A GENERALIZATION OF $\mathfrak{cP}$-BAER RINGS AND APP RINGS

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ABSTRACT. We introduce the concept of $\mathcal{ACP}$-Baer rings, and investigate its properties. We say a ring R is *right $\mathcal{ACP}$-Baer* if the right annihilator of every cyclic projective right R-module in R is pure as a right ideal. This class of rings generalizes the class of right APP-rings (Z. Liu and R. Zhao, A generalization of PP-rings and p.q.-Baer rings, *Glasg. Math. J.*, **48**(2) (2006)), and right $\mathfrak{cP}$-Baer rings (G. F. Birkenmeier and B. J. Heider, Annihilators and extensions of idempotent generated ideals, *Commun. Algebra*, **47**(3) (2019)), and is closed under direct products and forming upper triangular matrix rings. It is shown that the $\mathcal{ACP}$-Baer property is inherited by the polynomial and power series ring extensions. Connections to related classes of rings are also considered as well as relevant examples are included to illustrate and delimit the theory.

1. Introduction and Motivation

Throughout this paper, all rings are assumed to be associative with unity and all modules are assumed to be unital right R-modules unless explicitly stated otherwise. A ring R is said to be *reduced* if it does not have a non-zero nilpotent element. A ring R is called *abelian* if every idempotent in R is central (e.g., commutative rings, rings with no non-trivial idempotents and reduced rings). Let S be a non-empty subset of R. We denote

$$r_R(S) = \{a \in R \mid sa = 0, \forall s \in S\},$$

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and it is referred to as the *right annihilator* of S in R. Likewise,

$$\ell_R(S) = \{a \in R \mid as = 0, \forall s \in S\}$$

denotes the *left annihilator* of S in R.

Recall from [15] and [18] that R is a *Baer* (resp., *right Rickart* or also called *right p.p.*) ring if the right annihilator of every non-empty (resp., singleton) subset of R is generated by an idempotent. In [15] Kaplansky introduced Baer rings in order to view various properties of AW^*-algebras and von Neumann algebras. Note that the class of Baer rings includes the von Neumann algebras. In [12] Clark gave a generalization of the notion of a *Baer* ring – in fact, Clark defined the concept of a *quasi-Baer* ring as the ring in which the right annihilator of an ideal is generated by an idempotent as a right ideal.

In the same vein, Birkenmeier, Kim and Park introduced in [8] a *right principally quasi-Baer* ring (or simply *right p.q.-Baer* ring) as the ring in which the right annihilator of a principal right ideal is generated by an idempotent as a right ideal. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of any principal right ideal is projective. Moreover, some examples were given in [8] to show that the class of left p.q.-Baer rings is *not* contained in the class of right p.p.-rings and the class of right p.p.-rings is *not* contained in the class of left p.q.-Baer rings.

On the other hand, following Tominaga [27], a right ideal I of a ring R is said to be *left s-unital* if, for each $a \in I$, there exists an element $x \in I$ such that $xa = a$. According to Liu and Zhao [17], a ring R is called *right APP* if the right annihilator $r_R(aR)$ is left s-unital as an ideal of R for any element $a \in R$. As a common generalization of left and right p.q.-Baer rings, the authors in [19] introduced the concept of *weakly p.q.-Baer* rings thus: a ring R is weakly p.q.-Baer if, for each $a \in R$, there exists a non-empty subset E of left semi-central idempotents of R such that $r_R(aR) = \bigcup_{e \in E} eR$, where $r_R(-)$ denotes the right annihilator of $(-)$ in R. This implies that R modulo the right annihilator of any principal right ideal is flat. The class of weakly p.q.-Baer rings is a natural subclass of the class of APP rings and includes both left p.q.-Baer rings and right p.q.-Baer rings. Besides, Birkenmeier and LeBlanc [6] call a right R-module M *s.Baer* (resp., *s.Rickart*) if the right annihilator in R of a non-empty (resp., singleton) subset of M is generated by an idempotent of R.

These rings have a rich structure theory, because of the abundance of their idempotents. Moreover, these rings appear naturally not only in Ring Theory (e.g., right Noetherian right semi-hereditary rings are Baer rings), but too in other areas of Mathematics such as Operator Theory (e.g., von Neumann algebras are Baer rings and local multiplier algebras are quasi-Baer rings).

In [4], as a generalization of right p.q.-Baer rings, Birkenmeier and Heider define *right* $\mathfrak{cP}$*-Baer* rings as those rings in which every right annihilator of a cyclic projective right R-module is generated by an idempotent as a right ideal and show that every semi-prime ring has a $\mathfrak{cP}$-Baer hull. Among other important results, they also investigate the behavior of right $\mathfrak{cP}$-Baer rings with respect to the polynomial and formal power series ring extensions as well as to the generalized upper triangular matrix ring extensions.

In the present article, we introduce and study the notion of $\mathcal{ACP}$*-Baer* rings. In fact, we say that a ring R is *right* $\mathcal{ACP}$*-Baer* if, for each cyclic projective right R-module P, the right annihilator $r_R(P)$ is left s-unital. We know that, if P is a cyclic projective right R-module, then $P \cong eR$, where $e = e^2 \in R$. Hence,

$$r_R(P) = r_R(eR) = r_R(ReR) = r_R(tr(P)),$$

where $tr(-)$ designates the *trace* of $(-)$ in R. Therefore, R is a right $\mathcal{ACP}$-Baer ring if, for each $e = e^2 \in R$, $r_R(eR)$ is left s-unital. Thus, the class of $\mathcal{ACP}$-Baer rings (properly) includes right APP rings, $\mathfrak{cP}$-Baer rings and weakly p.q.-Baer rings, and hence Abelian rings, semi-simple Artinian rings, local rings, bi-regular rings, prime rings, and both right p.q.-Baer rings and left p.q.-Baer rings.

The remainder of the paper is organized as follows: In Section 2, we explore right $\mathcal{ACP}$-Baer rings and provide several basic results for them. In Section 3, we examine the transfer of the right $\mathcal{ACP}$-Baer property between the ring R, the polynomial ring $R[x]$ and the power series ring $R[[x]]$. Next, in Section 4, we study matrix extensions. Concretely, a complete characterization of when a 2-by-2 generalized upper triangular matrix ring is right $\mathcal{ACP}$-Baer is given. Using this result, it is established that the n-by-n upper triangular matrix ring over R is right $\mathcal{ACP}$-Baer if, and only if, R is right $\mathcal{ACP}$-Baer.

For more detailed information and examples of Baer and quasi-Baer rings we refer the interested reader to the valuable sources [3], [5], [7], [9], [10], [14], [20], [22], [21], [23] and [24], respectively.

2. Basic Results and Examples

An idempotent $e \in R$ is called *left* (resp., *right*) *semi-central* if $re = ere$ (resp., $er = ere$), for all $r \in R$. The set of all left (right) semi-central idempotents of R are denoted by $S_l(R)$ (resp., $S_r(R)$). Define $B(R) = S_l(R) \cap S_r(R)$ as the set of all central idempotents and if R is a semi-prime ring, then $S_l(R) = B(R) = S_r(R)$.

Our key instrument for the successful presentation is the following.

Definition 2.1. We say that an ideal I of R is *left s-unital* if, for every $a \in I$, $xa = a$ holds for some $x \in I$. The right mode is defined similarly.

It follows from [27, Theorem 1] that I is left s-unital if, and only if, for any finite set of elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $x \in I$ such that $a_i = xa_i$, $i = 1, 2, \ldots, n$. A submodule N of a right R-module M is called a *pure* submodule if $L_R \otimes N \longrightarrow L_R \otimes M$ is a monomorphism for every right R-module L. By virtue of [26, Proposition 11.3.13], an ideal I is left s-unital if, and only if, R/I is flat as a right R-module if, and only if, I is pure as a right ideal of R.

For completeness of the exposition, we recall once again the following.

Definition 2.2. A ring R is called a *right $\mathcal{ACP}$-Baer ring* (resp., *a left $\mathcal{ACP}$-Baer ring*) if, for any cyclic projective right (resp., left) R-module P, the right annihilator $r_R(P)$ (resp., the left annihilator $\ell_R(P)$) is left (resp., right) s-unital. Accordingly, the ring R is called *$\mathcal{ACP}$-Baer* if R is both right and left $\mathcal{ACP}$-Baer.

The next technicalities are worthwhile.

Lemma 2.3. *If $I_1, \ldots, I_n$ are left s-unital for some positive integer n, then so is $I = \cap_{i=1}^{n} I_n$.*

Proof. Let $a \in I = \bigcap_{i=1}^{n} I_n$. Thus, $a \in I_i$ for each $1 \leq i \leq n$. By hypothesis, there exists $x_i \in I_i$ such that $x_i a = a$. Setting $x := x_1 \cdots x_n$, then $x \in \bigcap_{i=1}^{n} I_n$ and $xa = a$. □

Proposition 2.4. *The following conditions are equivalent:*

(i) *R is a right $\mathcal{ACP}$-Baer ring;*

(ii) *$r_R(eR)$ is left s-unital for each $e = e^2 \in R$;*

(iii) *For any finitely many idempotent elements* $e_1, \ldots, e_n \in R$, $r_R(\sum_{i=1}^{n} e_iR)$ *is left s-unital.*

Proof. (i) $\Rightarrow$ (ii). Assume that R is right $\mathcal{ACP}$-Baer and $e = e^2 \in R$. As eR is a cyclic projective right R-module, then $r_R(eR)$ is left s-unital.
(ii) $\Rightarrow$ (iii). As e_i is an idempotent element of R for $1 \leq i \leq n$, then, by part (ii), $r_R(e_iR)$ is left s-unital. We know that

$$r_R(\sum_{i=1}^{n} e_iR) = \cap_{i=1}^{n} r_R(e_iR)$$

so $r_R(\sum_{i=1}^{n} e_iR)$ is left s-unital, as per Lemma 2.3.
(iii) $\Rightarrow$ (i). Letting P be a cyclic projective right R-module, then there exists $e = e^2 \in R$ such that $P \cong eR$. Therefore, by (iii), $r_R(P)$ is left s-unital, as required. ☐

Proposition 2.5. *For any ring, we have the following implications:*

(i) *right APP* $\Rightarrow$ *right* $\mathcal{ACP}$*-Baer.*
(ii) *weakly p.q.-Baer* $\Rightarrow$ *right* $\mathcal{ACP}$*-Baer.*
(iii) *right* $\mathfrak{cP}$*-Baer* $\Rightarrow$ *right* $\mathcal{ACP}$*-Baer.*

Proof. It follows immediately from the corresponding definitions. ☐

Examples of rings that are of the type "right $\mathcal{ACP}$-Baer rings include right $\mathfrak{cP}$-Baer rings", whence abelian rings, semi-simple Artinian rings, local rings, bi-regular rings and prime rings, will be constructed in the sequel. Indeed, it is clear that each right APP ring, and so weakly p.q.-Baer ring, is also a right (resp., left) $\mathcal{ACP}$-Baer. As a result, the class of right (resp., left) $\mathcal{ACP}$-Baer rings includes both left p.q.-Baer and right p.q.-Baer rings as shown in the diagram below.

However, the following two examples unambiguously show that all of the converses in Proposition 2.5 do not hold in general.

Example 2.6. We use the Dorroh extension [16, Example 1(1)]. To that goal, let $S_0 = \mathbb{Z}_2$, the ring of integers modulo 2, $S_1 = \mathbb{Z}_2 * \mathbb{Z}_2, S_2 = S_1 * \mathbb{Z}_2, \ldots, S_n = S_{n-1} * \mathbb{Z}_2, \ldots$, where the operations on S_n are as the following: for $(a, \bar{b}), (c, \bar{d}) \in S_n$ with $a, c \in S_{n-1}$, put

$$(a, \bar{b}) + (c, \bar{d}) = (a + c, b \bar{+} d) \text{ and } (a, \bar{b})(c, \bar{d}) = (ac + bc + da, \bar{bd}),$$

where $n = 1, 2, \ldots$. Then, each S_n is apparently a commutative ring of characteristic 2. Even more, each S_n is a Boolean ring, and so is commutative von Neumann regular.
In view of the ring-monomorphism $f : S_{n-1} \longrightarrow S_n$, defined by $f(x) =$

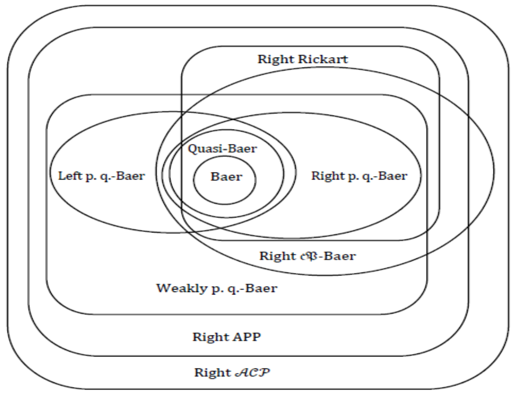

FIGURE 1. Venn diagram of subclasses of right $\mathcal{ACP}$-Baer rings

$(x, \bar{0})$, we can consider any S_{n-1} as a subring of S_n. We, thereby, can obtain a direct product $S = \prod_{n=1}^{\infty} S_n$ with $S_1 \subset S_2 \subset S_3 \subset \ldots$. Now, we consider the ring

$$R = \left\langle \bigoplus_{n=1}^{\infty} S_n, 1_S \right\rangle,$$

as a $\mathbb{Z}_2$-subalgebra of S, generated by $\bigoplus_{n=1}^{\infty} S_n$ and 1_S, where

$$1_S = ((0, \bar{1}), ((0, \bar{0}), \bar{1}), (((0, \bar{0}), \bar{0}), \bar{1}), \ldots).$$

Notice, by a simple inspection that, $(a, \bar{b}) \in S_1, (a, \bar{b}) = ((a, \bar{b}), \bar{0}) \in S_2, (a, \bar{b}) = (((a, \bar{b}), \bar{0}), \bar{0})$
$\in S_3$, etc., and R is a Boolean ring (hence it is commutative, reduced, abelian and a $\mathfrak{cP}$-Baer ring). Set $T = R[[x]]$. Thus, owing to [4], T is $\mathfrak{cP}$-Baer and so it is a $\mathcal{ACP}$-Baer ring.

Suppose now in a way of contradiction that T is an APP-ring. Assume also that

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots \in T,$$

with $f_0 = (1_{S_1}, 0, 0, \ldots), f_1 = (1_{S_1}, 0, 1_{S_3}, 0, \ldots), f_2 = (1_{S_1}, 0, 1_{S_3}, 0, 1_{S_5}, 0, \ldots), \ldots$. Letting

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots \in T,$$

where $g_0 = (0, 1_{S_2}, 0, 0, \ldots), g_1 = (0, 1_{S_2}, 0, 1_{S_4}, 0, \ldots)$,
$g_2 = (0, 1_{S_2}, 0, 1_{S_4}, 0, 1_{S_6}, 0, \ldots), \ldots$, we thus derive $g(x) \in r_T(f(x)T)$

and so there exists

$$h(x) = h_0 + h_1 x + h_2 x^2 + \cdots \in r_T(f(x)T)$$

such that $g(x) = h(x)g(x)$. Taking into account that $f(x)h(x) = 0$ and with [1, p. 2269] at hand, we infer that $f_i h_l = 0$ for all indices i and l. Hence, there exists $m_l \in \mathbb{N}$ such that h_l has the form $(a_{l1}, 0, a_{l3}, 0, \ldots, a_{l(2m_l+1)}, 0, 0, \ldots)$, where $a_{lk} \in S_k$ and $l = 1, 2, \ldots$.

On the other hand, $(1_T - h(x))g(x) = 0$ ensures that $(1_T - h_0)g_j = 0$ and $h_l g_j = 0$ for all indexes j and $l \geq 1$. So, there exists $n_l \in \mathbb{N}$ such that h_l has the form $(0, a_{l2}, 0, a_{l4}, 0, \ldots, a_{l(2n_l)}, 0, 0, \ldots)$, where $a_{lk} \in S_k$ and $l = 1, 2, \ldots$. Thus, $h_1 = h_2 = \ldots$, and so $h(x) = h_0$. This contradicts $g_j = h_0 g_j$, $j = 0, 1, \ldots$. Consequently, T is *not* APP, concluding the claim.

Example 2.7. (1) [5, Example 2.3]. For a given field F, let

$$R = \Big\{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid \ a_n \text{ is eventually constant}\Big\},$$

which is obviously a subring of $\prod_{n=1}^{\infty} F_n$, where $F := F_n$ for $n \in \mathbb{N}$. Thus, R is a commutative von Neumann regular ring (whence reduced). Hence, R is a $\mathfrak{cP}$-Baer ring. However, by [4, Theorem 4.6], $R[[x]]$ is a $\mathfrak{cP}$-Baer ring, which shows that it is right $\mathcal{ACP}$-Baer ring. Also, invoking [17, Example 2.4], $R[[x]]$ is *not* a weakly p.q.-Baer ring, as required.

(2) [8, Example 3.13]. For an integer $k > 1$, let W be the kth Weyl algebra over a field of characteristic zero. Then, W is simple, but neither right nor left hereditary. From [8], there exists a positive integer m such that the full $m \times m$ matrix ring $\mathrm{M}_m(W)$ over W is neither right nor left PP. Assume now that

$$S = \Big\{(a_n)_{n=1}^{\infty} \in \prod_{m=1}^{\infty} \mathrm{M}_m(W) \mid a_n \text{ is eventually constant}\Big\}.$$

Then, S is bi-regular (hence p.q.-Baer) and so it is weakly p.q.-Baer. Now, if we consider R as in point (1), then in accordance with Proposition 2.13 the direct sum $R \bigoplus S$ is a $\mathcal{ACP}$-Baer ring, but it is *not* weakly p.q.-Baer bearing in mind [19, Theorem 2.22], as required.

This completes the example.

Proposition 2.8. *Suppose that R is a regular ring. Then, the following conditions are equivalent:*

(1) *R is a right $\mathcal{ACP}$-Baer ring.*
(2) *R is a right APP ring.*
(3) *R is a weakly p.q.-Baer ring.*

Proof. These equivalencies follow immediately from [13, Theorem 1.1] combined with the corresponding definitions. ☐

Example 2.9. [8, Example 1.6]. For a field F, let us set

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \oplus_{n=1}^{\infty} F_n \\ \oplus_{n=1}^{\infty} F_n & \langle \oplus_{n=1}^{\infty} F_n, 1\rangle \end{pmatrix},$$

where $F_n := F$ for $n = 1, 2, \dots$ and $\langle \oplus_{n=1}^{\infty} F_n, 1\rangle$ is the subalgebra of $\prod_{n=1}^{\infty} F_n$ generated by $\oplus_{n=1}^{\infty} F_n$ and $1 \in \prod_{n=1}^{\infty} F_n$. Thus, it follows from [8] that R is a regular ring (hence, p.p.) and so it is a $\mathcal{ACP}$-Baer ring but manifestly *not* $\mathfrak{cP}$-Baer utilizing [4]. This concludes the example.

We continue our further work with a series of useful technicalities.

Proposition 2.10. *Suppose that R satisfies the ACC property on principal right ideals. Then R is right $\mathcal{ACP}$-Baer if and only if R is right $\mathfrak{cP}$-Baer.*

Proof. Assume that R is a right $\mathcal{ACP}$-Baer ring and $e = e^2 \in R$. We will prove that $r_R(eR) = cR$ for some $c = c^2 \in R$. To this target, let aR be the maximal principal right ideals contained in $r_R(eR)$. As $a \in r_R(eR)$, we have$ca = a$ for some $c \in r_R(eR)$. Thus, $a \in cR$ and so $aR \subseteq cR \subseteq r_R(eR)$. Therefore, the maximality of aR implies that $aR = cR$. Hence, $c \in aR$ and thus there exists $r \in R$ such that $c = ar$. We now deduce $ara = ca = a$ and, therefore, $c^2 = arar = ar = c$. Evidently,

$$r_R(eR) = cr_R(eR) + (1 - c)r_R(eR).$$

We claim that $(1 - c)r_R(eR) = \{0\}$. Indeed, if $x \in (1 - c)r_R(eR)$, then

$$cR \subseteq (c + x - cx)R \subseteq r_R(eR).$$

That is why, $cR = (c + x - cx)R$ in virtue of the maximality of cR.

On the other hand, $cR \cap (x - cx)R = \{0\}$ so that $x - cx = 0$. However, $(cx)^2 = 0$ forcing $x^2 = 0$. But, for every $u \in (1 - c)r_R(eR)$, we write $u = lu$ for some $l \in r_R(eR)$. Thus, $u = (1 - c)lu$. Set $w := (1 - c)l$ which allows us to deduce that $w \in (1 - c)r_R(eR)$ and $w^2 = 0$. Now, we have, $u = wu = w^2u = 0$ so that $(1-c)r_R(eR) = \{0\}$. Consequently, $r_R(eR) =$

cR and we finally conclude that R is right $\mathfrak{cP}$-Baer, as asserted. The converse implication is clear. $\square$

Proposition 2.11. *If R is a right $\mathcal{ACP}$-Baer ring, then eRe is a right $\mathcal{ACP}$-Baer ring for every idempotent $e \in R$.*

Proof. Let $e = e^2 \in R$ be a non-zero element. Also, let $x \in eRe$ be an arbitrary idempotent. We shall prove that $r_{eRe}(x(eRe))$ is left s-unital. To that purpose, as R is a right $\mathcal{ACP}$-Baer ring, we know by definition that $r_R(xR)$ is left s-unital. Assume now that $a \in r_{eRe}(x(eRe))$ is an arbitrary element. Since $a = eae$, $x = exe$ and $x(eRe)a = 0$, one sees that $xRa = 0$. So, by hypothesis, $a = fa$ for some $f \in r_R(xR)$. Set $f' := efe \in eRe$ Clearly, $a = f'a$. Hence, it is enough to show that $f' \in r_{eRe}(x(eRe))$. Letting $ere \in eRe$ be an arbitrary element, we derive

$$x(eRe)f' = x(eRe)fe = xRfe = 0,$$

which insures that eRe is a right $\mathcal{ACP}$-Baer ring, as claimed. $\square$

Proposition 2.12. *Let I be an index set, and R_i a ring for each $i \in I$. Then, the ring $R = \prod_{i \in I} R_i$ is right $\mathcal{ACP}$-Baer ring if, and only if, R_i is right $\mathcal{ACP}$-Baer ring for every $i \in I$.*

Proof. ($\Rightarrow$). Assume that $R = \prod_{i \in I} R_i$ is right $\mathcal{ACP}$-Baer ring and $e_i = e_i^2 \in R_i$ for some $i \in I$. We will prove that $r_{R_i}(e_iR_i)$ is left s-unital. To that aim, suppose $c_i \in r_{R_i}(e_iR_i)$. Set $e := \iota_i(e_i)$. By our hypothesis, $r_R(eR)$ is left s-unital. Put $c := \iota_i(c_i)$ and so, for an arbitrary element r, we have

$$cre = \iota_i(c_i)r\iota_i(e_i) = 0.$$

Thus, by definition, there exists $f \in r_R(eR)$ such that $fc = c$. Obviously, $f_ic_i = c_i$. Letting $r_i \in R_i$ be an arbitrary element, if we set $r := \iota_i(r_i)$, then

$$0 = ref = \iota_i(r_i)\iota_i(e_i)f.$$

Therefore, $r_ie_if_i = 0$ which guarantees that $f_i \in r_{R_i}(e_iR_i)$.

($\Leftarrow$). Let R_i be a right $\mathcal{ACP}$-Baer ring for any $i \in I$ and $e = e^2 \in R$. Thus, $e_i = \pi_i(e)$ remains an idempotent for every $i \in I$. Since R_i is a right $\mathcal{ACP}$-Baer ring, $r_{R_i}(e_iR_i)$ is left s-unital for all $i \in I$. Choose $c \in r_R(eR)$ to be an arbitrary element. Hence, $c_i = \pi_i(c) \in R_i$ and $c_i \in r_{R_i}(e_iR_i)$ for each $i \in I$. By definition, there exists $f_i \in r_{R_i}(e_iR_i)$ such that $f_ic_i = c_i$ for every $i \in I$. Setting $f := (f_i)_{i \in I}$, it is easy to verify that $fc = c$ and $f \in r_R(eR)$, as desired. $\square$

Proposition 2.13. *Let $R = \oplus_{i\in I} R_i$ (we emphasize that, when I is an infinite index set, R is a ring without identity), where R_i is a ring for each $i \in I$. Then, R is a right $\mathcal{ACP}$-Baer ring if, and only if, R_i is a right $\mathcal{ACP}$-Baer ring for each $i \in I$.*

Proof. A proof similar to that of Proposition 2.12 can successfully be applied to get the statement. □

Let A be a ring, let $B \leq A$ be a unitary subring (so it contains the same unity 1_A of A), let $\{A_i\}_{i=1}^{\infty}$ be a countable set of copies of A, let D be the direct product of all rings A_i, and let $R := R(A, B)$ be the subring of D generated by the ideal $\bigoplus_{i=1}^{\infty} A_i$ and by the subring $\{(b, b, \ldots) \mid b \in B\}$ (see, for instance, [28]).

We now arrive at the following assertion.

Proposition 2.14. *If A is an Abelian ring, then the ring $R(A, B)$ is a right $\mathcal{ACP}$-Baer ring if, and only if, both A and B are right $\mathcal{ACP}$-Baer rings.*

Proof. Choosing $R := R(A, B)$ as a right $\mathcal{ACP}$-Baer ring, then it is readily checked that A and B are both right $\mathcal{ACP}$-Baer rings.

For the converse, assume that $(e_i)_{i=1}^{\infty} \in R$ is an idempotent and $(a_i)_{i=1}^{\infty} \in r_R((e_i)_{i=1}^{\infty} R)$. So, e_i is an idempotent in A for each i. Also, by definition, there exists a positive integer n such that $e_n = e_{n+1} = \ldots \in B$ and $a_n = a_{n+1} = \ldots \in B$. As A is Abelian, we have that $e_i A a_i = \{0\}$ for all $1 \leq i \leq n$. Since A is a right $\mathcal{ACP}$-Baer ring, there is $b_i \in r_A(e_i A)$ such that $a_i = b_i a_i$ for any $1 \leq i \leq n-1$. Moreover, since B is a right $\mathcal{ACP}$-Baer ring, there is $b_n \in r_B(e_n B)$ such that $a_n = b_n a_n$. But as A is Abelian and $B \subseteq A$ is a unitary subring, we obtain that $b_n \in r_A(e_n A)$. Hence,

$$(b_1, b_2, \ldots, b_{n-1}, b_n, b_n, \ldots) \in r_R((e_i)_{i=1}^{\infty} R)$$

and, consequently,

$$(e_i)_{i=1}^{\infty} = (b_1, b_2, \ldots, b_{n-1}, b_n, b_n, \ldots)(e_i)_{i=1}^{\infty}.$$

Therefore, $R(A, B)$ is a right $\mathcal{ACP}$-Baer ring, as expected. □

As a consequence, we yield:

Corollary 2.15. *Let A be a right $\mathcal{ACP}$-Baer ring. If $r_B(eA) = 0$ for every $0 \neq e = e^2 \in B$, then $R(A, B)$ is a right $\mathcal{ACP}$-Baer ring.*

Proof. In the proof of Proposition 2.14, the following two cases occur:

(1) If $0 = e_n = e_{n+1} = \ldots \in B$, then $r_B(e_nB) = B$. Take $b_n = 1 \in B$.
(2) If $0 \neq e_n = e_{n+1} = \ldots \in B$, then $r_B(e_nA) = 0$ by assumption. Thus, $a_n = 0$. Take $b_n = 0 \in B$.

In both cases, we have $a_n = b_na_n$ and $e_nAb_n = 0$. So,

$$(e_i)_{i=1}^{\infty} = (b_1, b_2, \ldots, b_{n-1}, b_n, b_n, \ldots)(e_i)_{i=1}^{\infty}$$

and

$$(e_i)_{i=1}^{\infty} R(b_1, b_2, \ldots, b_{n-1}, b_n, b_n, \ldots) = 0.$$

Hence, R is a right $\mathcal{ACP}$-Baer ring, as promised. □

3. Polynomial Extensions of $\mathcal{ACP}$-Baer Rings

In this section, we will examine the polynomial and power series rings in relation to the right $\mathcal{ACP}$-Baer condition. We start with the following plain but useful claim.

Lemma 3.1. [2, Lemma 4.1] Let R be a ring. If $e(x) = \sum_{i=0}^{m} e_ix^i \in R[x]$ (resp., $e(x) = \sum_{i=0}^{\infty} e_ix^i \in R[[x]]$) is an idempotent, then $e_i \in Re_0R$ for any $i \geq 0$.

We are now in a position to establish the next two chief results.

Theorem 3.2. *For an arbitrary ring R, the following two conditions are equivalent:*

(i) *R is a right $\mathcal{ACP}$-Baer ring.*
(ii) *$R[x]$ is a right $\mathcal{ACP}$-Baer ring.*

Proof. (i) $\Rightarrow$ (ii). Let R be a right $\mathcal{ACP}$-Baer ring and $e(x) \in R[x]$ an arbitrary idempotent such that $e(x) = \sum_{i=0}^{m} e_ix^i$. Consulting with Lemma 3.1, $e_0 = e_0^2 \in R$. We will demonstrate that $r_{R[x]}(e(x)R[x])$ is left s-unital. To that aim, assume that $f(x) = \sum_{j=0}^{n} a_jx^j \in r_{R[x]}(e(x)R[x])$ is an arbitrary element. Thus, $e(x)R[x]f(x) = 0$. First, we shall prove that $e_0Ra_j = 0$ for each $0 \leq j \leq n$. We work by induction on j. In fact, since $e(x)Rf(x) = 0$, we have

$$0 = e(x)rf(x) = \sum_{i=0}^{m} (\sum_{j=0}^{n} e_ira_j)x^{i+j}$$

for any $r \in R$. If $j = 0$, then $0 = e_0ra_0$. Hence, $a_0 \in r_R(e_0R)$. Now, assume that the result is true for all $1 \leq t < j$. So, $e_0Ra_t = 0$ for each

$1 \le t < j$. We write

$$0 = e_0 r a_j + e_1 r a_{j-1} + \cdots + e_j r a_0.$$

Multiplying by e_0 from the left-hand side of the above equality, we obtain

$$0 = e_0 r a_j + e_0 e_1 r a_{j-1} + \cdots + e_0 e_j r a_0.$$

Next, with the aid of the induction hypothesis, we must have $e_0 R a_j = 0$. Therefore, $a_j \in r_R(e_0 R)$ for any $0 \le j \le n$. By hypothesis, $r_R(e_0 R)$ is left s-unital, and so there exists $c \in r_R(e_0 R)$ such that $c a_j = a_j$ for all j. That is why,

$$cf(x) = \sum_{j=0}^{n} c a_j x^j = f(x).$$

On the other hand, Lemma 3.1 tells us that $e_i \in Re_0R$ for all $0 \le i \le m$, so we derive $e_i R c = 0$. Letting $g(x) = \sum_{l=0}^{r} b_l x^l \in R[x]$ is an arbitrary element, we may write

$$e(x)g(x)c = (\sum_{i=0}^{m} e_i x^i)(\sum_{l=0}^{r} b_l x^l)c = \sum_{k=0}^{m+r}(\sum_{i+l=k} e_i b_l c)x^k = 0.$$

Consequently, $c \in r_{R[x]}(e(x)R[x])$, as needed.
(ii) $\Rightarrow$ (i). Assume that $R[x]$ is a right $\mathcal{ACP}$-Baer ring and $c = c^2 \in R$. Let $a \in r_R(cR)$. Thus, $a \in r_{R[x]}(cR[x])$, so there is $g(x) = \sum_{j=0}^{n} b_j x^j \in r_{R[x]}(cR[x])$ such that $g(x)a = a$. Then, $b_0 a = a$. As $cRg(x) = 0$, it must be that $cRb_0 = 0$. So, $b_0 \in r_R(cR)$ and, therefore, $r_R(cR)$ is left s-unital leading to R is a right $\mathcal{ACP}$-Baer ring, as required. □

Theorem 3.3. *Let R be a ring satisfying descending chain condition on left annihilators. Then, R is a right $\mathcal{ACP}$-Baer ring if, and only if, $R[[x]]$ is a right $\mathcal{ACP}$-Baer ring.*

Proof. ($\Rightarrow$). Let R be right $\mathcal{ACP}$-Baer. Assume that $e(x) = \sum_{i=0}^{\infty} e_i x^i \in R[[x]]$ is an idempotent and

$$f(x) = \sum_{j=0}^{\infty} a_j x^j \in r_{R[[x]]}(e(x)R[[x]]).$$

The utilization of Lemma 3.1 gives that $e_0 = e_0^2 \in R$. Moreover, $e(x)R[[x]]f(x) = 0$ whence $e(x)rf(x) = 0$ for every $r \in R$. Hence, $\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} e_i r a_j)x^{i+j} = 0$ and we get

$$e_0 r a_0 = 0 \qquad (1)$$

$$e_0ra_1 + e_1ra_0 = 0 \qquad (2)$$
$$\vdots$$
$$e_0ra_m + e_1ra_{m-1} + \cdots + e_mra_0 = 0 \qquad (3)$$
$$\vdots$$

Furthermore, since $e_0ra_0 = 0$, one has that $a_0 \in r_R(e_0R)$. Multiplying the equation (2) on the left side by e_0, we infer $e_0ra_1 + e_0e_1ra_0 = 0$ and using (1) we get $e_0ra_1 = 0$. So, $a_1 \in r_R(e_0R)$. If, however, we multiply equation (3) on the left side by e_0, then by the same argument we conclude that $a_m \in r_R(e_0R)$. This process can be continued to extract that $a_j \in r_R(e_0R)$ for all $j \geq 0$.

On the other hand, we consider the descending chain of left annihilators as follows:

$$\ell_R(a_0) \supseteq \ell_R(a_0, a_1) \supseteq \ell_R(a_0, a_1, a_2) \supseteq \cdots .$$

So, there exists n such that

$$\ell_R(a_0, a_1, \ldots, a_n) = \ell_R(a_0, a_1, \ldots, a_n, a_{n+1}) = \cdots .$$

Since $a_0, a_1, \ldots, a_n \in r_R(e_0R)$, by our assumption there is $c \in r_R(e_0R)$ such that $a_j = ca_j$ for $0 \leq j \leq n$. Therefore, $1 - c \in \ell_R(a_0, a_1, \ldots, a_n)$ and so $1-c \in \ell_R(a_0, a_1, \ldots, a_n, \ldots, a_l)$ for any $l \geq n$. Hence, $a_l = ca_l$ for any $l \geq n$. Moreover, it is routinely inspected that $c \in r_{R[[x]]}(e(x)R[[x]])$ and $f(x) = cf(x)$. Finally, $R[[X]]$ is a right $\mathcal{ACP}$-Baer ring, as stated.
($\Leftarrow$). Arguing as in the implication (ii) $\Rightarrow$ (i) of Theorem 3.2, we are done. □

4. Characterizations of Right $\mathcal{ACP}$-Baer Rings of Generalized Triangular Matrix Rings

We begin here with the following well-known technicality.

Lemma 4.1. [11, Lemma 3.1] Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ be the formal upper triangular matrix ring, where S,R are rings, and M is an (S, R)-bi-module and a unitary S-module. If $\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ is an ideal of T, then

$$r_T(\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}) = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap Ann_R(N) \end{pmatrix}.$$

As a consequence, we yield:

Corollary 4.2. *If* $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix} \in T$ *is an arbitrary idempotent, then*

$$r_T(eT) = \begin{pmatrix} r_S(e_1S) & r_M(e_1S) \\ 0 & r_R(e_2R) \cap r_R(e_1M + kR) \end{pmatrix}.$$

Proof. It follows immediately from Lemma 4.1. □

We are now prepared to prove the following major result.

Theorem 4.3. *Let* $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$*, where* S *and* R *are rings, and* M *is an* (S, R)*-bi-module. Then,* T *is right* $\mathcal{ACP}$*-Baer ring if, and only if, all of the following conditions are fulfilled:*

(1) *S and R are right $\mathcal{ACP}$-Baer rings;*
(2) *For each $e = e^2 \in S$, we have:*
  (2.1) *If $s \in r_S(eS)$ and $m \in r_M(eS)$, then there exists an element $c \in r_S(eS)$ such that $cs = s$ and $cm = m$;*
  (2.2) *The annihilator $r_R(eM)$ is left s-unital.*

Proof. ($\Rightarrow$). Assume T is a right $\mathcal{ACP}$-Baer ring. One sees that

(1) $S \cong \begin{pmatrix} 1_s & 0 \\ 0 & 0 \end{pmatrix} T \begin{pmatrix} 1_s & 0 \\ 0 & 0 \end{pmatrix}$ and $R \cong \begin{pmatrix} 0 & 0 \\ 0 & 1_R \end{pmatrix} T \begin{pmatrix} 0 & 0 \\ 0 & 1_R \end{pmatrix}$, where both $\begin{pmatrix} 1_s & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1_R \end{pmatrix}$ are idempotents of T. So, employing Proposition 2.11, R and S are both right $\mathcal{ACP}$-Baer rings.

(2.1) Let $e = e^2 \in S$, $s \in r_S(eS)$ and $m \in r_M(eS)$. Thus $e' = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \in T$ is an idempotent. As $e'T = \begin{pmatrix} eS & eM \\ 0 & 0 \end{pmatrix}$, $r_T(e'T) = \begin{pmatrix} r_S(eS) & r_M(eS) \\ 0 & 0 \end{pmatrix}$, by Lemma 4.1. Since, $\begin{pmatrix} s & m \\ 0 & 0 \end{pmatrix} \in r_T(e'T)$ we have,

$$\begin{pmatrix} c & n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s & m \\ 0 & 0 \end{pmatrix}$$

for some $\begin{pmatrix} c & n \\ 0 & 0 \end{pmatrix} \in r_T(e'T)$. Therefore, $cs = s$, $cm = m$.

(2.2) Let $e = e^2 \in S$. Thus, $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \in T$ is an idempotent and so $r_T(\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} T)$ is left s-unital. We also can easily verify that $r_R(eM)$ is left s-unital.

($\Leftarrow$). Suppose that both (1) and (2) are true. Let $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix} \in T$ be an arbitrary idempotent. So, $e_1 = e_1^2 \in S$, $e_2 = e_2^2 \in R$ and $k = e_1k + ke_2$. One observes that it suffices to show that $r_T(eT)$ is left s-unital. Now since $eT = \begin{pmatrix} e_1S & e_1M + kR \\ 0 & e_2R \end{pmatrix}$ by Lemma 4.1 we have,

$$r_T(eT) = \begin{pmatrix} r_S(e_1S) & r_M(e_1S) \\ 0 & r_R(e_2R) \cap r_R(e_1M + kR) \end{pmatrix}.$$

In fact, let

$$a = \begin{pmatrix} a_1 & n \\ 0 & a_2 \end{pmatrix} \in r_T(eT).$$

Thus,

$$a_2 \in r_R(e_2R) \cap r_R(e_1M + kR)$$

and so

$$a_2 \in r_R(e_2R) \cap r_R(e_1M).$$

Applying subsequently parts (1) and (2.2), one deduces that both $r_R(e_2R)$ and $r_R(e_1M)$ are left s-unital. Hence, $r_R(e_2R) \cap r_R(e_1M)$ is left s-unital by referring to Lemma 2.3. Therefore, $c_2a_2 = a_2$ for some $c_2 \in (r_R(e_2R) \cap r_R(e_1M))$. Since $a_1 \in r_S(e_1S)$ and $n \in r_M(e_1S)$, there is $c_1 \in r_S(e_1S)$ such that $c_1a_1 = a_1$, $c_1n = n$ in view of property (2.1). Put $c := \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$. Since

$$kRc_2 = e_1kRc_2 + ke_2Rc_2 = e_1kRc_2 = 0,$$

one concludes that

$$eTc = \begin{pmatrix} e_1Sc_1 & (e_1M + kR)c_2 \\ 0 & e_2Rc_2 \end{pmatrix} = 0.$$

Consequently, $f \in r_T(eT)$. Now, because $c_1n = n$, we obtain $ca = a$, finishing the arguments. □

An important assertion which can be deduced is the next one.

Corollary 4.4. *Let* $\mathrm{T}_n(R)$ *be the* $n \times n$ *upper triangular matrix ring over a ring* R*, where* $n \geq 1$ *is a positive integer. Then, the following items are equivalent:*

(1) R *is right* $\mathcal{ACP}$*-Baer ring.*
(2) $\mathrm{T}_n(R)$ *is right* $\mathcal{ACP}$*-Baer ring for every positive integer* n*.*
(3) $\mathrm{T}_k(R)$ *is right* $\mathcal{ACP}$*-Baer ring for some* $k > 1$*.*
(4) $\mathrm{T}_2(R)$ *is right* $\mathcal{ACP}$*-Baer ring.*

Proof. $(1) \Rightarrow (2)$. Let R be a right $\mathcal{ACP}$-Baer ring. We prove the required condition using induction on n. Indeed, it is obvious that $T_1(R)$ is a right $\mathcal{ACP}$-Baer ring. Since $T_2(R)$ satisfies the requested conditions of Theorem 4.3, it is necessarily a right $\mathcal{ACP}$-Baer ring. Assume that the result holds for all $1 \le l \le n$. We will establish now that $\mathrm{T}_{n+1}(R)$ is a right $\mathcal{ACP}$-Baer ring, as required. In fact, we know that

$$\mathrm{T}_{n+1}(R) \cong \begin{pmatrix} R & M \\ 0 & \mathrm{T}_n(R) \end{pmatrix},$$

where $M = (R, \dots, R)$ (n-tuple). To apply Theorem 4.3, let $e = e^2 \in R$, $a \in r_R(eR)$ and $\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \in r_M(eR)$. Thus,

$$0 = eR \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} = \begin{pmatrix} eRa_1 & eRa_2 & \dots & eRa_n \end{pmatrix}$$

and so $a_i \in r_R(eR)$ for all $1 \le i \le n$. Since $r_R(eR)$ is left s-unital, there is $c \in r_R(eR)$ such that $ca = a$ and $ca_i = a_i$ for all $1 \le i \le n$.
Further, let $e = e^2 \in R$. We will show that $r_{T_n(R)}(eM)$ is left s-unital. In this aspect, assume that $A \in r_{T_n(R)}(eM)$ such that $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$. Letting $\begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix} \in M$, we thereby extract that

$$\begin{aligned} e \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix} A &= \begin{pmatrix} er_1 & er_2 & \dots & er_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \\ &= \begin{pmatrix} er_1a_{11} & er_1a_{12} + er_2a_{22} & \dots & \sum_{i=1}^n er_ia_{in} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix}. \end{aligned}$$

Now, take $r_1 \neq 0$ and $r_i = 0$ for all $i \neq 1$. Then, $a_{1i} \in r_R(eR)$ for all $1 \le i \le n$. Similarly, for $r_2 \neq 0$ and $r_i = 0$ for all $i \neq 2$, we receive that $a_{2i} \in r_R(eR)$ for $2 \le i \le n$. However, by the induction argument, we have $a_{ij} \in r_R(eR)$ for all $1 \le i, j \le n$. As $r_R(eR)$ is left s-unital, there exists $f \in r_R(eR)$ such that $fa_{ij} = a_{ij}$ for all $1 \le i, j \le n$. Put $F := fI_n \in \mathrm{T}_n(R)$. It is now elementary that $FA = A$ and $F \in r_{T_n(R)}(eM)$, as required.
$(2) \Rightarrow (3)$. This implication is easy, so we omit its details.
$(3) \Rightarrow (4)$. Let $\mathrm{T}_k(R)$ be a right $\mathcal{ACP}$-Baer ring for some $k > 1$ and let

$e_{ij} \in T_k(R)$ be the matrix with 1 in the (i,j)-position and 0 elsewhere. Set $f := e_{11} + e_{22}$. Then, $f = f^2 \in \mathrm{T}_k(R)$ and $\mathrm{T}_2(R) \cong f\mathrm{T}_k(R)f$. Therefore, in virtue of Proposition 2.11, $\mathrm{T}_2(R)$ is a right $\mathcal{ACP}$-Baer ring.
(4) ⇒ (1). This implication is analogous to (iii) ⇒ (iv). □

The following statement is an automatic consequence of Theorem 4.3 and provides a rich source of rings which are right $\mathcal{ACP}$-Baer rings. Specifically, the following is valid:

Corollary 4.5. *Let S be a right $\mathcal{ACP}$-Baer ring and R a unitary subring of S such that $r_S(eR) = 0$ for any $0 \neq e = e^2 \in R$. Then, the ring $T = \begin{pmatrix} S & S \\ 0 & R \end{pmatrix}$ is a right $\mathcal{ACP}$-Baer ring.*

The next valuable consequence is the following one.

Corollary 4.6. *Let $T = \begin{pmatrix} \bar{S} & \bar{S} \\ 0 & S \end{pmatrix}$, where S is a right $\mathcal{ACP}$-Baer ring and $\bar{S} = S/P$ for a prime ideal P such that $r_S(eS) \nsubseteq P$, provided $e = e^2 \in P$. Then, T is right $\mathcal{ACP}$-Baer.*

Proof. Since $\bar{S}$ is a prime ring, [8, Lemma 1.2] informs us that $\bar{S}$ is a p.q.-Baer ring and so it is right $\mathcal{ACP}$-Baer. Thus, point (1) of Theorem 4.3 is true. Let $\bar{e} = \bar{e}^2 \in \bar{S}$ such that $\bar{e} = e + P$. Consider now the following two different cases:

(1) If $e \in P$, then $\bar{e} = P$ and so $r_{\bar{S}}(\bar{e}\bar{S}) = \bar{S}$ that is left s-unital. Thus, issue (2.1) of Theorem 4.3 holds. Besides, $r_S(\bar{e}\bar{S}) = r_S(P\bar{S}) = S$ is left s-unital, too.
(2) If $e \notin P$, then $r_{\bar{S}}(\bar{e}\bar{S}) = P$ that is left s-unital. So, issue (2.1) of Theorem 4.3 holds. Besides, $r_S(\bar{e}\bar{S}) = r_S(P\bar{S}) = P$ is left s-unital as well.

This substantiates our assertion. □

Example 4.7. (1) Assume that D is a domain, $R = \mathrm{M}_2(D)$ and $S = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in D\}$. Since D is a domain, R is $\mathcal{ACP}$-Baer. Also, $r_R(eS) = 0$ for any $0 \neq e = e^2 \in S$. Then, $T = \begin{pmatrix} S & S \\ 0 & R \end{pmatrix}$ is a $\mathcal{ACP}$-Baer ring just consulting with Corollary 4.5.

(2) Let F be a field and $F_n := F$ for $n = 1, 2, \ldots$. Set

$$S = \Big\{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid \ a_n \text{ is eventually constant}\Big\},$$

and

$$P = \Big\{(a_n)_{n=1}^{\infty} \in S \mid \ a_n = 0 \text{ eventually}\Big\}.$$

Thus, the ring $T = \begin{pmatrix} S/P & S/P \\ 0 & S \end{pmatrix}$ is right $\mathcal{ACP}$-Baer exploiting Corollary 4.6.

This finishes the example.

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