_{x=a},$$
$$I^\alpha_{b-}(D^\alpha_{b-}f(x)) = f(x) - \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\alpha}_{b-}f(x)|_{x=b}.$$

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(iii) If $f \in C[a,b]$, then

$$D^{c,\alpha}_{a+} I^{\alpha}_{a+} f(x) = f(x),$$
$$D^{c,\alpha}_{b-} I^{\alpha}_{b-} f(x) = f(x).$$

(iv) If $f \in AC[a,b]$, then

$$I^{\alpha}_{a+}(D^{c,\alpha}_{a+} f(x)) = f(x) - f(a),$$
$$I^{\alpha}_{b-}(D^{c,\alpha}_{b-} f(x)) = f(x) - f(b).$$

Property 2.3. *Let $0 < \alpha < 1$. The following identities hold:*

$$\begin{aligned} I^{\alpha}_{a+} C &= \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} C, \\ I^{\alpha}_{a+}(x-a)^{\alpha-1} &= \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(x-a)^{2\alpha-1}, \\ I^{\alpha}_{a+}(b-x)^{\alpha-1} &= \frac{(b-x)^{2\alpha-1}}{\Gamma(\alpha)} \left(B(\frac{b-a}{b-x};\alpha,\alpha) - B(1;\alpha,\alpha) \right), \end{aligned}$$

where C is a constant and $B(z;\alpha,\beta)$ is the "incomplete Beta function" defined by

$$B(z;\alpha,\beta) = \int_0^z u^{\alpha-1}(1-u)^{\beta-1} du.$$

Property 2.4. *(Integration by parts) The operators I^{α}_{a+}, I^{α}_{b-}, D^{α}_{a+}, D^{α}_{b-}, $D^{c,\alpha}_{a+}$, and $D^{c,\alpha}_{b-}$ satisfy the following:*

$$\begin{aligned} \int_a^b f(x) I^{\alpha}_{a+} g(x) dx &= \int_a^b g(x) I^{\alpha}_{b-} f(x) dx, \\ \int_a^b f(x) D^{\alpha}_{a+} g(x) dx &= \int_a^b g(x) D^{c,\alpha}_{b-} f(x) dx + f(x) I^{1-\alpha}_{a+} g(x)|_{x=a}^{x=b}, \\ \int_a^b f(x) D^{\alpha}_{b-} g(x) dx &= \int_a^b g(x) D^{c,\alpha}_{a+} f(x) dx - f(x) I^{1-\alpha}_{b-} g(x)|_{x=a}^{x=b}. \end{aligned}$$

Now, we recall the definition and some known properties for the Mittag-Leffler function. For more details, we refer the reader to [4, 8, 13]. The function $E_\delta(z)$ defined by

$$E_\delta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + 1)}, \quad (z \in \mathbb{C},\ \Re(\delta) > 0), \tag{2.1}$$

was introduced by Mittag-Leffler [12]. And the generalized Mittag-Leffler function $E_{\delta,\theta}(z)$ is defined by

$$E_{\delta,\theta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k+\theta)}, \quad (z, \theta \in \mathbb{C},\ \Re(\delta) > 0). \tag{2.2}$$

In particular, when $\delta = 1$ and $\delta = 2$, one has

$$E_1(z) = e^z, \qquad E_2(z) = \cosh(\sqrt{z}).$$

Two other particular cases of (2.2) are as follows:

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \qquad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}.$$

Next, we quote two results related to the Laplace transform.

Property 2.5. *([4]) For $\Re(q) > -1$, then*

$$\mathfrak{L}\{t^q\} = \frac{\Gamma(q+1)}{s^{q+1}} \quad \text{and} \quad \mathfrak{L}^{-1}\{s^q\} = \frac{1}{t^{q+1}\Gamma(q)}.$$

Property 2.6. *([4, 13]) For $\Re(\delta) > 0$ and $\theta \in \mathbb{C}$, then*

$$\mathfrak{L}^{-1}\{\frac{s^{\delta-\theta}}{s^\delta + \lambda}\} = t^{\theta-1}E_{\delta,\theta}(-\lambda t^\delta).$$

The following two results are related to the analysis of the generalized Mittag-Leffer function. They are crucial to the proof of Theorem 1.1.

Proposition 2.7. *For $\beta,\ \nu > 0$, the following formula for the generalized Mittag-Leffler function is valid,*

$$\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}dt = x^{\beta+\nu-1}E_{\alpha,\beta+\nu}(\lambda x^\alpha).$$

Note that the above can be obtained by the fractional-order term-by-term integration of (2.2) (the definition of $E_{\alpha,\beta}(z)$) and also be found in [13, p.25].

Theorem 2.8. *([4]) The distribution of real zeros of $E_{2\alpha,2}(-\lambda)$ has the following. Let $1/2 < \alpha < 1$ be fixed. There exists $N^* \in \mathbb{N}$ such that for each $n = 0, 1, 2, 3, \cdots, N^*-1$ the interval*

$$I_n(\alpha) := \left(\left(\frac{(2n+\frac{1}{2}+\frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})}\right)^{2\alpha}, \left(\frac{(2n+\frac{3}{2}+\frac{1}{2\alpha})\pi}{\sin(\frac{\pi}{2\alpha})}\right)^{2\alpha}\right), \tag{2.3}$$

contains two real zeros of $E_{2\alpha,2}(-\lambda)$.

Before to solve the fractional differential equations, we mention a recent result. In 2020, Dehghan and Mingarelli [4] applied the properties of fractional calculus and Laplace transform to derive the general solutions to three two-term fractional differential equations:

$$D^{c,\alpha}_{b-} D^{\alpha}_{a+} y(x) = 0, \quad D^{\alpha}_{b-} D^{c,\alpha}_{a+} y(x) = 0, \quad D^{c,\alpha}_{0+} D^{\alpha}_{0+} y(x) + \lambda y(x) = 0.$$

The authors also analyzed the Mittag-Leffler function to develop some properties related to real eigenvalues of the above third problem coupled with the Dirichlet type boundary conditions. Motivated by the above, we consider several features of differently defined fractional Sturm-Liouville operators, and deduce the general solutions.

Type 2.1 Consider the following Riemann-Liouville fractional differential equation:

$$D^{\alpha}_{b-} D^{\alpha}_{a+} y(x) = 0. \tag{2.4}$$

Applying the right and left fractional integrals on (2.4) and employing Property 2.2-2.3, one can obtain the general solution of (2.4) as follows:

$$\begin{aligned} y(x) \quad &= \quad \frac{\left(I^{1-\alpha}_{a+} y(x)\right)|_{x=a}}{\Gamma(\alpha)} (x-a)^{\alpha-1} \\ &\quad + \frac{\left(I^{1-\alpha}_{b-} D^{\alpha}_{a+} y(x)\right)|_{x=b}}{\Gamma^2(\alpha)} (b-x)^{2\alpha-1} \left(B(\frac{b-a}{b-x}; \alpha, \alpha) - B(1; \alpha, \alpha) \right). \end{aligned} \tag{2.5}$$

Type 2.2 Associated to (2.4) is another different composition,

$$D^{\alpha}_{a+} D^{c,\alpha}_{a+} y(x) = 0. \tag{2.6}$$

Applying the similar procedure as the above, one can obtain

$$\begin{aligned} y(x) \quad &= \quad y(a) + \frac{I^{1-\alpha}_{a+} \left(D^{c,\alpha}_{a+} y(x)\right)|_{x=a}}{\Gamma(\alpha)} I^{\alpha}_{a+} (x-a)^{\alpha-1} \\ &= \quad y(a) + \frac{I^{1-\alpha}_{a+} \left(D^{c,\alpha}_{a+} y(x)\right)|_{x=a}}{\Gamma(2\alpha)} (x-a)^{2\alpha-1}. \end{aligned} \tag{2.7}$$

Type 2.3 Consider the following fractional Sturm-Liouville equation:

$$D^{c,\alpha}_{b-} p(x) D^{\alpha}_{a+} y(x) = 0, \tag{2.8}$$

where p is a positive and C^1 function. Applying the right Riemann-Liouville fractional integral on

(2.8), one can obtain

$$p(x)D_{a+}^{\alpha}y(x) - p(b)\left(D_{a+}^{\alpha}y(x)|_{x=b}\right) = I_{b-}^{\alpha}(0).$$

i.e.,

$$D_{a+}^{\alpha}y(x) = [p(b)(D_{a+}^{\alpha}y(x)|_{x=b})]\frac{1}{p(x)}.$$

By Property 2.2, one can obtain

$$y(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}(I_{a+}^{1-\alpha}y(x))|_{x=a} + [p(b)(D_{a+}^{\alpha}y(x)|_{x=b})]I_{a+}^{\alpha}\left(\frac{1}{p(x)}\right). \tag{2.9}$$

Type 2.4 Consider

$$D_{b-}^{c,\alpha}D_{a+}^{c,\alpha}y(x) = 0. \tag{2.10}$$

By the similar manipulation as in the above, one can obtain

$$\begin{aligned} y(x) &= y(a) + \left(D_{a+}^{c,\alpha}y(x)|_{x=b}\right)I_{a+}^{\alpha}(1) \\ &= y(a) + \left(D_{a+}^{c,\alpha}y(x)|_{x=b}\right)\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{2.11}$$

3 Proof of Theorem 1.1

From the known properties and some preparation in the previous section, we are ready to give the proof of Theorem 1.1. First applying the left fractional integral on (1.3) and employing Property 2.2, one can obtain

$$D_{0+}^{c,\alpha}y(x) - \frac{x^{\alpha-1}}{\Gamma(\alpha)}(I_{0+}^{1-\alpha}D_{0+}^{c,\alpha}y(x))|_{x=0} = -\frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}y(s)ds. \tag{3.1}$$

Again, taking the left fractional integral on (3.1) and employing Property 2.3, one can get

$$\begin{aligned} y(x) &= y(0) + \frac{\left(I_{0+}^{1-\alpha}D_{0+}^{c,\alpha}y(x)\right)|_{x=0}}{\Gamma(2\alpha)}x^{2\alpha-1} - \frac{\lambda}{\Gamma^2(\alpha)}\int_0^x (x-t)^{\alpha-1}\left(\int_0^t (t-s)^{\alpha-1}y(s)ds\right)dt \\ &= y(0) + \frac{\left(I_{0+}^{1-\alpha}D_{0+}^{c,\alpha}y(x)\right)|_{x=0}}{\Gamma(2\alpha)}x^{2\alpha-1} - \frac{\lambda}{\Gamma^2(\alpha)}\int_0^x y(s)\left(\int_s^x (x-t)^{\alpha-1}(t-s)^{\alpha-1}dt\right)ds \\ &= y(0) + \frac{\left(I_{0+}^{1-\alpha}D_{0+}^{c,\alpha}y(x)\right)|_{x=0}}{\Gamma(2\alpha)}x^{2\alpha-1} - \frac{\lambda}{\Gamma^2(\alpha)}\int_0^x y(s)(x-s)^{2\alpha-1}B(\alpha,\alpha)ds \\ &= y(0) + \frac{\left(I_{0+}^{1-\alpha}D_{0+}^{c,\alpha}y(x)\right)|_{x=0}}{\Gamma(2\alpha)}x^{2\alpha-1} - \frac{\lambda}{\Gamma(2\alpha)}\left(y(x)*x^{2\alpha-1}\right) \\ &= c_0 + c_1\frac{x^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{\lambda}{\Gamma(2\alpha)}\left(y(x)*x^{2\alpha-1}\right), \end{aligned} \tag{3.2}$$

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where $c_0 = y(0)$, $c_1 = \left(I_{0+}^{1-\alpha} D_{0+}^{c,\alpha} y(x)\right)|_{x=0}$, the beta function $B(\cdot,\cdot)$ and "$*$" denotes the convolution of the two functions supported on $[0,\infty)$. Next, taking the Laplace transform and applying Property 2.5 on (3.2), one can obtain

$$\mathfrak{L}\{y(x)\} = \frac{c_0}{s} + \frac{c_1}{s^{2\alpha}} - \frac{\lambda}{\Gamma(2\alpha)} \mathfrak{L}\{y(x)\}\mathfrak{L}\{x^{2\alpha-1}\}.$$

Then,

$$\mathfrak{L}\{y(x)\} = \frac{c_0 s^{2\alpha-1}}{s^{2\alpha}+\lambda} + \frac{c_1}{s^{2\alpha}+\lambda}.$$

Applying Property 2.6, one can obtain the general solution of (1.3) as follows:

$$y(x) = c_0 E_{2\alpha,1}(-\lambda x^{2\alpha}) + c_1 x^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda x^{2\alpha}), \tag{3.3}$$

where $c_0 = y(0)$, $c_1 = \left(I_{0+}^{1-\alpha} D_{0+}^{c,\alpha} y(x)\right)|_{x=0}$. By imposing the boundary conditions (1.4) on (3.3) with $\left(I_{0+}^{1-\alpha} D_{0+}^{c,\alpha} y(x)\right)|_{x=0} \neq 0$, one can obtain

$$I_{0+}^{2-2\alpha} y(x)|_{x=1} = c_1 I_{0+}^{2-2\alpha}\left(x^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda x^{2\alpha})\right)|_{x=1} = 0. \tag{3.4}$$

Now employing Proposition 2.7, one can obtain

$$\begin{aligned} I_{0+}^{2-2\alpha}\left(x^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda x^{2\alpha})\right) &= \frac{1}{\Gamma(2-2\alpha)} \int_0^x (x-t)^{(2-2\alpha)-1} t^{2\alpha-1} E_{2\alpha,2\alpha}(-\lambda t^{2\alpha}) dt \\ &= \frac{1}{\Gamma(2-2\alpha)} x E_{2\alpha,2}(-\lambda x^{2\alpha}). \end{aligned} \tag{3.5}$$

Hence, the right endpoint boundary condition in (1.4) implies

$$E_{2\alpha,2}(-\lambda) = 0, \tag{3.6}$$

which defines the characteristic equation for the eigenvalues of (1.3)-(1.4). Therefore, Theorem 1.1 is valid by Theorem 2.8.

4 The FSLP (1.1)-(1.2)

In this section we will develop some elementary properties for the FSLP (1.1)-(1.2). We first present the general solution for this problem. By a solution of (1.1) is meant a function $y \in AC[a,b]$ such that $D_{a+}^{c,\alpha} y \in AC[a,b]$. Set $Y_y(x) := (q - \lambda\omega)y(x)$. Then, one can rewrite (1.1) as

$$D_{b-}^{c,\alpha} D_{a+}^{c,\alpha}[y(\cdot) + I_{a+}^{\alpha} I_{b-}^{\alpha} Y_y](x) = 0.$$

From (2.11), one can obtain

$$\begin{aligned} y(x) + I^{\alpha}_{a+} I^{\alpha}_{b-} Y_y(x) &= y(a) + [I^{\alpha}_{a+} I^{\alpha}_{b-} Y_y](a) \\ &\quad + D^{c,\alpha}_{a+} \left(y(\cdot) + I^{\alpha}_{a+} I^{\alpha}_{b-} Y_y] \right)(b) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \\ &= y(a) + [I^{\alpha}_{a+} I^{\alpha}_{b-} Y_y](a) \\ &\quad + [\left(D^{c,\alpha}_{a+} y(x)|_{x=b} \right) + \left(I^{\alpha}_{b-} Y_y|_{x=b} \right)] \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \\ &= y(a) + \left(D^{c,\alpha}_{a+} y(x)|_{x=b} \right) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

i.e.,

$$y(x) = y(a) + \left(D^{c,\alpha}_{a+} y(x)|_{x=b} \right) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} - I^{\alpha}_{a+} I^{\alpha}_{b-} Y_y(x). \tag{4.1}$$

Note that $D^{c,\alpha}_{a+} \left(\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \right) = 1$ and $D^{c,\alpha}_{a+} y(x) = D^{c,\alpha}_{a+} y(x)|_{x=b} - I^{\alpha}_{b-} Y_y(x)$ by (4.1). Now we have the following.

Lemma 4.1. *Let $Y_y(x) := (q - \lambda\omega) y(x)$. Then the regular fractional Sturm-Liouville problem* (1.1)-(1.2) *is equivalent to*

$$y(x) = -I^{\alpha}_{a+} I^{\alpha}_{b-} Y_y(x). \tag{4.2}$$

By the direct computation, one can obtain the following lemma.

Lemma 4.2. *For the Riemann-Loiuville fractional integrals on $[a, b]$, the following estimates are valid:*

$$I^{\alpha}_{a+}(1) \le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad I^{\alpha}_{b-}(1) \le \frac{b^{\alpha}}{\Gamma(\alpha+1)}, \quad I^{\alpha}_{a+} I^{\alpha}_{b-}(1) \le \frac{b^{\alpha}(b-a)^{\alpha}}{\Gamma^2(\alpha+1)}.$$

Later, we have the result related to the existence and uniqueness of solutions.

Theorem 4.3. *Let $\| \cdot \|$ denote the supremum norm on the space $C[a, b]$. Then, a unique continuous eigenfunction y_{λ} for* (1.1)-(1.2) *corresponding to each eigenvalue obeying*

$$\|q - \lambda\omega\| < \frac{\Gamma^2(\alpha+1)}{b^{\alpha}(b-a)^{\alpha}} \tag{4.3}$$

exists and such an eigenvalue is simple.

Proof. Set $C_{DN}[a, b] = \{ f \in C[a, b] : f(a) = D^{c,\alpha}_{a+} f(x)|_{x=b} = 0 \}$. Let us note that a solution of (1.1)-(1.2) can be interpreted as a fixed point of the mapping $T : C_{DN}[a, b] \to C_{DN}[a, b]$ defined by

$$Tf(x) = -I^{\alpha}_{a+} I^{\alpha}_{b-} Y_f(x).$$

For a pair of arbitrary $f,\ g \in C_{DN}[a,b]$, we calculate the distance between images Tf and Tg. First the inequality

$$\|Y_f - Y_g\| \leq \|q - \lambda\omega\| \cdot \|f - g\|$$

holds obviously. Also, one can get

$$|[I_{a+}^{\alpha} I_{b-}^{\alpha} Y_f](x) - [I_{a+}^{\alpha} I_{b-}^{\alpha} Y_g](x)| \leq \|Y_f - Y_g\| \cdot \left(I_{a+}^{\alpha} I_{b-}^{\alpha}(1)\right) \leq \frac{b^{\alpha}(b-a)^{\alpha}}{\Gamma^2(\alpha+1)} \|q - \lambda\omega\| \cdot \|f - g\|. \tag{4.4}$$

Hence,

$$\|Tf - Tg\| \leq \left(\frac{b^{\alpha}(b-a)^{\alpha}}{\Gamma^2(\alpha+1)} \|q - \lambda\omega\|\right) \|f - g\| < \|f - g\|$$

by the assumption. Thus, a unique fixed point denoted as $y_\lambda \in C[a,b]$ exists that solves (1.1) and satisfies the boundary conditions (1.2), provided (4.3) is fulfilled. Therefore such an eigenvalue is simple. □

Remark. *Let $1/2 < \alpha < 1$. Consider (1.1)-(1.2) where the coefficients are defined by $q \equiv 0$, $\omega \equiv 1$, and $[a,b] = [0,1]$. For $\lambda = 0$, the only solution is the trivial solution by (2.11). For $0 < \lambda < \Gamma^2(\alpha+1) < 1$, the integral form of solutions shall be $y(x) = \lambda I_{0+}^{\alpha} I_{1-}^{\alpha} y(x)$. For this case, the continuous eigenfunction y_λ exists and such an eigenvalue is simple by the above result.*

Next, let us define a fractional version of the Wronskian. For $y_1\ y_2 \in AC[a,b]$, the associated Wronskian is defined by

$$W_\alpha(y_1, y_2)(x) = y_1(x) I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha} y_2\right)(x) - y_2(x) I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha} y_1\right)(x). \tag{4.5}$$

For this, we have the following results.

Theorem 4.4. *Assume y_1 and y_2 are two solutions of (1.1)-(1.2). Then, the Wronskian defined as in (4.5) satisfies*

$$W_\alpha(y_1, y_2)(a) = W_\alpha(y_1, y_2)(x)$$

for all $x \in [a,b]$.

Proof. By assumption, one can obtain

$$y_2 D_{b-}^{c,\alpha} D_{a+}^{c,\alpha} y_1 + q y_1 y_2 = \lambda \omega y_1 y_2,$$

$$y_1 D_{b-}^{c,\alpha} D_{a+}^{c,\alpha} y_2 + q y_1 y_2 = \lambda \omega y_1 y_2.$$

Then, it implies that

$$y_1 D_{b-}^{c,\alpha} D_{a+}^{c,\alpha} y_2 - y_2 D_{b-}^{c,\alpha} D_{a+}^{c,\alpha} y_1 = 0. \tag{4.6}$$

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Also,

$$
\begin{aligned}
&DW_\alpha(y_1,y_2) = Dy_1 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_2\right) - y_1 D_{b-}^{\alpha}\left(D_{a+}^{c,\alpha}y_2\right) - Dy_2 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_1\right) + y_2 D_{b-}^{\alpha}\left(D_{a+}^{c,\alpha}y_1\right)\\
&= Dy_1 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_2\right) - y_1 D_{b-}^{\alpha}\left(D_{a+}^{c,\alpha}y_2 - D_{a+}^{c,\alpha}y_2(b)\right) - Dy_2 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_1\right) + y_2 D_{b-}^{\alpha}\left(D_{a+}^{c,\alpha}y_1 - D_{a+}^{c,\alpha}y_1(b)\right)\\
&= Dy_1 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_2\right) - y_1 D_{b-}^{c,\alpha}\left(D_{a+}^{c,\alpha}y_2\right) - Dy_2 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_1\right) + y_2 D_{b-}^{c,\alpha}\left(D_{a+}^{c,\alpha}y_1\right)\\
&= Dy_1 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_2\right) - Dy_2 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_1\right)
\end{aligned}
$$

by Definition 2.1 and (4.6). Then, for $x \in [a,b]$

$$
\begin{aligned}
\int_a^x DW_\alpha(y_1,y_2)dt &= \int_a^x Dy_1 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_2\right)dt - \int_a^x Dy_2 I_{b-}^{1-\alpha}\left(D_{a+}^{c,\alpha}y_1\right)dt\\
&= \int_a^x \left(D_{a+}^{c,\alpha}y_2\right) I_{a+}^{1-\alpha}\left(Dy_1\right)dt - \int_a^x \left(D_{a+}^{c,\alpha}y_1\right) I_{a+}^{1-\alpha}\left(Dy_2\right)dt\\
&= \int_a^x \left(D_{a+}^{c,\alpha}y_2\right)\left(D_{a+}^{c,\alpha}y_1\right)dt - \int_a^x \left(D_{a+}^{c,\alpha}y_1\right)\left(D_{a+}^{c,\alpha}y_2\right)dt\\
&= 0
\end{aligned}
$$

by Property 2.4 and Definition 2.1. Therefore, this completes the proof. □

5 Conclusions

In this article, we first consider a regular fractional Sturm-Loiuville equation (1.1) subject to the Dirichlet-Neumann type boundary conditions (1.2). We show that the continuous eigenfunction y_λ exists and such an eigenvalue is simple if some certain condition (4.3) holds. Next, we investigate a non-self-adjoint FSLP (1.3)-(1.4). We derive the existence and asymptotic behaviour of the real eigenvalues by using a known result related to an analysis of Mittag-Leffler functions.

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