DIGRAPHS, PRO-p GROUPS
AND MASSEY PRODUCTS IN GALOIS COHOMOLOGY

CLAUDIO QUADRELLI

ABSTRACT. Let p be a prime. We characterize the oriented right-angled Artin pro-p groups whose $\mathbb{F}_p$-cohomology algebra yields no essential n-fold Massey products for every $n > 2$, in terms of the associated digraph. Moreover, we show that the $\mathbb{F}_p$-cohomology algebra of such an oriented right-angled Artin pro-p group is isomorphic to the exterior Stanley-Reisner $\mathbb{F}_p$-algebra associated to the same digraph.

1. INTRODUCTION

1.1. **Framework.** Let X be a cochain complex of abelian groups and R a commutative ring. The R-cohomology groups $\mathrm{H}^i(X, R)$, $i \geq 0$, are equipped with the cup-product

$$\smile \colon \mathrm{H}^s(X, R) \otimes \mathrm{H}^t(X, R) \longrightarrow \mathrm{H}^{s+t}(X, R)$$

induced by the product of R, which turn the space $\coprod_i \mathrm{H}^i(X, R)$ into a ring. Massey products are multi-valued higher order cohomology operations of several variables which generalize the cup-product. More in detail, if $\alpha_1, \ldots, \alpha_n$, $n \geq 2$, is a sequence of length n of (non-necessarily distinct) elements of $\mathrm{H}^1(X, R)$, the "value" of the n-fold Massey product associated to the above sequence is a subset

$$\langle \alpha_1, \ldots, \alpha_n \rangle \subseteq \mathrm{H}^2(X, R),$$

which may be empty. If $n = 2$, then the 2-fold Massey product $\langle \alpha_1, \alpha_2 \rangle$ is the subset of $\mathrm{H}^2(X, R)$ containing only $\alpha_1 \smile \alpha_2$. If an n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is not empty and it does not contain 0, then it is said to be essential. (For an overview on Massey products see, e.g., [40].)

The presence of essential Massey products in cohomology reveals information which cannot be revealed by the ring structure. For example, if one considers singular cohomology of the complement of the Borromean rings, the ring structure tells just that the rings are pairwise disjoint, while the existence of essential 3-fold Massey products explains that the Borromean rings are not equivalent to three unconnected circles (see, e.g., [40, § 2.2]).

One of the topics which gained great interest in Galois theory, in recent years, is the study of Massey products in Galois cohomology (for a more detailed overview on Massey products in Galois cohomology we direct the reader to [8] and [27]). Given a prime number p, let $\mathbb{K}$ be a field containing a root of 1 of order p. Also, let $G_{\mathbb{K}}(p)$

---

denote the maximal pro-p Galois group of $\mathbb{K}$, namely, $G_{\mathbb{K}}(p)$ is the Galois group of the compositum of all Galois p-extensions of $\mathbb{K}$ — equivalently, $G_{\mathbb{K}}(p)$ is the maximal pro-p quotient of the absolute Galois group of $\mathbb{K}$. Consider the field with p-elements $\mathbb{F}_p$ as a trivial $G_{\mathbb{K}}(p)$-module, and the induced $\mathbb{F}_p$-cohomology algebra

$$\mathbf{H}^{\bullet}(G_{\mathbb{K}}(p), \mathbb{F}_p) = \coprod_{i \geq 0} \mathrm{H}^i(G_{\mathbb{K}}(p), \mathbb{F}_p).$$

The recent hectic research on Massey products in Galois cohomology started after the work [16] of M.J. Hopkins and K.G. Wickelgren, where they proved that if $\mathbb{K}$ is a global field of characteristic not 2, then in $\mathbf{H}^{\bullet}(G_{\mathbb{K}}(2), \mathbb{F}_2)$ there are no essential 3-fold Massey products, and moreover they conjectured that this is true for any field of characteristic not 2. Shortly after, in [25] J. Mináč and N.D. Tân conjectured that for every prime p and for any field $\mathbb{K}$ containing a root of 1 of order p, in $\mathbf{H}^{\bullet}(G_{\mathbb{K}}(p), \mathbb{F}_p)$ there are no essential n-fold Massey products for every $n > 2$. The main results obtained in this direction are the following:

(a) E. Matzri proved that Mináč-Tân's conjecture holds true for $n = 3$ (see the preprint [19], see also the published works [10, 26]);

(b) J. Mináč and N.D. Tân proved their own conjecture for local fields (see [27]);

(c) Y. Harpaz and O. Wittenberg proved Mináč-Tân's conjecture for number fields (see [15]);

(d) the author proved a strengthened version of Mináč-Tân's conjecture for fields satisfying I. Efrat's Elementary Type Conjecture (see [33, 34]);

(e) A. Merkurjev and F. Scavia proved that Mináč-Tân's conjecture holds true also for $n = 4$ (see [22]).

The absence of essential 3-fold Massey products in the $\mathbb{F}_p$-cohomology of maximal pro-p Galois groups provided new obstructions for the realization of pro-p groups as absolute Galois groups of fields (see [27, § 7]), which was a very remarkable achievement.

Moreover, there are number-theoretic analogues of the Borromean rings, giving essential Massey products in Galois cohomology (see, e.g., [12, 29, 39]). Further interesting results on Massey products in Galois cohomology have been obtained by various authors (see, e.g., [13, 14, 17, 20, 21, 23, 41]).

It is therefore of major interest, in current research in Galois theory, to study Massey products in the $\mathbb{F}_p$-cohomology of pro-p groups.

1.2. **Digraphs and oriented pro-p RAAGs.** Within this frame, we focus on Massey products in the $\mathbb{F}_p$-cohomology of oriented right-angled Artin pro-p groups associated to digraphs. By an digraph Γ we mean a pair of finite sets $\Gamma = (\mathcal{V}, \mathcal{E})$ — we tacitly assume that $\mathcal{V} \cap \mathcal{E} = \varnothing$ — where $\mathcal{V}$ is said to be the set of vertices of Γ, and $\mathcal{E}$ is the set of directed edges of Γ, which consists of ordered couples of distinct vertices — i.e.,

$$\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \smallsetminus \{ (v, v) \mid v \in \mathcal{V} \}.$$

In other words, all digraphs in this paper have no loops nor parallel directed edges — see, e.g., [5, § 1.10]. As an example, the diagram

(1.1)

$$v_1$$

$$v_2 \quad\quad v_3 \quad\quad v_4$$
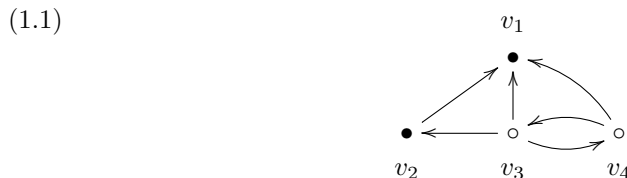
is the geometric representation of a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ with four vertices: the black ones are those vertices which are terminal vertices of directed edges whose inverse does not belong to $\mathcal{E}$.

A digraph with only undirected edges may be considered as an "undirected" graph, where the couples of edges $(v, w), (w, v)$ are identified with the subset $\{v, w\} \subseteq \mathcal{V}$ (see Remark 2.3 below).

Now for a prime p put $q = p^f$ for some $f \in \mathbb{N} \smallsetminus \{0\}$ — we require also $f \geq 2$ in the case $p = 2$. The oriented right-angled Artin pro-*p* group (oriented pro-*p* RAAG for short) associated to a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ and to q is the pro-*p* group with pro-*p* presentation

$$G = \left\langle\, v \in \mathcal{V} \mid wuw^{-1} = \begin{cases} u^{1+q} & \text{if } (w, u) \notin \mathcal{E}, \\ u & \text{if } (w, u) \in \mathcal{E}, \end{cases} \forall\, (u, w) \in \mathcal{E} \,\right\rangle.$$

Observe that if a digraph Γ has only undirected edges, then G is the pro-*p* RAAG (namely, the pro-*p* completion of the discrete RAAG) associated to Γ seen a graph (and it does not depend on q).

Oriented pro-*p* RAAGs associated to digraphs have been studied in [3], where a digraph is called an "oriented graph" (though, not every digraph is an oriented graph as defined in [5, § 1.10]). The family of oriented pro-*p* RAAGs associated to digraphs is extremely rich, and it includes free pro-*p* groups, free abelian pro-*p* groups, certain families of *p*-adic analytic pro-*p* groups and even some finite *p*-groups (see, e.g., [3, § 1]). For this reason, pro-*p* groups associated to digraphs have been object of study in recent times, especially from a Galois-theoretic perspective (see, e.g., [2–4, 35, 38]).

1.3. **Main results.** Our main goal is to characterize oriented pro-*p* RAAGs whose $\mathbb{F}_p$-cohomology algebra yields no essential Massey products in terms of the associated digraph.

Theorem 1.1. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph, and let G be the oriented pro-p RAAG associated to Γ and to a p-power p^f (with $f \geq 2$ in case $p = 2$). Then the following are equivalent:*

(i) *for every $n > 2$ there are no essential n-fold Massey products in $\mathbf{H}^\bullet(G, \mathbb{F}_p)$;*

(ii) *for every $n > 2$, if $\alpha_1, \ldots, \alpha_n$ is a sequence of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$ satisfying*

$$\alpha_1 \smile \alpha_2 = \alpha_2 \smile \alpha_3 = \ldots = \alpha_{n-1} \smile \alpha_n = 0,$$

then the associated n-fold Massey product contains 0;

(iii) *the digraph Γ is special-clique.*

(The property described in item (ii) of Theorem 1.1 is called strong n-Massey vanishing property, and it was introduced by A. Pál and E. Szabó in [31], see also Remark 5.3 below.)

Here a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is said to be special-clique if it satisfies the following two conditions:

(a) every vertex which is the terminal vertex of a directed edge (i.e., which is black) is a sinkhole;

(b) the initial vertices of directed edges with the same terminal vertex are all joined to each other.

If Γ satisfies (at least) condition (a), then it is said to be a special digraph. (For the detailed definitions see Definition 2.4 below). For example, the digraph represented in (1.1) is not special-clique, nor special: both conditions (a)–(b) are not satisfied, as the vertex v_1 is a sinkhole, but v_2 is not — as for condition (a) —, and the vertices v_2, v_4 are origins of directed edges pointing both at the sinkhole v_1 but are not joined to each other — as for condition (b).

We prove Theorem 1.1 using the "pro-p translation" of a result of W. Dwyer (see [7]) which interprets the existence of Massey products in the $\mathbb{F}_p$-cohomology of a group G in terms of the existence of certain representations from G to the group of upper unitriangular matrices with entries in $\mathbb{F}_p$ (see Proposition 5.4 below).

Also, we exploit the fact that the $\mathbb{F}_p$-cohomology groups of degree 1 and 2 of an oriented pro-p RAAG are completely described in terms of the incidence structure of the associated digraph: namely, if G is the oriented pro-p RAAG associated to a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ and to a p-power q, then $\mathrm{H}^1(G, \mathbb{F}_p) \simeq \Lambda_1(\Gamma^*)$ and $\mathrm{H}^2(G, \mathbb{F}_p) \simeq \Lambda_2(\Gamma^*)$, where $\Lambda_i(\Gamma^*)$ is the subspace of degree i of the exterior Stanley-Reisner $\mathbb{F}_p$-algebra $\mathbf{\Lambda}_\bullet(\Gamma^*)$ associated to Γ.

It is worth underlining that Theorem 1.1 presents a phenomenon which is analogous to the example of the Borromean rings: Massey products in the $\mathbb{F}_p$-cohomology of oriented pro-p RAAGs detect a combinatorial property of the underlying graphs — being special-clique —, while the $\mathbb{F}_p$-cohomology groups of degree 1 and 2 (which depend only on the incidence structure of the underlying graphs, as stated above) do not.

It is well-known that the cohomology algebra with coefficients in an arbitrary field of the discrete RAAG associated to a graph (with only undirected edges) is the associated exterior Stanley-Reisner algebra over that field (cf., e.g., [1, Thm. 1.2] or [32, § 3.2]). Moreover, by a result of K. Lorensen, one knows that the $\mathbb{F}_p$-cohomology of the pro-p RAAG associated to a graph (with only undirected edges) is isomorphic to the associated exterior Stanley-Reisner $\mathbb{F}_p$-algebra (see [18, Thm. 2.6]). On the other hand, it is an open problem to determine which digraphs yield oriented pro-p RAAGs whose whole $\mathbb{F}_p$-cohomology algebra is isomorphic to the whole associated exterior Stanley-Reisner $\mathbb{F}_p$-algebra (see [35, § 5.6]). We prove that this happens for oriented pro-p RAAGs whose $\mathbb{F}_p$-cohomology has no essential Massey products.

Theorem 1.2. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a special-clique digraph, and let G the oriented pro-p RAAG associated to Γ and to a p-power $q = p^f$ (with $f \geq 2$ in case $p = 2$). Then*

$$\mathbf{H}^\bullet(G, \mathbb{F}_p) \simeq \mathbf{\Lambda}_\bullet(\Gamma^*).$$

We prove Theorem 1.2 following the strategy outlined in [35, § 5.6], and employing the combinatorial properties of special-clique digraphs.

Theorems 1.1–1.2 provide concrete examples of pro-p groups whose $\mathbb{F}_p$-cohomology: gives rise to no essential Massey products — and, complementarily, with essential Massey products —; is a quadratic $\mathbb{F}_p$-algebra (see § 2.4 below). As it often happens in profinite group theory, it is very important to find concrete examples of profinite groups satisfying certain given properties, especially when there there is an astounding lack of such examples (as in the case of pro-p groups with quadratic $\mathbb{F}_p$-cohomology, see [35, § 1.1]). At this aim, it is worth stressing that the examples of pro-p groups, whose $\mathbb{F}_p$-cohomology gives rise to essential Massey products, studied so far have presentations whose relations involve higher commutators (see, e.g., [27, § 7]); while oriented pro-p RAAGs have relations involving only elementary commutators times, possibly, powers of generators.

One of the aims of the present work is to provide an introduction to the study of Massey products in the $\mathbb{F}_p$-cohomology of pro-p groups — with a mostly group-theoretic perspective and quite a concrete approach — for non-specialists in Galois cohomology, accessible (and, hopefully, appealing) to a broad audience, in particular to graduate students working in profinite group theory. Moreover, we believe that the techniques used to prove Theorem 1.1 may be useful for further future investigations on Massey products in Galois cohomology.

1.4. **Structure of the paper.** In § 2 we recall some definitions on digraphs (cf. § 2.1), and we study some properties of special-clique graphs (cf. § 2.2–2.3). Also, we recall the definition of the exterior Stanley-Reisner algebra associated to a digraph (cf. § 2.4).

In § 3, after introducing some notation (cf. § 3.1), we recall some properties of oriented pro-p RAAGs, providing several concrete examples (cf. § 3.2–3.3).

In § 4 we provide a short vademecum on $\mathbb{F}_p$-cohomology of pro-p groups (cf. § 4.1), for the convenience of the reader, and we recall some facts on the $\mathbb{F}_p$-cohomology of oriented pro-p RAAGs (cf. § 4.2). Then, we prove Theorem 1.2 (cf. § 4.3).

In § 5 we recall some definitions on Massey products, and the "translation" of Massey products in terms of upper unitriangular representations. Also, we prove some facts on upper unitriangular matrices with entries in $\mathbb{F}_p$ which will be used to prove Theorem 1.1.

Finally, § 6 is devoted to the proof of Theorem 1.1: the equivalence between statement (i) and statement (iii) is proved in § 6.1, while the equivalence between statement (ii) and statement (iii) is proved in § 6.2.

2. Digraphs and pro-p groups

2.1. **Digraphs.** The formal definition of a digraph — as it may be found, e.g., in [5, § 1.10] — is the following: a pair of disjoint sets $\Gamma = (\mathcal{V}, \mathcal{E})$ together with two maps $i \colon \mathcal{E} \to \mathcal{V}$ and $t \colon \mathcal{E} \to \mathcal{V}$ — the first one gives the initial vertex of a directed edge, and the second one gives the terminal vertex of a directed edge Since we assume throughout the paper that every digraph has no loops — i.e., no directed edges $\mathbf{e} \in \mathcal{E}$ satisfying $i(\mathbf{e}) = t(\mathbf{e})$ —, nor parallel directed edges — i.e., two directed edges $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}$ are parallel if $i(\mathbf{e}_1) = i(\mathbf{e}_2)$ and $t(\mathbf{e}_1) = t(\mathbf{e}_2)$ —, we can consider $\mathcal{E}$ as a subset of $\mathcal{V} \times \mathcal{V} \smallsetminus \{(v, v)\}$, as done in the Introduction.

In the following we recall some definitions which generalize some well-known notions on graphs (cf., e.g., [5, pp. 3–4]).

Definition 2.1. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph.

(a) Γ is said to be complete if for any couple $v, w \in \mathcal{V}$, one has $(v, w) \in \mathcal{E}$ or $(w, v) \in \mathcal{E}$.

(b) An induced subdigraph of Γ is a digraph $\Gamma' = (\mathcal{V}', \mathcal{E}')$ such that $\mathcal{V}' \subseteq \mathcal{V}$ and
$$\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}' \times \mathcal{V}');$$
it is said to be proper if $\mathcal{V}' \subsetneq \mathcal{V}$ and $\mathcal{V}' \neq \varnothing$. In particular, an induced subdigraph of Γ which is a complete digraph is called a clique of Γ.

(c) The star of a vertex $v \in \mathcal{V}$ is the induced subdigraph $\mathrm{St}(v) = (\mathcal{V}_v, \mathcal{E}_v)$ of Γ whose vertices are v and all other vertices of Γ which are adjacent to v, i.e.,
$$\mathcal{V}_v = \{\, v \,\} \cup \{\, w \mid (v, w) \in \mathcal{E} \text{ or } (w, v) \in \mathcal{E} \,\}.$$

For digraphs, we will make use of the following.

Definition 2.2. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph.

(a) We call a vertex $w \in \mathcal{V}$ a special vertex if there exists at least another vertex $v \in \mathcal{V}$ such that $(v, w) \in \mathcal{E}$ but $(w, v) \notin \mathcal{E}$; otherwise we call v an ordinary vertex.

(b) We call a special vertex w a sinkhole if for any other vertex $v \in \mathcal{V}$ such that $(v, w) \in \mathcal{E}$ one has $(w, v) \notin \mathcal{E}$.

From now on, if $(v, w), (w, v) \in \mathcal{E}$ for two vertices $v, w \in \mathcal{V}$, we will identify these two directed edges and (with an abuse of notation) we will consider them as a single, undirected, edge; while by a directed edge we will mean only an edge $(v, w) \in \mathcal{E}$ such that $(w, v) \notin \mathcal{E}$. We will represent a digraph as follows: special and ordinary vertices are respectively black and white dots, while if $(v, w), (w, v) \in \mathcal{E}$ we draw a single unoriented arc joining the vertices v and w. For example, the three diagrams

(2.1)
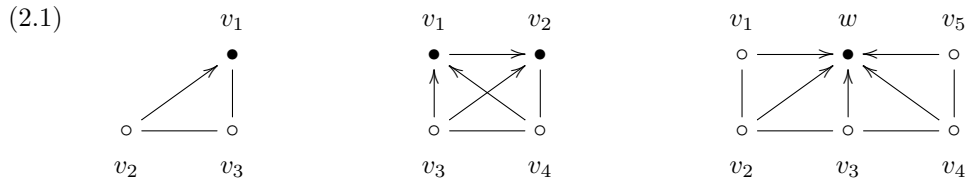
represent three digraphs, respectively with 3, 4 and 6 vertices. Observe that a special vertex may be an end of an undirected edge (as v_1 in the first diagram), or even the

first coordinate of a directed edge (as v_1 in the second diagram, or v_2 in (1.1)); while the vertex w in the third diagram (and also v_1 in (1.1)) is a sinkhole.

Remark 2.3. Given a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$, set

$$|\mathcal{E}| = \{ \{v, w\} \in \mathcal{P}_2(\mathcal{V}) \mid (v, w) \in \mathcal{E} \text{ or } (w, v) \in \mathcal{E} \}.$$

Then the pair $|\Gamma| = (\mathcal{V}, |\mathcal{E}|)$ is a graph (in the sense of [5, § 1.1]). Conversely, from a graph $G = (\mathcal{V}, E)$ — where $E \subseteq \mathcal{P}_2(\mathcal{V})$ — one may construct the digraph $G_{or} = (\mathcal{V}, E_{or})$, with

$$E_{or} = \{ (v, w), (w, v) \in \mathcal{V} \times \mathcal{V} \mid \{v, w\} \in E \}.$$

Clearly, $|G_{or}| = G$; on the other hand, given a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$, one has $|\Gamma|_{or} = \Gamma$ if, and only if, Γ has only undirected edges. Henceforth, we will identify graphs and digraphs with only undirected edges via the functors $\_{or}$ and $|\_|$, and we will call the latter "undigraphs".

2.2. **Special digraphs and special-clique digraphs.**

**Definition 2.4.** Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph.

(a) Γ is said to be special if every special vertex is a sinkhole.
(b) Γ is said to be special-clique if it is special and moreover the star of every special vertex is a clique of Γ.

The former definition was introduced in [3, § 2.3]. It is straightforward to see that both properties are inherited by induced subdigraphs.

Remark 2.5. (a) An undigraph is always special and special-clique, as it has no special vertices (and thus the conditions in Definition 2.4 are trivially satisfied).
(b) If $\Gamma = (\mathcal{V}, \mathcal{E})$ is a special digraph and $v_1, v_2 \in \mathcal{V}$ are two distinct special vertices, then v_1, v_2 are disjoint — namely, $(v_1, v_2), (v_2, v_1) \notin \mathcal{E}$.
(c) A complete digraph is special (and special-clique) if, and only if, it has at most a special vertex, and such a vertex is a sinkhole.

For example, the first two digraphs represented in (2.1) are not special — and thus neither special-clique —, as their directed edges are not sinkholes. On the other hand, the third diagram represents a special digraph, as the only special vertex is a sinkhole, but it is not special-clique, as the star of the only special vertex (which is the whole digraph) is not complete.

Example 2.6. Consider the digraphs with geometric representations

(2.2)
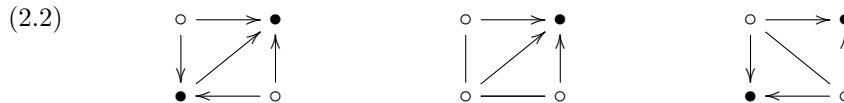
The left-one is not special as the bottom special vertex is not a sinkhole; the center-one is special but not special-clique as the star of the only special vertex is not a clique; the right-one is special-clique.

It is easy to see that a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is special if, and only if, it contains no induced subdigraphs with three vertices whose geometric representation is

(2.3)

— no matter whether the bottom vertices are joined or not (here we use $\circledast$ to represent vertices which are not necessarily ordinary nor special). Moreover, a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is special-clique if, and only if, it contains no induced subdigraphs with three vertices whose geometric representation is as in (2.3), nor

(2.4)

2.3. **Patching of digraphs.** a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is said to be the patching of two induced subdigraphs Γ_1, Γ_2 along a common subdigraph Γ' if there are three induced subdigraphs $\Gamma_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $\Gamma_2 = (\mathcal{V}_2, \mathcal{E}_2)$ and $\Gamma' = (\mathcal{V}', \mathcal{E}')$ of Γ such that

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \qquad \mathcal{V}' = \mathcal{V}_1 \cap \mathcal{V}_2, \qquad \mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$$

(cf. [3, pp. 4–5]).

Not every digraph may be constructed as the patching of two proper induced subdigraphs, as, for example, one has the following fact (whose proof is left to the reader).

Fact 2.7. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a complete digraph. Then Γ may not be constructed as the patching of two proper induced subdigraphs.*

Example 2.8.   (a) The digraph represented in (1.1) is the patching of the induced subdigraphs with vertices respectively $\{v_1, v_2, v_3\}$ and $\{v_1, v_3, v_4\}$, along the common subdigraph with vertices $\{v_1, v_3\}$.
   (b) The third digraph represented in (2.1) is the patching of the two "square" induced subdigraphs — with vertices respectively $\{w, v_1, v_2, v_3\}$ and $\{w, v_3, v_4, v_5\}$ —, which are equal, along the common subdigraph with vertices $\{w, v_3\}$; while the second digraph represented in (2.1) is not the patching of any pair of induced subdigraphs, as it is complete (cf. Lemma 2.7).
   (c) Each of the three graphs in Example 2.6 is the patching of the two "triangles" subdigraphs along the subdigraph which consists of the diagonal of the square.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a special-clique digraph, and let $\Gamma_0 = (\mathcal{V}_0, \mathcal{E}_0)$ be the induced subdigraph of Γ whose vertices $\mathcal{V}_0$ are all the ordinary vertices of Γ — roughly speaking, Γ_0 is the maximal induced subdigraph of Γ which is an undigraph. It follows from the definition of special-clique digraphs that Γ may be constructed by iterating the patching of Γ_0 with $\mathrm{St}(v) = (\mathcal{V}_v, \mathcal{E}_v)$, for every special vertex v of Γ, along the common clique with vertices $\mathcal{V}_v \cap \mathcal{V}_0$.

Example 2.9. The digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ with geometric representation
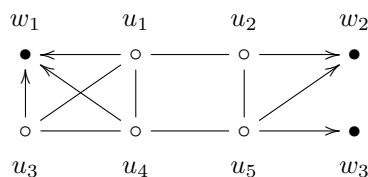
is special-clique. Let $\Gamma_0 = (\mathcal{V}_0, \mathcal{E}_0)$ be the induced subdigraph with vertices $\mathcal{V}_0 = \{u_1, \ldots, u_5\}$, and let $\Delta_1, \Delta_2, \Delta_3$ be the cliques of Γ (and of Γ_0) with vertices respectively $\{u_1, u_3, u_4\}$, $\{u_2, u_5\}$ and $\{u_5\}$. Then Γ may be constructed by patching Γ_0 with $\mathrm{St}(w_1)$ along Δ_1; and then patching the resulting digraph with $\mathrm{St}(w_2)$ along Δ_2; and finally patching the resulting digraph with $\mathrm{St}(w_3)$ along Δ_3.

2.4. The exterior Stanley-Reisner $\mathbb{F}_p$-algebra.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph, and let V^* be the $\mathbb{F}_p$-vector space with basis $\mathcal{V}^* = \{v^* \colon \mathcal{V} \to \mathbb{F}_p \mid v \in \mathcal{V}\}$, where

$$v^*(u) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

The exterior Stanley-Reisner $\mathbb{F}_p$-algebra $\boldsymbol{\Lambda}_\bullet(\Gamma^*)$ associated to Γ is the graded $\mathbb{F}_p$-algebra

$$(2.5) \qquad \boldsymbol{\Lambda}_\bullet(\Gamma^*) = \frac{\boldsymbol{\Lambda}_\bullet(V^*)}{(v^* \wedge w^* \mid \{v, w\} \notin |\mathcal{E}|)},$$

where $\boldsymbol{\Lambda}_\bullet(V^*)$ denotes the exterior $\mathbb{F}_p$-algebra generated by V^*. Namely, the exterior Stanley-Reisner $\mathbb{F}_p$-algebra associated to the digraph Γ is defined to be equal to the exterior Stanley-Reisner $\mathbb{F}_p$-algebra associated to the graph $|\Gamma|$ (see, e.g., [32, § 3.2]).

It is easy to see that for each $n \geq 0$, the space $\Lambda_n(\Gamma^*)$ has a basis which is in 1-to-1 correspondence with the cliques of Γ with n vertices — e.g., for $n = 2$ a basis is given by $\{v^* \wedge w^* \mid \{v, w\} \in |\mathcal{E}|\}$. For example, $\Lambda_n(\Gamma^*)$ is trivial if Γ has less than n vertices.

Remark 2.10. The exterior Stanley-Reisner $\mathbb{F}_p$-algebra is a quadratic graded $\mathbb{F}_p$-algebra, which means that the whole structure as a graded $\mathbb{F}_p$-algebra is determined by the spaces of degree 1 and 2 (cf., e.g., [35, § 2.1]), as the ideal we quotient on in (2.5) is generated by elements of degree 2.

3. Oriented pro-p RAAGs

3.1. Notation.

Given an arbitrary group H, the commutator of two elements $x, y \in H$ is

$$[x, y] = xyx^{-1}y^{-1}.$$

Given three elements $x, y, z \in H$, one has the well-known equalities

$$(3.1) \qquad [xy, z] = [x, [y, z]][y, z][x, x] \qquad \text{and} \qquad [x, yz] = [x, y][y, [x, z]][x, z].$$

from which one deduces

$$(3.2) \qquad [x^n, y] = [x, y]^n \qquad \text{and} \qquad [x, y^n] = [x, y]^n$$

if $x, y \in H$ satisfy, respectively, $[x, [x, y]] = 1$ and $[[x, y], y] = 1$.

Given a pro-p group G, every subgroup will be implicitly assumed to be closed, and the generators of G, or of a closed subgroup, are to be intended in the topological sense. In particular,

    (a) $G' = [G, G]$ will denote the closed derived subgroup of G, namely, G' is the closed subgroup of G generated by the commutators $[x, y]$ with x, y running through all elements of G;

    (b) for every $n \geq 1$, G^{p^n} will denote the closed subgroup generated by the elements x^{p^n}, with x running through all elements of G;

(c) the Frattini subgroup $\Phi(G)$ of G is the closed subgroup of G generated by G' and G^p.

All the above subgroups are normal subgroups; in particular the quotient $G/\Phi(G)$ is an elementary abelian p-group.

3.2. **Oriented pro-p RAAGs.** From now on, q will denote a power p^f of the prime number p, with $f \geq 1$ — we will tacitly assume that $f \geq 2$ in case $p = 2$.

Given a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$, let $F_\mathcal{V}$ be the free pro-p group generated by $\mathcal{V}$. Given q, let $R_{\mathcal{E},q}$ be the normal subgroup of F generated as a normal subgroup by the elements $r_{v,w}$ with $\{v, w\}$ running through the elements of $|\mathcal{E}|$, with

$$r_{v,w} = \begin{cases} [v,w] & \text{if } (v,w) \text{ is an undirected edge,} \\ [w,v]v^{-q} & \text{if } (v,w) \text{ is a directed edge.} \end{cases}$$

The oriented pro-p RAAG associated to Γ and to q is $G = F_\mathcal{V}/R_{\mathcal{E},q}$. Observe that if Γ is an undigraph, then G coincides with the pro-p RAAG associated to the graph $|\Gamma|$, i.e., the pro-p completion of the discrete RAAG associated to $|\Gamma|$ (and obviously it does not depend on the choice of q).

Remark 3.1. The normal subgroup $R_{\mathcal{E},q}$ is contained in $\Phi(F_\mathcal{V})$, whence $G = F_\mathcal{V}/R_{\mathcal{E},q}$ is a minimal presentation of G (cf., e.g., [30, Ch. III, § 9, pp. 224–226]). In particular, $\mathcal{V}$ is a minimal generating set of G.

Example 3.2. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a complete special digraph, with $\mathcal{V} = \{w, v_1, \ldots, v_d\}$, $d \geq 1$ and w the only special vertex (cf. Remark 2.5–(c)). Then the oriented pro-p RAAG associated to Γ and q is the pro-p group

$$G = \left\langle w, v_1, \ldots, v_d \mid [v_i, v_j] = 1, \ wv_iw^{-1} = v_i^{1+q} \ \forall\, i = 1, \ldots, d \right\rangle \simeq \mathbb{Z}_p^d \rtimes \mathbb{Z}_p.$$

Remark 3.3. In [3], the oriented pro-p RAAG associated to a digraph Γ and to q is defined to be the pair (G, θ), where G is the pro-p group defined above, and $\theta\colon G \to 1+p\mathbb{Z}_p$ is a homomorphism of pro-p groups, the orientation of the oriented pro-p RAAG. Since we will make no use of such homomorphism, for simplicity we define an oriented pro-p RAAG to be only the pro-p group defined above, without considering the associated homomorphism θ.

The pro-p RAAGs associated to undigraphs are torsion-free pro-p groups; in fact, more generally, if Γ is an undigraph and Γ' is an induced subdigraph of Γ, the pro-p RAAG associated to Γ' is a subgroup of the pro-p RAAG associated to Γ. This may not occur for an oriented pro-p RAAGs associated to a digraph, as shown by the following examples.

Example 3.4. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be the digraph with geometric representation
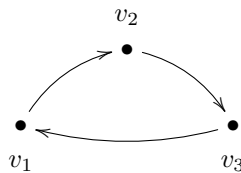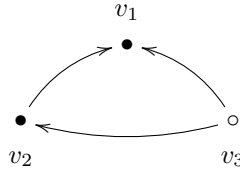
For $q = p$, the associated oriented pro-p RAAG is

$$G = \langle\, v_1, v_2, v_3 \ \mid\ [v_1, v_2] = v_2^p, [v_2, v_3] = v_3^p, [v_3, v_1] = v_1^p \,\rangle,$$

which is a finite p-group, as shown by J. Mennike (cf. [37, Ch. I, § 4.4, Ex. 2(e)]). In particular, the oriented pro-p RAAG associated to a single vertex (which is isomorphic to $\mathbb{Z}_p$) and the oriented pro-p RAAG associated to an edge (which is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_p$, cf. Example 3.2) are not subgroups of G (cf. [3, Ex. 4.7]).

Example 3.5. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be the digraph with geometric representation

For $q = p$, the associated oriented pro-p RAAG is

$$G = \langle\, v_1, v_2, v_3 \ \mid\ [v_1, v_2] = v_2^p,\ [v_1, v_3] = [v_2, v_3] = v_3^p \,\rangle.$$

Then $[v_2^q, v_3] = v_3^{(1+q)^q - 1}$; on the other hand, one computes

$$[[v_2^q, v_3]] = \left[v_1 v_2 v_1^{-1} v_2^{-1}, v_3\right] = v_1 \left(v_2 \left(v_1^{-1} \left(v_2^{-1} v_3 v_2\right) v_1\right) v_2^{-1}\right) v_1^{-1} \cdot v_3^{-1}$$

$$= \left(\left(\left(v_3^{(1+q)^{-1}}\right)^{\epsilon(1+q)^{-1}}\right)^{1+q}\right)^{\epsilon(1+q)} \cdot v_3^{-1}$$

$$= v_3^1 \cdot v_3^{-1} = 1.$$

Since $(1+q)^q - 1 = (1 + q^2 + q^3(q-1)/2 + \ldots) - 1 \neq 0$, the vertex v_3 yields non-trivial torsion. In particular, the oriented pro-p RAAG associated to the digraph consisting of the single vertex v_3 is not a subgroup of G.

Observe that the digraphs in Examples 3.4–3.5 are not special. Indeed, for special digraphs one has the following result (cf. [3, Prop. 4.11]).

Proposition 3.6. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a special digraph, and let G be the oriented pro-p RAAG associated to Γ and to q. If $\Delta = (\mathcal{V}_\Delta, \mathcal{E}_\Delta)$ is a clique of Γ and G_Δ the oriented pro-p RAAG associated to Δ, then the inclusion $\mathcal{V}_\Delta \hookrightarrow \mathcal{V}$ induces a monomorphism of pro-p groups $G_\Delta \hookrightarrow G$.*

Consequently, if $\Gamma = (\mathcal{V}, \mathcal{E})$ is a special-clique graph, and G is an associated oriented pro-p RAAG, then for every special vertex $v \in \mathcal{V}$ both the oriented pro-p RAAG $G_{\mathrm{St}(v)}$ associated to the star $\mathrm{St}(v) = (\mathcal{V}_v, \mathcal{E}_v)$, and the oriented pro-$p$ RAAG A associated to the clique with vertices $\mathcal{V}_v \smallsetminus \{v\}$ (which is a free abelian pro-p group), are subgroups of G.

Remark 3.7. In [35, § 5], the authors introduced the notion of generalized p-RAAGs associated to p-labeled oriented graphs. The family of oriented pro-p RAAGs associated to digraphs is a subfamily of the family of generalized p-RAAGs (cf. [3, § 4.2]), thus all results on generalized p-RAAGs obtained in [35] apply also to oriented pro-p RAAGs associated to digraphs.

3.3. **Oriented pro-p RAAGs, free products and amalgams.** Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph, and for q a p-power let G be the associated pro-p RAAG. It follows from the definition of G that if Γ has connected components $\Gamma_1, \ldots, \Gamma_r$, then G is the free pro-p product of the oriented pro-p RAAGs $G_1, \ldots, G_r$ associated respectively to the digraphs $\Gamma_1, \ldots, \Gamma_r$, i.e., $G = G_1 \amalg \ldots \amalg G_r$ (for the definition of free pro-p product of pro-p groups see, e.g., [36, § 9.1]).

For patching of digraphs we may have a similar phenomenon. Recall that the the amalgamated free pro-p product G of two pro-p groups G_1, G_2 with amalgam a common subgroup $H \subseteq G_1, G_2$, is the push-out

$$
\begin{array}{ccc}
 & G_1 & \\
 & \overset{\psi_1}{\searrow} & \\
H \hookrightarrow & & G \\
 & \searrow \nearrow & \\
 & G_2 & \overset{\psi_2}{}
\end{array}
$$

in the category of pro-p groups, which is unique (cf. [36, § 9.2]). We write $G = G_1 \amalg_H G2$. An amalgamated free pro-p product $G = G_1 \amalg_H G_2$ is said to be proper if the homomorphisms ψ_1, ψ_2 are monomorphisms. In that case one identifies G_1, G_2, H with their images in G.

In case of special digraphs one has the following result, which is the "oriented version" of [35, Prop. 5.22], which deals with a more general case (cf. Remark 3.7).

Lemma 3.8. *Let $\Gamma_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\Gamma_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be two special digraphs, and let $\Delta = (\mathcal{V}_\Delta, \mathcal{E}_\Delta)$ be a clique of both Γ_1 and Γ_2 whose vertices are ordinary. Moreover, for $q = p^f$ let G_1, G_2, G_Δ be the oriented pro-p RAAGs associated respectively to $\Gamma_1, \Gamma_2, \Delta$. Then G_Δ is a subgroup of both G_1 and G_2, and the amalgamated free pro-p product*

$$G = G_1 \amalg_{G_\Delta} G_2$$

is proper.

Proof. Since Δ is a complete undigraph, G_Δ is the free abelian pro-p group generated by the set of vertices $\mathcal{V}_\Delta$. Moreover, G_Δ is a subgroup of both G_1, G_2 by Proposition 3.6.

Now fix $i = 1$ or $i = 2$, and let $\mathcal{V}_0$ be the set of ordinary vertices of Γ_i — thus $\mathcal{V}_0 \supseteq \mathcal{V}_\Delta$ —, and let $\mathcal{Y}$ be the set $\mathcal{Y} = \{y_u \mid u \in \mathcal{V}_0\}$. Moreover, let A be the free abelian pro-p group generated by $\mathcal{Y}$, and let $\bar{G}$ be the pro-p group

$$\bar{G} = A \rtimes X = \langle \, \mathcal{Y}, x^\epsilon \mid [y_u, y_{u'}] = 1, [x, y_u] = y_u^{\epsilon q} \, \forall \, u, u' \in \mathcal{V}_0 \, \rangle,$$

where X is either trivial — and hence $\epsilon = 0$ — if Γ_i has no special vertices; or X is the pro-p-cyclic group generated by an element x such that $xyx^{-1} = y^{1+q}$ for every $y \in A$ — and hence $\epsilon = 1$ — if Γ_i has at least a special vertex. By construction, one has

$$(3.3) \qquad\qquad \bar{G}^{p^n} = A^{p^n} \rtimes X^{p^n}.$$

(Observe that $\bar{G}$ is isomorphic to the oriented pro-p RAAG associated to a complete special digraph whose ordinary vertices are $\mathcal{Y}$, and whose only special vertex — if X is not trivial — is x, cf. Example 3.2.) Now, the assignment

$$u \longmapsto y_u \qquad \text{and} \qquad w \longmapsto x$$

for every $u \in \mathcal{V}_0$ and for every special vertex $w \in \mathcal{V}$ — if there is any, and thus if $X \neq \{1\}$ — induces a homomorphism of pro-p groups $\pi \colon G_i \to \bar{G}$, as

$$\pi([u, u']) = [y_u, y_{u'}] = 1 \qquad \text{and} \qquad \pi([w, u]) = [x, y_u] = y_u^q = \pi(u^q)$$

for every $u, u' \in \mathcal{V}_0$, and for every special vertex $w \in \mathcal{V}$ (recall that there are no relations between the special vertices of Γ_i, as they are disjoint, for Γ is special). Moreover, π is clearly surjective. On the other hand, $\pi(G_\Delta) = A_\Delta$, where $A_\Delta \subseteq A$ is the free abelian pro-p group generated by $\{y_u \mid u \in \mathcal{V}_\Delta\}$. Thus, the restriction $\pi|_{G_\Delta} \colon G_\Delta \to A_\Delta$ is an isomorphism of (abelian) pro-p groups.

We claim that

$$(3.4) \qquad\qquad G_\Delta^{p^n} = G_\Delta \cap G_i^{p^n} \qquad \text{for every } n \geq 1.$$

The inclusion $G_\Delta^{p^n} \subseteq G_\Delta \cap G_i^{p^n}$ is obvious. Conversely, let z be an element of G_Δ such that $z \in G_i^{p^n}$. Then $\pi(z) \in \bar{G}^{p^n}$, and since $\pi(z) \in A_\Delta \subseteq A$, by (3.3) one has $\pi(z) \in A_\Delta^{p^n}$. Finally,

$$z = \pi^{-1}(\pi(z)) \in G_\Delta^{p^n},$$

as $\pi|_{G_\Delta} \colon G_\Delta \to A_\Delta$ is an isomorphism of abelian pro-p groups.

Since i was arbitrarily chosen, (3.4) holds for both $i = 1, 2$. Then [36, Thm. 9.2.4] implies that the amalgamated free pro-p product is proper. $\qquad\square$

It may happen that an oriented, but not special, graph decomposes as patching of two induced subdigraphs, but the associated oriented pro-p RAAG is not the amalgamated free pro-p product of the two oriented pro-p RAAGs associated to the two induced subdigraphs, as shown by the following.

Example 3.9. Let $\tilde{\Gamma} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ be the digraph with geometric representation (1.1) — namely,
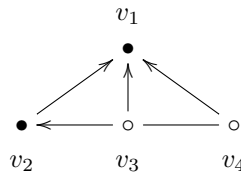
—, and let $\Gamma = (\mathcal{V}, \mathcal{E})$, $\Gamma' = (\mathcal{V}', \mathcal{E}')$ and $\Delta = (\mathcal{V}_\Delta, \mathcal{E}_\Delta)$ be the induced subdigraphs of $\tilde{\Gamma}$ with vertices respectively

$$\mathcal{V} = \{v_1, v_2, v_3\}, \qquad \mathcal{V}' = \{v_1, v_3, v_4\}, \qquad \mathcal{V}_\Delta = \{v_2, v_3\}.$$

Hence Γ is as in Example 3.5, and $\tilde{\Gamma}$ is the patching of Γ and Γ' along Δ.
For $q = p$ let $\tilde{G}$, G, H and G_Δ be the oriented pro-p RAAGs associated respectively to $\tilde{\Gamma}$, Γ, Γ' and Δ. Hence G is as in Example 3.5, while $H \simeq \mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$ and $G_\Delta \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_p$ (cf. Example 3.2). Then $\tilde{G}$ is not an amalgamated free pro-p product of G and H: the subgroup of H generated by v_1, v_3 is G_Δ by Proposition 3.6, and it is torsion-free, while the subgroup of G generated by v_1, v_3 has non-trivial torsion by Example 3.5.

4. The $\mathbb{F}_p$-cohomology of oriented pro-p RAAGs

4.1. $\mathbb{F}_p$-cohomology of pro-p groups in a nutshell. Throughout this subsection, G will denote a pro-p group, and the finite field $\mathbb{F}_p$ will be considered as a trivial G-module. For $n \geq 0$, we will denote the nth $\mathbb{F}_p$-cohomology group $\mathrm{H}^n(G, \mathbb{F}_p)$ simply by $\mathrm{H}^n(G)$. For the reader's convenience, in this subsection (which may be safely skipped by who is accustomed with group cohomology) we recall briefly some basic facts on the cohomology of G with coefficients in $\mathbb{F}_p$ — for a more detailed account see [37, Ch. I, §§ 2–4] or [30, Ch. I], to which we make reference.

4.1.1. *Degree 0 and 1.* The 0th $\mathbb{F}_p$-cohomology group $\mathrm{H}^0(G)$ is just $\mathbb{F}_p$, while one has an equality of discrete $\mathbb{F}_p$-vector spaces

$$(4.1) \qquad\qquad \mathrm{H}^1(G) = \mathrm{Hom}(G, \mathbb{F}_p),$$

where the former is the group of homomorphisms of pro-p groups $G \to \mathbb{F}_p$, where the latter is considered as a cyclic group of order p, cf. [37, Ch. I, § 4.2].

If G is finitely generated, then (4.1) induces an isomorphism of finite $\mathbb{F}_p$-vector spaces

$$\mathrm{H}^1(G) \simeq (G/\Phi(G))^*,$$

and $\lrcorner^*$ denotes the $\mathbb{F}_p$-dual. In particular, the dimension of $\mathrm{H}^1(G)$ as a $\mathbb{F}_p$-vector space is equal to the minimum number of generators of G (cf. [37, p. 31]).

4.1.2. *Higher degrees.* For $n \geq 1$, the nth $\mathbb{F}_p$-cohomology group $\mathrm{H}^n(G)$ is a subquotient of the discrete $\mathbb{F}_p$-vector space

$$C^n(G) = \left\{ f \colon \underbrace{G \times \ldots \times G}_{n \text{ times}} \to \mathbb{F}_p \mid f \text{ is a continuous map} \right\},$$

where the Cartesian product $G \times \ldots \times G$ has the product topology, and $\mathbb{F}_p$ is a discrete space (cf. [37, Ch. I, § 2.2]). (If $n = 0$, $\mathrm{H}^0(G) = \mathbb{F}_p$ may be seen as the space of constant functions.)

4.1.3. *Cup-product.* Pick two elements $\alpha_1 \in \mathrm{H}^{n_1}(G)$ and $\alpha_2 \in \mathrm{H}^{n_2}(G)$, for some $n_1, n_2 \geq 0$, and let $f_1 \in C^{n_1}(G)$ and $f_2 \in C^{n_2}(G)$ be representatives respectively of α_1 and α_2. Then the continuous map $f_1 \cdot f_2 \in C^{n_1+n_2}(G)$, defined pointwise via the product of $\mathbb{F}_p$ as a field, belongs to some element of $\mathrm{H}^{n_1+n_2}(G)$, which is called the cup-product of α_1 and α_2, and it is denoted by $\alpha_1 \smile \alpha_2$ (cf. [30, Ch. I, § 4]). Moreover, one has

$$\alpha_2 \smile \alpha_1 = (-1)^{n_1 n_2} \alpha_1 \smile \alpha_2$$

(cf. [30, Prop. 1.4.4]). Altogether, the space $\mathbf{H}^\bullet(G) = \coprod_{n \geq 0} \mathrm{H}^n(G)$ endowed with the cup-product, is a graded, and graded-commutative, $\mathbb{F}_p$-algebra.

4.1.4. *Functoriality.* Let $\psi \colon G_1 \to G_2$ be a homomorphism of pro-p groups. In cohomology, ψ induces a homomorphism $\psi^n \colon \mathrm{H}^2(G_2) \to \mathrm{H}^n(G_1)$ for every $n \geq 1$, where

$$\psi^1(\alpha) = \alpha \circ \psi \colon G_1 \longrightarrow \mathbb{F}_p$$

for every $\alpha \in \mathrm{H}^1(G_2)$ (cf. [37, Ch. I, § 2.4]). If ψ is surjective, then one has an exact sequence

$$
\begin{array}{ccccc}
0 \longrightarrow & \mathrm{H}^1(G_2) & \xrightarrow{\psi^1} & \mathrm{H}^1(G_1) & \longrightarrow & \mathrm{H}^1(\mathrm{Ker}(\psi))^{G_1} \\
& & & & & \\
& & \mathrm{H}^2(G_2) & \xrightarrow{\psi^2} & \mathrm{H}^2(G_1)
\end{array}
$$

(4.2)

where $\mathrm{H}^1(\mathrm{Ker}(\psi))^{G_1}$ denotes the subspace of $\mathrm{H}^1(\mathrm{Ker}(\psi))$ of the elements fixed by the action given by $g.\alpha(h) = \alpha(g^{-1}hg)$ for every $g \in G_1$, $h \in \mathrm{Ker}(\psi)$ and $\alpha \in \mathrm{H}^1(\mathrm{Ker}(\psi))$ (cf. [37, Ch. I, § 2.6–(b)]).

On the other hand, if $H = G_1$ is a subgroup of $G = G_2$, and $\psi\colon H \to G$ is the inclusion, for every $n \geq 1$ the map ψ^n is called the restriction (of degree n), and denoted by $\mathrm{res}_{G,H}^n$ (cf. [37, p. 12]). If $\alpha \in \mathrm{H}^1(G) = \mathrm{Hom}(G, \mathbb{F}_p)$, we write simply $\mathrm{res}_{G,H}^1(\alpha) = \alpha|_H$. For every $\alpha_1 \in \mathrm{H}^{n_1}(G)$ and $\alpha_2 \in \mathrm{H}^{n_2}(G)$, with $n_1, n_2 \geq 0$ and $n = n_1 + n_2$, one has

$$(4.3) \qquad \mathrm{res}_{G,H}^n(\alpha_1 \smile \alpha_2) = \mathrm{res}_{G,H}^{n_2}(\alpha_1) \smile \mathrm{res}_{G,H}^{n_2}(\alpha_2)$$

(cf. [37, p. 15]).

4.1.5. *Degree 2.* Let $G = F/R$ be a minimal presentation of G, and suppose that G (and thus also F) is finitely generated, and R is a finitely generated as a normal subgroup of F. By (4.1) the canonical projection $\pi\colon F \to G$ yields an isomorphism in cohomology $\pi^1\colon \mathrm{H}^1(G) \simeq \mathrm{H}^1(F)$; on the other hand $\mathrm{H}^2(G)$ is trivial as F is free (cf. [37, Ch. I, § 3.4, Cor. p. 23]). Finally, the vector space $\mathrm{H}^1(R)^F$ is isomorphic to $(R/R^p[R,F])^*$. Altogether, (4.2) applied to $\psi = \pi$ implies that π^2 is an isomorphism, too, and thus one has an isomorphism

$$(4.4) \qquad \mathrm{trg}\colon \left(\frac{R}{R^p[R,F]}\right)^* \xrightarrow{\sim} \mathrm{H}^2(G),$$

called transgression (cf. [37, Ch. I, § 4.3]). In particular, the dimension of $\mathrm{H}^2(G)$ as a $\mathbb{F}_p$-vector space is equal to the minimal number of generators of R as a normal subgroup of F.

4.2. **$\mathbb{F}_p$-cohomology of oriented pro-p RAAGs.** Let G be the oriented pro-p RAAG associated to a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ and to q. Since G is minimally generated by $\mathcal{V}$ (cf. Remark 3.1), for every $v \in \mathcal{V}$ the map v^* induces a homomorphism of pro-p groups $G \to \mathbb{F}_p$, which we will denote (with a slight abuse of notation) by v^* as well. Then by (4.1) one has

$$(4.5) \qquad \mathrm{H}^1(G) = \Lambda_1(\Gamma^*) = V^*.$$

This equality extends to an isomorphism

$$(4.6) \qquad \lambda_2\colon \Lambda_2(\Gamma^*) \xrightarrow{\sim} \mathrm{H}^2(G),$$

where $\lambda_2(v^* \wedge w^*) = v^* \smile w^*$ (cf. [35, Lemma 5.8]) — observe that

$$\dim\left(\mathrm{H}^2(G)\right) = \mathrm{card}(|\mathcal{E}|) = \dim\left(\Lambda_2(\Gamma^*)\right).$$

In particular, if one considers the minimal presentation $G = F_{\mathcal{V}}/R_{\mathcal{E},q}$, then the $\mathbb{F}_p$-vector space $R_{\mathcal{E},q}/N$ — where we set $N = (R_{\mathcal{E},q})^p[R_{\mathcal{E}_q}, F_{\mathcal{V}}]$ — has a basis $\{r_{v,w}N \mid \{v,w\} \in |\mathcal{E}|\}$, and the isomorphism λ_2 and the transgression (4.4) are described by

$$
v^* \wedge w^* \quad \xrightleftharpoons[\lambda_2^{-1}]{\lambda_2} \quad v^* \smile w^* \quad \xrightleftharpoons[\mathrm{trg}]{\mathrm{trg}^{-1}} \quad \pm (r_{v,w}N)^*
$$

for every $\{v,w\} \in |\mathcal{E}|$ (the sign of the right-side term is minus if (v,w) is special, as in this case the commutator showing up in $r_{v,w}$ is $[w,v] = [v,w]^{-1}$).

Remark 4.1. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph, and let G be the oriented pro-p RAAG associated to Γ and to q. Suppose that Γ has an induced subdigraph $\Gamma' = (\mathcal{V}', \mathcal{E}')$ such that the oriented pro-p RAAG H associated to Γ' and to q is a subgroup of G via the inclusion $\mathcal{V}' \hookrightarrow \mathcal{V}$ (e.g., Γ is special and Γ' is a clique, cf. Proposition 3.6). Then the map $\mathrm{res}^1_{G,H} \colon \mathrm{H}^1(G) \to \mathrm{H}^1(H)$ is given by

$$
\mathrm{res}^1_{G,H}(v^*) = \begin{cases} v^*|_H & \text{if } v \in \mathcal{V}', \\ 0 & \text{if } v \notin \mathcal{V}'; \end{cases}
$$

while the map $\mathrm{res}^2_{G,H} \colon \mathrm{H}^2(G) \to \mathrm{H}^2(H)$ is given by

$$
\mathrm{res}^2_{G,H}(v^* \smile w^*) = \begin{cases} (v^*|_H) \smile (w^*|_H) & \text{if } v, w \in \mathcal{V}', \\ 0 & \text{if } \{v,w\} \notin \mathcal{E}', \end{cases}
$$

and its kernel is

$$
\begin{aligned}
\mathrm{Ker}(\mathrm{res}^2_{G,H}) &= \mathrm{Span}_{\mathbb{F}_p}\left(v^* \smile w^* \mid \{v,w\} \notin \mathcal{E}' \right) \\
&= \mathrm{H}^1(G) \smile \mathrm{Ker}(\mathrm{res}^1_{G,H}),
\end{aligned}
$$

where the latter is the subspace of $\mathrm{H}^2(G)$ generated by the cup-products of elements of the first factor with elements of the second factor.

By the result of K. Lorensen [18, Thm. 2.6], if Γ is an undigraph and G is the associated pro-p RAAG, then (4.5)–(4.6) extend to an isomorphism of graded $\mathbb{F}_p$-algebras $\mathbf{\Lambda}_\bullet(\Gamma^*) \simeq \mathbf{H}^\bullet(G)$. In general, this is not the case for oriented pro-p RAAGs associated to digraphs.

Example 4.2. It is well-known that if a pro-p group G has non-trivial torsion, then $\mathrm{H}^n(G) \neq 0$ for every $n \geq 0$. Therefore, if Γ is a digraph as in Example 3.4, or as in Example 3.5, and G is the associated oriented pro-p RAAG, for $q = p^f$, then $\mathbf{\Lambda}_\bullet(\Gamma^*) \not\simeq \mathbf{H}^\bullet(G)$, as $\Lambda_n(\Gamma^*) = 0$ for $n \geq 4$.

One knows that a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ yields oriented pro-p RAAGs whose $\mathbb{F}_p$-cohomology algebra is isomorphic to the associated exterior Stanley-Reisner $\mathbb{F}_p$-algebra in the following cases:

(a) if Γ is an undigraph, as proved by K. Lorensen;
(b) if Γ is triangle-free (cf. [35, Thm. F]);
(c) if Γ is special and chordal (cf. [35, Thm. H] and [3, Thm. 1.3–(iii)]);

(d) if Γ is obtained by mirroring a digraph yielding an oriented pro-p RAAG (cf. [35, Rem. 5.25–(b)]).

Remark 4.3. Let G be the oriented pro-p RAAG associated to a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ and to q. If the $\mathbb{F}_p$-cohomology algebra $\mathbf{H}^\bullet(G)$ is a quadratic $\mathbb{F}_p$-algebra (cf. § 2.4), then it is necessarily isomorphic to $\boldsymbol{\Lambda}_\bullet(\Gamma^*)$, as the latter is the quadratic $\mathbb{F}_p$-algebra generated by $\Lambda_1(\Gamma^*) = \mathrm{H}^1(G)$, and with space of degree 2 $\Lambda_2(\Gamma^*) \simeq \mathrm{H}^2(G)$ (cf. [35, Thm. E]).

4.3. **Cohomology and special-clique digraphs.** We are ready to prove Theorem 1.2. We follow the strategy used to prove [35, Thm. H] (see also [35, Rem. 2.25–(c)]).

Theorem 4.4. *Let* $\Gamma = (\mathcal{V}, \mathcal{E})$ *be a special-clique special digraph, and for* $q = p^f$ *let* G *be the associated oriented pro-p RAAG. Then* $\mathbf{H}^\bullet(G) \simeq \boldsymbol{\Lambda}_\bullet(\Gamma^*)$.

Proof. We proceed by induction on the number of special vertices of Γ. If Γ has no special vertices, then Γ is an undigraph, and thus $\mathbf{H}^\bullet(G) \simeq \boldsymbol{\Lambda}_\bullet(\Gamma^*)$ (cf. § 4.2).

Now suppose that Γ has n special vertices, $n \geq 1$, and that the statement holds for every special-clique special digraph with at most $n-1$ special vertices.

Let $w \in \mathcal{V}$ be a special vertex, and put

$$\mathrm{St}(w) = (\mathcal{V}_w, \mathcal{E}_w), \qquad \Delta = (\mathcal{V}_\Delta, \mathcal{E}_\Delta), \qquad \Gamma' = (\mathcal{V}', \mathcal{E}'),$$

where Δ and Γ' are the induced subdigraphs of Γ whose vertices are respectively $\mathcal{V}_\Delta = \mathcal{V}_w \smallsetminus \{w\}$ and $\mathcal{V}' = \mathcal{V} \smallsetminus \{w\}$. Then $\mathrm{St}(s)$ and Δ are cliques of Γ, while Γ' is a special-clique special digraph with $n-1$ special vertices. Moreover, Γ is the patching of $\mathrm{St}(w)$ and Γ' along Δ.

By Proposition 3.6, the subgroups G_w and G_Δ of G generated respectively by $\mathcal{V}_w$ and $\mathcal{V}_\Delta$ are isomorphic to the oriented pro-p RAAGs associated respectively to $\mathrm{St}(w)$ and Δ. Analogously, G_Δ — which is the free abelian pro-p group generated by $\mathcal{V}_\Delta$ — is a subgroup of both G_w and $G_{\Gamma'}$, where $G_{\Gamma'}$ is the oriented pro-p RAAG associated to Γ'. By Lemma 3.8, the amalgamated free pro-p product

$$(4.7) \qquad \tilde{G} = G_w \amalg_{G_\Delta} G_{\Gamma'}$$

is proper.

By induction, the three subgroups $G_w, G_{\Gamma'}, G_\Delta$ have quadratic $\mathbb{F}_p$-cohomology, which we identify respectively with $\boldsymbol{\Lambda}_\bullet(\mathrm{St}(w)^*)$, $\boldsymbol{\Lambda}_\bullet((\Gamma')^*)$ and $\boldsymbol{\Lambda}_\bullet(\Delta^*)$. In order to apply [35, Thm. B] to the proper amalgamated free pro-p product (4.7), we need to check that the restriction maps

$$\mathrm{res}^1_{G_w, G_\Delta} \colon \mathrm{H}^1(G_w) \to \mathrm{H}^1(G_\Delta) \qquad \text{and} \qquad \mathrm{res}^1_{G_{\Gamma'}, G_\Delta} \colon \mathrm{H}^1(G_{\Gamma'}) \to \mathrm{H}^1(G_\Delta)$$

are surjective; and moreover that

$$\mathrm{Ker}(\mathrm{res}^2_{G_w, G_\Delta}) = \mathrm{Ker}(\mathrm{res}^1_{G_w, G_\Delta}) \wedge \mathrm{H}^1(G_w),$$
$$\mathrm{Ker}(\mathrm{res}^2_{G_{\Gamma'}, G_\Delta}) = \mathrm{Ker}(\mathrm{res}^1_{G_{\Gamma'}, G_\Delta}) \wedge \mathrm{H}^1(G_{\Gamma'})$$

But these condition are satisfied by Remark 4.1, and thus we may apply [35, Thm. B], which implies that the $\mathbb{F}_p$-algebra $\mathbf{H}^\bullet(\tilde{G})$ is quadratic. In particular, for every $n \geq 1$ one has a short exact sequence

$$0 \longrightarrow \mathrm{H}^n(\tilde{G}) \xrightarrow{f^n_{\tilde{G}}} \Lambda_n(\mathrm{St}(w)^*) \oplus \Lambda_n((\Gamma')^*) \xrightarrow{f^n_{G_\Delta}} \Lambda_n(\Delta^*) \longrightarrow 0 \ ,$$

where $f_{\tilde{G}}^n = (\mathrm{res}_{\tilde{G},G_w}^n, \mathrm{res}_{\tilde{G},G_{\Gamma'}}^n)$, and $f_{G_\Delta}^n$ extends (via multiplication) to degree n the map $f_{G_\Delta}^1(v_1^*, v_2^*) = v_1^*|_{G_\Delta} - v_2^*|_{G_\Delta}$ for $v_1 \in \mathcal{V}_w$ and $v_2 \in \mathcal{V}'$ (cf. [35, p. 653], see also [36, Prop. 9.2.13]). Again by Remark 4.1, $\mathrm{Ker}(f_{\tilde{G}}^1)$ is easily seen to be isomorphic to $V^* = \Lambda_1(\Gamma^*)$, as $\mathcal{V} = \{w\} \cup \mathcal{V}_\Delta \cup (\mathcal{V}' \smallsetminus \mathcal{V}_\Delta)$; while kernel of $f_{\tilde{G}}^2$ is
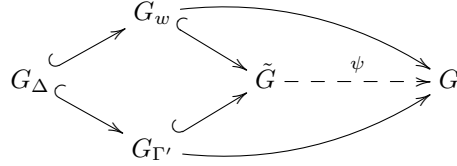
$$(w^* \wedge \Lambda_1(\mathrm{St}(w)^*)) \oplus (\Lambda_2(\Delta^*), -\Lambda_2(\Delta^*)) \oplus \left(\mathrm{Ker}(\mathrm{res}_{G_{\Gamma'},G_\Delta}^1) \wedge \Lambda_1((\Gamma')^*) \right),$$

which is isomorphic to $\Lambda_2(\Gamma^*)$, as

$$|\mathcal{E}| = \{\, \{w,u\} \mid u \in \mathcal{V}_w \,\} \cup |\mathcal{E}_\Delta| \cup \{\, \{v,u\} \mid v \in \mathcal{V}' \smallsetminus \mathcal{V}_\Delta, u \in \mathcal{V}' \,\}.$$

Altogether, $\mathbf{H}^\bullet(\tilde{G}) \simeq \mathbf{\Lambda}_\bullet(\Gamma^*)$ (cf. Remark 4.3).

Finally, we claim that $\tilde{G} \simeq G$. Indeed, by the universal property of push-out's, the monomorphism $G_w \to G$ and the homomorphism $G_{\Gamma'} \to G$ induced by $\mathcal{V}' \hookrightarrow \mathcal{V}$ yield the commutative diagram

Consider the homomorphisms $\psi^n \colon \mathrm{H}^n(G) \to \mathrm{H}^n(\tilde{G})$, $n \geq 1$ (cf. § 4.1.4). Since ψ^1 and ψ^2 are isomorphisms, $\mathrm{H}^1(\mathrm{Ker}(\psi))^{\tilde{G}} = 0$ by (4.2). Hence

$$0 = \left(\mathrm{H}^1(\mathrm{Ker}(\psi))^{\tilde{G}} \right)^* \simeq \frac{\mathrm{Ker}(\psi)}{\mathrm{Ker}(\psi)^p \left[\mathrm{Ker}(\psi), \tilde{G} \right]}$$

(here the isomorphism is an isomorphism of $\mathbb{F}_p$-vector spaces), and also the latter is trivial. Therefore, also $\mathrm{Ker}(\psi)$ is trivial, as the above quotient gives a set of generators of $\mathrm{Ker}(\psi)$ as a normal subgroup of $\tilde{G}$. $\qquad\square$

It remains an open problem to determine which digraphs yield oriented pro-p RAAGs whose $\mathbb{F}_p$-cohomology is isomorphic to the exterior Stanley-Reisner $\mathbb{F}_p$-algebra associated to the digraph. In particular, one has the following conjecture, which is the "directed version" of the conjecture formulated in [35, p. 672].

Conjecture 4.5. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a special digraph, and let G be the oriented pro-p RAAG associated to Γ and to a p-power q ($q \neq 2$ in case $p = 2$). Then*

$$\mathbf{H}^\bullet(G) \simeq \mathbf{\Lambda}_\bullet(\Gamma^*).$$

5. Massey products

5.1. **Massey products in $\mathbb{F}_p$-cohomology.** In the following two subsections, we recall some basic definitions and properties of Massey products in the $\mathbb{F}_p$-cohomology of pro-p groups, and the "translation" of Massey products in terms of upper unitriangular representations — this is all we will need to prove Theorem 1.1. For the formal definition of Massey products in $\mathbb{F}_p$-cohomology of pro-p groups in terms of cochains see, e.g., [33, § 2.1].

Definition 5.1. Let G be a pro-p group, let n be a positive integer, $n \geq 2$, and let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $\mathrm{H}^1(G)$.

(a) The n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is said to be defined if it is non-empty.
(b) The n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is said to vanish if $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$.
(c) The n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is said to be essential if it is defined but it does not vanish.

Massey products satisfy the following (see, e.g., [33, Rem. 2.2 and Prop. 2.6]).

Proposition 5.2. *Let G be a pro-p group, and let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $\mathrm{H}^1(G)$. Then one has the following.*

(i) *if the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined then*

$$(5.1) \qquad \alpha_1 \smile \alpha_2 = \alpha_2 \smile \alpha_3 = \ldots = \alpha_{n-1} \smile \alpha_n = 0;$$

(ii) *if $\alpha_i = 0$ for some i, then the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ vanishes.*

A pro-p group G is said to satisfy the n-Massey vanishing property, $n \geq 2$, with respect to $\mathbb{F}_p$, if every defined n-fold Massey product in $\mathbf{H}^\bullet(G)$ vanishes. For this reason, Minač-Tân's conjecture is also called the "Massey vanishing conjecture". Moreover, G is said to satisfy the strong n-Massey vanishing property, with respect to $\mathbb{F}_p$, if every sequence $\langle \alpha_1, \ldots, \alpha_n \rangle$ of length n of elements of $\mathrm{H}^1(G)$ satisfying condition (5.1) yields a vanishing n-fold Massey product (cf. [31, Def. 1.2]). Clearly, the strong Massey vanishing property is stronger than the Massey vanishing property.

Remark 5.3. In [28], J. Minač and N.D. Tân asked whether the maximal pro-p Galois group of a field containing a root of 1 of order p satisfies the strong n-Massey vanishing property, with respect to $\mathbb{F}_p$, for every $n > 2$ (see also [31] and [33, Rem. 2.5])— thus, strengthening the Massey vanishing conjecture. Later, it was shown that there exist fields whose maximal pro-p Galois group satisfies the Massey vanishing conjecture, but not the strong one: namely, $\mathbb{Q}$, by Y. Harpaz and O. Wittenberg (cf. [13, Example A.15]); and the function field of a certain variety over a given field, by A. Merkurjev and F. Scavia (cf. [22, 24]).

5.2. **Upper unitriangular matrices.** For $n \geq 1$ let

$$\mathbb{U}_{n+1} = \left\{ \begin{pmatrix} 1 & a_{1,2} & \cdots & & a_{1,n+1} \\ & 1 & a_{2,3} & \cdots & a_{2,n+1} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & a_{n,n+1} \\ & & & & 1 \end{pmatrix} \mid a_{i,j} \in \mathbb{F}_p \right\} \subseteq \mathrm{GL}_{n+1}(\mathbb{F}_p)$$

be the p-group of upper unitriangular matrices with entries in $\mathbb{F}_p$. The center of $\mathbb{U}_{n+1}$ is

$$\mathrm{Z}(\mathbb{U}_{n+1}) = \{ I_{n+1} + aE_{1,n+1} \mid a \in \mathbb{F}_p \},$$

where I_{n+1} denote the $(n+1) \times (n+1)$-identity matrix, and for $1 \leq i < j \leq n+1$ let $E_{i,j}$ denote the $(n+1) \times (n+1)$-matrix whose (i,j)-entry is 1, and all other entries are 0. The projection on the $(1, n+1)$-entry yields an isomorphism (of cyclic groups of order p) $\mathrm{Z}(\mathbb{U}_{n+1}) \simeq \mathbb{F}_p$. We put

$$\bar{\mathbb{U}}_{n+1} = \mathbb{U}_{n+1} / \mathrm{Z}(\mathbb{U}_{n+1}).$$

5.3. **Upper unitriangular matrices and Massey products.** Let G be a pro-p group, and let $\rho\colon G \to \mathbb{U}_{n+1}$ be a homomorphism of pro-p groups. For every $i = 1, \dots, n$ the $(i, i+1)$-entry of ρ, denoted by $\rho_{i,i+1}$, is a homomorphism $G \to \mathbb{F}_p$, and thus it may be considered as an element of $\mathrm{H}^1(G)$. Analogously, if $n \geq 2$ and $\bar{\rho}\colon G \to \bar{\mathbb{U}}_{n+1}$ is a homomorphism of pro-p groups, then for every $i = 1, \dots, n$ the $(i, i+1)$-entry of $\bar{\rho}$, denoted by $\bar{\rho}_{i,i+1}$, is a homomorphism $G \to \mathbb{F}_p$, and thus it may be considered as an element of $\mathrm{H}^1(G)$ as well.

The following is the pro-p version of the "translation" of Massey products in $\mathbb{F}_p$-cohomology in terms of upper unitriangular representations due to W. Dwyer (cf., e.g., [11, Lemma 9.3], see also [8, § 8]).

Proposition 5.4. *Let G be a pro-p group and let $\alpha_1, \dots, \alpha_n$ be a sequence of elements of $\mathrm{H}^1(G)$, with $n \geq 2$.*

  (i) *The n-fold Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined if, and only if, there exists a continuous homomorphism $\bar{\rho}\colon G \to \bar{\mathbb{U}}_{n+1}$ such that $\bar{\rho}_{i,i+1} = \alpha_i$ for every $i = 1, \dots, n$.*

 (ii) *The n-fold Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ vanishes if, and only if, there exists a continuous homomorphism $\rho\colon G \to \mathbb{U}_{n+1}$ such that $\rho_{i,i+1} = \alpha_i$ for every $i = 1, \dots, n$.*

5.4. **Four lemmata on upper unitriangular matrices.** Here we provide four technical lemmata on upper unitriangular matrices which will be used to prove Theorem 1.1.

Lemma 5.5. *Let G be a pro-p group, and let $\rho\colon G \to \mathbb{U}_{n+1}$ be a homomorphism of pro-p groups for some $n \geq 3$. Suppose that $\rho_{i,i+1}(x) = 1$ for every $i = 1, \dots, n$ for some element $x \in G$. Then there exists a homomorphism of (pro-p) groups $\rho'\colon G \to \mathbb{U}_{n+1}$ such that $\rho'_{i,i+1}(y) = \rho_{i,i+1}(y)$ for all $i = 1, \dots, n$ and $y \in G$, and $\rho'(x) = A$, where*

$$
(5.2) \qquad A = I_{n+1} + E_{1,2} + \dots + E_{n,n+1} = \begin{pmatrix} 1 & 1 & 0 & & 0 \\ & 1 & 1 & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.
$$

Proof. Set $A' = \rho(x)$. We claim that there exists a basis $\{v_1, \dots, v_{n+1}\}$ of $\mathbb{F}_p^{n+1}$ such that

$$
(5.3) \qquad A' v_i = v_i + v_{i-1} \qquad \text{and} \qquad v_i = \begin{pmatrix} b_{i,1} \\ \vdots \\ b_{i,i-1} \\ 1 \\ 0 \\ \vdots \end{pmatrix}
$$

— namely, the i-th coordinate of v_i is, and the coordinates from the $(i+1)$-th on are 0 — for every $i = 1, \dots, n+1$.

Clearly, we may set $v_1 = (1, 0, \dots)^T$ and $v_2 = (0, 1, 0 \dots)^T$. Now suppose that $\{v_1, \dots, v_m\}$ is a set of vectors satisfying (5.3), for $m \leq n$. Then the linear system

$Aw = w + v_m$, with $w = (x_1, \ldots, x_m, x_{m+1}, 0, \ldots)^T$, — explicitly

$$\begin{cases} x_1 + x_2 + \ \ldots \ \ldots \ + \ a_{1,m+1} x_{m+1} \ = \ x_1 \ + \ b_{m,1} \\ \qquad x_2 + \ x_3 \ + \ldots + \ a_{2,m+1} x_{m+1} \ = \ x_2 \ + \ b_{m,2} \\ \qquad\qquad\qquad\qquad \vdots \\ \quad x_{m-1} + x_m + a_{m-1,m+1} x_{m+1} = x_{m-1} + b_{m,m-1} \\ \qquad\qquad x_m + \qquad x_{m+1} \qquad = x_m \ + \quad 1 \\ \qquad\qquad\qquad\quad x_{m+1} \qquad = x_{m+1} \end{cases}$$

— has a solution with $x_m = v_{m+1} = 1$, and we may set $v_{m+1} := w$.

Let $M \in \mathrm{GL}_{n+1}(\mathbb{F}_p)$ be the matrix whose columns are the vectors $v_1, \ldots, v_{n+1}$. Then $A = M^{-1} A' M$. Moreover, $M \in \mathbb{U}_{n+1}$, and therefore the conjugation of $\mathbb{U}_{n+1}$ with M is an inner automorphism of $\mathbb{U}_{n+1}$. Hence, we set $\rho' = M^{-1} \rho M$, so that $\mathrm{Im}(\rho') \subseteq \mathbb{U}_{n+1}$, and one has $\rho'(x) = A$. Finally, the equality $\rho'_{i,i+1}(y) = \rho_{i,i+1}(y)$ is satisfied for all $i = 1, \ldots, n$, as for any $B, C \in \mathbb{U}_{n+1}$, the $(i, i+1)$-entries of $C^{-1} B C$ and of B are equal. $\qquad\square$

Lemma 5.6. *For $n \geq 3$, let $A \in \mathbb{U}_{n+1}$ be as in (5.2), and let $B \in \mathbb{U}_{n+1}$ be a matrix (with entries $b_{i,j}$ for $1 \leq i < j \leq n-1$) such that $[A', B] \in \mathrm{Z}(\mathbb{U}_{n+1})$. Then $b_{i,i+k} = b_{1,1+k}$ for every $k = 1, \ldots, n-2$ and $i = 1, \ldots, n+1-k$ — namely,*

(5.4)
$$B = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{n-2} & b_{1,n} & b_{1,n+1} \\ & 1 & b_1 & \ddots & & b_{n-2} & b_{2,n+1} \\ & & 1 & \ddots & \ddots & & b_{n-2} \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & 1 & b_1 & b_2 \\ & & & & & 1 & b_1 \\ & & & & & & 1 \end{pmatrix},$$

where $b_k = b_{1,1+k}$ for $k = 1, \ldots, n-1$.

Proof. Since $AB \equiv BA \bmod \mathrm{Z}(\mathbb{U}_{n+1})$, for $k = 2, \ldots, n-1$ and $i = 1, \ldots, n+1-k$ the $(i, i+k)$-entries of the matrices AB and BA are equal — observe that the the $(i, i+1)$-entries of the two products are always equal, as the $(i, i+1)$-entries of $[A, B]$ are 0. Therefore,

$$b_{i,i+k} + b_{i+1,i+k} = b_{i,i+k-1} + b_{i,i+k},$$

which implies $b_{i,i+(k-1)} = b_{i+1,i+1+(k-1)}$. $\qquad\square$

Lemma 5.7. *For $n \geq 3$, let $B, C \in \mathbb{U}_{n+1}$ be matrices with B as in (5.4), and*

(5.5)
$$C = \begin{pmatrix} 1 & 1 & c_{1,3} & \cdots & c_{1,n} & c_{1,n+1} \\ & 1 & 0 & \ddots & & c_{2,n+1} \\ & & 1 & \ddots & \ddots & \vdots \\ & & & \ddots & 0 & c_{n-1,n+1} \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix},$$

and suppose that $[C, B] \in \mathrm{Z}(\mathbb{U}_{n+1})$. *Then* $b_1 = \ldots = b_{n-2} = 0$.

Proof. Since $CB \equiv BC \bmod \mathrm{Z}(\mathbb{U}_{n+1})$, for $k = 2, \ldots, n-1$ and the $(1, 1+k)$-entries of the matrices CB and BC are equal — observe that the the $(1,2)$-entries of the two products are always equal, as the $(1,2)$-entry of $[C, B]$ is 0. Therefore,

$$\begin{aligned}
(5.6) \quad & b_k + b_{k-1} + c_{1,3}b_{k-2} + \ldots + c_{1,1+(k-2)}b_2 + c_{1,1+(k-1)}b_1 + c_{1,1+k} = \\
& = c_{1,1+k} + b_1 c_{2,1+k} + b_2 c_{3,1+k} + \ldots + b_{k-2}c_{k-1,1+k} + b_{k-1} \cdot 0 + b_k.
\end{aligned}$$

If $k = 2$, then equality (5.6) is $b_2 + b_1 + c_{1,3} = c_{1,3} + b_2$, which implies that $b_1 = 0$. For arbitrary k, if $b_1 = \ldots = b_{k-2} = 0$, then equality (5.6) is $b_k + b_{k-1} + c_{1,1+k} = c_{1,1+k} + b_k$, which implies that also b_{k-1} is 0. $\qquad\square$

Lemma 5.8. *For $n \geq 3$ and $c \in \mathbb{F}_p$, $a \neq 0$, let $C \in \mathbb{U}_{n+1}$ be a matrix — with entries $c_{i,j}$ for $1 \leq i < j \leq n+1$ —, such that $c_{i,i+1} \in \{0, a\}$ for every $i = 1, \ldots, n$, and $c_{i,j} = 0$ for $j - i \geq 2$. Then for every $f \geq 1$ ($f \geq 2$ if $p = 2$) there exists a matrix $B \in \mathbb{U}_{n+1}$ — with entries $b_{i,j}$ for $1 \leq i < j \leq n+1$ — such that $b_{1,2} = \ldots = b_{n,n+1} = 0$ and*

$$[B, C] = C^{p^f}.$$

Proof. Set $q = p^f$. First, observe that for every $1 \leq i, j \leq n+1$, $j - i \geq 2$, one has

$$[I_{n+1} + bE_{i,j}, C] = I_{n+1} - \epsilon_1 ab E_{i-1,j} + \epsilon_2 ab E_{i,j+1},$$

where

$$\epsilon_1 = \begin{cases} 1 & \text{if } b_{j-1,j} = a, \\ 0 & \text{if } b_{j-1,j} = 0, \end{cases} \qquad \text{and} \qquad \epsilon_2 = \begin{cases} 1 & \text{if } b_{i,i+1} = a, \\ 0 & \text{if } b_{j-1,j} = 0. \end{cases} \qquad\square$$

6. ORIENTED PRO-p RAAGS AND MASSEY PRODUCTS

6.1. **Digraphs that are not special-clique.** Recall that a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is special if, and only if, it does not contain induced subdigraphs as in (2.3), and it is also special-clique if, and only if, one excludes also induced subdigraphs as in (2.4). For our purposes, it is convenient to reformulate the above conditions in the following way: a digraph $\Gamma = (\mathcal{V}, \mathcal{E})$ is not a special-clique special digraph if, and only if, it contains an induced subdigraph Γ' with geometric representation

(6.1)

(in the right-side representation $(w, v) \in \mathcal{E}$, and possibly also $(v, w) \in \mathcal{E}$); or an induced subdigraph Γ' with geometric representation

(6.2)

where the two-headed arrows between u and v mean that at least one of (u, v) and (v, u) — possibly both — are edges.

Our goal is to show that in both cases, the $\mathbb{F}_p$-cohomology of an associated oriented pro-p RAAGs gives rise to essential Massey products.

6.1.1. *First case: induced subdigraph of type* (6.1).

Proposition 6.1. *Let* $\Gamma = (\mathcal{V}, \mathcal{E})$ *be a digraph as in* (6.1)*, and let* G *be the oriented pro-p RAAG associated to* Γ *and to* q*. Then the q-fold Massey product*

$$(6.3) \qquad \langle \alpha, \underbrace{\beta, \ldots, \beta}_{(q-2)\ times}, \alpha \rangle,$$

with $\alpha = u^* + v^*, \beta = u^* \in \mathrm{H}^1(G)$*, is essential.*

Proof. First observe that $\alpha \smile \beta = v^* \smile u^* = 0$, as u, v are disjoint, while clearly $\beta \smile \beta = 0$.

The vertices u, v, w are subject to the relations $r_{u,w}$ and $r_{v,w}$, which are

$$(6.4) \qquad [w, u] = u^q \qquad \text{and} \qquad [w, v] = \begin{cases} v^q & \text{if } (v, w) \text{ is a directed edge} \\ 1 & \text{if } (v, w), (w, v) \in \mathcal{E}, \\ w^{-q} & \text{if } (w, v) \text{ is a directed edge.} \end{cases}$$

To show that the q-fold Massey product (6.3) is defined, we produce a homomorphism $\bar{\rho} \colon G \to \bar{\mathbb{U}}_{q+1}$ satisfying $\bar{\rho}_{1,2} = \rho_{q,q+1} = \alpha$ and $\bar{\rho}_{i,i+1} = \beta$ for $i = 2, \ldots, q-1$. Take the matrices $A, C \in \mathbb{U}_{q+1}$, with A as in (5.2), and

$$C = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ & 1 & 0 & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

Then $A^q = I_{q+1} + E_{1,q+1}$ and $C^q = I_{q+1}$. The assignment

$$u \mapsto A \cdot \mathrm{Z}(\mathbb{U}_{q+1}), \qquad v \mapsto C \cdot \mathrm{Z}(\mathbb{U}_{q+1}), \qquad w, w' \mapsto I_{q+1} \cdot \mathrm{Z}(\mathbb{U}_{q+1})$$

for any $w' \in \mathcal{V}, w' \neq u, v$, induces a homomorphism $\bar{\rho} \colon G \to \bar{\mathbb{U}}_{q+1}$: indeed one has

$$[I_{q+1}, A] = I_{q+1} \equiv A^q \bmod \mathrm{Z}(\mathbb{U}_{q+1}) \qquad \text{and} \qquad [I_{q+1}, C] = I_{q+1} = C^q = I_{q+1}^{-q},$$

as prescribed by (6.4); and moreover

$$[\bar{\rho}(w'), \bar{\rho}(z)] = I_{q+1} = C^q \equiv A^q \mod \mathrm{Z}(\mathbb{U}_{q+1})$$

for $z = u, v, w$, so that the relation $r_{w',z}$, occurring whenever a vertex w' is joined to one of u, v, w, is satisfied. Altogether, the q-fold Massey product (6.3) is defined by Proposition 5.4–(i).

Now suppose there exists a homomorphism $\rho \colon G \to \mathbb{U}_{q+1}$ satisfying $\rho_{1,2} = \rho_{q,q+1} = \alpha$ and $\rho_{i,i+1} = \beta$ for $i = 2, \ldots, q-1$. By Lemma 5.5, we may suppose that $\rho(u) = A$, with A as above — and as in (5.2). Put $B = \rho(w)$ and $C = \rho(v)$. Then the entries of the 1st upper diagonal of B are 0, while C is as in Lemma 5.7. In particular,

$$B^q = C^q = I_{q+1}.$$

By (6.4), one has $[B, A] = A^q \in Z(\mathbb{U}_{q+1})$, so that Lemma 5.6 implies that B as in (5.4). Moreover, by (6.4), $[B, C] \in Z(\mathbb{U}_{q+1})$, and thus Lemma 5.7 implies that $b_2 = \ldots = b_{q-2} = 0$ — namely, the non-0 entries of B are concentrated in the $(q-1)$th and in the qth upper diagonals (and in the main diagonal, of course). Hence, one computes

$$[B, A] = [B, C] = I_{q+1} + (b_{1,q} - b_{2,q+1})E_{1,q+1},$$

but by (6.4) the former commutator should be equal to $A^q = I_{q+1} + E_{1,1+q}$, while the latter should be equal to I_{q+1}, a contradiction. Therefore, a homomorphism

$$\rho \colon G \longrightarrow \mathbb{U}_{q+1}$$

with the prescribed properties cannot exist, and thus the q-fold Massey product (6.3) does not vanish by Proposition 5.4–(ii). $\qquad\square$

Remark 6.2. Let G be the oriented pro-p RAAG as in Proposition 6.1. By [3, Prop. 5.4, Prop. 6.5], G cannot occur as the maximal pro-p Galois group of a field containing a root of 1 of order p. If $p = q = 3$, then G yields essential 3-fold Massey products, and thus by E. Matzri's result [19] (cf. § 1.1), Proposition 6.1 provides an alternative proof of the fact that, in this case, G cannot occur as the maximal pro-3 Galois group of a field containing a root of 1 of order 3.

6.1.2. *Second case: induced subdigraph of type* (6.2).

Proposition 6.3. *Let* $\Gamma = (\mathcal{V}, \mathcal{E})$ *be a digraph as in* (6.2), *and let* G *be the oriented pro-p RAAG associated to* Γ *and to* q. *Then the q-fold Massey product*

$$(6.5) \qquad\qquad \underbrace{\langle \alpha, \ldots, \alpha \rangle}_{q \ times},$$

with $\alpha = u^* + v^* \in \mathrm{H}^1(G)$, *is essential.*

Proof. The vertices u, v, w are subject to the three relations $r_{u,v}$, $r_{v,w}$, $r_{w,u}$, which are

$$[w, u] = u^q,$$

$$[v, w] = \begin{cases} 1 & \text{if } (v, w), (w, v) \in \mathcal{E}, \\ w^q & \text{if } (w, v) \text{ is a directed edge,} \end{cases}$$

(6.6)

$$[v, u] = \begin{cases} 1 & \text{if } (v, u), (u, v) \in \mathcal{E}, \\ u^q & \text{if } (u, v) \text{ is a directed edge,} \\ v^{-q} & \text{if } (v, u) \text{ is a directed edge.} \end{cases}$$

Put $\alpha = u^* + v^* \in \mathrm{H}^1(G)$. Clearly $\alpha \smile \alpha = 0$. We claim that the q-fold Massey product (6.5) is defined but does not vanish.

To show that the q-fold Massey product (6.5) is defined, we produce a homomorphism $\bar{\rho} \colon G \to \bar{\mathbb{U}}_{q+1}$ satisfying $\bar{\rho}_{i,i+1} = \alpha$ for $i = 1, \ldots, q$. Take the matrix $A \in \mathbb{U}_{q+1}$ as in (5.2). Then $A^q = I_{q+1} + E_{1,q+1}$. The assignment

$$u, v \mapsto A \cdot Z(\mathbb{U}_{q+1}), \qquad w, w' \mapsto I_{q+1} \cdot Z(\mathbb{U}_{q+1})$$

for $w' \in \mathcal{V}, w' \neq u, v$, induces a homomorphism $\bar{\rho} \colon G \to \bar{\mathbb{U}}_{q+1}$: indeed one has

$$[I_{q+1}, A] = I_{q+1} \equiv A^q \bmod Z(\mathbb{U}_{q+1}) \qquad \text{and} \qquad [A, A] = I_{q+1} \equiv A^{\pm q} \bmod Z(\mathbb{U}_{q+1}),$$

as prescribed by (6.6); and moreover

$$[\rho(w'), \rho(z)] = I_{q+1} \equiv A^q \mod \mathrm{Z}(\mathbb{U}_{q+1}),$$

so that the relation $r_{w',z}$, occurring whenever any vertex w' is joined to one of $z = u, v, w$, is satisfied. Hence, the q-fold Massey product (6.5) is defined by Proposition 5.4–(i).

Now suppose there exists a homomorphism $\rho\colon G \to \mathbb{U}_{q+1}$ satisfying $\rho_{i,i+1} = \alpha$ for $i = 1, \ldots, q$. By Lemma 5.5, we may suppose that $\rho(u) = A$, with A as above — and as in (5.2). Put $B = \rho(w)$ and $C = \rho(v)$. Then the entries of the 1st upper diagonal of B are 0, while all entries of the 1st upper diagonal of C are equal to 1. In particular,

$$B^q = I_{q+1} \qquad \text{and} \qquad C^q = A^q = I_{q+1} + E_{1,q+1}.$$

By (6.6), one has $[B, A], [A, C] \in \mathrm{Z}(\mathbb{U}_{q+1})$, so that Lemma 5.6 implies that B, C are as in (5.4) (where we call c_i, $i = 2, \ldots, q - 2$, the entries of the ith upper diagonal of C). Then one computes

$$[B, A] = I_{q+1} + (b_{1,q} - b_{2,q+1})E_{1,q+1},$$

and $[B, A] = A^q = I_{q+1} + E_{1,q+1}$ implies $b_{1,q} - b_{2,q+1} = 1$. On the other hand, the $(1, q + 1)$-entry of $[C, B]$ is

$$(b_{1,q+1} + b_{2,q+1} + b_{q-2}c_2 + \ldots + b_2 c_{q-2} + 0 + c_{1,q+1}) -$$
$$- (c_{1,1+q} + 0 + b_2 c_{q-2} + \ldots + b_{q-2}c_2 + b_{1,q} + b_{1,q+1}) = b_{2,q+1} - b_{1,q},$$

and $[C, B] = B^{\epsilon q} = I_{q+1}$ (with $\epsilon = 0, 1$ depending on whether (v, w) is an edge) implies $b_{2,q+1} - b_{1,q} = 0$, a contradiction. Therefore, a homomorphism $\rho\colon G \to \mathbb{U}_{q+1}$ with the prescribed properties cannot exist, and thus the q-fold Massey product (6.5) does not vanish by Proposition 5.4–(ii). $\qquad\square$

6.2. **Special-clique digraphs and Massey products.** Here we prove that condition (iii) in Theorem 1.1 implies condition (ii). We will proceed as follows: given a sequence $\alpha_1, \ldots, \alpha_n$ of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$ satisfying (5.1), we will construct explicitly a homomorphism $\rho\colon G \to \mathbb{U}_{n+1}$ such that $\rho_{i,i+1} = \alpha_i$ for every $i = 1, \ldots, n$, so that the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ contains 0 by Proposition 3.6–(ii). At this aim, we need the following lemma, which is a consequence of [33, Prop. 2.10].

Lemma 6.4. *For $a \in \mathbb{F}_p$, $a \neq 0$, let $A \in \mathbb{U}_{n+1}$ be a matrix whose non-0 entries are concentrated in the main diagonal and in the 1st upper diagonal — namely,*

$$A = I_{n+1} + a_{1,2}E_{1,2} + a_{2,3}E_{2,3} + \ldots + a_{n,n+1}E_{n,n+1}.$$

Then there exists a matrix $B \in \mathbb{U}_{n+1}$ whose entries in the 1st upper diagonal are 0 and such that $[B, A] = A^q$.

Proof. First, we write A as a block diagonal matrix,

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}$$

where for each $h = 1, \ldots, r$ one has $A_h \in \mathbb{U}_{m_h+1}$ for some $m_h \geq 0$, and either $A_h = I_{m_h+1}$, or the entries of the 1st upper diagonal of A_h are all not 0. For each block,

one has either $A_h^q = I_{m_h+1}$, if $A_h = I_{m_h+1}$ or $m_h < q$; or the non-0 entries of A_h^q are concentrated in the main diagonal and in the qth upper diagonal — namely,

$$A_h^q = I_{m_h+1} + a(E_{1,1+q} + \ldots + E_{m_h+1-q,m_h+1}) = \begin{pmatrix} 1 & \cdots & * & & 0 \\ & 1 & 0 & \ddots & \\ & & \ddots & 0 & * \\ & & & 1 & \vdots \\ & & & & 1 \end{pmatrix},$$

if the entries of the 1st upper diagonal of A_h are all not 0 and $m_h \geq q$. Altogether, one has

$$A^q = \begin{pmatrix} A_1^q & & & \\ & A_2^q & & \\ & & \ddots & \\ & & & A_r^q \end{pmatrix}.$$

Now, if A_h is a block whose entries in the 1st upper diagonal of A_h are not 0, and $m_h \geq q$, by [33, Prop. 2.10] there exists a matrix $B_h \in \mathbb{U}_{m_h+1}$, whose entries in the 1st diagonal are 0, such that $[B_h, A_h] = A_h^q$. If instead $A_h^q = I_{m_h+1}$, we put $B_h = I_{m_h+1}$, so that $[B_h, A_h] = I_{m_h+1} = A_h^q$ anyway.

Finally, let $B \in \mathbb{U}_{n+1}$ be the diagonal block matrix whose blocks are $B_1, \ldots, B_r$. Then

$$[B, A] = \begin{pmatrix} [B_1, A_1] & & & \\ & [B_2, A_2] & & \\ & & \ddots & \\ & & & [B_r, A_r] \end{pmatrix}$$

$$= \begin{pmatrix} A_1^q & & & \\ & A_2^q & & \\ & & \ddots & \\ & & & A_r^q \end{pmatrix} = A^q. \qquad \square$$

Proposition 6.5. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a special-clique special digraph, and let G be the oriented pro-p RAAG associated to Γ and to q. Then G satisfies the strong n-Massey vanishing property for every $n \geq 3$.*

Proof. If Γ has no special vertices, then Γ is an undigraph, and the associated pro-p RAAG satisfies the n-Massey vanishing property for every $n \geq 3$ by [2, Thm. 1.1]. So, we suppose that Γ has at leas a special vertex.

Let $\alpha_1, \ldots, \alpha_n$ be a sequence of elements of $\mathrm{H}^1(G, \mathbb{F}_p)$ satisfying (5.1). For every vertex $v \in \mathcal{V}$, set

$$A(v) = I_{n+1} + \alpha_1(v)E_{1,2} + \alpha_2(v)E_{2,3} + \ldots + \alpha_n(v)E_{n,n+1}$$

$$= \begin{pmatrix} 1 & \alpha_1(v) & 0 & & 0 \\ & 1 & \alpha_2(v) & \ddots & \\ & & \ddots & & 0 \\ & & & 1 & \alpha_n(v) \\ & & & & 1 \end{pmatrix} \in \mathbb{U}_{n+1}.$$

By the proof of [2, Thm. 1.1], one has

(6.7) $\qquad \alpha_i(v)\alpha_{i+1}(v') - \alpha_i(v')\alpha_{i+1}(v) = 0 \qquad$ for every $i = 1, \ldots, n-1$,

whenever v and v' are adjacent ordinary vertices, which implies that

(6.8) $\qquad\qquad\qquad\qquad [A(v), A(v')] = I_{n+1} \qquad$ for every $v, v' \in \mathcal{V}_0$

(cf. [2, p. 13]).

Now let $w \in \mathcal{V}_s$ be a special vertex of Γ, and put $\mathrm{St}(w) = (\mathcal{W}, \mathcal{E}_w)$. Since Γ is special-clique, $\mathrm{St}(w)$ is a clique of Γ, and by Proposition 3.6 the associated oriented pro-p RAAG G_w is a subgroup of G via the inclusion $\mathcal{W} \hookrightarrow \mathcal{V}$ Moreover, the $\mathbb{F}_p$-cohomology algebra of G_w is the exterior Stanley-Reisner $\mathbb{F}_p$-algebra

$$\mathbf{\Lambda}_\bullet(\mathrm{St}(w)) = \mathbf{\Lambda}_\bullet(\mathbb{F}_p\mathcal{W}^*) = \mathbf{\Lambda}_\bullet(\mathrm{H}^1(G_w))$$

(for example, by Theorem 1.2). For every $i = 1, \ldots, n-1$, one has

(6.9) $\qquad\qquad\qquad 0 = \mathrm{res}^2_{G,G_w}(\alpha_i \smile \alpha_{i+1}) = (\alpha_i|_{G_w}) \smile (\alpha_{i+1}|_{G_w}),$

and since $\mathrm{H}^2(G_w) = \Lambda_2(\mathrm{H}^1(G_2))$, (6.9) implies that $\alpha_i|_{G_w}$ and $\alpha_{i+1}|_{G_w}$ are $\mathbb{F}_p$-linearly dependent. Therefore, there exists $\bar{\alpha} \in \mathrm{H}^1(G_w)$ and $a_1, \ldots, a_n \in \mathbb{F}_p$ such that

$$\alpha_i|_{G_w} = a_i\bar{\alpha} \qquad \text{for all } i = 1, \ldots, n.$$

Here we have two cases.

Case 1. Suppose that $\bar{\alpha}(u) = 0$ for every ordinary vertex u of $\mathrm{St}(w)$. Then $A(u) = I_{n+1}$, and thus $[A(v), A(u)] = I_{n+1}$ for every other ordinary vertex v of Γ, and

$$[A(w'), A(u)] = I_{n+1} = A(u)^q$$

for every special vertex w' of Γ.

Case 2. Suppose now that $\bar{\alpha}(u) \neq 0$ for some ordinary vertex u of $\mathrm{St}(w)$. Then for every vertex $u' \in \mathcal{W}$ (including $u' = u, w$) one has

(6.10) $\qquad \alpha_i(u') = a_i\bar{\alpha}(u') = \dfrac{\bar{\alpha}(u')}{\bar{\alpha}(u)} \cdot a_i\bar{\alpha}(u) = \dfrac{\bar{\alpha}(u')}{\bar{\alpha}(u)} \cdot \alpha_i(u) \qquad$ for all $i = 1, \ldots, n.$

For every $u' \in \mathcal{W}$, $u' \neq w$, replace the matrix $A(u')$ with $A'(u') = A(u)^{\bar{\alpha}(u')/\bar{\alpha}(u)}$. Observe that for every $i = 1, \ldots, n$ the $(i, i+1)$-entry of the latter matrix is precisely $\frac{\bar{\alpha}(u')}{\bar{\alpha}(u)}\bar{\alpha}(u)$, which is equal to $\alpha_i(u')$ by (6.10). In particular, if $\bar{\alpha}(u') = 0$ then $A'(u') = I_{n+1} = A(u')$, while clearly $A'(u) = A(u)$.

If $v \in \mathcal{V}_0$ is an ordinary vertex of Γ adjacent to an ordinary vertex u' of $\mathrm{St}(w)$, then by (6.7) and by (6.10) for every $i = 1, \ldots, n$ one has

$$
\begin{aligned}
0 &= \alpha_i(v)\alpha_{i+1}(u') - \alpha_i(u')\alpha_{i+1}(v) \\
&= \alpha_i(v)\frac{\bar{\alpha}(u')}{\bar{\alpha}(u)}\alpha_{i+1}(u) - \frac{\bar{\alpha}(u')}{\bar{\alpha}(u)}\alpha_i(u)\alpha_{i+1}(v) \\
&= \frac{\bar{\alpha}(u')}{\bar{\alpha}(u)} \cdot (\alpha_i(v)\alpha_{i+1}(u) - \alpha_i(u)\alpha_{i+1}(v))
\end{aligned}
$$

Hence, if $\bar{\alpha}(u') \neq 0$ then $\alpha_i(v)\alpha_{i+1}(u') = \alpha_i(u')\alpha_{i+1}(v)$, and (6.8) implies the equality $[A(v), A(u)] = I_{n+1}$, and thus also

$$
[A(v), A'(u')] = \left[A(v), A(u)^{\bar{\alpha}(u')/\bar{\alpha}(u)} \right] = I_{n+1}.
$$

On the other hand, if $\bar{\alpha}(u') = 0$, then $A'(u') = I_{n+1}$, and $[A(v), A'(u')] = I_{n+1}$ trivially.

Finally, let $B \in \mathbb{U}_{n+1}$ be a matrix whose entries in the 1st upper-diagonal are 0 and such that $[B, A(u)] = A(u)^q$ — cf. Lemma 6.4 —, and set $A'(w) = BA(u)^{\bar{\alpha}(w)/\bar{\alpha}(u)}$. Then for every $i = 1, \ldots, n$ the $(i, i+1)$-th entry of $A'(w)$ is $\frac{\bar{\alpha}(w)}{\bar{\alpha}(u)}\bar{\alpha}(u)$, which is equal to $\alpha_i(w)$; moreover, for every $u' \in \mathcal{W}$, $u' \neq w$ (but possibly $u' = u$) one computes, applying (3.1)–(3.2),

$$
\begin{aligned}
[A'(w), A'(u')] &= \left[BA(u)^{\bar{\alpha}(w)/\bar{\alpha}(u)}, A(u)^{\bar{\alpha}(u')/\bar{\alpha}(u)} \right] \\
&= \left[B, A(u)^{\bar{\alpha}(u')/\bar{\alpha}(u)} \right] \\
&= [B, A(u)]^{\bar{\alpha}(u')/\bar{\alpha}(u)} = \left(A(u)^{\bar{\alpha}(u')/\bar{\alpha}(u)} \right)^q = A(u')^q.
\end{aligned}
$$

Altogether, the assignment $v \mapsto A(v)$ or $v \mapsto A'(v)$ induces a homomorphism $\rho \colon G \to \mathbb{U}_{n+1}$ satisfying $\rho_{i,i+1} = \alpha_i$ for all $i = 1, \ldots, n$, and thus the n-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ vanishes by Proposition 5.4–(ii). $\square$

Remark 6.6. In the case $\Gamma = (\mathcal{V}, \mathcal{E})$ is an undigraph (which is trivially special-clique), from Proposition 6.5 one recovers [2, Thm. 1.1], which states that the pro-p RAAG associated to an undigraph satisfies the strong n-Massey vanishing property for every $n > 2$.

Now we are ready to prove Theorem 1.1, by putting together Propositions 6.1–6.3 and Proposition 6.5.

Proof of Theorem 1.1. In order to prove the implication (i)$\Rightarrow$(iii), suppose that $\Gamma = (\mathcal{V}, \mathcal{E})$ is not a special-clique special digraph, and let G be the oriented pro-p RAAG associated to Γ and to q. Then there exists an induced subdigraph $\Gamma' = (\mathcal{V}', \mathcal{E}')$ with three vertices, which is as in (6.1) or in (6.2), and Propositions 6.1–6.3 imply that there exist essential q-fold Massey products in $\mathbf{H}^\bullet(G)$. This completes the proof of (i)$\Rightarrow$(iii).

The implication (iii)$\Rightarrow$(ii) is provided by Proposition 6.5.

Finally, the implication (ii)$\Rightarrow$(i) follows by definition. $\square$

References

[1] L. Bartholdi, H. Härer, and Th. Schick, *Right angled Artin groups and partial commutation, old and new*, Enseign. Math. **66** (2020), no. 1-2, 33–61.

[2] S. Blumer, A. Cassella, and C. Quadrelli, *Groups of p-absolute Galois type that are not absolute Galois groups*, J. Pure Appl. Algebra **227** (2023), no. 4, Paper No. 107262.

[3] S. Blumer, C. Quadrelli, and Th.S. Weigel, *Oriented right-angled Artin pro-p groups and maximal pro-p Galois groups*, Int. Math. Res. Not. **2024** (2024), no. 8, 6790–6819.

[4] A. Cassella and C. Quadrelli, *Right-angled Artin groups and enhanced Koszul properties*, J. Group Theory **24** (2021), no. 1, 17–38.

[5] R. Diestel, *Graph theory*, 5th ed., Graduate Texts in Mathematics, vol. 173, Springer, Berlin, 2017.

[6] J.D. Dixon, M.P.F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999.

[7] W.G. Dwyer, *Homology, Massey products and maps between groups*, J. Pure Appl. Algebra **6** (1975), no. 2, 177–190.

[8] I. Efrat, *The Zassenhaus filtration, Massey products, and representations of profinite groups*, Adv. Math. **263** (2014), 389–411.

[9] ———, *The lower p-central series of a free profinite group and the shuffle algebra*, J. Pure Appl. Algebra **224** (2020), no. 6, 106260, 13.

[10] I. Efrat and E. Matzri, *Triple Massey products and absolute Galois groups*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 12, 3629–3640.

[11] I. Efrat and C. Quadrelli, *The Kummerian property and maximal pro-p Galois groups*, J. Algebra **525** (2019), 284–310.

[12] J. Gärtner, *Higher Massey products in the cohomology of mild pro-p-groups*, J. Algebra **422** (2015), 788–820.

[13] P. Guillot, J. Minač, and A. Topaz, *Four-fold Massey products in Galois cohomology*, Compos. Math. **154** (2018), no. 9, 1921–1959. With an appendix by O. Wittenberg.

[14] P. Guillot and J. Minač, *Extensions of unipotent groups, Massey products and Galois theory*, Adv. Math. **354** (2019), article no. 106748.

[15] Y. Harpaz and O. Wittenberg, *The Massey vanishing conjecture for number fields*, Duke Math. J. **172** (2023), no. 1, 1–41.

[16] M.J. Hopkins and K.G. Wickelgren, *Splitting varieties for triple Massey products*, J. Pure Appl. Algebra **219** (2015), no. 5, 1304–1319.

[17] Y.H.J. Lam, Y. Liu, R.T. Sharifi, P. Wake, and J. Wang, *Generalized Bockstein maps and Massey products*, Forum Math. Sigma **11** (2023), Paper No. e5.

[18] K. Lorensen, *Groups with the same cohomology as their pro-p completions*, J. Pure Appl. Algebra **214** (2010), no. 1, 6–14.

[19] E. Matzri, *Triple Massey products in Galois cohomology*, 2014. Preprint, available at `arXiv:1411.4146`.

[20] ———, *Triple Massey products of weight $(1, n, 1)$ in Galois cohomology*, J. Algebra **499** (2018), 272–280.

[21] ———, *Higher triple Massey products and symbols*, J. Algebra **527** (2019), 136–146.

[22] A. Merkurjev and F. Scavia, *The Massey Vanishing Conjecture for fourfold Massey products modulo 2*, 2023. Preprint, available at `arXiv:2301.09290`.

[23] ———, *On the Massey Vanishing Conjecture and Formal Hilbert 90*, 2023. Preprint, available at `arXiv:2308.13682`.

[24] ———, *Non-formality of Galois cohomology modulo all primes*, 2023. Preprint, available at `arXiv:2309.17004`.

[25] J. Minač and N.D. Tân, *The kernel unipotent conjecture and the vanishing of Massey products for odd rigid fields*, Adv. Math. **273** (2015), 242–270.

[26] ———, *Triple Massey products vanish over all fields*, J. London Math. Soc. **94** (2016), 909–932.

[27] ———, *Triple Massey products and Galois theory*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 1, 255–284.

[28] ———, *Counting Galois $\mathbb{U}_4(\mathbb{F}_p)$-extensions using Massey products*, J. Number Theory **176** (2017), 76–112.

[29] M. Morishita, *Milnor's link invariants attached to certain Galois groups over* **Q**, Proc. Japan Acad. Ser. A Math. Sci. **76** (2000), no. 2, 18–21.

[30] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 323, Springer-Verlag, Berlin, 2008.

[31] A. Pál and E. Szabó, *The strong Massey vanishing conjecture for fields with virtual cohomological dimension at most 1*, 2020. Preprint, available at `arXiv:1811.06192`.

[32] S. Papadima and A.I. Suciu, *Algebraic invariants for right-angled Artin groups*, Math. Ann. **334** (2006), no. 3, 533–555.

[33] C. Quadrelli, *Massey products in Galois cohomology and the elementary type conjecture*, J. Number Theory **258** (2024), 40–65.

[34] ———, *Massey products in Galois cohomology and Pythagorean fields*, Comm. Algebra **52** (2024), no. 10, 4399–4414.

[35] C. Quadrelli, I. Snopce, and M. Vannacci, *On pro-p groups with quadratic cohomology*, J. Algebra **612** (2022), 636–690.

[36] L. Ribes and P.A. Zalesskiĭ, *Profinite groups*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 40, Springer-Verlag, Berlin, 2010.

[37] J-P. Serre, *Galois cohomology*, Corrected reprint of the 1997 English edition, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author.

[38] I. Snopce and P. Zalesskii, *Right-angled Artin pro-p groups*, Bull. Lond. Math. Soc. **54** (2022), no. 5, 1904–1922.

[39] D. Vogel, *Massey products in the Galois cohomology of number fields*, 2004, `http://www.ub.uni-heidelberg.de/archiv/4418`. PhD thesis, University of Heidelberg.

[40] H. Wang, *Massey product and its application*, 2012, `http://hewang.sites.northeastern.edu/research`. Preprint.

[41] K. Wickelgren, *Massey products $\langle y, x, x, \ldots, x, x, y \rangle$ in Galois cohomology via rational points*, J. Pure Appl. Algebra **221** (2017), no. 7, 1845–1866.

DEPARTMENT OF SCIENCE & HIGH-TECH, UNIVERSITY OF INSUBRIA, ITALY EU

Email address: `claudio.quadrelli@uninsubria.it`