ON S-NOETHERIAN MODULES AND S-STRONG MORI MODULES

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ABSTRACT. In this paper, we study some properties of S-Noetherian modules and S-strong Mori modules. Among other things, we prove the Hilbert basis theorem for S-Noetherian modules and S-strong Mori modules.

1. INTRODUCTION

In this paper, R always denotes a commutative ring with identity, S is a (not necessarily saturated) multiplicative subset of R and M stands for a unitary R-module. (For the sake of avoiding the confusion, we use D instead of R when R is an integral domain.)

Recall that M is a *Noetherian module* if every submodule of M is finitely generated (or equivalently, the ascending chain condition on submodules of M holds) and R is a *Noetherian ring* if R is a Noetherian R-module. In [19], Wang and McCasland introduced new algebraic objects whose classes contain those with Noetherian property. They defined a w-module M to be a *strong Mori module* (SM-module) if M satisfies the ascending chain condition on w-submodules of M (or equivalently, each w-submodule of M is w-finite), where w denotes the so-called w-operation on M. (Recall that a w-module M is *w-finite* if there exists a finitely generated submodule F of M such that $M = F_w$.) Also, D is said to be a *strong Mori domain* (SM-domain) if D is an SM-module as a D-module.

In [1], Anderson and Dumitrescu generalized the concepts of the Noetherian rings and the Noetherian modules using multiplicative sets. Authors defined a submodule N of M to be *S-finite* if there exist an $s \in S$ and a finitely generated submodule F of M such that $Ns \subseteq F \subseteq N$, while an ideal I of R is *S-finite* if I is S-finite as an R-module. Also, M is *S-Noetherian* if every submodule of M is S-finite, while R is an *S-Noetherian ring* if R is S-Noetherian as an R-module. The readers can refer to [1, 7, 12, 13, 14, 15] for S-Noetherian rings and S-Noetherian modules. In [11], Kim, Kim and Lim generalized the concepts of SM-domains and SM-modules using multiplicative sets. They defined a submodule N of M to be *S-w-finite* if there exist an $s \in S$ and a finitely generated submodule F of M such that $Ns \subseteq F_w \subseteq N_w$, while an ideal I of D is *S-w-finite* if I is S-w-finite as a D-module. Also, a w-module M is an *S-strong Mori module* (S-SM-module) if every w-submodule of M is S-w-finite; and D is an *S-strong Mori domain* (S-SM-domain)

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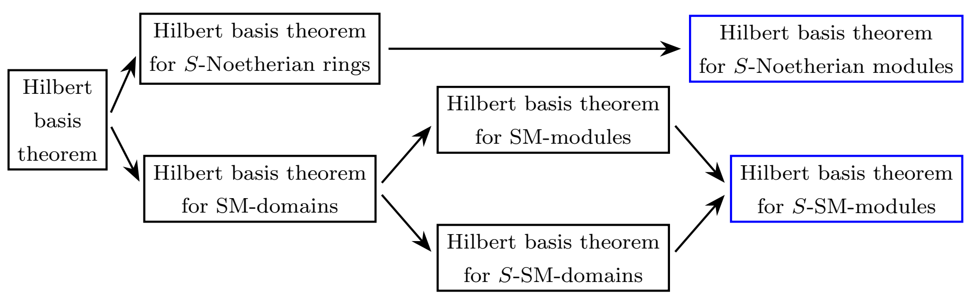
if D is an S-SM-module over D. The readers can refer to [5, 6, 8, 11, 18, 19] for (S-)SM-domains and (S-)SM-modules.

Recall that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring; and M is Noetherian if and only if $M[X]$ is Noetherian [4, Theorem 7.5 and Chapter 7, Exercise 10]. This is well known as Hilbert basis theorem. In [1], Anderson and Dumitrescu proved the Hilbert basis theorem for S-Noetherian rings, which states that if S is an anti-Archimedean subset of R, then R is an S-Noetherian ring if and only if $R[X]$ is an S-Noetherian ring [1, Proposition 9]. Also, Chang proved the Hilbert basis theorem for SM-domains and SM-modules in [5, 6]; that is, D is an SM-domain if and only if $D[X]$ is an SM-domain [5, Theorem 2.2]; and for w-module M, M is an SM-module over D if and only if $M[X]$ is an SM-module over $D[X]$ [6, Theorem 2.5]. In [11], the authors proved the Hilbert basis theorem for S-SM-domain, which states that if S is an anti-Archimedean subset of D, then D is an S-SM-domain if and only if $D[X]$ is an S-SM-domain [11, Theorem 2.8]. The main purpose of this paper is to prove the Hilbert basis theorem for S-Noetherian modules and S-SM-modules. To summarize, we present the following diagram.



This paper consists of three sections including introduction. In Section 2, we investigate some basic properties of quotient modules and S-Noetherian modules. We define a module which has finite character and show that if M is a locally S-Noetherian module which has finite character, then M is an S-Noetherian module (Proposition 2.4). We also show that the Hilbert basis theorem for S-Noetherian module when S is an anti-Archimedean subset of R, *i.e.*, M is an S-Noetherian R-module if and only if $M[X]$ is an S-Noetherian $R[X]$-module if and only if $M[X]_N$ is an S-Noetherian $R[X]_N$-module (Theorem 2.6). In Section 3, we study some properties of w-submodules and S-SM-modules. Also, we define a module which has finite w-character and then we show that if M is an S-SM-module, then M is a w-locally S-Noetherian module; and if M is a w-locally S-Noetherian module which has finite w-character, then M is an S-SM-module (Proposition 3.6). Finally, we show that the Hilbert basis theorem for S-SM-modules when S is an anti-Archimedean subset of D, *i.e.*, M is an S-SM-module over D if and only if $M[X]$ is an S-SM-module over $D[X]$ if and only if $M[X]_{N_v}$ is an S-SM-module over $D[X]_{N_v}$ if and only if $M[X]_{N_v}$ is an S-Noetherian $D[X]_{N_v}$-module (Theorem 3.8).

To help readers better understanding this paper, we review some definitions and notations related to star-operations. Let D be an integral domain with quotient field K, $\mathbf{F}(D)$ the set of nonzero fractional ideals of D and $\mathbf{T}(D)$ the set of nonzero torsion-free D-modules. For an $I \in \mathbf{F}(D)$, set $I^{-1} := \{a \in K \mid aI \subseteq D\}$.

The mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_v := (I^{-1})^{-1}$ is called the *v-operation*, and the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_t := \bigcup\{J_v \,|\, J$ is a nonzero finitely generated fractional subideal of $I\}$ is called the *t-operation*. An ideal I of D is a *v-ideal* (respectively, t-ideal) if $I_v = I$ (respectively, $I_t = I$). An ideal J of D is a *Glaz–Vasconcelos ideal* (GV-ideal), and denoted by $J \in \mathrm{GV}(D)$ if J is finitely generated and $J_v = D$. For each $M \in \mathbf{T}(D)$, *w-envelop* of M is the set $M_{w_D} := \{x \in M \otimes K \,|\, xJ \subseteq M$ for some $J \in \mathrm{GV}(D)\}$. If there is no confusion, we simply write w for w_D. The mapping on $\mathbf{T}(D)$ defined by $M \mapsto M_w$ is called the *w-operation*. An element $M \in \mathbf{T}(D)$ is a *w-module* if $M_w = M$, while an ideal I of D is a *w-ideal* if I is a w-module as a D-module. Let $*$ be the t-operation or the w-operation on D. Then a proper ideal I of D is said to be a *maximal $*$-ideal* of D if there does not exist a proper $*$-ideal which properly contains I. Let $*$-Max(D) be the set of maximal $*$-ideals of D. Then it is easy to see that if D is not a field, then $*$-Max$(D) \neq \emptyset$. The useful facts in this paper, t-Max$(D) = w$-Max(D) [2, Theorem 2.16] and $M_w = \bigcap_{\mathfrak{m} \in t\text{-Max}(D)} M_{\mathfrak{m}}$ for all nonzero D-modules M [2, Theorem 4.3]. The readers can refer to [2, 10, 17] for star-operations.

2. S-Noetherian modules

Let R be a commutative ring with identity and let S and T be multiplicative subsets of R. Then $S_T = \{\frac{s}{t} \,|\, s \in S$ and $t \in T\}$ is a multiplicative subset of R_T. We start this section with simple results for a submodule of quotient modules and a quotient module of S-Noetherian modules.

Lemma 2.1. *Let R be a commutative ring with identity and let S and T be multiplicative subsets of R. Let M be a unitary R-module. Then the following assertions hold.*

(1) *If A is an R_T-submodule of M_T, then $A = L_T$ for some R-submodule L of M.*
(2) *If M is an S-Noetherian R-module, then M_T is an S_T-Noetherian R_T-module. Furthermore, if T consists of regular elements of R, then M_T is an S-Noetherian R_T-module.*

Proof. (1) Suppose that A is an R_T-submodule of M_T and let t be any element of T. Note that M_T is an R-module and the map $\varphi_t : M \to M_T$ given by $\varphi_t(m) = \frac{mt}{t}$ is an R-module homomorphism. Let $L = \varphi_t^{-1}(A)$. Then L is an R-submodule of M. Let $\frac{\ell}{v} \in L_T$, where $\ell \in L$ and $v \in T$. Then $\varphi_t(\ell) \in A$, so $\frac{\ell}{v} = \frac{\ell t}{t}\frac{t}{tv} = \varphi_t(\ell)\frac{t}{tv} \in A$. Hence $L_T \subseteq A$. For the reverse containment, let $\ell \in M$ and $v \in T$ with $\frac{\ell}{v} \in A$. Then $\varphi_t(\ell) = \frac{\ell t}{t} = \frac{\ell}{v}\frac{tv}{t} \in A$, so $\ell \in \varphi_t^{-1}(A) = L$. This implies that $\frac{\ell}{v} \in L_T$. Hence $A \subseteq L_T$, and thus $A = L_T$.

(2) Let A be an R_T-submodule of M_T. Then by (1), $A = L_T$ for some R-submodule L of M. Since M is an S-Noetherian R-module, there exist $s \in S$ and $\ell_1, \dots, \ell_n \in L$ such that

$$Ls \subseteq \ell_1 R + \cdots + \ell_n R.$$

Fix an element $t \in T$. Note that $(Ls)_T = L_T\frac{s}{t}$ and $(\ell_k R)_T = \frac{\ell_k}{t}R_T$ for all $1 \leq k \leq n$, so we have

$$A\tfrac{s}{t} = L_T\tfrac{s}{t} = (Ls)_T \subseteq (\ell_1 R + \cdots + \ell_n R)_T = \tfrac{\ell_1}{t}R_T + \cdots + \tfrac{\ell_n}{t}R_T \subseteq L_T = A.$$

Hence A is an S_T-finite R_T-submodule of M_T, which means that M_T is an S_T-Noetherian R_T-module.

Note that if T consists of regular elements of R, then R can be naturally embedded in R_T. Hence we may assume that S is a multiplicative subset of R_T. Thus the second argument holds. □

Let R be a commutative ring with identity and let P be a prime ideal of R. Then $S := R \setminus P$ is a (saturated) multiplicative subset of R. Let M be a unitary R-module. We say that M is *P-finite* if M is S-finite; and M is a *P-Noetherian module* if M is an S-Noetherian module. For an element $r \in R$ and an R-submodule L of M, we set $L : r = \{x \in M \,|\, xr \in L\}$. It is easy to see that $L : r$ is an R-submodule of M containing L.

Proposition 2.2. *Let R be a commutative ring with identity, $\mathfrak{m}$ a maximal ideal of R and M a torsion-free unitary R-module. Then the following assertions are equivalent.*

1. *M is an $\mathfrak{m}$-Noetherian module.*
2. *$M_{\mathfrak{m}}$ is a Noetherian $R_{\mathfrak{m}}$-module and every nonzero finitely generated R-submodule L of M, there exists an element $s \in R \setminus \mathfrak{m}$ such that $L_{\mathfrak{m}} \cap M = L : s$.*

Proof. (1) ⇒ (2) Let A be a nonzero $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$. Then by Lemma 2.1(1), $A = B_{\mathfrak{m}}$ for some R-submodule B of M. Since M is an $\mathfrak{m}$-Noetherian module, there exist $s \in R \setminus \mathfrak{m}$ and $b_1, \ldots, b_n \in B$ such that $Bs \subseteq b_1 R + \cdots + b_n R$, so we obtain

$$\begin{aligned} B_{\mathfrak{m}} &= (Bs)_{\mathfrak{m}} \\ &\subseteq b_1 R_{\mathfrak{m}} + \cdots + b_n R_{\mathfrak{m}} \\ &\subseteq B_{\mathfrak{m}}. \end{aligned}$$

Hence $B_{\mathfrak{m}} = b_1 R_{\mathfrak{m}} + \cdots + b_n R_{\mathfrak{m}}$. Thus $M_{\mathfrak{m}}$ is a Noetherian $R_{\mathfrak{m}}$-module. For the second argument, let L be a nonzero finitely generated R-submodule of M. Since $L_{\mathfrak{m}} \cap M$ is an R-submodule of M, there exist $u \in R \setminus \mathfrak{m}$ and $c_1, \ldots, c_m \in L_{\mathfrak{m}} \cap M$ such that

$$(L_{\mathfrak{m}} \cap M)u \subseteq c_1 R + \cdots + c_m R.$$

For each $i = 1, \ldots, m$, take an element $t_i \in R \setminus \mathfrak{m}$ such that $c_i t_i \in L$. Let $t = t_1 \cdots t_m$ and let $s = tu$. Then $(c_1 R + \cdots + c_m R)t \subseteq L$. Hence we obtain

$$(L_{\mathfrak{m}} \cap M)s \subseteq (c_1 R + \cdots + c_m R)t \subseteq L.$$

Thus $L_{\mathfrak{m}} \cap M = L : s$.

(2) ⇒ (1) Let L be a nonzero R-submodule of M. Then $L_{\mathfrak{m}}$ is an $R_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$. Since $M_{\mathfrak{m}}$ is a Noetherian $R_{\mathfrak{m}}$-module, $L_{\mathfrak{m}} = a_1 R_{\mathfrak{m}} + \cdots + a_n R_{\mathfrak{m}}$ for some $a_1, \ldots, a_n \in L$. Therefore by the assumption, we have

$$\begin{aligned} L &\subseteq L_{\mathfrak{m}} \cap M \\ &= (a_1 R_{\mathfrak{m}} + \cdots + a_n R_{\mathfrak{m}}) \cap M \\ &= (a_1 R + \cdots + a_n R) : s \end{aligned}$$

for some $s \in R \setminus \mathfrak{m}$. Hence $Ls \subseteq a_1R + \cdots + a_nR$, which means that L is $\mathfrak{m}$-finite. Thus M is an $\mathfrak{m}$-Noetherian module. □

Proposition 2.3. *Let R be a commutative ring with identity and let M be a torsion-free unitary R-module. Then the following conditions are equivalent.*

(1) *M is a Noetherian module.*
(2) *M is a P-Noetherian module for all $P \in \mathrm{Spec}(R)$.*
(3) *M is an $\mathfrak{m}$-Noetherian module for all $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. (1) ⇒ (2) ⇒ (3) These implications are obvious.

(3) ⇒ (1) Suppose that M is an $\mathfrak{m}$-Noetherian module for all $\mathfrak{m} \in \mathrm{Max}(R)$ and let L be an R-submodule of M. Then for each $\mathfrak{m} \in \mathrm{Max}(R)$, there exist an element $s_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ and a finitely generated R-submodule $F_{\mathfrak{m}}$ of L such that $Ls_{\mathfrak{m}} \subseteq F_{\mathfrak{m}}$. Let $S = \{s_{\mathfrak{m}} \,|\, \mathfrak{m} \in \mathrm{Max}(R)\}$. Then S is not contained in any maximal ideal of R, so there exist $s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n} \in S$ such that $(s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n}) = R$. Therefore we obtain

$$\begin{aligned} L &= L(s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n}) \\ &\subseteq F_{\mathfrak{m}_1} + \cdots + F_{\mathfrak{m}_n} \\ &\subseteq L. \end{aligned}$$

Hence $L = F_{\mathfrak{m}_1} + \cdots + F_{\mathfrak{m}_n}$. Note that $F_{\mathfrak{m}_1} + \cdots + F_{\mathfrak{m}_n}$ is finitely generated. Thus M is a Noetherian module. □

Let D be an integral domain, S a multiplicative subset of D and M a unitary D-module. We define M to be *locally S-Noetherian* if for each maximal ideal $\mathfrak{m}$ of D, $M_{\mathfrak{m}}$ is an S-Noetherian $D_{\mathfrak{m}}$-module. Let L be a D-submodule of M. Then it is easy to see that $(L : M) = \{d \in D \,|\, Md \subseteq L\}$ is an ideal of D. Recall that D has *finite character* if every nonzero nonunit in D belongs to only finitely many maximal ideals of D (equivalently, each nonzero proper ideal of D is contained in only finitely many maximal ideals of D). This concept can be generalized to the module version as follows: M has *finite character* if for each nonzero element a of M with $(aD : M) \neq D$, $(aD : M)$ is contained in only finitely many maximal ideals of D. It is easy to show that M has finite character if and only if for each nonzero proper D-submodule L of M, $(L : M)$ is contained in only finitely many maximal ideals of D.

Proposition 2.4. *Let D be an integral domain, S a multiplicative subset of D and M a torsion-free unitary D-module. Then the following assertions hold.*

(1) *If M is an S-Noetherian module, then M is a locally S-Noetherian module.*
(2) *If M is a locally S-Noetherian module which has finite character, then M is an S-Noetherian module.*

Proof. (1) This is an immediate consequence of Lemma 2.1(2).

(2) Let A be a D-submodule of M and let a be a nonzero element of A such that $(aD : M) \neq D$. Since M has finite character, $(aD : M)$ is contained in only finitely many maximal ideals of D, say $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. Since $M_{\mathfrak{m}_1}, \dots, M_{\mathfrak{m}_n}$ are S-Noetherian, for each $i \in \{1, \dots, n\}$, there exist an element $s_i \in S$ and a finitely generated D-submodule F_i of A such that $A_{\mathfrak{m}_i} s_i \subseteq (F_i)_{\mathfrak{m}_i}$. Let $s = s_1 \cdots s_n$ and let $F = aD + F_1 + \cdots + F_n$. Then $A_{\mathfrak{m}_i} s \subseteq F_{\mathfrak{m}_i}$ for all $i = 1, \dots, n$. Let $\mathfrak{m}'$ be a

maximal ideal of D which is distinct from $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. Then $(aD : M) \nsubseteq \mathfrak{m}'$, so we can pick an element $r \in (aD : M) \setminus \mathfrak{m}'$. Therefore $\frac{m}{1} = \frac{mr}{r} \in (aD)_{\mathfrak{m}'}$ for all $m \in M$. This shows that $(aD)_{\mathfrak{m}'} = M_{\mathfrak{m}'}$, which indicates that $A_{\mathfrak{m}'} = M_{\mathfrak{m}'} = F_{\mathfrak{m}'}$. Hence $A_{\mathfrak{m}} s \subseteq F_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of D. Consequently, we have

$$\begin{aligned} As &= \left(\bigcap_{\mathfrak{m} \in \mathrm{Max}(D)} A_{\mathfrak{m}} \right) s \\ &\subseteq \bigcap_{\mathfrak{m} \in \mathrm{Max}(D)} A_{\mathfrak{m}} s \\ &\subseteq \bigcap_{\mathfrak{m} \in \mathrm{Max}(D)} F_{\mathfrak{m}} \\ &= F, \end{aligned}$$

where the equalities follow from [2, Theorem 4.3]. Since F is a finitely generated D-submodule of A, A is S-finite. Thus M is an S-Noetherian D-module. □

The next example shows that the converse of Proposition 2.4(1) does not generally hold.

Example 2.5. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and let $R = \prod_{i \in \mathbb{N}} \mathbb{Z}_2$.

(1) Note that $\mathbb{Z}_2 \times \{0\} \times \{0\} \times \cdots \subsetneq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \{0\} \times \{0\} \times \cdots \subsetneq \cdots$ is a strict ascending chain of ideals of R, so R is not a Noetherian ring.

(2) Note that $\mathrm{Max}(R) = \{\prod_{i \in \mathbb{N}} A_i \,|\, \text{for each } j \in \mathbb{N},\ A_j = \{0\} \text{ and } A_i = \mathbb{Z}_2 \text{ for all } i \neq j\}$, so for all $M \in \mathrm{Max}(R)$, R_M has only two elements. Hence R is a locally Noetherian ring.

Let R be a commutative ring with identity and let M be a unitary R-module. For an element $f \in M[X]$, the *content module* $c(f)$ of f is defined to be the R-submodule of M generated by the coefficients of f. In particular, if $M = R$, then $c(f)$ is called the *content ideal* of R. Let $N = \{f \in R[X] \,|\, c(f) = R\}$. Then N is a (saturated) regular multiplicative subset of $R[X]$ [16, page 17] (or [3, page 559]). The quotient module $M[X]_N$ of $M[X]$ by N is usually called the *Nagata module* of M. Recall that a multiplicative subset S of R is *anti-Archimedean* if $\bigcap_{n \geq 1} s^n R \cap S \neq \emptyset$. Now, we give the main result in this section which involves the Hilbert basis theorem and the Nagata module extension for S-Noetherian modules.

Theorem 2.6. *Let R be a commutative ring with identity, S an anti-Archimedean subset of R and M a unitary R-module. Then the following statements are equivalent.*

(1) *M is an S-Noetherian R-module.*
(2) *$M[X]$ is an S-Noetherian $R[X]$-module.*
(3) *$M[X]_N$ is an S-Noetherian $R[X]_N$-module.*

Proof. (1) $\Rightarrow$ (2) Let A be an $R[X]$-submodule of $M[X]$. For each $k \geq 0$, let B_k be the set consisting of zero and the leading coefficients of the polynomials in A of degree less than or equal to k and let $B = \bigcup_{k \geq 0} B_k$. Then each B_k and B are R-submodules of M such that $B_k \subseteq B_{k+1}$ for all $k \geq 0$. Since M is an S-Noetherian R-module, there exist $t \in S$ and $b_1, \dots, b_n \in B$ such that $Bt \subseteq b_1 R + \cdots + b_n R$.

Take an integer d so that $b_1, \dots, b_n \in B_d$. Then $Bt \subseteq b_1R + \cdots + b_nR \subseteq B_d$. Again, since M is an S-Noetherian R-module, for each $j \in \{0, \dots, d\}$, there exist $s_j \in S$ and $b_{j1}, \dots, b_{jk_j} \in B_j$ such that $B_j s_j \subseteq b_{j1}R + \cdots + b_{jk_j}R$. Let $s = s_0 \cdots s_d t$. Then we obtain

$$Bs \subseteq b_1R + \cdots + b_nR \subseteq B_d$$

and for all $j \in \{0, \dots, d\}$,

$$B_j s \subseteq b_{j1}R + \cdots + b_{jk_j}R.$$

For each $j \in \{0, \dots, d\}$ and $\ell \in \{1, \dots, k_j\}$, let $f_{j\ell} = b_{j\ell}X^j$+(lower terms) $\in A$.

Now, we claim that $Au \subseteq \sum_{0 \le i \le d} \sum_{1 \le j \le k_i} f_{ij}R[X]$ for some $u \in S$. Let $f = aX^m$+(lower terms) $\in A$. First, we suppose that $m \ge d+1$. Then $a \in B$, so $as \in B_d$, which implies that $as^2 \in b_{d1}R + \cdots + b_{dk_d}R$. Therefore $as^2 = b_{d1}r_1 + \cdots + b_{dk_d}r_{k_d}$ for some $r_1, \dots, r_{k_d} \in R$. Let $\alpha = fs^2 - \sum_{\ell=1}^{k_d} f_{d\ell}r_\ell X^{m-d}$. Then $\alpha \in A$ with $\deg(\alpha) \le m-1$. By repeating this process, we have $q_1 \in \mathbb{N}$ and $g_1, \dots, g_{k_d} \in R[X]$ such that $\beta := fs^{q_1} - \sum_{\ell=1}^{k_d} f_{d\ell}g_\ell \in A$ and $\deg(\beta) \le d$. Since the leading coefficient of β belongs to $B_{\deg(\beta)}$, there exist $r'_1, \dots, r'_{k_{\deg(\beta)}} \in R$ such that $\gamma := \beta s - \sum_{\ell=1}^{k_{\deg(\beta)}} f_{\deg(\beta)\ell}r'_\ell \in A$ and $\deg(\gamma) \le \deg(\beta) - 1$. If we still have $\gamma \ne 0$, then we repeat the same process. After finitely many steps, we obtain

$$fs^{q_2} \in \textstyle\sum_{0 \le i \le d} \sum_{1 \le j \le k_i} f_{ij}R[X]$$

for some $q_2 \in \mathbb{N}$. Second, we suppose that $m \le d$. Then a similar argument as in the previous case shows that

$$fs^{q_3} \in \textstyle\sum_{0 \le i \le d} \sum_{1 \le j \le k_i} f_{ij}R[X]$$

for some $q_3 \in \mathbb{N}$. Since S is an anti-Archimedean subset of R, there exists an element $u \in \bigcap_{n \ge 1} s^nR \cap S$, so we have

$$fu \in \textstyle\sum_{0 \le i \le d} \sum_{1 \le j \le k_i} f_{ij}R[X].$$

Since f was arbitrarily chosen in A, we obtain

$$Au \subseteq \textstyle\sum_{0 \le i \le d} \sum_{1 \le j \le k_i} f_{ij}R[X].$$

Hence A is an S-finite $R[X]$-submodule of $M[X]$. Thus $M[X]$ is an S-Noetherian $R[X]$-module.

(2) $\Rightarrow$ (3) This implication follows directly from Lemma 2.1(2).

(3) $\Rightarrow$ (1) Let A be an R-submodule of M. Then $A[X]_N$ is an $R[X]_N$-submodule of $M[X]_N$. Since $M[X]_N$ is an S-Noetherian $R[X]_N$-module, there exist $s \in S$, $f_1, \dots, f_n \in A[X]$ and $g_1, \dots, g_n \in N$ such that

$$A[X]_N s \subseteq \tfrac{f_1}{g_1}R[X]_N + \cdots + \tfrac{f_n}{g_n}R[X]_N.$$

Let $a \in A$. Then we can find $h_1, \dots, h_n \in R[X]$ and $\alpha_1, \dots, \alpha_n \in N$ such that $as = \frac{f_1}{g_1}\frac{h_1}{\alpha_1} + \cdots + \frac{f_n}{g_n}\frac{h_n}{\alpha_n}$. Let $\alpha = \prod_{i=1}^n g_i\alpha_i$ and for each $i = 1, \dots, n$, let $\beta_i = \frac{\alpha h_i}{g_i\alpha_i}$. Then $as = \frac{f_1\beta_1 + \cdots + f_n\beta_n}{\alpha}$, so we have

$$\begin{aligned} as\alpha &= f_1\beta_1 + \cdots + f_n\beta_n \\ &\in (c(f_1) + \cdots + c(f_n))[X]. \end{aligned}$$

Since $\alpha \in N$, $as \in c(f_1) + \cdots + c(f_n)$. Therefore $As \subseteq c(f_1) + \cdots + c(f_n)$. Note that $c(f_1) + \cdots + c(f_n)$ is a finitely generated R-submodule of A. Hence A is an S-finite R-submodule of M. Thus M is an S-Noetherian R-module. □

3. S-strong Mori modules

We start this section with some observations for S-w-finite modules.

Remark 3.1. Let D be an integral domain, S a multiplicative subset of D and M a torsion-free w-module as a D-module.

(1) Let L be a nonzero D-submodule of M. Then L_w is a w-submodule of M. If L_w is S-w-finite, then there exist an element $s \in S$ and a w-finite type submodule F of L_w such that $L_w s \subseteq F$, so $Ls \subseteq F$. Conversely, if there exist an element $s \in S$ and a w-finite type submodule F of L_w such that $Ls \subseteq F$, then $L_w s \subseteq F$. Hence we may extend the concept of S-w-finite modules to any nonzero submodule of a w-module as follows: A nonzero submodule L of M is *S-w-finite* if there exist an element $s \in S$ and a w-finite type submodule F of L_w such that $Ls \subseteq F$.

(2) By (1), M is an S-SM-module if and only if every nonzero submodule of M is S-w-finite.

(3) Suppose that L is an S-w-finite submodule of M. Then we can find $s \in S$ and $a_1, \dots, a_n \in L_w$ such that $Ls \subseteq (a_1D + \cdots + a_nD)_w$, so for each $i = 1, \dots, n$, there exists an element $J_i \in \mathrm{GV}(D)$ such that $a_iJ_i \subseteq L$. Let $J = J_1 \cdots J_n$. Then $J \in \mathrm{GV}(D)$ [19, Lemma 1.1] (or [9, Lemma 2.3(3)]) and $a_iJ \subseteq L$ for all $i \in \{1, \dots, n\}$, so we obtain

$$\begin{aligned}(a_1D + \cdots + a_nD)_w &= ((a_1D + \cdots + a_nD)J)_w \\ &= (a_1J + \cdots + a_nJ)_w,\end{aligned}$$

where the first equality follows from [19, Proposition 2.7]. Hence $Ls \subseteq (a_1J + \cdots + a_nJ)_w$. Note that for all $i \in \{1, \dots, n\}$, a_iJ is a finitely generated submodule of L. Thus we may assume that $a_1, \dots, a_n \in L$ by replacing $a_1D + \cdots + a_nD$ by $a_1J + \cdots + a_nJ$.

Lemma 3.2. *Let D be an integral domain and let S be a multiplicative subset of D. Let M be a torsion-free D-module, w the w-operation on M and $\overline{w}$ the w-operation on M_S as a D_S-module. Suppose that M is a w-module. If L is a nonzero D-submodule of M, then the following assertions hold.*

(1) *If L is a w-submodule of M, then $L_S \cap M$ is a w-submodule of M.*
(2) *If L_S is a $\overline{w}$-submodule of M_S, then $L_S \cap M$ is a w-submodule of M.*
(3) *$(L_w)_S \subseteq (L_S)_{\overline{w}}$ and $((L_w)_S)_{\overline{w}} = (L_S)_{\overline{w}}$.*

Proof. (1) Let $x \in (L_S \cap M)_w$. Then $xJ \subseteq L_S \cap M$ for some $J \in \mathrm{GV}(D)$. Since J is finitely generated, $xsJ \subseteq L$ for some $s \in S$, so $xs \in L_w = L$. Also, $x \in M_w = M$. Hence $x \in L_S \cap M$. Thus $L_S \cap M$ is a w-submodule of M.

(2) Let $x \in (L_S \cap M)_w$. Then $xJ \subseteq L_S \cap M$ for some $J \in \mathrm{GV}(D)$, so $xJD_S \subseteq L_S$. Note that $JD_S \in \mathrm{GV}(D_S)$ (cf. [10, Lemma 3.4(1)]), so $x \in (L_S)_{\overline{w}} = L_S$. Also, $x \in M_w = M$. Hence $x \in L_S \cap M$. Thus $L_S \cap M$ is a w-submodule of M.

(3) Let $x \in (L_w)_S$. Then $xs \in L_w$ for some $s \in S$, so there exists an element $J \in \mathrm{GV}(D)$ such that $xsJ \subseteq L$. Since $xJD_S \subseteq L_S$ and $JD_S \in \mathrm{GV}(D_S)$, $x \in (L_S)_{\overline{w}}$.

Hence $(L_w)_S \subseteq (L_S)_{\overline{w}}$. Also, by the previous inclusion, $((L_w)_S)_{\overline{w}} \subseteq ((L_S)_{\overline{w}})_{\overline{w}} = (L_S)_{\overline{w}}$. Thus $((L_w)_S)_{\overline{w}} = (L_S)_{\overline{w}}$. □

Let D be an integral domain and let P be a prime ideal of D. Then $S := D \setminus P$ is a (saturated) multiplicative subset of D. Let M be a w-module and let L be a nonzero submodule of M. We say that L is *P-w-finite* if L is S-w-finite; and M is a *P-strong Mori module* (P-SM-module) if M is an S-SM-module.

Proposition 3.3. *Let D be an integral domain, $\mathfrak{m}$ a maximal w-ideal of D and M a torsion-free w-module as a D-module. Then the following assertions are equivalent.*

(1) *M is an $\mathfrak{m}$-SM-module.*
(2) *$M_{\mathfrak{m}}$ is a Noetherian $D_{\mathfrak{m}}$-module and for every nonzero finitely generated D-submodule L of M, there exists an element $s \in D\setminus\mathfrak{m}$ such that $(L_w)_{\mathfrak{m}} \cap M = L_w : s$.*

Proof. (1) ⇒ (2) Let A be a nonzero $D_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$. Then by Lemma 2.1(1), $A = B_{\mathfrak{m}}$ for some D-submodule B of M. Since M is an $\mathfrak{m}$-SM-module, there exist $s \in D \setminus \mathfrak{m}$ and $b_1, \dots, b_n \in B$ such that $Bs \subseteq (b_1D + \cdots + b_nD)_w$, so we obtain

$$\begin{aligned} B_{\mathfrak{m}} &= (Bs)_{\mathfrak{m}} \\ &\subseteq ((b_1D + \cdots + b_nD)_w)_{\mathfrak{m}} \\ &= b_1D_{\mathfrak{m}} + \cdots + b_nD_{\mathfrak{m}} \\ &\subseteq B_{\mathfrak{m}}, \end{aligned}$$

where the second equality comes from [19, Remark before Proposition 4.6] (or [2, Theorem 4.3]) since t-Max$(D) = w$-Max(D) [2, Theorem 2.16]. Hence $B_{\mathfrak{m}} = b_1D_{\mathfrak{m}} + \cdots + b_nD_{\mathfrak{m}}$. Therefore $M_{\mathfrak{m}}$ is a Noetherian $D_{\mathfrak{m}}$-module. For the remaining argument, let L be a nonzero finitely generated D-submodule of M. Then $(L_w)_{\mathfrak{m}} \cap M$ is a w-submodule of M by Lemma 3.2(1), so there exist $t \in D\setminus\mathfrak{m}$ and $c_1, \dots, c_m \in (L_w)_{\mathfrak{m}} \cap M$ such that

$$((L_w)_{\mathfrak{m}} \cap M)t \subseteq (c_1D + \cdots + c_mD)_w$$

and $c_1t_1, \dots, c_mt_m \in L_w$ for some $t_1, \dots, t_m \in D \setminus \mathfrak{m}$. Let $t' = t_1 \cdots t_m$. Then $(c_1D + \cdots + c_mD)_w t' \subseteq L_w$. Therefore

$$((L_w)_{\mathfrak{m}} \cap M)tt' \subseteq (c_1D + \cdots + c_mD)_w t' \subseteq L_w.$$

This fact implies that $(L_w)_{\mathfrak{m}} \cap M = L_w : s$, where $s = tt'$.

(2) ⇒ (1) Let L be a nonzero D-submodule of M. Then $L_{\mathfrak{m}}$ is a $D_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$, so $L_{\mathfrak{m}} = a_1D_{\mathfrak{m}} + \cdots + a_nD_{\mathfrak{m}}$ for some $a_1, \dots, a_n \in L$. Hence

$$\begin{aligned} L &\subseteq L_{\mathfrak{m}} \cap M \\ &= (a_1D_{\mathfrak{m}} + \cdots + a_nD_{\mathfrak{m}}) \cap M \\ &= ((a_1D + \cdots + a_nD)_w)_{\mathfrak{m}} \cap M \\ &= (a_1D + \cdots + a_nD)_w : s \end{aligned}$$

for some $s \in D \setminus \mathfrak{m}$, where the third equality comes from [19, Remark before Proposition 4.6], which means that $Ls \subseteq (a_1D + \cdots + a_nD)_w$. Thus L is $\mathfrak{m}$-w-finite. Consequently, M is an $\mathfrak{m}$-SM-module. □

Proposition 3.4. *Let D be an integral domain and let M be a torsion-free w-module as a D-module. Then the following conditions are equivalent.*

1. *M is an SM-module.*
2. *M is a P-SM-module for all $P \in w\text{-Spec}(D)$.*
3. *M is an $\mathfrak{m}$-SM-module for all $\mathfrak{m} \in w\text{-Max}(D)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These implications are obvious.

(3) $\Rightarrow$ (1) Suppose that M is an $\mathfrak{m}$-SM-module for all $\mathfrak{m} \in w\text{-Max}(D)$ and let L be a w-submodule of M. Then for each $\mathfrak{m} \in w\text{-Max}(D)$, there exist an element $s_{\mathfrak{m}} \in D \setminus \mathfrak{m}$ and a finitely generated D-submodule $F_{\mathfrak{m}}$ of L such that $Ls_{\mathfrak{m}} \subseteq (F_{\mathfrak{m}})_w$. Let $S = \{s_{\mathfrak{m}} \mid \mathfrak{m} \in w\text{-Max}(D)\}$. Then S is not contained in any maximal w-ideal of D, so there exist $s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n} \in S$ such that $(s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n})_w = D$. Hence we obtain

$$\begin{aligned} L &= (L(s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n})_w)_w \\ &= (L(s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n}))_w \\ &\subseteq (F_{\mathfrak{m}_1} + \cdots + F_{\mathfrak{m}_n})_w \\ &\subseteq L. \end{aligned}$$

Thus $L = (F_{\mathfrak{m}_1} + \cdots + F_{\mathfrak{m}_n})_w$. Consequently, M is an SM-module. □

Recall that an integral domain D has *finite w-character* if for each nonzero nonunit in D belongs to only finitely many maximal w-ideals of D, or equivalently, for each nonzero proper ideal of D is contained in only finitely many maximal w-ideals of D. Generalizing this, a finite w-character can be defined in the module as follows: A D-module M has *finite w-character* if for each nonzero element a of M with $(aD : M) \neq D$, $(aD : M)$ is contained in only finitely many maximal w-ideals of D. It is easy to show that M has finite w-character if and only if for each nonzero proper D-submodule L of M, $(L : M)$ is contained in only finitely many maximal w-ideals of D. Also, it is easy to show that every commutative ring with identity which has finite w-character has finite w-character as module. Recall that a D-module M is a *w-locally S-Noetherian D-module* if for each maximal w-ideal $\mathfrak{m}$, $M_{\mathfrak{m}}$ is an S-Noetherian $D_{\mathfrak{m}}$-module.

Proposition 3.5. *Let D be an integral domain, S a multiplicative subset of D and M a torsion-free w-module as a D-module. Then the following assertions hold.*

1. *If M is an S-SM-module, then M is a w-locally S-Noetherian module.*
2. *If M is a w-locally S-Noetherian module which has finite w-character, then M is an S-SM-module.*

Proof. (1) Let $\mathfrak{m}$ be a maximal w-ideal of D and let A be a $D_{\mathfrak{m}}$-submodule of $M_{\mathfrak{m}}$. Then by Lemma 2.1(1), $A = B_{\mathfrak{m}}$ for some D-submodule B of M, so there exist $s \in S$ and $b_1, \dots, b_n \in B$ such that $A's \subseteq (b_1D + \cdots + b_nD)_w$. Therefore we obtain

$$As = B_{\mathfrak{m}}s \subseteq ((b_1D + \cdots + b_nD)_w)_{\mathfrak{m}} = b_1D_{\mathfrak{m}} + \cdots + b_nD_{\mathfrak{m}},$$

where the last equality comes from [19, Remark before Proposition 4.6]. Hence A is S-finite. Thus $M_{\mathfrak{m}}$ is an S-Noetherian $D_{\mathfrak{m}}$-module for each $\mathfrak{m} \in w\text{-Max}(D)$. Consequently, M is a w-locally S-Noetherian module.

(2) Suppose that M is a w-locally S-Noetherian module which has finite w-character and let A be a D-submodule of M. Let a be a nonzero element of A such that $(aD : M) \neq D$. Then $(aD : M)$ is contained in only finitely many maximal w-ideals of D, say $\mathfrak{m}_1, \dots, \mathfrak{m}_m$. Since for each $i = 1, \dots, m$, $M_{\mathfrak{m}_i}$ is S-Noetherian, we obtain that there exist $s_i \in S$ and a finitely generated D-submodule F_i of A such that $A_{\mathfrak{m}_i} s_i \subseteq (F_i)_{\mathfrak{m}_i}$. Let $s = s_1 \cdots s_m$ and $F = aD + F_1 + \cdots + F_m$. Then $A_{\mathfrak{m}_i} s \subseteq F_{\mathfrak{m}_i}$ for all $i = 1, \dots, m$. Let $\mathfrak{m}' \neq \mathfrak{m}_i$ for all $i = 1, \dots, m$. Then $(aD : M) \nsubseteq \mathfrak{m}'$. Hence there exists $x \in (aD : M)$ such that $x \notin \mathfrak{m}'$; that is, for all $m \in M$, $mx \in aD$, but $x \notin \mathfrak{m}'$. Hence $\frac{m}{1} = \frac{mx}{x} \in (aD)_{\mathfrak{m}'}$. Therefore $(aD)_{\mathfrak{m}'} = M_{\mathfrak{m}'}$; that is, $F_{\mathfrak{m}'} = M_{\mathfrak{m}'}$. This fact implies that $A_{\mathfrak{m}} s \subseteq F_{\mathfrak{m}}$ for each $\mathfrak{m} \in w\text{-Max}(D)$, so we have

$$\begin{aligned} A_w s &= \Big(\bigcap_{\mathfrak{m} \in w\text{-Max}(D)} A_{\mathfrak{m}} \Big) s \\ &\subseteq \bigcap_{\mathfrak{m} \in w\text{-Max}(D)} A_{\mathfrak{m}} s \\ &\subseteq \bigcap_{\mathfrak{m} \in w\text{-Max}(D)} F_{\mathfrak{m}} \\ &= F_w, \end{aligned}$$

where the equalities follow from [17, Theorem 7.3.6]. Since F is finitely generated and $F \subseteq A$, A is S-w-finite type. Thus M is an S-SM-module. □

Let D be an integral domain and let M be a w-module as a D-module. We say that M is a *DW-module* if every nonzero D-submodule of M is a w-module. Let $N_v = \{f \in D[X] \,|\, c(f)_v = D\}$. Then N_v is a (saturated) multiplicative subset of $D[X]$ [10, Proposition 2.1]; and the quotient module $M[X]_{N_v}$ of $M[X]$ by N_v is called the *t-Nagata module* of M.

Lemma 3.6. *Let D be an integral domain and let M be a nonzero D-module. Then $M[X]_{N_v}$ is a DW-module.*

Proof. Suppose that A is a $D[X]_{N_v}$-submodule of $M[X]_{N_v}$. Let $f \in A_w$. Then $fJ \in A$ for some $J \in \text{GV}(D[X]_{N_v})$. Note that $\text{GV}(D[X]_{N_v}) = \{D[X]_{N_v}\}$ [17, Theorems 6.3.12 and 6.6.18], so $J = D[X]_{N_v}$. Hence $f \in A$, which indicates that $A_w = A$. Thus $M[X]_{N_v}$ is a DW-module. □

Lemma 3.7. (cf. [17, Proposition 6.6.13]) *Let D be an integral domain and let M be a torsion-free D-module. Denote that $M[X]_W$ is the w-envelop of a $D[X]$-module $M[X]$. Then the following assertions hold.*

(1) $M_w[X] = (M[X])_W$.
(2) *If M is a w-module, then $M[X]$ is a w-$D[X]$-module.*

Proof. (1) Let $f := a_0 + a_1X + \cdots + a_nX^n \in M_w[X]$. Then $a_i \in M_w$ for all $0 \leq i \leq n$, so for each $0 \leq i \leq n$, there exists an element $J_i \in \text{GV}(D)$ such that $a_iJ_i \subseteq M$. Let $J = J_0 \cdots J_n$. Then $a_iJ \subseteq M$ for all $0 \leq i \leq n$. Hence $(a_0D[X]+\cdots+a_nD[X])JD[X] \subseteq M[X]$. Since $JD[X] \in \text{GV}(D[X])$, $a_0D[X]+\cdots+a_nD[X] \subseteq (M[X])_W$. It follows that $f \in (M[X])_W$. For the reverse containment, let $f \in (M[X])_W$. Then there exists $J := (f_1, \dots, f_n) \in \text{GV}(D)$ such that $fJ \subseteq$

$M[X]$. Note that for each $1 \le i \le n$, there exists a positive integer m such that $c(f)c(f_i)^{m+1} = c(ff_i)c(f_i)^m$ [17, Theorem 1.7.16]. Hence there exists a positive integer k such that $c(f)c(f_i)^{k+1} = c(ff_i)c(f_i)^k$ for all $1 \le i \le n$. Therefore

$$c(f)(c(f_1)^{k+1} + \cdots + c(f_n)^{k+1}) = c(ff_1)c(f_1)^k + \cdots + c(ff_n)c(f_n)^k \subseteq M.$$

Since $c(f_1)^{k+1} + \cdots + c(f_n)^{k+1} \in \mathrm{GV}(D)$, $c(f) \subseteq M_w$. Thus $f \in M_w[X]$. Consequently, $M_w[X] = (M[X])_W$.

(2) This is an immediate consequence of the previous result. □

Now, we are ready to prove the Hilbert basis theorem and the t-Nagata module extension for S-SM-modules.

Theorem 3.8. *Let D be an integral domain, S an anti-Archimedean subset of D, $N_v = \{f \in D[X] \,|\, c(f)_v = D\}$ and M a torsion-free w-module as a D-module. Then the following statements are equivalent.*

1. *M is an S-SM-module.*
2. *$M[X]$ is an S-SM-module.*
3. *$M[X]_{N_v}$ is an S-SM-module.*
4. *$M[X]_{N_v}$ is an S-Noetherian module.*

Proof. (1) ⇒ (2) First, note that $M[X]$ is a W-module by Lemma 3.7(2). Let A be a w-submodule of $M[X]$ and let B be the set consisting of zero and the leading coefficients of the polynomials in A. Then B is a D-submodule of M. Since M is an S-SM-module, B is S-w-finite, so there exist $s \in S$ and $b_1, \dots, b_m \in B$ such that $Bs \subseteq (b_1D + \cdots + b_mD)_w$. For each $i \in \{1, \dots, m\}$, write $f_i = b_iX^{n_i} +$ (lower terms) $\in A$. Let $n = \max\{n_1, \dots, n_m\}$ and let $C = f_1D[X] + \cdots + f_mD[X]$. Let $f = aX^k +$ (lower terms) $\in A$. Then $a \in B$, so $as \in (b_1D + \cdots + b_mD)_w$. Therefore there exists an element $J \in \mathrm{GV}(D)$ such that $asJ \subseteq b_1D + \cdots + b_mD$. Let $J = (d_1, \dots, d_t)$. Then for each $j \in \{1, \dots, t\}$, $asd_j = \sum_{i=1}^m b_ir_{ji}$ for some $r_{j1}, \dots, r_{jm} \in D$. If $k \ge n$, then for each $j \in \{1, \dots, t\}$, let $g_j = fsd_j - \sum_{i=1}^m f_ir_{ji}X^{k-n_i}$. Then for all $j \in \{1, \dots, t\}$, $g_j \in A$ with $\deg(g_j) < k$. If we still have some $j \in \{1, \dots, t\}$ such that $\deg(g_j) \ge n$, then we repeat the same process. Let b be the leading coefficient of g_j. Then $b \in B$, so $bsJ_1 \subseteq b_1D + \cdots + b_mD$ for some $J_1 := (d'_1, \dots, d'_{t'}) \in \mathrm{GV}(D)$. Hence for each $\ell \in \{1, \dots, t'\}$, $bsd'_\ell = \sum_{i=1}^m b_ir'_{\ell i}$. Let $g'_j = g_jsd'_\ell - \sum_{i=1}^m f_ir'_{\ell i}X^{\deg(g_j)-n_i}$. Then $g'_j \in A$, $\deg(g'_j) < \deg(g_j)$ and $g'_j = (fsd_j - \sum_{i=1}^m f_ir_{ji}X^{k-n_i})sd'_\ell - \sum_{i=1}^m f_ir'_{\ell i}X^{\deg(g_j)-n_i}$. After finitely many steps, we get $J' \in \mathrm{GV}(D)$ and an integer $q \ge 1$ such that $fs^qJ' \subseteq (A \cap L) + C$, where $L = M \oplus MX \oplus \cdots \oplus MX^{n-1}$. Since L is an S-SM-module [11, Lemma 2.7(2)], $(A \cap L)_w$ is S-w-finite, so there exist $t \in S$ and $h_1, \dots, h_s \in A \cap L$ such that $(A \cap L)_wt \subseteq (h_1D + \cdots + h_sD)_w \subseteq (h_1D[X] + \cdots + h_sD[X])_W$. Let $u \in \bigcap_{i \ge 1} s^iD \cap S$. Then we have

$$\begin{aligned} futJ'D[X] &\subseteq ((A \cap L) + C)tD[X] \\ &= (A \cap L)tD[X] + CtD[X] \\ &\subseteq (h_1D[X] + \cdots + h_sD[X])_W + C. \end{aligned}$$

Since $J'D[X] \in \mathrm{GV}(D[X])$, $fut \in ((h_1D[X] + \cdots + h_sD[X])_W + C)_W$. Hence we have

$$\begin{aligned} Aut &\subseteq ((h_1D[X] + \cdots + h_sD[X])_W + C)_W \\ &= (h_1D[X] + \cdots + h_sD[X] + C)_W. \end{aligned}$$

Thus A is S-w-finite. Consequently, $M[X]$ is an S-SM-module.

(2) $\Rightarrow$ (4) Let A be a nonzero $D[X]_{N_v}$-submodule of $M[X]_{N_v}$. Then by Lemma 2.1(1), $A = A'_{N_v}$ for some nonzero $D[X]$-submodule A' of $M[X]$. Since A'_w is S-w-finite, there exist $s \in S$ and $f_1, \ldots, f_n \in A'$ such that $A'_w s \subseteq (f_1D[X] + \cdots + f_nD[X])_w$. Let $f \in A$. Then $fg \in A'$ for some $g \in N_v$, so we have

$$fgsJ \subseteq f_1D[X] + \cdots + f_nD[X]$$

for some $J \in \mathrm{GV}(D[X])$. Write $J = (h_1, \ldots, h_m)$ for some $h_1, \ldots, h_m \in D[X]$ and let $h = h_1 + h_2X^{\deg(h_1)+1} + \cdots + h_mX^{\sum_{j=1}^{m} \deg(h_j)+m-1}$. Then $c(h)_v = D$ and $fsgh \in f_1D[X] + \cdots + f_nD[X]$. Since $gh \in N_v$, we obtain

$$fs \in (f_1D[X] + \cdots + f_nD[X])_{N_v}.$$

Hence $As \subseteq (f_1D[X]+\cdots+f_nD[X])_{N_v}$. Thus $M[X]_{N_v}$ is an S-Noetherian $D[X]_{N_v}$-module.

(4) $\Rightarrow$ (1) Let A be a w-submodule of M. Then $A[X]_{N_v}$ is a $D[X]_{N_v}$-submodule of $M[X]_{N_v}$. Since $M[X]_{N_v}$ is an S-Noetherian module, there exist $s \in S$ and $f_1, \ldots, f_n \in A[X]$ such that $A[X]_{N_v}s \subseteq (f_1D[X] + \cdots + f_nD[X])_{N_v}$, so we obtain

$$A[X]_{N_v}s \subseteq (c(f_1) + \cdots + c(f_n))[X]_{N_v}.$$

Let $a \in A$. Then $asg \in (c(f_1) + \cdots + c(f_n))[X]$ for some $g \in N_v$, so $asc(g) \subseteq c(f_1)+\cdots+c(f_n)$. Since $c(g) \in \mathrm{GV}(D)$, $as \in (c(f_1)+\cdots+c(f_n))_w$ [20, Proposition 3.5] (or [9, Lemma 2.4]). Hence $As \subseteq (c(f_1) + \cdots + c(f_n))_w$. Thus A is S-w-finite. Consequently, M is an S-SM-module.

(3) $\Leftrightarrow$ (4) This equivalence comes directly from Lemma 3.6. □

The following result has already been proved in [11], but we can prove it in a different way from the proof in [11] using Theorem 3.8.

Corollary 3.9. ([11, Theorem 2.11(2)]) *Let D be an integral domain and let S be a multiplicative subset of D. Then D is an S-SM-domain if and only if every S-w-finite torsion-free w-module is an S-SM-module.*

Proof. Suppose that D is an S-SM-domain and let M be an S-w-finite torsion-free w-module as a D-module. Then there exist $s \in S$ and a finitely generated D-submodule L of M such that $Ms \subseteq L_w$, so $M[X]_{N_v}s \subseteq L_w[X]_{N_v} = L[X]_{N_v}$, where the equality comes from [6, Lemma 2.4(3)]. Hence $M[X]_{N_v}$ is S-finite. Since $D[X]_{N_v}$ is an S-Noetherian domain [11, Theorem 2.8], $M[X]_{N_v}$ is an S-Noetherian $D[X]_{N_v}$-module [7, Proposition 2.1]. Thus by Theorem 3.8, M is an S-SM-module. The converse is obvious. □

The next result recovers the fact that every surjective endomorphism of an SM-module is an isomorphism [6, Theorem 2.10].

Proposition 3.10. *Let D be an integral domain, S a multiplicative subset of D and M a torsion-free w-module as a D-module. If M is an S-SM-module and $\varphi : M \to M$ is a D-module epimorphism, then φ is an isomorphism.*

Proof. For each $n \geq 2$, let $\varphi^n = \varphi^{n-1} \circ \varphi$. Then φ^n is a D-module homomorphism, so $\text{Ker}(\varphi^n)$ is a w-submodule of M for all $n \geq 2$ [6, Lemma 2.9]. Hence we obtain an ascending chain $\text{Ker}(\varphi) \subseteq \text{Ker}(\varphi^2) \subseteq \cdots$ of w-submodules of M. Since M is an S-SM-module, there exist $s \in S$ and $k \in \mathbb{N}$ such that $\text{Ker}(\varphi^k)s \subseteq \text{Ker}(\varphi^n)$ for all $k \geq n$ [8, Theorem 1]. Let $x \in \text{Ker}(\varphi)$. Since φ^n is surjective, there exists an element $m \in M$ such that $\varphi^n(m) = x$, so $\varphi^{n+1}(m) = \varphi(x) = 0$. Therefore $m \in \text{Ker}(\varphi^{n+1})$, which implies that $ms \in \text{Ker}(\varphi^n)$. Hence $\varphi^n(m)s = \varphi^n(ms) = 0$. Since M is torsion-free, $\varphi^n(m) = 0$. Thus φ is an isomorphism. □

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