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AN INTEGRAL FORMULA FOR COMPLEX AFFINE ALGEBRAIC HYPERSURFACES

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ABSTRACT. It's well known that the integral of the top exterior power of the first Chern form over a closed complex manifold is an integer, called a Chern number, which is the Euler number χ in the case of complex dimension one. Now we show that its integral over an irreducible algebraic hypersurface of degree d in complex Euclidean space C^{n+1} is also an integer, with absolute value between 0 and $d(d-1)^n$; and the upper bound is reached by an affine hypersurface whose projective completion is nonsingular and transverse to the hyperplane at infinity. In particular, the total Gauss curvature of a general smooth affine plane curve is equal to $2\pi(\chi - 1 - \sqrt{1-\chi})$.

1. Introduction

There is a way of establishing Gauss-Bonnet theorem for hypersurfaces in Euclidean space in terms of the degree of the Gauss map. We also have a notion of degree of a rational map between algebraic varieties. In light of these, consider a complex algebraic hypersurface

$$V = \{f(z_0, \cdots, z_n) = 0, f \text{ prime}\} \subset C^{n+1}$$

and its rational Gauss map

$$\Phi : V^0 \to P^n,\ z \to [f_{,0}(z) : \cdots : f_{,n}(z)]$$

where $V^0 = \{\nabla f = (f_{,0}, \cdots, f_{,n}) \neq 0\} \cap V$ is the smooth locus of V. Coincidentally, the Kahler form η of the Fubini–Study metric of P^n pulls back to -1 times the first Chern form c_1 of V^0 with the induced metric from ambient Euclidean space (Lemma 2.1). By Wirtinger Theorem, the volume form is pulled back as

$$\Phi^*(dP^n) = \Phi^*(\eta^n/n!) = \Phi^*(\eta)^n/n! = \tfrac{(-1)^n}{n!} c_1^n \,.$$

If Φ is dominant with degree m, then there is a nonempty Zariski open subset $U \subset P^n$ such that Φ restricts to a m-sheeted smooth covering map, which is proper, from Zariski open subset $\Phi^{-1}(U) \subset V^0$ onto U connected. By virtue of the relation between degree of proper maps and integration of top forms (Theorem 2.3),

$$\int_{\Phi^{-1}(U)} \Phi^*(dP^n) = m \int_U dP^n \,,$$

provided the latter integral exists. Since a nowhere dense analytic subset of a complex manifold has measure zero (Proposition 2.4, Remark 2.5), which does not affect integrability or the value of integral,

$$\int_{V^0} \Phi^*(dP^n) = m \int_{P^n} dP^n \,.$$

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Substituting $1/n!$ for the volume of P^n ([6], page 297), we obtain the formula (Theorem 2.9)

$$\int_{V^0} c_1{}^n = (-1)^n \deg(\Phi), \tag{1}$$

which can be interpreted as the first Chern number of V. We will show that (Theorem 2.7)

$$m \leq d(d-1)^n \quad \text{where } d = \deg f,$$

and the upper bound is almost always reached (Theorem 3.4). Because

$$KdV = 2\pi c_1 \qquad K : \text{Gauss (sectional) curvature} \tag{2}$$

for a Hermitian Riemann surface, we develop the Gauss-Bonnet formula for affine plane curves under some additional assumptions (Theorem 4.3).

2. For arbitrary affine hypersurfaces

The following observation first appeared in ([1], prop 3, page 819), as far as I know. However, I give another proof, which also provides an expression (3) for the first Chern form of a hypersurface in complex Euclidean space without using the metric.

Lemma 2.1. *Suppose a hypersurface M is the zero locus of a holomorphic function f defined in an open subset of Euclidean space C^{n+1} whose gradient vanishes nowhere on M. Let η be the Kahler form of P^n, whose homogeneous coordinate representation (i.e., pull back via the projection $\pi : C^{n+1}\backslash\{0\} \to P^n$) is $\pi^*\eta = (2\pi)^{-1} i\partial\bar{\partial}\log(|W_0|^2 + \cdots + |W_n|^2)$, then η pulls back to minus the first Chern form c_1 associated with the induced metric of M via the holomorphic Gauss map $\Phi : M \to P^n$, $z \to [f_{,0}(z) : \cdots : f_{,n}(z)]$.*

Proof. Because Φ is the composition

$$\begin{array}{l}\Phi : M \to C^{n+1} \to P^n \\ z \to \nabla f(z) \to [f_{,0}(z) : \cdots : f_{,n}(z)],\end{array}$$

and pull back by a holomorphic map commutes with $\partial, \overline{\partial}$; η is first pulled back to

$$(2\pi)^{-1} i\partial\bar{\partial}\log(|W_0|^2 + \cdots + |W_n|^2)$$

on $C^{n+1}\backslash\{0\}$ and then to

$$(2\pi)^{-1} i\partial\bar{\partial}\log(|f_{,0}|^2 + \cdots + |f_{,n}|^2) = \omega$$

on a neighborhood of M. Locally M is the graph of a holomorphic function, say, $z_0 = h(z_1, ..., z_n)$, and

$$\varphi(z_1, ..., z_n) = (h(z_1, ..., z_n), z_1, ..., z_n)$$

is a coordinate patch of M with Jacobian matrix

$$D\varphi = \begin{bmatrix} h_{,1} & \cdots & h_{,n} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}.$$

Let b_j $(j = 0, \ldots, n)$ be the determinant of the submatrix of $D\varphi$ obtained by deleting the jth row. Suppose the induced metric on M has Hermitian components $[g_{k\bar{l}}]_{n\times n}$, then

$$\begin{aligned}
\det[g_{k\bar{l}}] &= |D\varphi^T\overline{D\varphi}| \quad \text{(up to a constant factor)} \\
&= \sum_{j=0}^{n} b_j\bar{b}_j = 1 + \sum_{i=1}^{n} |h_{,i}|^2 \quad \text{(Cauchy} - \text{Binet formula)} \\
&= 1 + \sum_{i=1}^{n} |f_{,i}/f_{,0}|^2 = |\nabla f|^2/|f_{,0}|^2 \,.
\end{aligned}$$

So $\omega|_M$ has coordinate representation

$$(2\pi)^{-1} i\partial\bar{\partial}\log(|f_{,0}\circ\varphi|^2 + \cdots + |f_{,n}\circ\varphi|^2).$$

Since $c_1 = -(2\pi)^{-1} i\partial\bar{\partial}\log\det[g_{k\bar{l}}]$, we see that

$$\omega|_M = -c_1 \tag{3}$$

□

Proposition 2.2. *In the same setting as above, the top form ${c_1}^n$ vanishes at a point of M if and only if* $\begin{vmatrix} 0 & f_{,l} \\ f_{,k} & f_{,kl} \end{vmatrix} = 0$ $(k, l = 0, \ldots, n)$ *at that point.*

Proof. In the notation of the previous proposition, on the graph of the holomorphic function $z_0 = h(z_1, \ldots, z_n)$, the Kahler form

$$\lambda = ig_{j\bar{k}}dz^j \wedge d\bar{z}^k \quad (j, k = 1, \ldots, n)$$

satisfies Wirtinger Theorem (here $z_j = z^j$ and repeated indices are summed over)

$$\lambda^n = n!dV^0 = n!i^n|g_{j\bar{k}}|dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \,.$$

Likewise, the Ricci form

$$\Re = (2\pi)c_1 = iR_{j\bar{k}}dz^j \wedge d\bar{z}^k = i\bar{\partial}\partial\log|g_{k\bar{l}}| = i\bar{\partial}\partial\log a$$

where $a_k = f_{,k}(h(z_1, \ldots, z_n), z_1, \ldots, z_n)$, $\;a = \sum_{k=0}^{n} |a_k|^2$,satisfies

$$\Re^n = n!i^n|\Re_{j\bar{k}}|dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \,.$$

We calculate that

$$|-R_{j\bar{k}}| = |(\log a)_{,\bar{k}j}| = a^{-2n}\det[a_{,\bar{k}j}a - a_{,\bar{k}}a_{,j}]$$

$$= a^{-2n}\begin{vmatrix} a_{,\bar{1}1}a - a_{,\bar{1}}a_{,1} & \cdots & a_{,\bar{1}n}a - a_{,\bar{1}}a_{,n} \\ \vdots & \ddots & \vdots \\ a_{,\bar{n}1}a - a_{,\bar{n}}a_{,1} & \cdots & a_{,\bar{n}n}a - a_{,\bar{n}}a_{,n} \end{vmatrix}$$

$$
\begin{aligned}
&= a^{-2n}\Big(\begin{vmatrix} a_{,\bar{1}1}a & \cdots & a_{,\bar{1}n}a \\ \vdots & \ddots & \vdots \\ a_{,\bar{n}1}a & \cdots & a_{,\bar{n}n}a \end{vmatrix} + \begin{vmatrix} -a_{,\bar{1}}a_{,1} & a_{,\bar{1}2}a & \cdots & a_{,\bar{1}n}a \\ \vdots & \vdots & \ddots & \vdots \\ -a_{,\bar{n}}a_{,1} & a_{,\bar{n}2}a & \cdots & a_{,\bar{n}n}a \end{vmatrix} \\
&+ \begin{vmatrix} a_{,\bar{1}1}a & -a_{,\bar{1}}a_{,2} & \cdots & a_{,\bar{1}n}a \\ \vdots & \vdots & \ddots & \vdots \\ a_{,\bar{n}1}a & -a_{,\bar{n}}a_{,2} & \cdots & a_{,\bar{n}n}a \end{vmatrix} + \ldots + \begin{vmatrix} a_{,\bar{1}1}a & \cdots & a_{,\bar{1}n-1}a & -a_{,\bar{1}}a_{,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{,\bar{n}1}a & \cdots & a_{,\bar{n}n-1}a & -a_{,\bar{n}}a_{,n} \end{vmatrix} \Big) \\
&= a^{-2n}a^{n-1}\Big(a \begin{vmatrix} a_{,\bar{1}1} & \cdots & a_{,\bar{1}n} \\ \vdots & \ddots & \vdots \\ a_{,\bar{n}1} & \cdots & a_{,\bar{n}n} \end{vmatrix} - a_{,1} \begin{vmatrix} a_{,\bar{1}} & a_{,\bar{1}2} & \cdots & a_{,\bar{1}n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{,\bar{n}} & a_{,\bar{n}2} & \cdots & a_{,\bar{n}n} \end{vmatrix} \\
&+ a_{,2} \begin{vmatrix} a_{,\bar{1}1} & -a_{,\bar{1}} & \cdots & a_{,\bar{1}n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{,\bar{n}1} & -a_{,\bar{n}} & \cdots & a_{,\bar{n}n} \end{vmatrix} - \ldots + a_{,n} \begin{vmatrix} a_{,\bar{1}1} & \cdots & a_{,\bar{1}n-1} & -a_{,\bar{1}} \\ \vdots & \ddots & \vdots & \vdots \\ a_{,\bar{n}1} & \cdots & a_{,\bar{n}n-1} & -a_{,\bar{n}} \end{vmatrix} \Big) \\
&= a^{-(n+1)} \begin{vmatrix} a & a_{,1} & \cdots & a_{,n} \\ a_{,\bar{1}} & a_{,\bar{1}1} & \cdots & a_{,\bar{1}n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{,\bar{n}} & a_{,\bar{n}1} & \cdots & a_{,\bar{n}n} \end{vmatrix} \\
&= a^{-(n+1)} \begin{vmatrix} \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \\ \bar{a}_{0,\bar{1}} & \bar{a}_{1,\bar{1}} & & \bar{a}_{n,\bar{1}} \\ \vdots & & \ddots & \\ \bar{a}_{0,\bar{n}} & \bar{a}_{1,\bar{n}} & & \bar{a}_{n,\bar{n}} \end{vmatrix} \cdot \begin{vmatrix} a_0 & a_{0,1} & \cdots & a_{0,n} \\ a_1 & a_{1,1} & & a_{1,n} \\ \vdots & & \ddots & \\ a_n & a_{n,1} & & a_{n,n} \end{vmatrix};
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{vmatrix} a_0 & a_{0,1} & \cdots & a_{0,n} \\ a_1 & a_{1,1} & & a_{1,n} \\ \vdots & & \ddots & \\ a_n & a_{n,1} & & a_{n,n} \end{vmatrix} = \begin{vmatrix} a_k & a_{k,j} \end{vmatrix} = \begin{vmatrix} f_{,k} & f_{,kj} + f_{,k0}h_{,j} \end{vmatrix} \quad (k = 0, \ldots, n, \ j = 1, \ldots, n) \\
&= \begin{vmatrix} f_{,0} & f_{,01} + f_{,00}h_{,1} & \cdots & f_{,0n} + f_{,00}h_{,n} \\ f_{,1} & f_{,11} + f_{,10}h_{,1} & \cdots & f_{,1n} + f_{,10}h_{,n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{,n} & f_{,n1} + f_{,n0}h_{,1} & \cdots & f_{,nn} + f_{,n0}h_{,n} \end{vmatrix}
\end{aligned}
$$

$$= \left(\begin{vmatrix} f_{,0} & f_{,01} & \cdots & f_{,0n} \\ f_{,1} & f_{,11} & \cdots & f_{,1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{,n} & f_{,n1} & \cdots & f_{,nn} \end{vmatrix} + \begin{vmatrix} f_{,0} & f_{,00}h_{,1} & f_{,02} & \cdots & f_{,0n} \\ f_{,1} & f_{,10}h_{,1} & f_{,12} & \cdots & f_{,1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{,n} & f_{,n0}h_{,1} & f_{,n2} & \cdots & f_{,nn} \end{vmatrix} + \right.$$

$$\left. \ldots + \begin{vmatrix} f_{,0} & f_{,01} & \cdots & f_{,0(n-1)} & f_{,00}h_{,n} \\ f_{,1} & f_{,11} & \cdots & f_{,1(n-1)} & f_{,10}h_{,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{,n} & f_{,n1} & \cdots & f_{,n(n-1)} & f_{,n0}h_{,n} \end{vmatrix} \right)$$

$$= -\frac{1}{f_{,0}} \begin{vmatrix} 0 & f_{,0} & \cdots & f_{,n} \\ f_{,0} & f_{,00} & \cdots & f_{,0n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{,n} & f_{,n0} & \cdots & f_{,nn} \end{vmatrix} \quad \text{(expand according to the 0th row and use } h_{,j} = -\frac{f_{,j}}{f_{,0}}\text{)} .$$

□

Recall that the degree of a proper smooth map $f : M \to N$ between oriented boundaryless manifolds with N connected is the integer

$$\deg(f) = \sum_{f(p)=q} \operatorname{sgn}(df_p) \quad (0 \text{ if } f^{-1}(q) = \emptyset)$$

for any regular value $q \in N$. Given a compactly supported differential form ω on N, if M is also connected, we know that

$$\int_M f^*\omega = \deg(f) \int_N \omega, \tag{4}$$

which can be proved using de Rham cohomology ([4], Theorem 2.3, page 105). Now we generalize it to integration of forms with closed support.

Theorem 2.3. *Let $f : M \to N$ be a proper smooth map of oriented boundaryless manifolds with N connected, ω a top form on N. Assume further that the number of preimage points of any regular value is bounded above by the same positive integer. If ω is integrable on N then $f^*\omega$ is integrable on M, and Equation (4) holds.*

Proof. The regular values of f with nonempty preimage constitute an open set $Q \subset N$, and f restricts to a finite sheeted smooth covering map from $f^{-1}(Q)$ onto Q, with the number of sheets bounded above by l. Every point $q \in Q$ has a neighborhood $V_q \subset Q$ whose preimage is a finite disjoint union of open sets $U_{iq}{}'s$ in M and the restriction of f on each is a diffeomorphism. All such open sets $V_q{}'s$ cover Q, and there is a partition of unity $\{\phi_j\}$ subordinate to the cover, with $\operatorname{supp}\phi_j \subseteq V_j$. Since $\phi_j\omega$ is compactly supported in V_j, and f is diffeomorphic on U_{ij}, we have

$$\int_{U_{ij}} |f^*(\phi_j\omega)| = \int_{V_j} |\phi_j\omega|$$

and summing over i,

$$\int_{f^{-1}(Q)} |f^*(\phi_j\omega)| = \int_{f^{-1}(V_j)} |f^*(\phi_j\omega)| \leq l\int_{V_j} |\phi_j\omega| = l\int_Q |\phi_j\omega|.$$

Suppose ω is integrable on N hence also on Q,

$$\sum_j \int_Q |\phi_j\omega| = \int_Q |\omega| < \infty$$

by monotone convergence theorem. Therefore

$$\sum_j \int_{f^{-1}(Q)} |f^*(\phi_j\omega)| = \int_{f^{-1}(Q)} |f^*\omega| = \int_M |f^*\omega| < \infty$$

since $\sum_j \phi_j \circ f = 1$ on $f^{-1}(Q)$ and $f^*\omega$ is zero on $M \backslash f^{-1}(Q)$. That is, $f^*\omega$ is integrable on M. Now taking orientation into account,

$$\int_{U_{ij}} f^*(\phi_j\omega) = \pm\int_{V_j} \phi_j\omega,$$

according to whether f is orientation preserving or reversing on U_{ij}, assuming that $V_q{}'s$ are connected. Summing over i, since all points of Q have the same signed degree $m = \deg(f)$,

$$\int_{f^{-1}(Q)} f^*(\phi_j\omega) = \int_{f^{-1}(V_j)} f^*(\phi_j\omega) = m\int_{V_j} \phi_j\omega = m\int_Q \phi_j\omega\ .$$

Assume integrability and sum over j, the right hand side becomes $m\int_Q \omega$ (dominant convergence theorem), while the left hand side is

$$\sum_j \int_{f^{-1}(Q)} (\phi_j{}^\circ f) f^*\omega = \int_{f^{-1}(Q)} f^*\omega = \int_M f^*\omega.$$

If $m \neq 0$, then f is surjective, $N\backslash Q$ is the set of critical values which has measure zero, so

$$\int_M f^*\omega = m\int_N \omega.$$

If $m = 0$, this equation also holds. □

Proposition 2.4. *A nowhere dense analytic subset of a complex manifold has measure zero.*

Proof. This is a straightforward consequence of the fact that the zero set of a holomorphic function not identically zero on a connected open set of C^n has (Lebesgue) measure 0 (Corollary 10, page 9 of [3]), because a nowhere dense analytic subset A of a complex manifold M can be covered by connected holomorphic coordinate charts $\{U_\alpha, \varphi_\alpha\}$ such that $U_\alpha \cap A$ is the common zeros of a finite number of holomorphic functions not identically zero on U_α. □

Remark 2.5. If M is equiped with a volume density which induces a positive regular Borel measure μ on M by Riesz representation theorem, then A also has μ measure 0.

Proposition 2.6. *The dominance and degree of the rational Gauss map of an affine hypersurface is invariant under nonsingular affine transformations.*

Proof. Given an affine transformation

$$\phi(x) = Ax + b, \ \ x, b \in K^n, \ \ A \text{ invertible}$$

and a hypersurface $\{f = 0\} \subset K^n$, let $g(x) = f \circ \phi(x)$, then the hypersurface $\{g = 0\} \subset K^n$ is mapped by ϕ isomorphically onto $\{f = 0\}$, in such a way that their gradient vectors at corresponding points are related by

$$(g_{,1}(x), ..., g_{,n}) = (f_{,1}(\phi(x)), ..., f_{,n})A.$$

If there is a Zariski open set $Q \subset P^{n-1}$ such that each $q \in Q$ has exactly D preimage points in $\{f = 0\}$ under the map $x \to [f_{,1}(x) : \cdots : f_{,n}]$, then the Zariski open set $QA = \{qA | q \in Q\} \subset P^{n-1}$ has the property that each $qA \in QA$ has precisely D preimage points in $\{g = 0\}$ under the map $x \to [g_{,1}(x) : \cdots : g_{,n}]$. That is, their Gauss maps have the same degree. □

Theorem 2.7. *If the Gauss map Φ of a hypersurface $V = \{f(z_0, ..., z_n) = 0, f \text{ prime with degree } d\} \subset C^{n+1}$ is dominant, then $\deg(\Phi) \le d(d-1)^n$.*

Proof. The image of the dominant rational map Φ contains a nonempty Zariski open set whose points have precisely $m = \deg(\Phi)$ preimage points in V^0. Since critical values have measure zero, one of them, say $q \in P^n$, must be a regular value. After an affine transformation which does not change the degree by the proposition above, we may assume $q = [1 : 0 : \cdots : 0]$. Each of the m points satisfies the system of equations

$$\{f = 0, \ \ f_{,j} = 0, \ \ j = 1, ..., n\}.$$

By Proposition 2.2, the following determinants are nonzero at these regular points

$$\begin{vmatrix} 0 & f_{,k} \\ f_{,j} & f_{,jk} \end{vmatrix} \ (j,k = 0, ..., n) = \begin{vmatrix} 0 & f_{,0} & 0 \\ f_{,0} & f_{,00} & f_{,0k} \\ 0 & f_{,j0} & f_{,jk} \end{vmatrix} \ (j,k = 1, ..., n)$$
$$= -f_{,0} \begin{vmatrix} f_{,0} & 0 \\ f_{,j0} & f_{,jk} \end{vmatrix} \ (j,k = 1, ..., n),$$

so all of the m solutions are nonsingular in the sense of Lemma 2.8 below. Since $f_{,j}$ has degree at most $d-1$, it follows from the lemma that $m \le d(d-1)^n$. □

Lemma 2.8. *Let $f_1, ..., f_n \in C[z_1, ..., z_n]$ be polynomials of degrees $d_1, ..., d_n$ respectively, then the number of nonsingular common zeros of them (i.e. at which $[f_{i,j}]_{1 \le i,j \le n}$ has full rank) is at most the product $d_1 \cdots d_n$.*

Proof. Let $F_1, ..., F_n$ be the homogenizations of $f_1, ..., f_n$, respectively, whose degrees do not change. Then $(z_1, ..., z_n)$ is a nonsingular solution of the system $\{f_1 = \cdots = f_n = 0\}$ if and only if $[1 : z_1 : \cdots : z_n]$ is a nonsingular projective solution of the system $\{F_1 = \cdots = F_n = 0\}$, thus Lemma 11.5.1 of [2] applies. □

Theorem 2.9. *Given an algebraic hypersurface $V = \{f(z_0, ..., z_n) = 0, f \text{ prime with degree } d\} \subset C^{n+1}$, let $\omega = (2\pi)^{-1} i\partial\bar{\partial}\log(|\nabla f|^2)$ defined in a neighborhood of the smooth locus V^0 of V. If*

$\begin{vmatrix} 0 & f_{,k} \\ f_{,j} & f_{,jk} \end{vmatrix}$ $(j,k = 0,\dots,n)$ *does not vanish identically on* V*, then the Gauss map* Φ *is dominant andd*

$$\int_{V^0} \omega^n = \deg(\Phi) \leq d(d-1)^n \ .$$

Proof. Φ is dominant if and only if the smooth map Φ is regular at some point $p \in V^0$, in which case the determinant in the premise of the theorem is nonzero at p by Proposition 2.2. So Theorem 2.7 holds. Moreover, the smooth locus V^0 of an irreducible complex variety is connected in the analytic topology, thus a proper closed algebraic subset of V^0 is nowhere dense by the identity theorem of holomorphic functions on connected complex manifolds, so it has measure zero according to Proposition 2.4. Therefore the argument before Formula (1) goes through. □

Remark 2.10. Let $S = C(f_{,0}/f_{,j},\dots,f_{,n}/f_{,j}),\ f_{,j} \neq 0 \in C(V)$ be the subfield of the rational function field $C(V)$ generated over C by these n functions, then Φ is dominant if and only if the degree of the field extension $C(V) \supseteq S$ is finite, in which case $\deg(\Phi) = [C(V):S]$.

Example 2.11. nondegenerate quadrics in affine space
A nondegenerate quadric $V \subseteq C^{n+1}$ is equivalent under the group of affine transformations to either of the two canonical ones:

$$z_0 + z_1^2 + \cdots + z_n^2 = 0,$$
$$z_0^2 + z_1^2 + \cdots + z_n^2 = 1.$$

Since the index is affine invariant by Proposition 2.6, we need only to calculate it for the two canonical quadrics. For the former,

$$S = C(f_{,1}/f_{,0},\dots,f_{,n}/f_{,0}) = C(z_1,\dots,z_n),$$
$$C(V) = C(z_0,\dots,z_n) = S(z_0).$$

Since

$$z_0 = -z_1^2 - \cdots - z_n^2 \in S,$$

we get

$$[C(V):S] = 1.$$

For the latter,

$$S = C(f_{,1}/f_{,0},\dots,f_{,n}/f_{,0}) = C(z_1/z_0,\dots,z_n/z_0),$$
$$C(V) = C(z_0,\dots,z_n) = S(z_0),$$
$$z_0^2 = (1 + (z_1/z_0)^2 + \cdots + (z_n/z_0)^2)^{-1} \in S,\ z_0 \notin S,$$

hence

$$[C(V):S] = 2.$$

Example 2.12. the surface $V = \{z^{p+2} = x^2 + y^2, p \text{ prime}\} \subseteq C^3$
Let $f = x^2 + y^2 - z^{p+2}$, then

$$S = C(f_x/f_z, f_y/f_z) = C(x/z^{p+1}, y/z^{p+1}),$$

and $C(V) = C(x,y,z) = S(z)$.
Dividing through the equatipon by z^{2p+2}, we get

$$1/z^p = x^2/z^{2p+2} + y^2/z^{2p+2},$$

or

$$z^p = ((x/z^{p+1})^2 + (y/z^{p+1})^2)^{-1} = a \in S.$$

Since S contains pth roots of unity, and p is prime, $C(V)/S$ is Kummer extension with degree p.

3. For general smooth affine hypersurfaces

For smooth projective hypersurfaces $W = \{F(Z_0,...,Z_n) = 0\} \subset P^n$, there is a nice formula which states that the integral of the top exterior power of the Kahler form η equals (up to a constant factor) the degree d of the hypersurface ([5], page 227)

$$\int_W \eta^{n-1} = d. \tag{5}$$

One can check that the constant factor is correct by choosing V to be a hyperplane $Z_0 = 0 \subset P^n$, then the formula is consistent with the volume of P^{n-1}.

Let $V = \{F(1,z_1,...,z_n) = f(z_1,...,z_n) = 0\} \subset C^n$ be its affine part, in a neighborhood of which the form $\omega = (2\pi)^{-1} i\partial\bar{\partial}\log(|\nabla f|^2)$ is defined. Since $|\nabla f(z_1,...,z_n)|^2 = \sum_{j=1}^{n} |F_{,j}(1,z_1,...,z_n)|^2$, the expression

$$(2\pi)^{-1} i\partial\bar{\partial}\log(|F_{,1}(Z_0,Z_1,...,Z_n)|^2 + \cdots + |F_{,n}|^2) = \widetilde{\omega}$$

seems to extend ω, but it needs some interpretation. $\bigcup_{j=1}^{n} \{F_{,j} \neq 0\} = A$ is a nonempty Zariski open subset of P^n whose preimage under $\pi : C^{n+1} \backslash \{0\} \to P^n$ is the domain of definition of $\widetilde{\omega}$.

Proposition 3.1. *$\widetilde{\omega}$ is the pull back via π of a unique form ξ on A that extends ω.*

Proof. Recall that given a surjective submersion from manifold M to N with connected fibers, a basic differential form on M is one that vanishes on any vertical vector, and so does its exterior derivative; it is the pull back of a unique form on N. Now we show that $\widetilde{\omega}$ is a basic form. Since $\widetilde{\omega}$ is closed, we need only to show that the form vanishes on any vertical vector. Any vertical vector is a linear combination of Z, $\overline{Z}$. Write (repeated indices indicate summation)

$$\begin{aligned}\widetilde{\omega} &= (2\pi)^{-1} i\partial\bar{\partial}\log(|F_{,1}|^2 + \cdots + |F_{,n}|^2)\\ &= (2\pi)^{-1} i\partial\bar{\partial}\log g = (2\pi)^{-1} i(g^{-1}g_{,j})_{,\bar{k}} dZ_j \wedge d\overline{Z}_k\ .\end{aligned}$$

The contraction

$$Z_j(g^{-1}g_{,j})_{,\bar{k}} = (g^{-1}Z_j g_{,j})_{,\bar{k}} = 0$$

because

$$Z_j g_{,j} = Z_j F_{,lj}\overline{F_{,l}} = (d-1)F_{,l}\overline{F_{,l}} = (d-1)g. \ \ (l = 1,...,n, j = 0,...,n)$$

Similar calculations show that its contraction with $\overline{Z}$ is also zero. So $\widetilde{\omega} = \pi^*\xi$ for some ξ defined on A. To see what ξ looks like in an affine chart $\{Z_j \neq 0\}$ of P^n, note that the chart map is the composition $\varphi = \pi \circ \tau$:

$$(z_0, ..\widehat{z}_j.., z_n) \to (z_0, .., 1, .., z_n) \to [z_0 : \cdots : 1 : \cdots : z_n]\ ;$$

so

$$\varphi^*\xi = \tau^*\pi^*\xi,$$

which means that we simply substitute $(z_0,..,1,..,z_n)$ for $(Z_0,...,Z_n)$ in the expression of $\pi^*\xi$. For example, in the case $j=0$,

$$\tau^*\pi^*\xi = (2\pi)^{-1} i\partial\bar{\partial}\log(|F_{,1}(1,z_1,...,z_n)|^2+\cdots+|F_{,n}|^2)$$
$$= (2\pi)^{-1} i\partial\bar{\partial}\log(|f_{,1}(z_1,...,z_n)|^2+\cdots+|f_{,n}|^2),$$

which coincides with ω defined on $A\cap\{Z_0\neq 0\}\subset P^n = \{\nabla f\neq 0\}\subset C^n$. □

Proposition 3.2. *ξ defined above is cohomologous to $d-1$ times the Kahler form η of P^n on their common domain of definition.*

Proof. Replace F by $Z_0^2+\cdots+Z_n^2$ and let l run through $0,...,n$ in the proof of Proposition 3.1, we get that $\pi^*\eta$ is also a basic form. We have

$$\pi^*\xi-(d-1)\pi^*\eta = d(i\partial\log(\sum_{j=0}^{n}|Z_j|^{2(d-1)}/\sum_{j=1}^{n}|F_{,j}|^2)) = d\beta$$

on $\pi^{-1}(A)$. β is a basic form with respect to the submersion π onto A because

$$\beta = i\partial\log f = if^{-1}f_{,j}dZ_j\ ,$$

vertical vector $Z=Z_j\partial_j$, any vertical vector is a linear combination of Z, $\overline{Z}$,

$$\beta(Z) = if^{-1}f_{,j}Z_j = if^{-1}\tfrac{d}{dt}|_{t=1}f(tZ_0,...,tZ_n)$$
$$= if^{-1}\tfrac{d}{dt}|_{t=1}f(Z_0,...,Z_n) = 0 = \beta(\overline{Z}).$$

Therefore, $\beta=\pi^*\alpha$, and

$$\pi^*(\xi-(d-1)\eta) = \pi^*\xi-(d-1)\pi^*\eta = d\beta = d\pi^*\alpha = \pi^*d\alpha,$$

and by uniqueness, $\xi-(d-1)\eta = d\alpha$ on A, which proves the claim. □

Sometimes ω can extend to a neighborhood of the whole of W, this happens when A contains W.

Proposition 3.3. *If the degree d hypersurface $W=\{F=0\}\subset P^n$ ($d\geq 2$, $n\geq 2$) is nonsingular and transverse to the hyperplane $L=\{Z_0=0\}$, then ξ can be defined on a neighborhood of the whole of W. Furthermore, the premise is a Zariski open condition and satisfied by a general degree d hypersurface.*

Proof. The transversality condition can be expressed as the Jacobian matrix having full rank at nonzero points where $F(0,Z_1,...,Z_n)=0$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ F_{,0} & F_{,1} & \cdots & F_{,n} \end{bmatrix},$$

or equivalently, the system of equations

$$\{F=F_{,1}=\cdots=F_{,n}=0\}$$

has no nonzero solution with $Z_0=0$, so $L\cap W$ is contained in A. On the other hand, on $\{Z_0\neq 0\}\cap W$, by Euler's identity,

$$Z_0F_{,0}+Z_1F_{,1}\cdots+Z_nF_{,n}=0,$$

$F_{,1}=\cdots=F_{,n}=0$ would imply $F_{,0}=0$, contradicting the smoothness hypothesis. So A contains W. Now assume nonsingularity, which is a Zariski open condition as we know. That is, the projective space P^m that parameterizes all degree d hypersurfaces in P^n contains a Zariski open subset U such that that a hypersurface is nonsingular if and only if it corresponds to a parameter value in U. The system

$$\{F = F_{,1} = \cdots = F_{,n} = 0\}$$

cannot have projective solution with $Z_0 \neq 0$ by the argument above, so the hypersurface is transversal to L if and only if the system has no projective solution regardless of Z_0. To show that this is also a Zariski open condition, we work with a generic $F \in Z[Y_0,\ldots,Y_m,Z_0,\ldots,Z_n]$ with $m+1$ indeterminate coefficients. The system above defines a Zariski closed set in $P^n \times P^m$ which projects to a Zariski closed set $B \subset P^m$, because the projection $P^n \times P^m \to P^m$ is a closed map in the Zariski topology. Points in $U \cap B^c$ correspond to those hypersurfaces that are both nonsingular and transverse to L. To show that $U \cap B^c$ is nonempty, we need only to find a particular hypersurface satisfying the condition. Set $F = \sum_{j=0}^{n} a_j Z_j^d$, $a_j \neq 0$, whose gradient never vanishes in P^n. If $\nabla F(Z_0,\ldots,Z_n)$ is proportional to $(1,0,\ldots,0)$, then $Z_1 = \cdots = Z_n = 0$; there is no such point on $\{F=0\} \subset P^n$, so it is transverse to L. ☐

Theorem 3.4. *For a nonsingular hypersurface $V = \{f(z_1,\ldots,z_n) = 0, f$ prime with degree $d\} \subset C^n$, let $\omega = (2\pi)^{-1} i\partial\bar{\partial}\log(|\nabla f|^2)$ defined in a neighborhood of V. If the projective completion W of V is also nonsingular and transverse to $\{Z_0 = 0\} \subset P^n$, then $\int_V \omega^{n-1} = d(d-1)^{n-1}$.*

Proof. The formula is trivially true in the case of $d = 1$, henceforth we assume $d \geq 2$. By Proposition 3.3, ω extends to a form ξ defined on a neighborhood of W, such that ξ is cohomologous to $(d-1)\eta$, so

$$\int_V \omega^{n-1} = \int_W \xi^{n-1} = (d-1)^{n-1}\int_W \eta^{n-1} = d(d-1)^{n-1}$$

by Proposition 2.4 and Formula (5). ☐

Remark 3.5. If we parameterize affine hypersurfaces by the same parameter that parameterizes their projective completions, then the theorem holds for a general affine hypersurface by Proposition 3.3.

4. For general smooth affine plane curves and beyond

Cohn-Vossen's inequality ([9], section 4, Theorem 10 or [8]) states that a complete boundaryless finitely connected (i.e. homeomorphic to a closed surface N minus a finite number l of points) Riemannian 2-manifold M whose curvature integral exists satisfies

$$\int_M K dM \leq 2\pi\chi(M),$$

where the Euler number

$$\chi(M) = \chi(N) - l.$$

By relation (2), we apply Theorem 2.9 in the case of irreducible affine plane curve V of degree d,

$$\int_{V^0} K dV^0 = -\int_V \omega \geq 2\pi d(1-d).$$

Corollary 4.1. *For a nonsingular irreducible complex affine plane curve V of degree d,*

$$d(1-d) \leq \tfrac{1}{2\pi}\textstyle\int_V K dV \leq \chi(V).$$

Proof. $V = \{f = 0\}$ is a complete Riemann surface with the induced metric tensor because it is a properly embedded submanifold of the complete Euclidean space C^2. It is biholomorphic to a Zariski open subset of a smooth projective curve, so V is finitely connected. Integrability of ω means that V has finite total absolute curvature, hence Cohn-Vossen's inequality applies. □

Example 4.2. Euler number of the affine part of a complex nonsingular irreducible projective plane curve

Suppose the curve $V' \subset P^2$ has degree d, then its Euler number $\chi(V') = d(3-d)$. Its affine part V is V' minus a finite number $l \leq d$ of points, so

$$\chi(V) = \chi(V') - l \geq d(3-d) - d = d(2-d),$$

consistent with the inequality above.

Theorem 4.3. *If the projective completion $\overline{V}$ of an affine plane curve $V \subset C^2$ is smooth and transverse to the line at infinity (this is satisfied by a general affine hypersurface by Remark 3.5), then*

$$\frac{1}{2\pi}\int_V K dV = 2\chi(V) - \chi(\overline{V}) = \chi(V) - 1 - \sqrt{1-\chi(V)}\ .$$

Proof. Suppose the defing polynomial $f(z_1, z_2)$ of V is irreducible and has degree $d \geq 1$, the highest degree terms H of f is homogeneous of degree d, so the homogenization F of f can be written as

$$F(Z_0, Z_1, Z_2) = H(Z_1, Z_2) + Z_0 G(Z_0, Z_1, Z_2)\ .$$

Since

$$\nabla F(0, Z_1, Z_2) = (G(0, Z_1, Z_2), \nabla H(Z_1, Z_2)),$$

$\overline{V}$ is transverse to $\{Z_0 = 0\}$ in P^2, or equivalently,

$$(a_1, a_2) \neq (0,0),\ \ H(a_1, a_2) = 0 \ \ \Rightarrow \ \ \nabla H(a_1, a_2) \neq 0.$$

H is a product of linear factors, and if one of them, say $a_1 Z_1 + a_2 Z_2$ has multiplicity bigger than one, then

$$H(a_2, -a_1) = 0,\ \ \nabla H(a_2, -a_1) = 0,$$

a contradiction. So F has exactly d distinct projective zeros in $\{Z_0 = 0\}$. Therefore the Euler number

$$\chi(V) = \chi(\overline{V}) - d = d(3-d) - d = d(2-d).$$

Theorem 3.4 and relation (2) tell us that

$$\tfrac{1}{2\pi}\textstyle\int_V K dV = d(1-d) = \chi(V) - d = 2\chi(V) - \chi(\overline{V}).$$

Solve the equation $\chi(V) = d(2-d)$ for d, we get

$$d = 1 + \sqrt{1-\chi(V)},$$

where we have dropped the solution $d = 1 - \sqrt{1-\chi(V)}$ since $d \geq 1$. Finally,

$$\frac{1}{2\pi}\int_V K dV = \chi(V) - d = \chi(V) - 1 - \sqrt{1-\chi(V)}\ .$$

□

Moreover, Yau and Yang proposed some generalizations of Cohn-Vossen inequality to higher dimensional complete Riemannian and Kahler manifolds respectively ([10], Questions 1.1 and 1.2), later Liu solved Yang's question under some additional assumptions ([7], Theorem 1.6) such as nonnegative bisectional curvature. But a complex hypersurface in complex Euclidean space has nonpositive holomorphic sectional and bisectional curvature ([1], page 816, prop 1). Thus our integral formula in this paper suggests that Yang's question in the special case of integrating the top exterior power of the Ricci form (proportional to the first Chern form) over complete noncompact Kahler manifolds might also have a positive answer under other or no curvature assumptions.

References

[1] Albert Vitter, *On the curvature of complex hypersurfaces*, Indiana Univ. Math. J., Vol. 23, No. 9, pp. 813-826, (1974), https://www.jstor.org/stable/24890781

[2] Bochnak, J., Coste, M., Roy, M. F., *Real algebraic geometry*, Springer, (1998)

[3] Gunning R. C., Rossi H., *Analytic functions of several complex variables*, Prentice Hall, Inc, (1965)

[4] Enrique Outerelo, Jesus M. Ruiz, *Mapping degree theory*, AMS, (2009)

[5] Joe Harris, *Algebraic geometry a first course*, Springer-Verlag

[6] Jean-Pierre Demailly, *Complex analytic and differential geometry*, Université de Grenoble I. Institut Fourier, version of June, 2012, https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf

[7] Liu, G.,*Cohn–Vossen inequality on certain noncompact Kahler manifolds*, Math. Z. 302, 1025–1034 (2022), https://doi.org/10.1007/s00209-022-03105-5

[8] S. Cohn-Vossen, *K ¨urzeste Wege und Totalkr ¨ummung auf Fl ¨achen*, Compositio Math. 2, 63–113 (1935)

[9] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. 32, 13-72 (1952)

[10] Yang, B., *On a problem of Yau regarding a higher dimensional generalization of the Cohn–Vossen inequality*, Math. Ann. 355, 765–781 (2013), https://doi.org/10.1007/s00208-012-0803-3

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