GROUPOID ACTIONS AND KOOPMAN REPRESENTATIONS

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Abstract. We study the C^*-algebra $C^*(\kappa)$ generated by the Koopman representation $\kappa = \kappa^\mu$ of a locally compact groupoid G acting on a measure space (X, μ), where μ is quasi-invariant for the action. We interpret κ as an induced representation and we prove that if the groupoid $G \ltimes X$ is amenable, then κ is weakly contained in the regular representation $\rho = \rho^\mu$ associated to μ, so we have a surjective homomorphism $C_r^*(G) \to C^*(\kappa)$. We consider the particular case of Renault-Deaconu groupoids $G = G(X, T)$ acting on their unit space X and show that in some cases $C^*(\kappa) \cong C^*(G)$.

1. introduction

The concept of a group action on a space was generalized to a groupoid action and it has applications to dynamical systems, representation theory and operator algebras. If groups can roughly be described as the set of symmetries of certain objects, then groupoids can be thought as the set of symmetries of fibered objects.

A unitary representation of a locally compact groupoid G endowed with a Haar system is a triple $L = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ consisting of a quasi-invariant measure μ on the unit space $G^{(0)}$ of G, a Borel Hilbert bundle $G^{(0)} * \mathcal{H}$ over $G^{(0)}$, and a Borel homomorphism $\hat{L} : G \to \operatorname{Iso}(G^{(0)} * \mathcal{H})$ such that $\hat{L}(g) = (r(g), L_g, s(g))$ and $L_g : \mathcal{H}(s(g)) \to \mathcal{H}(r(g))$ is a Hilbert space isomorphism between fibers (cf. [25, Definition II.1.6]; see also [36, Definition 7.7],[26, 27]).

A Koopman representation of G is a unitary representation of G determined by a pair (X, μ) consisting of a locally compact space X on which G acts leaving the measure μ quasi-invariant. That is, X is fibered over $G^{(0)}$ by a continuous open surjection $\omega : X \to G^{(0)}$, and μ admits a disintegration $d\mu(\cdot) = \int_{X_u} d\mu_u(\cdot) d\tilde{\mu}(u)$ where each μ_u is a probability measure supported on $X_u := \omega^{-1}(u)$ and $\tilde{\mu} = \omega_*(\mu)$ is a probability measure on $G^{(0)}$

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that is quasi-invariant in the usual sense. The Hilbert bundle $\mathcal{H}$ for the Koopman representation determined by (X, μ) is $\{L^2(X_u, \mu_u)\}_{u \in G^{(0)}}$ and the representation L, denoted here by κ^μ, is given by

$$\kappa^\mu_g : L^2(X_{s(g)}, \mu_{s(g)}) \to L^2(X_{r(g)}, \mu_{r(g)}),$$

$$\kappa^\mu_g \xi(x) := D(g^{-1}, x)^{\frac{1}{2}} \xi(g^{-1}x),$$

where $D(g, \cdot)$ is the Radon-Nikodym derivative $dg\mu_{s(g)}/d\mu_{r(g)}$ (see Section 4 for details). It is our main goal to study the C^*-algebra $C^*(\kappa^\mu)$ generated by the Koopman representation.

We begin by fixing some notation associated with a locally compact Hausdorff groupoid G with a Haar sytem. We recall the definition of a groupoid action $G \curvearrowright X$ on a locally compact space X fibered over the unit space of G and of additional concepts like orbits, stabilizers and transitive actions. We illustrate with several examples of actions, including the cases $X = G^{(0)}, X = G$, $X = G/H$ for H a closed subgroupoid and $X = \bigcup_{x \in G^{(0)}} G(x, S)$, where $G(x, S)$ is the Cayley graph for a generating set S. We also review the definition of the action groupoid $G \ltimes X$ and of the concepts of groupoid fibration and groupoid covering.

We continue with quasi-invariant measures on X for $G \curvearrowright X$ and relate them to measures for the action groupoid $G \ltimes X$. We recall some facts about groupoid representations, induced representations and amenability. The Koopman representation $\kappa^\mu : G \to \mathcal{B}(L^2(X, \mu))$ associated to a quasi-invariant probability measure μ on X can be understood as the induced representation of the trivial representation ι of the groupoid $G \ltimes X$. Induced representations in the case G is a Borel transformation group groupoid already appeared in Definition 3.5 of [23]. When G acts on itself by left multiplication and the support of the quasi-invariant measure is full, the Koopman representation is just the left regular representation.

We prefer to work with unitary representations of groupoids which appear in a natural way in our context rather than with the integrated forms at the level of C^*-algebras. Most of our results could be recast in terms of Hilbert modules à la Rieffel. Holkar has already shown in [12] that Rieffel's construction of induced representations is valid for topological groupoid correspondences. We believe that Renault's perspective from [32] of inducing unitary representation at the groupoid level is better suited for examples and to illustrate how one can recover the classical definitions and results going back to Mackey's work on induced representations of groups.

We define $C^*(\kappa^\mu)$ to be the closure of $\kappa^\mu(C_c(G))$ in $\mathcal{B}(L^2(X, \mu))$ and we try to relate it to $C^*_r(G)$. In our main result, Theorem 5.3, we prove that if the action groupoid $G \ltimes X$ is σ-compact and amenable, and the measure μ has full support, then the Koopman representation is weakly contained in the left regular representation associated to μ, so we have a surjective

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homomorphism $C_r^*(G) \to C^*(\kappa^\mu)$. In some cases (see the examples involving graph C^*-algebras in section 6), this is an isomorphism.

In the case when the Renault-Deaconu groupoid $G(X,T)$ associated to a local homeomorphism $T : X \to X$ acts on a space Y, it is known that the action groupoid is isomorphic to another Renault-Deaconu groupoid, see [15]. The form of quasi-invariant measures for $G(X,T)$ with given Radon-Nikodym derivative is studied in several papers, like [18, 15, 30]. We illustrate the theory with several examples in the last section of the paper.

Since we rely on [32] for some of our constructions and definitions, we are assuming that our groupoids and spaces are Hausdorff, locally compact and second countable.

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2. Groupoid actions

A groupoid G is a small category with inverses. We will use s and r for the source and range maps $s, r : G \to G^{(0)}$, where $G^{(0)}$ is the unit space. We always assume that G has a locally compact Hausdorff topology compatible with the algebraic structure. The set of composable pairs is denoted by $G^{(2)}$. Let G_u be the set of $g \in G$ with $s(g) = u$, let G^v be the set of $g \in G$ with $r(g) = v$, and let $G_u^v = G_u \cap G^v$. Two units $x, y \in G^{(0)}$ belong to the same G-orbit if there exists $g \in G$ such that $s(g) = x$ and $r(g) = y$. When every G-orbit is dense in $G^{(0)}$, the groupoid G is called minimal. The isotropy group of a unit $x \in G^{(0)}$ is the group

$$G_x^x := \{g \in G \mid s(g) = r(g) = x\},$$

and the isotropy bundle is

$$G' := \{g \in G \mid s(g) = r(g)\} = \bigcup_{x \in G^{(0)}} G_x^x.$$

A groupoid G is said to be principal if all isotropy groups are trivial, or equivalently, $G' = G^{(0)}$.

To construct C^*-algebras from a groupoid G, we will assume that G is second countable with a Haar system. Recall that a Haar system is given by a family of Radon measures $\{\lambda^u\}_{u \in G^{(0)}}$ such that $\mathrm{supp}(\lambda^u) = G^u$ and for any $f \in C_c(G)$, the map

$$u \mapsto \int_G f(g) d\lambda^u(g)$$

is continuous and we have

$$\int_G f(hg)d\lambda^{s(h)}(g) = \int_G f(g)d\lambda^{r(h)}(g).$$

It is known that the existence of a Haar system implies that the range and the source maps are open. We assume throughout the paper that our groupoids have Haar systems.

An étale groupoid is a topological groupoid where the range map r (and necessarily the source map s) is a local homeomorphism. The unit space $G^{(0)}$ of an étale groupoid is always an open subset of G and a Haar system is given by the counting measures.

Definition 2.1. Let G be a topological groupoid. A bisection is a subset $U \subseteq G$ such that s and r are both injective when restricted to U.

An open bisection U determines a homeomorphism $\pi_U = (r|_U) \circ (s|_U)^{-1} : s(U) \to r(U), \pi_U(x) = r(s^{-1}(x))$. An étale groupoid has sufficiently many open bisections which generate its topology.

Example 2.2. Let X be a locally compact Hausdorff space and let $T : X \to X$ be a local homeomorphism. The Renault-Deaconu groupoid associated to T is

$$G(X,T) = \{(x, m-n, y) \in X \times \mathbb{Z} \times X : T^m(x) = T^n(y)\}$$

with operations

$$(x,k,y)(y,\ell,z) = (x,k+\ell,z),\ (x,k,y)^{-1} = (y,-k,x).$$

We identify the unit space of $G(X,T)$ with X via the map $(x,0,x) \mapsto x$. The range and source maps are then

$$r(x,k,y) = x,\ s(x,k,y) = y.$$

A basis for the topology consists of sets of the form

$$Z(U,m,n,V) = \{(x,m-n,y) : T^m(x) = T^n(y), x \in U, y \in V\},$$

where U, V are open subsets of X such that $T^m|_U$ and $T^n|_V$ are one-to-one and $T^m(U) = T^n(V)$. These are bisections for $G(X,T)$, and with this topology, $G(X,T)$ becomes an étale groupoid.

We now recall the definition of a groupoid action on a space given in [36, Definition 2.1] or [2, Definition 4.1]:

Definition 2.3. A topological groupoid G is said to act (on the left) on a locally compact space X, if there are given a continuous surjection $\omega : X \to G^{(0)}$, called the anchor or moment map, and a continuous map

$$G * X \to X, \quad \text{write} \quad (g,x) \mapsto g \cdot x = gx,$$

where

$$G * X = \{(g,x) \in G \times X \mid s(g) = \omega(x)\},$$

that satisfy

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i) $\omega(g \cdot x) = r(g)$ for all $(g, x) \in G * X$,

ii) $(g_2, x) \in G * X,\ (g_1, g_2) \in G^{(2)}$ implies $(g_1g_2, x), (g_1, g_2 \cdot x) \in G * X$ and

$$g_1 \cdot (g_2 \cdot x) = (g_1g_2) \cdot x,$$

iii) $\omega(x) \cdot x = x$ for all $x \in X$.

We denote by X_u the fiber $\omega^{-1}(u)$ over $u \in G^{(0)}$.

We should mention that in [20, Section 2] the authors required that the anchor map is open as well. This assumption was dropped in [36, Definition 2.1], see the comments in [36, Remark 2.2].

The action of G on X is called transitive if given $x, y \in X$, there is $g \in G$ with $g \cdot x = y$ and is free if $g \cdot x = x$ for some x implies $g = \omega(x) \in G^{(0)}$.

The set of fixed points in X is defined as

$$X^G = \{x \in X : g \cdot x = x \text{ for all } g \in G_{\omega(x)}^{\omega(x)}\}.$$

If G has trivial isotropy, then $X^G = X$.

The orbit of $x \in X$ is

$$Gx = \{g \cdot x : g \in G,\ s(g) = \omega(x)\}.$$

The set of orbits is denoted by $G\backslash X$ and has the quotient topology. The action of G on X is called minimal if every orbit Gx is dense in X. For a transitive action there is a single orbit.

For $x \in X$, its stabilizer group is

$$G(x) = \{g \in G : g \cdot x = x\},$$

which is a subgroup of G_u^u for $u = \omega(x)$.

Remark 2.4. Note that if the action of G on X is transitive, then $G(x) \cong G(y)$ for $x, y \in X$. Indeed, if $h \cdot x = y$, then $g \mapsto hgh^{-1}$ is an isomorphism $G(x) \to G(y)$.

Example 2.5. A groupoid G with open source and range maps acts on its unit space $G^{(0)}$ by $g \cdot s(g) = r(g)$. In this case, $\omega = id$. The groupoid is called transitive if this action is transitive. Notice that $g \cdot u = u$ for all $g \in G_u^u$, in particular $G(u) = G_u^u$ and $(G^{(0)})^G = G^{(0)}$. A transitive groupoid with discrete unit space is of the form $G^{(0)} \times K \times G^{(0)}$, where K is a copy of the isotropy group. The operations are

$$(x, g, y)(y, h, z) = (x, gh, z),\ (x, g, y)^{-1} = (y, g^{-1}, x),$$

where $x, y, z \in X$ and $g, h \in K$.

Example 2.6. A groupoid G acts on itself by left multiplication with $\omega(g) = r(g)$. More generally, if G is a groupoid and H is a closed subgroupoid with the same unit space, we can form the set of left cosets G/H since H acts on G on the right via multiplication. Since we assume that G and H have Haar systems, the corresponding range and source maps are open. Therefore G/H is locally compact and Hausdorff because H acts properly on G (see [20, §2]). The groupoid G acts on the G/H by left multiplication. Here $\omega : G/H \to G^{(0)}$, $\omega(gH) = r(g)$ is an open map (see [20, §2]). Since $g_1H = g_2H$ if and only if $g_1^{-1}g_2 \in H$, if follows that $r(g_1) = r(g_2)$ and ω is well defined. Note that this action is not necessarily transitive, since given $g_1H, g_2H \in G/H$, the element $g_2g_1^{-1}$ is defined only for $s(g_1) = s(g_2)$.

Remark 2.7. If G acts on X, then the fibered product

$$G * X = \{(g, x) \in G \times X \mid s(g) = \omega(x)\}$$

has a natural structure of groupoid, called the semi-direct product or action groupoid and is denoted by $G \ltimes X$, where

$$(G \ltimes X)^{(2)} = \{((g_1, x_1), (g_2, x_2)) \mid \ x_1 = g_2 \cdot x_2\},$$

with operations

$$(g_1, g_2 \cdot x_2)(g_2, x_2) = (g_1g_2, x_2), \ \ (g, x)^{-1} = (g^{-1}, g \cdot x).$$

The source and range maps of $G \ltimes X$ are

$$s(g, x) = (s(g), x) = (\omega(x), x), \quad r(g, x) = (r(g), g \cdot x) = (\omega(g \cdot x), g \cdot x),$$

and the unit space $(G \ltimes X)^{(0)}$ may be identified with X via the map

$$i : X \to G \ltimes X, \ \ i(x) = (\omega(x), x).$$

Note that the source and range maps defined above are open even if the anchor map ω is not assumed to be open (see [2, page 10]).

Remark 2.8. Recall that if $\lambda = \{\lambda^u\}_{u\in G^{(0)}}$ is a Haar system for G then $\overline{\lambda} = \{\overline{\lambda}^x\}_{x\in X}$ defined via

$$\int_{(G\ltimes X)^x} f(g, y)\, d\overline{\lambda}^x(g, y) := \int_{G^{\omega(x)}} f(g, g^{-1} \cdot x)\, d\lambda^{\omega(x)}(g)$$

for all $f \in C_c(G \ltimes X)$ and $x \in X$ is a Haar system on $G \ltimes X$. We will use this Haar system for the action groupoid, see Ex. 2.1.7 on page 37 in [36] and [2, page 10].

Recall from [7, Definition 2.1] and [9, Definition 3.2] that a *groupoid fibration* is a morphism of locally compact groupoids $\pi : G \to H$ with the property that the map

$$G \to H * G^{(0)}, g \mapsto (\pi(g), s(g))$$

is open and surjective, where $H * G^{(0)} = \{(h, x) \in H \times G^{(0)} \ \mid \ s(h) = \pi(x)\}$. In particular, for any $h \in H$ and any $x \in G^{(0)}$ with $\pi(x) = s(h)$ there is $g \in G$ with $s(g) = x$ and $\pi(g) = h$. If g is unique for any such h and x, then π is called a *groupoid covering*. Note that for a groupoid covering we have $\pi^{-1}(H^{(0)}) = G^{(0)}$.

For G acting on X with $\omega : X \to G^{(0)}$ open, the projection map

$$\pi : G \ltimes X \to G, \ \ \pi(g, x) = g$$

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is a covering of groupoids. Conversely, given a covering of groupoids $\pi : G \to H$, there is an action of H on $X = G^{(0)}$ with $\omega = \pi|_{G^{(0)}} : G^{(0)} \to H^{(0)}$ and $G \cong H \ltimes X$. The action is defined by $h \cdot x = r(g)$, where $g \in G$ is unique with $\pi(g) = h$ and the isomorphism is given by $g \mapsto (\pi(g), s(g))$.

Note that for G acting on $X = G^{(0)}$ by $g \cdot s(g) = r(g)$, we get $G \ltimes X \cong G$.

Example 2.9. Consider $E = (E^0, E^1, r, s)$ a topological graph and G a topological groupoid. Recall that E^0, E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is continuous and $s : E^1 \to E^0$ is a local homeomorphism. Let $c : E^0 \cup E^1 \to G$ be a continuous function such that $c(E^0) \subset G^{(0)}$, $c(s(e)) = s(c(e)),\ \ c(r(e)) = r(c(e))$ and such that $(c(e_1), c(e_2)) \in G^{(2)}$ for $e_1e_2 \in E^2$. The map c is called a cocycle and it can be extended to finite paths by $c(e_1e_2 \cdots e_k) = c(e_1)c(e_2) \cdots c(e_k)$.

The skew-product graph $E \times_c G$ has vertices

$$E^0 \times_c G = \{(v, g) : c(v) = s(g)),$$

edges

$$E^1 \times_c G = \{(e, g) : (g, c(e)) \in G^{(2)}\}$$

and incidence maps

$$\tilde{r}(e, g) = (r(e), gc(e)),\ \ \tilde{s}(e, g) = (s(e), g).$$

Then $(E^0 \times_c G, E^1 \times_c G, \tilde{r}, \tilde{s})$ becomes a topological graph since $\tilde{s}$ is a local homeomorphism and $\tilde{r}$ is continuous. Moreover, G acts freely on $E^0 \times_c G$ by $h\cdot(v, g) = (v, hg)$, where $\omega : E^0 \times_c G \to G^{(0)}, \omega(v, g) = r(g)$. Similarly, G acts freely on $E^1 \times_c G$ by $h\cdot(e, g) = (e, hg)$ with $\omega : E^1 \times_c G \to G^{(0)}, \omega(e, g) = r(g)$. The action commutes with the incidence maps and the quotient graph is isomorphic to E.

Example 2.10. Let G be a topological groupoid. We say that $Y \subset G^{(0)}$ is a topological transversal if Y contains an open transversal (recall that a transversal intersects every orbit). A compact generating pair (S, Y) of G is made of a compact subset $S \subset G$ and a compact topological transversal Y such that for every $g \in G|_Y = \{g \in G : s(g), r(g) \in Y\}$ there exists n such that $\bigcup_{0 \le k \le n} (S \cup S^{-1})^k$ is a neighborhood of g in $G|_Y$. Here, for a subset $A \subset G$, A^k is the set of all products $a_1a_2 \cdots a_k$ where $a_i \in A$.

If (S, Y) is a compact generating pair for G and $x \in Y$, the Cayley graph $G(x, S)$ is the directed graph with vertex set $G^x_Y = \{g \in G : s(g) \in Y, r(g) = x\}$ such that there is an edge from g_1 to g_2 whenever there is $h \in S$ with $g_2 = g_1h$. If $G^{(0)}$ is compact, then the groupoid G with generating set $(S, G^{(0)})$ acts freely on the union of Cayley graphs $\bigcup_{x \in G^{(0)}} G(x, S)$ by left multiplication.

In particular, if $\sigma : \mathbb{T} \to \mathbb{T}, \sigma(z) = z^d$ for $d \ge 2$ and $G = G(\mathbb{T}, \sigma)$ is the groupoid of germs of the pseudogroup generated by σ (see section 2 in [29]), then we can take the generating set S to be a finite set of germs of maps $\sigma^{-1} : \sigma(U) \to U$, where $U \subseteq \mathbb{T}$ is an open set such that $\sigma : U \to \sigma(U)$ is a homeomorphism. Then the Cayley graphs $G(z, S)$ are regular trees of degree $d+1$ and the groupoid G acts on their union. For more on Cayley graphs of groupoids, see [21].

3. Quasi-invariant measures and representations

Let G be a locally compact groupoid with left Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ and let μ be a measure on $G^{(0)}$. The measure $\nu = \mu \circ \lambda$ on G induced by μ is defined via

$$\int_G f(g)\, d\nu(g) = \int_{G^{(0)}} \int_{G^u} f(g)\, d\lambda^u(g) d\mu(u)$$

for all $f \in C_c(G)$. Let ν^{-1} be the push-forward of ν under the inverse map.

Definition 3.1. A measure μ on $G^{(0)}$ is called quasi-invariant ([25, Definition I.3.2]) if its induced measure ν is equivalent to its inverse ν^{-1}, i.e. they have the same nullsets (we write $\nu \sim \nu^{-1}$ in this case).

Remark 3.2. For an étale groupoid G, a Radon measure μ on $G^{(0)}$ is quasi-invariant if for all open bisections U, the measures $(\pi_{U*}\mu)|_{s(U)}$ and $\mu|_{r(U)}$ are equivalent. Here $\pi_U : s(U) \to r(U)$, $\pi_U(x) = r(s^{-1}(x))$ and $(\pi_{U*}\mu)(B) = \mu(\pi_U^{-1}(B))$ for $B \subset G^{(0)}$ a Borel set (see Definition 2.3.8 and Exercise 2.3.9 in [31]).

Remark 3.3. Recall from [25, Proposition I.3.3] that if $\Delta_\mu : G \to (0, \infty)$ is a Radon-Nikodym derivative such that

$$\int_{G^{(0)}} \int_{G^u} f(g)\, d\lambda^u(g) d\mu(u) = \int_{G^{(0)}} \int_{G_u} f(g)\Delta_\mu(g)\, d\lambda_u(g) d\mu(u),$$

where λ_u is the push forward of λ^u under the inversion map for all $u \in G^{(0)}$, then Δ_μ is a cocycle a.e. Moreover, [24, Theorem 3.2] implies that one can choose Δ_μ to be a strict cocycle: $\Delta_\mu(gh) = \Delta_\mu(g)\Delta_\mu(h)$ for all $(g, h) \in G^{(2)}$ and $\Delta_\mu(g^{-1}) = \Delta_\mu(g)^{-1}$ for all $g \in G$.

Example 3.4. Let $\sigma : \mathbb{T} \to \mathbb{T}$ be $\sigma(z) = z^d$ for $d \geq 2$ an integer. Let $G(\mathbb{T}, \sigma)$ be the associated Renault-Deaconu groupoid, isomorphic to the groupoid of germs of the pseudogroup generated by σ. Then the Haar measure μ on $\mathbb{T}$ is quasi-invariant and $\Delta_\mu(z, k-l, w) = (1/d)^{k-l}$ for all $(z, k-l, w) \in G(\mathbb{T}, \sigma)$. This fact follows from [14, §4.5] and [31, Proposition 3.4.1].

We assume now that the topological groupoid G acts on the space X via $\omega : X \to G^{(0)}$. Recall that if μ' is a finite nontrivial measure on X then there is a probability measure μ on X such that $\mu' \sim \mu$, i.e. they have the same nullsets.

Definition 3.5. Suppose that μ is a Radon probability measure on X and let $\tilde{\mu} := \omega_*(\mu)$ on $G^{(0)}$. That is, $\tilde{\mu}(B) = \mu(\omega^{-1}(B))$ for all Borel sets $B \subset G^{(0)}$. A decomposition of μ relative to ω is a disintegration of μ along ω (see, for example, [35, §I.2]):

(1) $\operatorname{supp} \mu_u \subseteq X_u = \omega^{-1}(u)$ for $\tilde{\mu}$-a.e. $u \in G^{(0)}$ and
(2) for all bounded Borel functions f on X, the map $u \mapsto \int_{X_u} f(x)\, d\mu_u(x)$ is bounded and Borel on $G^{(0)}$ and
$$\int_X f(x)\, d\mu(x) = \int_{G^{(0)}} \int_{X_u} f(x)\, d\mu_u(x)\, d\tilde{\mu}(u).$$

Definition 3.6. Let μ be a Radon probability measure on X. We say that μ is G-quasi-invariant for the action of G on X if it admits a decomposition $\{\mu_u\}$ relative to ω such that both of the following conditions hold:

(1) For $\tilde{\mu} \circ \lambda$-a.e. $g \in G$, the measure $g\mu_{s(g)}$ is equivalent with $\mu_{r(g)}$, where $g\mu_{s(g)}(B) := \mu_{s(g)}(g^{-1}B)$ for any Borel set $B \subseteq X_{r(g)}$; and
(2) the measure $\tilde{\mu} = \omega_*(\mu)$ on $G^{(0)}$ is quasi-invariant for the groupoid G.

Both of the two conditions in the definition are needed as the following examples show.

Example 3.7. (1) Assume that G is a locally compact *group* acting on a locally compact Hausdorff space. Therefore $G^{(0)} = \{e\}$ and $\omega(x) = e$ for all $x \in X$. Let μ be a probability measure on X. Then $\tilde{\mu} = \delta_e$, the point mass at e, and $\mu_e = \mu$. Hence μ is G-quasi-invariant in the sense of Definition 3.6 if and only if it is G-quasi-invariant in the classical sense: the measure $g\mu$ is equivalent to μ for all $g \in G$, where $g\mu(B) = \mu(g^{-1}B)$.
(2) Assume that G is a locally compact Hausdorff groupoid that acts on its unit space $X = G^{(0)}$ as in Example 2.5. Thus $\omega(x) = x$ for all $x \in G^{(0)}$, $\tilde{\mu} = \mu$, $X_x = \{x\}$, and, hence, $\mu_x = \delta_x$. Therefore a measure μ on $G^{(0)}$ is G-quasi-invariant in the sense of Definition 3.6 if and only if μ is a quasi-invariant measure for G in the usual sense.

Renault defined in [32, Definition 2.2] a G-quasi-invariant measure to be a quasi-invariant measure for the action groupoid. The following theorem proves that the two definitions are equivalent. For the case of Borel groupoids, this result is Corollary 5.3.11 in [3] and a similar result appears in [32, Proposition 3.1].

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Theorem 3.8. *If the groupoid G acts on X, then a measure μ on X is G-quasi-invariant iff μ is quasi-invariant for the action groupoid $G \ltimes X$ with unit space X.*

Proof. Assume that μ is a G-quasi-invariant measure on X. For $g \in G$, let $D(g, \cdot)$ be the Radon-Nikodym derivative $d(g\mu_{s(g)})/d\mu_{r(g)}$. Note that since μ is a G-quasi-invariant measure and $\{\lambda^u\}$ is a Haar system on G, $(G \ltimes X, \mu \circ \overline{\lambda})$ is a measured groupoid (see [22],[24]); the Haar system $\overline{\lambda}$ on $G \ltimes X$ was introduced in Remark 2.8. Therefore, using virtually the same arguments as in the proof of [35, Corollary D.34], we can choose D to be Borel and $D(g_1g_2, x) = D(g_1, g_2 \cdot x)D(g_2, x)$ for all $(g_1, g_2) \in G^{(2)}$ and μ-almost all x. Let $\Delta_{\tilde{\mu}}$ be the modular function associated with $\tilde{\mu} = \omega_*(\mu)$ and set $\Delta_\mu(g, x) := D(g, x)\Delta_{\tilde{\mu}}(g)$ for all $(g, x) \in G \ltimes X$. Let $\overline{\nu} := \mu \circ \overline{\lambda}$. We prove that $\overline{\nu} \sim \overline{\nu}^{-1}$ and that a Radon-Nikodym derivative is given by $\Delta_\mu(g, x)$. Let $f)$ be a bounded Borel function on $G \rtimes X$. We have

$$\begin{aligned}\int_{G\ltimes X} f(g,x)\,d\overline{\nu}(g,x) &= \int_X \int_{G^{\omega(x)}} f(g, g^{-1}\cdot x)\,d\lambda^{\omega(x)}(g)\,d\mu(x)\\ &= \int_{G^{(0)}}\int_{X_u}\int_{G^u} f(g, g^{-1}\cdot x)\,d\lambda^u(g)\,d\mu_u(x)\,d\tilde{\mu}(u)\end{aligned}$$

which, by Fubini's theorem,

$$= \int_{G^{(0)}}\int_{G^u}\int_{X_u} f(g, g^{-1}\cdot x)\,d\mu_u(x)\,d\lambda^u(g)\,d\tilde{\mu}(u)$$

which, since $g\mu_{s(g)} \sim \mu_{r(g)}$,

$$= \int_{G^{(0)}}\int_{G^u}\int_{X_{s(g)}} f(g, x)D(g,x)\,d\mu_{s(g)}(x)\,d\lambda^u(g)\,d\tilde{\mu}(u)$$

which, since $\tilde{\mu}$ is quasi-invariant for G,

$$= \int_{G^{(0)}}\int_{G_u}\int_{X_u} f(g, x)D(g,x)\,d\mu_u(x)\Delta_{\tilde{\mu}}(g)\,d\lambda_u(g)\,d\tilde{\mu}(u)$$

which, using Fubini's theorem again,

$$\begin{aligned}&= \int_{G^{(0)}}\int_{X_u}\int_{G_u} f(g, x)D(g,x)\Delta_{\tilde{\mu}}(g)\,d\lambda_u(g)\,\mu_u(x)\,d\tilde{\mu}(u)\\ &= \int_X\int_{G_u} f(g,x)D(g,x)\Delta_{\tilde{\mu}}(g)\,d\lambda_u(g)\,d\mu(x)\\ &= \int_{G\ltimes X} f(g,x)\Delta_\mu(g,x)\,d\overline{\nu}^{-1}(g,x).\end{aligned}$$

Thus μ is quasi-invariant for $G \ltimes X$.

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Assume now that μ is a quasi-invariant measure on X for the action groupoid $G \ltimes X$ and let $\Delta_\mu(g,x)$ be the associated Radon-Nikodym derivative. Using [35, Theorem I.5] we disintegrate μ with respect to $\tilde{\mu} = \omega_*(\mu)$,

$$\int_X f(x)\, d\mu(x) = \int_{G^{(0)}} \int_{X_u} f(x)\, d\mu_u(x)\, d\tilde{\mu}(u),$$

where $\{\mu_u\}$ is a family of Radon probability measures with $\operatorname{supp} \mu_u \subseteq X_u$ for all f bounded Borel functions on X. We prove first that $\tilde{\mu}$ is a quasi-invariant measure for G. Let f be a bounded Borel function on G. Then

$$\tilde{\mu} \circ \lambda(f) = \int_{G^{(0)}} \int_{G^u} f(g)\, d\lambda^u(g)\, d\tilde{\mu}(u) = \int_X \int_{G^{\omega(x)}} f(g)\, d\lambda^{\omega(x)}(g)\, d\mu(x)$$

which by the quasi-invariance of μ

$$= \int_X \int_{G_{\omega(x)}} f(g)\Delta_\mu(g,x)\, d\lambda_{\omega(x)}(g)\, d\mu(x)$$

$$= \int_{G^{(0)}} \int_{X_u} \int_{G_u} f(g)\Delta_\mu(g,x)\, d\lambda_u(x)\, d\mu_u(x)\, d\tilde{\mu}(u)$$

which by Fubini's theorem

$$= \int_{G^{(0)}} \int_{G_u} f(g) \left(\int_{X_u} \Delta_\mu(g,x)\, d\mu_u(x) \right) d\lambda_u(g)\, d\tilde{\mu}(u)$$

which, by defining $\Delta_{\tilde{\mu}}(g) := \int_{X_{s(g)}} \Delta_\mu(g,x)\, d\mu_{s(g)}(x)$,

$$= \int_{G^{(0)}} \int_{G_u} f(g)\Delta_{\tilde{\mu}}(g)\, d\lambda_u(g)\, d\tilde{\mu}(u) = \int_G f(g)\Delta_{\tilde{\mu}}(g)\, d(\tilde{\mu} \circ \lambda)^{-1}(g).$$

Therefore $\tilde{\mu} \circ \lambda \sim (\tilde{\mu} \circ \lambda)^{-1}$ and, thus, $\tilde{\mu}$ is quasi-invariant for G.

Let $g \in G$. Then the set $U := \{(g,x) : x \in X_{s(g)}\}$ is a measurable bisection with respect to $\overline{\nu} = \mu \circ \overline{\lambda}$. Note that $s(U) = X_{s(g)}$ and $r(U) = X_{r(g)}$. Using the fact that $\{\lambda^u\}$ is a Haar system for G, one can check that $\overline{\nu}$ is quasi-invariant under U in the sense of [25, Definition I.3.18 i)]. Since μ is quasi-invariant for $G \ltimes X$, Proposition I.3.20 of [25] implies that μ is quasi-invariant under U in the sense of [25, Definition I.3.18 ii)]. Thus, by definition, $g\mu_{s(g)} \sim \mu_{r(g)}$. □

Definition 3.9. Given a groupoid G with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$, if μ is any Radon measure on $G^{(0)}$, then the regular representation on μ, denoted $\operatorname{Ind}\mu$, acts on $L^2(G, \nu^{-1})$ via

$$\operatorname{Ind}\mu(f)(\xi)(g) = \int_G f(h)\xi(h^{-1}g)\, d\lambda^{r(g)}(h) \tag{3.0.1}$$

for all $f \in C_c(G)$, $\xi \in L^2(G, \nu^{-1})$, and $g \in G$ ([25, Definition II.1.8]; see also [36, Proposition 1.41]).

If $f \in C_c(G)$, then its reduced norm is

$$\|f\|_r := \sup\{\, \|\operatorname{Ind}\delta_u(f)\| \,:\, u \in G^{(0)}\,\}, \tag{3.0.2}$$

where δ_u is the point mass at $u \in G^{(0)}$. The reduced C^*-algebra of G, $C_r^*(G)$, is the completion of $C_c(G)$ under the reduced norm. If μ is any Radon measure on $G^{(0)}$ with full support then $\|f\|_r = \|(\operatorname{Ind}\mu)(f)\|$ for all $f \in C_c(G)$ ([36, Corollary 5.23]).

Recall ([25, Definition II.1.6]; see also [36, Definition 7.7],[26, 27]) that a unitary representation of a groupoid G with Haar system $\lambda = \{\lambda^u\}_{u\in G^{(0)}}$ is a triple $L := (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ consisting of a quasi-invariant measure μ on $G^{(0)}$, a Borel Hilbert bundle $G^{(0)} * \mathcal{H}$ over $G^{(0)}$, and a Borel homomorphism $\hat{L} : G \to \operatorname{Iso}(G^{(0)} * \mathcal{H})$ such that $\hat{L}(g) = (r(g), L_g, s(g))$ and $L_g : \mathcal{H}(s(g)) \to \mathcal{H}(r(g))$ is a Hilbert space isomorphism. Here $\mathcal{H}(u)$ denotes the fiber over $u \in G^{(0)}$.

Given a Borel Hilbert bundle $G^{(0)} * \mathcal{H}$ and a measure μ on $G^{(0)}$, we can define the Hilbert space

$$L^2(G^{(0)} * \mathcal{H}, \mu) = \{f \in B(G^{(0)} * \mathcal{H}) \,:\, u \mapsto \|f(u)\|^2_{\mathcal{H}(u)} \text{ is } \mu-\text{integrable}\},$$

where $B(G^{(0)} * \mathcal{H})$ is the set of Borel sections of the bundle (see [36, Section 3.5] for an outline of Borel bundles and [35, Appendix F] for a detailed study of them).

Given a unitary representation $L = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ of G there is an I-norm bounded representation L of $C_c(G)$ on $L^2(G^{(0)} * \mathcal{H}, \mu)$ via the vector-valued integral

$$L(f)\xi(u) = \int_G f(g) L_g \xi(s(g)) \Delta_\mu(g)^{-1/2}\, d\lambda^u(g)$$

for $\xi \in L^2(G^{(0)} * \mathcal{H}, \mu)$, where Δ_μ is the modular function defined by μ. The representation L is called the *integrated form* of the unitary representation (see, for example, [36, Definition 7.14]). Moreover, by the powerful disintegration theorem of Renault ([26]; see also [36, Theorem 8.2]) any such representation of $C_c(G)$ is equivalent to the integrated form of a unitary representation of the second countable groupoid G.

The integrated form of a groupoid representation extends to a representation of the C^*-algebra of the groupoid ([36, Corollary 8.5]).

Remark 3.10. Two unitary representations $L = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ and $L' = (\mu, G^{(0)} * \mathcal{H}', \hat{L}')$ of G having the same quasi-invariant measure μ are equivalent, $L \cong L'$, if $G^{(0)} * \mathcal{H}$ and $G^{(0)} * \mathcal{H}'$ are isomorphic as Hilbert bundles (see, for example, [35, Definition F.22]) via a Borel bundle map $U : G^{(0)} * \mathcal{H} \to G^{(0)} * \mathcal{H}'$ which intertwines L and L' in the sense that $U(r(g)) \circ L_g = L'_g \circ U(s(g))$ for all $g \in G$ ([25, Definition II.1.6]). Recall that U is determined by a family of unitaries $U(u) : \mathcal{H}(u) \to \mathcal{H}'(u)$ for all $u \in G^{(0)}$.

Given two unitary representations $L = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ and $L' = (\mu, G^{(0)} * \mathcal{H}', \hat{L}')$, we can construct their direct sum $L \oplus L' = (\mu, G^{(0)} * (\mathcal{H} \oplus \mathcal{H}'), \widehat{L \oplus L'})$ and their tensor product $L \otimes L' = (\mu, G^{(0)} * (\mathcal{H} \otimes \mathcal{H}'), \widehat{L \otimes L'})$ by taking

$$\widehat{L \oplus L'}(g) = (r(g), L_g \oplus L'_g, s(g)), \quad \widehat{L \otimes L'}(g) = (r(g), L_g \otimes L'_g, s(g)),$$

where the direct sums and the tensor products of the Hilbert bundles are done fiberwise.

Example 3.11. The trivial representation $\iota = (\mu, G^{(0)} \times \mathbb{C}, \hat{\iota})$ on μ, where μ is a quasi-invariant measure, $G^{(0)} \times \mathbb{C}$ is the trivial one-dimensional line bundle and $\iota_g(z) = z$ for all $z \in \mathbb{C}$. Note that $L \otimes \iota \cong L$ for all unitary representations L of G with the same quasi-invariant measure μ.

Its integrated form acts on $L^2(G^{(0)}, \mu)$ via

$$\iota(f)(\xi)(u) = \int_G f(\gamma)\xi(s(\gamma))\Delta_\mu(\gamma)^{-1/2}\, d\lambda^u(\gamma)$$

for all $f \in C_c(G)$ and $\xi \in L^2(G^{(0)}, \mu)$.

Example 3.12. Assume that μ is a quasi-invariant measure on $G^{(0)}$. Let $L^2(G, \lambda) := \{L^2(G^u, \lambda^u)\}_{u \in G^{(0)}}$. The (left) regular representation ρ of G on μ is the unitary representation $(\mu, G^{(0)} * L^2(G, \lambda), \hat{\rho})$, where

$$\rho_g : L^2(G^{s(g)}, \lambda^{s(g)}) \to L^2(G^{r(g)}, \lambda^{r(g)})$$

is defined via $\rho_g(\xi)(h) = \xi(g^{-1}h)$ for all $\xi \in L^2(G^{s(g)}, \lambda^{s(g)})$ and $h \in G^{r(g)}$. Even though in general ρ depends on μ, to ease the notation we write ρ instead of ρ^μ, especially when the measure μ is fixed.

Its integrated form is called the (left) regular representation of $C_c(G)$ on μ and it is unitarily equivalent with $\operatorname{Ind}\mu$ defined in (3.0.1) via $W : L^2(G, \nu) \to L^2(G, \nu^{-1})$, $W\xi = \xi\Delta_\mu^{1/2}$ ([25, Proposition II.1.10]; see also [19, Definition 3.29 and Exercise 3.30]). Therefore, if μ has full support, $\|f\|_r = \|\rho(f)\|$ for all $f \in C_c(G)$.

Recall that if A is a C^*-algebra, π is a representation of A and S is a set of representations of A, the following assertions are equivalent:

(1) $\ker \pi \supseteq \bigcap\{\ker \sigma \mid \sigma \in S\}$;

(2) each vector state associated with π is a weak-$*$ limit of states that are sums of vector functionals associated to representations in S.

If either assertion holds, we say that π is weakly contained in S and write $\pi \prec S$. If $S = \{\sigma\}$ has only one element, we say that π is weakly contained in σ and write $\pi \prec \sigma$. In this case there is a surjective homomorphism $C^*(\sigma) \to C^*(\pi)$ given by $\sigma(a) \mapsto \pi(a)$, where $C^*(\pi)$ is the C^*-algebra generated by $\pi(a)$ for $a \in A$. We say that π and σ are weakly equivalent if and only if $\pi \prec \sigma$ and $\sigma \prec \pi$; this happens if and only if $\ker \pi = \ker \sigma$, and in this case $C^*(\pi) \cong C^*(\sigma)$.

A unitary groupoid representation π is wekly contained in a unitary representation σ if the integrated form of π is weakly contained in the integrated form of σ.

The following definitions and results about amenability are taken from [3] and [4]; see also chapter 9 in [36].

Definition 3.13. (see Definition 2.6 in [4])

Let G be a locally compact groupoid with Haar system $\{\lambda^u\}$. A quasi-invariant measure μ on $G^{(0)}$ is amenable if there is a net $\{f_i\}$ of non-negative measurable functions on G such that

(1) For all i and a.e. $u \in G^{(0)}$ we have $\displaystyle\int_G f_i d\lambda^u = 1$;

(2) The functions $g \mapsto \displaystyle\int_G |f_i(g^{-1}h) - f_i(h)| d\lambda^{r(g)}(h)$ tend to the zero function in the weak-$*$-topology of $L^\infty(G, \mu \circ \lambda)$.

The groupoid is called measurewise amenable in case each quasi-invariant measure on $G^{(0)}$ is amenable ([3, §3.3]).

Definition 3.14. ([3, Definition 2.2.2]) We say that a locally compact groupoid G (maybe without a Haar system) is topologically amenable if it admits a continuous approximate invariant mean, i.e. a net $\{m_i\}$ of continuous systems of probability measures for r which is approximately invariant, in the sense that the function $g \mapsto \|g m_i^{s(g)} - m_i^{r(g)}\|_1$ tends to zero uniformly on the compact subsets of G, where $\|\cdot\|_1$ denotes the total variation norm.

When G admits a continuous Haar system $\{\lambda^u\}$, the following Proposition follows from [3, Proposition 2.2.6] since the action of G on itself is proper.

Proposition 3.15. *A locally compact groupoid G with Haar system $\{\lambda^u\}$ is topologically amenable if and only if there exists a net $\{f_i\}$ of non-negative compactly supported continuous functions on G such that*

(i) For all i and $u \in G^{(0)}$ we have $\displaystyle\int_G f_i d\lambda^u = 1$;

(ii) The functions $g \mapsto \displaystyle\int_G |f_i(g^{-1}h) - f_i(h)| d\lambda^{r(g)}(h)$ tend to the zero function uniformly on the compact sets of G.

Topological amenability implies measurewise amenability ([3, Proposition 3.3.5]). The converse is true under some additional hypotheses. For example, [3, Theorem 3.3.7] proves that topological amenability is equivalent to measurewise amenability if G has a continuous Haar system and has countable orbits.

Example 3.16. The Renault-Deaconu groupoid $G(X,T)$ constructed from a local homeomorphism $T : X \to X$ as in Example 2.2 is topologically amenable, see [29, Proposition 2.9] and [33, Proposition 3.1].

Note that amenability for groupoids is equivalent to the weak containment of the trivial representation $\iota = (\mu, G^{(0)} \times \mathbb{C}, \hat{\iota})$ in the regular representation

$\rho = (\mu, G^{(0)} * L^2(G,\lambda), \hat{\rho})$, see Proposition 3.4 in [28]. In particular, for amenable groupoids there is a surjective homomorphism $C^*_r(G) \to C^*(\iota)$.

4. Inducing representations from $G \ltimes X$ to G and the Koopman representation

Assume now that the groupoid G acts (on the left) on X and let $G \ltimes X$ be the action groupoid. We assume as in the previous section that $\lambda = \{\lambda^u\}_{u\in G^{(0)}}$ is a Haar system on G and the corresponding Haar system on $G \ltimes X$ is denoted by $\overline{\lambda} = \{\overline{\lambda}^x\}_{x\in X}$ (see Remark 2.8). Following the well known case of the group action groupoid (see, for example, [1, Page 5] for group actions and [27, Page 17] for groupoid dynamical systems) we define the induction map from unitary representations of $G \ltimes X$ to unitary representations of G. Let $L = (\mu, X * \mathcal{H}, \hat{L})$ be a unitary representation of $G \ltimes X$ and let $\tilde{\mu} = \omega_*(\mu)$. Since μ is quasi-invariant for $G \ltimes X$, Theorem 3.8 implies that $\tilde{\mu}$ is quasi-invariant for G and there is a decomposition of μ relative to ω such that $g\mu_{s(g)} \sim \mu_{r(g)}$. Let $D : G \ltimes X \to \mathbb{R}^+$ be a Borel choice of the Radon-Nikodym derivative $\dfrac{dg\mu_{s(g)}}{d\mu_{r(g)}}$ such that $D(g_1g_2, x) = D(g_1, g_2\cdot x)D(g_2, x)$ for all $(g_1, g_2) \in G^{(2)}$ and μ-almost all x (see the proof of Theorem 3.8). For each $u \in G^{(0)}$ define

$$\mathcal{K}(u) := L^2(X_u, \mu_u) := \int_{X_u}^{\oplus} \mathcal{H}(x)\, d\mu_u(x),$$

where $X_u = \omega^{-1}(u)$. The Borel structure on $\mathcal{H}$ defines a natural Borel structure on $\mathcal{K} := \{\mathcal{K}(u)\}_{u\in G^{(0)}}$ making $G^{(0)} * \mathcal{K}$ a Borel Hilbert bundle. Specifically, let $\{f_n\}$ be a fundamental sequence for $X * \mathcal{H}$ (see, for example, [35, Definition F.1]). Then one can define a sequence $\{g_n\}$ of sections of $G^{(0)} * \mathcal{K}$ via $g_n(u)(x) = f_n(x)$. This sequence satisfies the hypotheses of [35, Proposition F.8] and, thus, there is a unique analytic Borel structure on $G^{(0)} * \mathcal{K}$ such that $G^{(0)} * \mathcal{K}$ becomes an analytic Hilbert bundle and $\{g_n\}$ is a fundamental sequence.

Definition 4.1. The *induced representation* of a unitary representation $L = (\mu, X * \mathcal{H}, \hat{L})$ of $G \ltimes X$ to G is the unitary representation $\operatorname{Ind} L = (\tilde{\mu}, G^{(0)} * \mathcal{K}, \operatorname{Ind}\hat{L})$ of G, where $\operatorname{Ind}\hat{L} : G \to \operatorname{Iso}(G^{(0)} * \mathcal{K})$, $\operatorname{Ind}\hat{L}_g = (r(g), \operatorname{Ind} L_g, s(g))$ and, for $g \in G$, $\operatorname{Ind} L_g : \mathcal{K}(s(g)) \to \mathcal{K}(r(g))$ is defined via

$$\operatorname{Ind} L_g\xi(x) = D(g^{-1}, x)^{1/2} L_{(g,g^{-1}\cdot x)}\big(\xi(g^{-1}\cdot x)\big)$$

for all $\xi \in \mathcal{K}(s(g))$.

Remark 4.2. Note that the above Definition can be deduced from [32, Sections 3.2 and 3.3] with a bit of effort. Indeed, let $Z = G \ltimes X$ viewed as a topological space. Then G acts properly on the left on Z via the natural action, and $G \ltimes X$ acts properly and freely on the right on Z, since the action of any groupoid on itself is free and proper. Moreover, one can check that Z is a groupoid correspondence in the sense of Holkar (see, for example, Definition 2.3 of [32]), with the cocycle Δ in the definition being the Radon-Nikodym derivative on $G \ltimes X$ corresponding to μ and the system of measures $\alpha = \{\alpha_u\}$ given by $\alpha_u = \lambda_u$ for all $u \in G^{(0)}$. Then, following the steps in Section 3.2 and 3.3 of [32], one can recover our Definition 4.1.

As discussed above, the induced representation of G extends to an I-bounded representation $\operatorname{Ind} L : C_c(G) \to \mathcal{B}(L^2(G^{(0)} * \mathcal{K}, \tilde{\mu}))$ via the vector integral

$$\operatorname{Ind} L(f)\xi(u) = \int_{G^u} f(g) \operatorname{Ind} L_g \xi(s(g)) \Delta_{\tilde{\mu}}(g)^{-1/2} \, d\lambda^u(g)$$

for all $\xi \in L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$. Equivalently, the induced representation is characterized by (see [36, Proposition 7.12])

$$\begin{aligned}
&\langle \operatorname{Ind} L(f)\xi \,,\, \eta \rangle \\
&\quad = \int_{G^{(0)}} \int_{G^u} f(g) \big\langle \operatorname{Ind} L_g \xi(s(g)) \,,\, \eta(r(g)) \big\rangle \Delta_{\tilde{\mu}}(g)^{-1/2} \, d\lambda^u(g) \, d\tilde{\mu}(u) \\
&= \int_{G^{(0)}} \int_{G^u} f(g) \int_{X_u} D(g^{-1}, x)^{1/2} \big\langle L_{(g, g^{-1}\cdot x)} \xi(s(g))(g^{-1} \cdot x) \,,\, \eta(r(g))(x) \big\rangle \\
&\qquad\qquad d\mu_u(x) \Delta_{\tilde{\mu}}(g)^{-1/2} \, d\lambda^u(g) \, d\tilde{\mu}(u) \\
&= \int_{G^{(0)}} \int_{X_u} \int_{G^u} f(g) \Delta_\mu(g^{-1}, x)^{1/2} \big\langle L_{(g, g^{-1}\cdot x)} \xi(s(g))(g^{-1} \cdot x) \,,\, \eta(r(g))(x) \big\rangle \\
&\qquad\qquad d\lambda^u(g) d\mu_u(x) d\tilde{\mu}(u) \\
&= \int_X \int_{G^{\omega(x)}} f(g) \Delta_\mu(g^{-1}, x)^{1/2} \big\langle L_{(g, g^{-1}\cdot x)} \xi(s(g))(g^{-1} \cdot x) \,,\, \eta(r(g))(x) \big\rangle \\
&\qquad\qquad d\lambda^{\omega(x)}(g) \, d\mu(x),
\end{aligned}$$

for all $\xi, \eta \in L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$, where $\langle \cdot \,,\, \cdot \rangle$ represent the inner products in the corresponding Hilbert spaces and $\Delta_\mu(g^{-1}, x) = D(g^{-1}, x)\Delta_{\tilde{\mu}}(g^{-1})$ is the Radon-Nikodym derivative on $G \ltimes X$ corresponding to μ.

Definition 4.3. Assuming that the groupoid G acts on X, let μ be a G-quasi-invariant measure on X or, equivalently, a quasi-invariant measure for $G \ltimes X$. We define the Koopman representation κ^μ of G to be the induced representation of the trivial representation $\iota = (\mu, X \times \mathbb{C}, \hat{\iota})$ of $G \ltimes X$, where, recall from Example 3.11, $\iota_{(g,x)}(z) = z$ for all $(g, x) \in G \ltimes X$ and $z \in \mathbb{C}$. Since the measure μ is typically fixed, we write shortly κ for κ^μ when there is no possibility for confusion. In general, the relationship between κ and μ is complicated, and we plan to address this issue in a future project. In the last section, we illustrate this relationship in some particular examples.

Therefore the Koopman representation κ is given by $(\tilde{\mu}, G^{(0)} * \mathcal{K}, \hat{\kappa})$, where $\mathcal{K} = \{L^2(X_u, \mu_u)\}_{u \in G^{(0)}}$ and, for $g \in G$,

$$\kappa_g : L^2(X_{s(g)}, \mu_{s(g)}) \to L^2(X_{r(g)}, \mu_{r(g)})$$

is given by

$$\kappa_g \xi(x) = D(g^{-1}, x)^{1/2} \xi(g^{-1} \cdot x),$$

which recovers the classical definition for group actions (see Definition 13.A.5 in [5] for example).

Hence κ extends to an I-bounded representation of $C_c(G)$ on $L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$ via

$$\kappa(f)\xi(u) = \int_{G^u} f(g) \kappa_g \xi(s(g)) \Delta_{\tilde{\mu}}(g)^{-1/2} \, d\lambda^u(g).$$

Note that we can identify $L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$ with $L^2(X, \mu)$ via the unitary $V : L^2(G^{(0)} * \mathcal{K}, \tilde{\mu}) \to L^2(X, \mu)$, $V(\xi)(x) = \xi(\omega(x))(x)$. Therefore we can view the Koopman representation as a representation of $C_c(G)$ on $L^2(X, \mu)$ via

$$\begin{aligned} \kappa(f)\xi(x) &= \int_{G^{\omega(x)}} f(g) \kappa_g(\xi)(x) \Delta_{\tilde{\mu}}(g)^{-1/2} \, d\lambda^{\omega(x)}(g) \\ &= \int_{G^{\omega(x)}} f(g) \xi(g^{-1} \cdot x) \Delta_\mu(g^{-1}, x)^{1/2} \, d\lambda^{\omega(x)}(g), \end{aligned}$$

where recall that $\Delta_\mu(g^{-1}, x) = D(g^{-1}, x) \Delta_{\tilde{\mu}}(g^{-1})$. Equivalently, κ is characterized by

$$\begin{aligned} \langle \kappa(f)\xi \, , \, \eta \rangle &= \int_X \int_{G^{\omega(x)}} f(g) \xi(g^{-1} \cdot x) \overline{\eta(x)} \Delta_\mu(g^{-1}, x)^{1/2} d\lambda^{\omega(x)}(g) d\mu(x) \\ &= \int_{G^{(0)}} \int_{X_u} \int_{G^u} f(g) \xi(g^{-1} \cdot x) \overline{\eta(x)} \Delta_\mu(g^{-1}, x)^{1/2} d\lambda^u(g) d\mu_u(x) d\tilde{\mu}(u) \end{aligned}$$

for all $\xi, \eta \in L^2(X, \mu)$.

We denote by $C^*(\kappa)$ the closure of $\kappa(C_c(G))$ in the operator norm of $\mathcal{B}(L^2(X, \mu))$.

Example 4.4. Let (G, E) be a level transitive self-similar groupoid action (see [8]) such that $|uE^1| = p \geq 2$ is constant for all $u \in E^0$. Then G acts on $X = E^\infty$ and the uniform probability measure ν on E^∞ is G-invariant. Then the C^*-algebra $C^*(\kappa)$ of the Koopman representation of G on $L^2(X, \nu)$ is residually finite dimensional and it has a normalized trace τ_0. For G an amenable group and for an essentially free self-similar action, it is proved in Theorem 9.14 of [10] that $C^*(\kappa) \cong C^*_r(G)$. We believe that this isomorphism holds true for level transitive self-similar amenable groupoid actions.

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Remark 4.5. If $X = G^{(0)}$ and G acts on X via $g \cdot s(g) = r(g)$, then, since $G \ltimes G^{(0)} \cong G$, the Koopman representation κ^μ of G associated to a G-quasi-invariant measure μ on $G^{(0)}$ is given by the trivial representation $\iota = (\mu, G^{(0)} \times \mathbb{C}, \hat{\iota})$.

If $X = G$ and G acts on itself by left multiplication, then assuming that μ has full support, the Koopman representation κ^μ is just the left regular representation $\rho = \rho^G$. If $X = G/H$ where H is a closed subgroupoid, the Koopman representation is the quasi-regular representation $\rho^{G/H}$.

Remark 4.6. Given a closed subgroupoid H of G with the same unit space such that H is endowed with a Haar system β and G is endowed with a Haar system λ, recall that G acts on G/H with $\omega : G/H \to G^{(0)}, \omega(gH) = r(g)$. Given a unitary representation $L = (\nu, H^{(0)} * \mathcal{K}, \hat{L})$ of (H, β), one can induce it to a representation of (G, λ) following the steps in [32, Section 3]. Specifically, under the assumption that $H^{(0)} = G^{(0)}$, one can define a groupoid correspondence from (G, λ) to (H, β) in the sense of [32, Definition 2.3] by setting $X = G$ and $\alpha_u = \lambda_u$ for all $u \in G^{(0)}$. Therefore, ν defines a G-quasi-invariant measure μ on G/H as in [32, Section 3.2]. Let $\tilde{\mu}$ and $\{\mu_u\}_{u \in G^{(0)}}$ as in Definition 3.6. Then the induced representation $\operatorname{Ind}_H^G L$ of G is $(\tilde{\mu}, G^{(0)} * \mathcal{H}, \operatorname{Ind}_H^G \hat{L})$, where $\mathcal{H}$ is the Hilbert bundle obtained from the completion of

$$\{\xi : G \to \mathcal{K} : \xi(g) \in \mathcal{K}(s(g)) \text{ and } \xi(gh) = L_{h^{-1}}\xi(g)\}$$

and

$$(\operatorname{Ind}_H^G L)_g \xi(x) = D(g^{-1}, xH)^{1/2} \xi(g^{-1}x).$$

We have $\mathcal{H}(u) = L^2((G/H)_u * \mathcal{K}, \mu_u) = \int_{(G/H)_u}^{\oplus} \mathcal{K}(x) d\mu_u(x)$, where $(G/H)_u = \{gH \in G/H : r(g) = u\}$. The induced representation of a direct sum is the direct sum of induced representations.

Example 4.7. If $H = G^{(0)}$ and $\iota = (\mu, G^{(0)} \times \mathbb{C}, \hat{\iota})$ is the trivial representation of H with $\iota_u(z) = z$, then $\operatorname{Ind}_H^G \iota$ is the left regular representation ρ^G of G. For a general closed subgroupoid H with a Haar system, $\operatorname{Ind}_H^G \iota$ is the quasi-regular representation $\rho^{G/H}$ of G on $L^2(G/H, \mu)$.

The following result is inspired from the similar result in the case of groups, see [6, Appendix E].

Proposition 4.8. *Suppose H is a closed subgroupoid of G with the same unit space. Let $L = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ be a unitary representation of G and let $M = (\mu, H^{(0)} * \mathcal{K}, \hat{M})$ be a unitary representation of H. Then $L \otimes \operatorname{Ind}_H^G M$ is equivalent to $\operatorname{Ind}_H^G((L|_H) \otimes M)$.*

Proof. If $G^{(0)} * \mathcal{M}$ and $G^{(0)} * \mathcal{L}$ are the Hilbert bundles of $\operatorname{Ind}_H^G M$ and $\operatorname{Ind}_H^G((L|_H) \otimes M)$ respectively, define a Borel bundle map $U : G^{(0)} * (\mathcal{H} \otimes$

$\mathcal{M}) \to G^{(0)} * \mathcal{L}$ by

$$U(u)(\xi \otimes \eta)(x) = L_{x^{-1}}\xi \otimes \eta(x), \ \forall u \in G^{(0)} \text{ and } x \in G_u$$

and verify that $U(u)$ is unitary for each u. Moreover, U intertwines $\operatorname{Ind}((L|_H)\otimes M)$ and $L \otimes \operatorname{Ind}_H^G M$ since

$$\left(\left(\operatorname{Ind}_H^G(L|_H \otimes M)_g\right) U(s(g))(\xi \otimes \eta)\right)(x) = D(g^{-1}, xH)^{1/2} L_{x^{-1}g}\xi \otimes \eta(g^{-1}x) =$$

$$= U(r(g))\left(L_g\xi \otimes (\operatorname{Ind}_H^G M)_g\eta\right)(x).$$

□

Corollary 4.9. *Let G be a locally compact groupoid and let H be a closed subgroupoid with the same unit space. If $L = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ is a representation of G, then $\operatorname{Ind}_H^G(L|_H)$ is equivalent to $L \otimes \rho^{G/H}$, where $\rho^{G/H}$ is the quasi-regular representation of G on $L^2(G/H, \mu)$. In particular, for $H = G^{(0)}$, $L \otimes \rho^G$ is equivalent to $(\dim L) \otimes \rho^G$, where $(\dim L)_u = id_{\mathcal{H}(u)}$.*

Proof. For the first part, we apply Proposition 4.8 for M the trivial representation of H. For the second part, $L|_{G^{(0)}}$ is a direct sum of trivial representations on $G^{(0)} * \mathcal{C}$, where for $u \in G^{(0)}$, $\mathcal{C}(u) = \mathbb{C}^{n(u)}$ if $n(u) = \dim \mathcal{H}(u)$ is finite and $\mathcal{C}(u)$ is infinite dimensional otherwise. □

5. Properties of the Koopman representation

We still assume that the groupoid G acts (on the left) on X and let $G \ltimes X$ denote the action groupoid.

Lemma 5.1. *Let $L = (\mu, X * \mathcal{H}, \hat{L})$ be a unitary representation of $G \ltimes X$. Then for all non-negative $f \in C_c(G)$ we have*

$$\| \operatorname{Ind} L(f)\| \le \|\kappa^\mu(f)\|.$$

Proof. Recall that $\mathcal{K}(u) := L^2(X_u, \mu_u) := \int_{X_u}^{\oplus} \mathcal{H}(x)\, d\mu_u(x)$, where $X_u = \omega^{-1}(u)$. For $\xi \in L^2(G^{(0)} * \mathcal{K})$ define $\tilde{\xi}(x) = \|\xi(\omega(x))(x)\|$. Then $\tilde{\xi} \in L^2(X, \mu)$ and $\|\tilde{\xi}\| = \|\xi\|$.

Let $f \in C_c(G)$ be a non-negative function, and let $\xi, \eta \in L^2(G^{(0)} * \mathcal{K})$. We have

$$|\langle \operatorname{Ind} L(f)\xi\ ,\ \eta\rangle| = \left|\int_{G^{(0)}}\int_{G^u}\int_{X_u} f(g)\Delta_\mu(g^{-1},x)^{1/2}\big\langle L_{(g,g^{-1}\cdot x)}\xi(s(g))(g^{-1}\cdot x)\ ,\ \eta(r(g))(x)\big\rangle \, d\mu_u(x)d\lambda^u(g)d\tilde{\mu}(u)\right|$$
$$\le \int_{G^{(0)}}\int_{G^u}\int_{X_u} f(g)\Delta_\mu(g^{-1},x)^{1/2}\left|\big\langle L_{(g,g^{-1}\cdot x)}\xi(s(g))(g^{-1}\cdot x)\ ,\ \eta(r(g))(x)\big\rangle\right| \, d\mu_u(x)d\lambda^u(g)d\tilde{\mu}(u)$$

which, since $L_{(g,g^{-1}\cdot x)}$ is a Hilbert space isomorphism,

$$\le \int_{G^{(0)}}\int_{G^u}\int_{X_u} f(g)\Delta_\mu(g^{-1},x)^{1/2}\|\xi(s(g))(g^{-1}\cdot x)\|\|\eta(r(g))(x)\| \, d\mu_u(x)d\lambda^u(g)d\tilde{\mu}(u)$$
$$= \int_{G^{(0)}}\int_{G^u}\int_{X_u} f(g)\Delta_\mu(g^{-1},x)^{1/2}\tilde{\xi}(g^{-1}\cdot x)\tilde{\eta}(x)d\mu_u(x)d\lambda^u(g)d\tilde{\mu}(u)$$
$$= \langle \kappa_\mu(f)\tilde{\xi}\ ,\ \tilde{\eta}\rangle \le \|\kappa_\mu(f)\|\|\tilde{\xi}\|\|\tilde{\eta}\| = \|\kappa_\mu(f)\|\|\xi\|\|\eta\|$$

The result follows. □

Theorem 5.2. *With the notation as above, assume that μ is a G-quasi-invariant probability measure on X with full support and let $\tilde{\mu}$ be the push-forward quasi-invariant measure on $G^{(0)}$. Then, for all non-negative $f \in C_c(G)$ we have $\|\operatorname{Ind}\tilde{\mu}(f)\| \le \|\kappa^\mu(f)\|$.*

Proof. Recall from Example 3.12 that $\operatorname{Ind}\tilde{\mu}$ is unitarily equivalent with the integrated form of the unitary representation $\rho = (\tilde{\mu}, G^{(0)} * L^2(\lambda), \hat{\rho})$ of G, where $L^2(\lambda) = \{L^2(G^u,\lambda^u)\}_{u\in G^{(0)}}$, and, for $g \in G$, $\rho_g(\xi)(h) = \xi(g^{-1}h)$ for all $h \in G^{r(g)}$.

Consider the unitary representation $L = (\mu, X * \mathcal{H}, \hat{L})$ of $G \ltimes X$, where $\mathcal{H}(x) = L^2(G^{\omega(x)}, \lambda^{\omega(x)})$ for all $x \in X$, and, for $(g,x) \in G \ltimes X$, $L_{(g,x)} : \mathcal{H}(x) \to \mathcal{H}(g\cdot x)$ is given by

$$L_{(g,x)}(\xi)(h) = \xi(g^{-1}h) \text{ for all } \xi \in \mathcal{H}(x) \text{ and } h \in G^{r(g)}.$$

Let $f \in C_c(G)$ be a non-negative function. By Lemma 5.1, $\|\operatorname{Ind} L(f)\| \le \|\kappa_\mu(f)\|$. We prove next that $\|\rho(f)\| \le \|\operatorname{Ind} L(f)\|$. This implies the result.

By definition, $\operatorname{Ind} L$ is given by $(\tilde{\mu}, G^{(0)} * \mathcal{K}, \operatorname{Ind}\hat{L})$, where

$$\mathcal{K}(u) = \int_{X_u}^{\oplus} L^2(G^u,\lambda^u)\, d\mu_u(x).$$

While $\mathcal{K}(u) \cong L^2(X_u, \mu_u) \otimes L^2(G^u, \lambda^u)$, we prefer to view elements of $\mathcal{K}$ as sections $\xi : X_u \to L^2(G^u, \lambda^u)$ endowed with the norm

$$\|\xi\|^2 = \int_{X_u} \int_{G^u} |\xi(x)(g)|^2 \, d\lambda^u(g) d\mu_u(x).$$

Then, for $g \in G$, $\xi \in \mathcal{K}(s(g))$, $x \in X_{r(g)}$ and $h \in G^{r(g)}$,

$$\begin{aligned}\operatorname{Ind} L_g(\xi)(x)(h) &= D(g^{-1}, x)^{1/2} L_{(g, g^{-1}\cdot x)}(\xi(g^{-1} \cdot x))(h) \\ &= D(g^{-1}, x)^{1/2} \xi(g^{-1} \cdot x)(g^{-1}h),\end{aligned}$$

for all $\xi \in \mathcal{K}(s(g))$. Therefore, for $f \in C_c(G)$, $\operatorname{Ind} L(f)$ acts on $L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$ via

$$\begin{aligned}\langle \operatorname{Ind} L(f)\xi \,,\, \eta\rangle &= \int_{G^{(0)}} \int_{G^u} f(g) \big\langle \operatorname{Ind} L_g \xi(s(g)) \,,\, \eta(r(g))\big\rangle \Delta_{\tilde{\mu}}(g)^{-1/2} \, d\lambda^u(g) d\tilde{\mu}(u) \\ &= \int_{G^{(0)}} \int_{G^u} \int_{X_u} \int_{G^u} f(g)\xi(s(g))(g^{-1} \cdot x)(g^{-1}h)\overline{\eta(u)(x)(h)} \Delta_\mu(g^{-1}, x)^{1/2} \\ &\qquad d\lambda^u(h) \, d\mu_u(x) \, d\lambda^u(g) d\tilde{\mu}(u),\end{aligned}$$

for all $\xi, \eta \in L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$.

Let $f \in C_c(G)$ be a non-negative function and let $\xi, \eta \in L^2(G^{(0)} * L^2(\lambda), \tilde{\mu})$. Then ξ defines an element $\tilde{\xi} \in L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$ via $\tilde{\xi}(u)(x)(h) := \xi(u)(h)$ and $\|\xi\| = \|\tilde{\xi}\|$ since μ_u is a probability measure for all $u \in G^{(0)}$. Similarly η defines $\tilde{\eta} \in L^2(G^{(0)} * \mathcal{K}, \tilde{\mu})$ such that $\|\eta\| = \|\tilde{\eta}\|$. We have

$$\begin{aligned}|\langle \rho(f)\xi \,,\, \eta\rangle| &= \left| \int_{G^{(0)}} \int_{G^u} f(g) \big\langle \rho_g(\xi(s(g))) \,,\, \eta(u)\big\rangle \Delta_{\tilde{\mu}}(g)^{-1/2} \, d\lambda^u(g) \, d\tilde{\mu}(u) \right| \\ &= \left| \int_{G^{(0)}} \int_{G^u} \int_{G^u} f(g)\xi(s(g))(g^{-1}h)\overline{\eta(u)(h)} \Delta_{\tilde{\mu}}(g)^{-1/2} d\lambda^u(h) d\lambda^u(g) d\tilde{\mu}(u) \right|\end{aligned}$$

which, since $\Delta_{\tilde{\mu}}(g^{-1}) = \int_{X_u} \Delta_\mu(g^{-1}, x) d\mu_u(x)$ a.e. and $\Delta_{\tilde{\mu}}$ is a cocycle,

$$\begin{aligned}&= \left| \int_{G^0} \int_{G^u} \int_{X_u} \int_{G^u} f(g)\tilde{\xi}(s(g))(g^{-1} \cdot x)(g^{-1}h)\overline{\tilde{\eta}(u)(x)(h)} \Delta_\mu(g^{-1}, x)^{1/2} \right. \\ &\qquad \left. d\lambda^u(h) d\mu_u(x) d\lambda^u(g) d\tilde{\mu}(u) \right| \\ &= \left| \langle \operatorname{Ind} L(f)\tilde{\xi} \,,\, \tilde{\eta}\rangle \right| \le \|\operatorname{Ind} L(f)\| \|\tilde{\xi}\| \|\tilde{\eta}\| = \|\operatorname{Ind} L(f)\| \|\xi\| \|\eta\|.\end{aligned}$$

It follows that $\|\rho(f)\| \le \|\operatorname{Ind} L(f)\| \le \|\kappa^\mu(f)\|$. □

Theorem 5.3. *Assume that the action groupoid $(G \ltimes X, \overline{\lambda})$ is topologically amenable. Assume also that μ has full support. Then the Koopman representation κ^μ is weakly contained in the left regular representation ρ. In particular, we have a surjection $C^*_r(G) \to C^*(\kappa^\mu)$.*

We will write in the following ρ^G for the left regular representation on $\tilde{\mu}$ of G and $\rho^{G\ltimes X}$ for the left regular representation on μ of $G \ltimes X$. We break the proof of the theorem into two parts. First we prove that the Koopman representation κ is weakly contained into the induced representation $\operatorname{Ind}\rho^{G\ltimes X}$. In the second part we prove that $\|\operatorname{Ind}\rho^{G\ltimes X}(f)\| \leq \|\rho^G(f)\|$ for all $f \in C_c(G)$. This implies the result.

Using [3, Proposition 2.2.7] (see the discussion following Definition 2.6 of [33] for the equivalence between the various definitions of amenability in the σ-compact case), there is a sequence of functions $\{f_n\} \in C_c(G \ltimes X)$ such that the following conditions hold:

$$\int_{G^{\omega(x)}} |f_n(g, g^{-1}x)|^2 \, d\lambda^{\omega(x)}(g) = 1 \quad \text{for all } x \in X \tag{5.0.1}$$

and

$$\lim_{n\to\infty} \int_{G^{r(h)}} |f_n(h^{-1}g, g^{-1}hx) - f_n(g, g^{-1}hx)|^2 \, d\lambda^{r(h)}(g) = 0 \tag{5.0.2}$$

uniformly on compact subsets of $G \ltimes X$.

Proposition 5.4. *Assume the hypotheses of Theorem 5.3. Then the Koopman representation is weakly contained in* $\operatorname{Ind}\rho^{G\ltimes X}$. *Therefore,* $\|\kappa^\mu(f)\| \leq \|\operatorname{Ind}\rho^{G\ltimes X}(f)\|$ *for all* $f \in C_c(G)$.

Proof. The left regular representation $\rho^{G\ltimes X}$ on μ of $G \ltimes X$ is the unitary representation $(\mu, X * L^2(G \ltimes X, \overline{\lambda}), \hat{\rho}^{G\ltimes X})$, where the fiber over $x \in X$ of the Hilbert bundle is $L^2((G \ltimes X)^x, \overline{\lambda}^x)$. We will write $\langle \cdot, \cdot \rangle_x$ for the inner product in the fiber over x. It is useful to keep in mind that

$$(G \ltimes X)^x = \{(g, g^{-1}x) \,:\, g \in G^{\omega(x)}\}.$$

If $(h, x) \in G \ltimes X$ then $\rho^{G\ltimes X}_{(h,x)} : L^2((G \ltimes X)^x, \overline{\lambda}^x) \to L^2((G \ltimes X)^{hx}, \overline{\lambda}^{hx})$ is given by

$$\rho^{G\ltimes X}_{(h,x)}\xi(g, g^{-1}hx) = \xi(h^{-1}g, g^{-1}hx).$$

Therefore if $\xi \in L^2((G \ltimes X)^x, \overline{\lambda}^x)$ and $\eta \in L^2((G \ltimes X)^{hx}, \overline{\lambda}^{hx})$,

$$\langle \rho^{G\ltimes X}_{(h,x)}\xi, \eta\rangle_x = \int_{G^{r(h)}} \xi(h^{-1}g, g^{-1}hx)\overline{\eta(g, g^{-1}hx)} \, d\lambda^{r(h)}(g).$$

Then the induced representation $\operatorname{Ind}\rho^{G\ltimes X}$ is the unitary representation $(\tilde{\mu}, G^{(0)} * \mathcal{L}, \operatorname{Ind}\hat{\rho}^{G\ltimes X})$ of G, where

$$\mathcal{L}(u) = \int_{X_u}^{\oplus} L^2((G \ltimes X)^x, \overline{\lambda}^x) \, d\mu_u(x) \cong L^2(X_u, \mu_u) \otimes L^2(G^u, \lambda^u) \cong \mathcal{K}(u).$$

Thus, if $\xi, \eta \in \mathcal{L}(u)$

$$\begin{aligned}\langle \xi\,,\,\eta\rangle_u &= \int_{X_u} \langle \xi(x), \eta(x)\rangle_x \, d\mu_u(x) \\ &= \int_{X_u}\int_{G^u} \xi(x)(g, g^{-1}x)\overline{\eta(x)(g, g^{-1}x)} \, d\lambda^u(g) \, d\mu_u(x).\end{aligned}$$

If $h \in G$, $\operatorname{Ind}\rho_h^{G\ltimes X} : \mathcal{L}(s(h)) \to \mathcal{L}(r(h))$ is given via

$$(\operatorname{Ind}\rho_h^{G\ltimes X}\xi)(x)(g, g^{-1}x) = D(h^{-1}, x)^{1/2}\xi(h^{-1}x)(h^{-1}g, g^{-1}x)$$

for all $\xi \in \mathcal{L}(s(h))$, $x \in X_{r(h)}$ and $g \in G^{r(h)}$. Therefore, if $\xi \in \mathcal{L}(s(h))$ and $\eta \in \mathcal{L}(r(h))$ we have

$$\begin{aligned}&\langle \operatorname{Ind}\rho_h^{G\ltimes X}\xi\,,\,\eta\rangle_{r(h)}\\ &= \int_{X_{r(h)}}\int_{G^{r(h)}} D(h^{-1}, x)^{1/2}\xi(h^{-1}x)(h^{-1}g, g^{-1}x)\overline{\eta(x)(g, g^{-1}x)}\,d\lambda^{r(h)}(g)d\mu_{r(h)}(x).\end{aligned}$$

The integrated form of $\operatorname{Ind}\rho^{G\ltimes X}$ of $C_c(G)$ acts on $L^2(G^{(0)} * \mathcal{L}, \tilde{\mu})$ via

$$\begin{aligned}&\langle \operatorname{Ind}\rho^{G\ltimes X}(f)\xi\,,\,\eta\rangle\\ &\quad= \int_{G^{(0)}}\int_{G^u} f(h)\int_{X_u}\int_{G^u} D(h^{-1}, x)^{1/2}\xi(s(h))(h^{-1}x)(h^{-1}g, g^{-1}x)\\ &\qquad\qquad\cdot\overline{\eta(u)(x)(g, g^{-1}x)}\,d\lambda^u(g)d\mu_u(x)\Delta_{\tilde{\mu}}(g)^{-1/2}\,d\lambda^u(h)d\tilde{\mu}(u)\end{aligned}$$

for all $f \in C_c(G)$ and $\xi, \eta \in L^2(G^{(0)} * \mathcal{L}, \tilde{\mu})$.

Let κ^μ be the Koopman representation acting on $L^2(X, \mu)$ and let $\xi \in L^2(X, \mu)$. For $n \in \mathbb{N}$ define $\xi_n \in L^2(G^{(0)} * \mathcal{L}, \tilde{\mu})$ via

$$\xi_n(u)(x)(g, g^{-1}x) = \xi(x)f_n(g, g^{-1}x).$$

We check that indeed $\xi_n \in L^2(G^{(0)} * \mathcal{L}, \tilde{\mu})$ for all $n \in \mathbb{N}$ and $\|\xi_n\| = \|\xi\|$:

$$\begin{aligned}\|\xi_n\|^2 &= \int_{G^{(0)}} \|\xi(u)\|_u^2\,d\tilde{\mu}(u)\\ &= \int_{G^{(0)}}\int_{X_u}\int_{G^u} |\xi_n(u)(x)(g, g^{-1}x)|^2\,d\lambda^u(g)d\mu_u(x)d\tilde{\mu}(u)\\ &= \int_{G^{(0)}}\int_{X_u} |\xi(x)|^2\left(\int_{G^u} |f_n(g, g^{-1}x)|^2\,d\lambda^u(g)\right)d\mu_u(x)d\tilde{\mu}(x)\\ &= \int_{G^{(0)}}\int_{X_u} |\xi(x)|^2\,d\mu_u(x)d\tilde{\mu}(u) = \|\xi\|^2.\end{aligned}$$

We used (5.0.1) in the second to last equality.

Next we prove that $\lim_{n\to\infty}\langle \operatorname{Ind}\rho^{G\ltimes X}(f)\xi_n\,,\,\xi_n\rangle = \langle \kappa^\mu(f)\xi, \xi\rangle$ for all $f \in C_c(G)$. This implies the weak containment of κ^μ in $\operatorname{Ind}\rho^{G\ltimes X}$. We have

$$\begin{aligned}&\langle \operatorname{Ind}\rho^{G\ltimes X}(f)\xi_n\,,\,\xi_n\rangle = \int_{G^{(0)}}\int_{G^u} f(h)\int_{X_u} \xi(h^{-1}x)\overline{\xi(x)}D(h^{-1}, x)\\ &\cdot\left(\int_{G^u} f_n(h^{-1}g, g^{-1}x)\overline{f_n(g, g^{-1}x)}\,d\lambda^u(g)\right)d\mu_u(x)\Delta_{\tilde{\mu}}(h)^{-1/2}d\lambda^u(h)d\tilde{\mu}(u)\end{aligned}$$

Equation 5.0.2 implies that (see the proof of [3, Proposition 2.2.7])

$$2\lim_{n\to\infty}\int_{G^u} f_n(h^{-1}g, g^{-1}x)\overline{f_n(g,g^{-1}x)}d\lambda^u(g) =$$
$$\lim_{n\to\infty}\left(\int_{G^u}|f_n(h^{-1}g,g^{-1}x)|^2 d\lambda^u(h) + \int_{G^u}|f_n(g,g^{-1}x)|^2 d\lambda^u(g)\right) = 2$$

uniformly on compact subsets of $G \ltimes X$. Therefore

$$\lim_{n\to\infty}\langle \operatorname{Ind}\rho^{G\ltimes X}(f)\xi_n\,,\,\xi_n\rangle$$
$$=\int_{G^{(0)}}\int_{G^u} f(h)\int_{X_u}\xi(h^{-1}x)\overline{\xi(x)}D(h^{-1},x)d\mu_u(x)\Delta_{\tilde{\mu}}(h)^{-1/2}d\lambda^u(h)d\tilde{\mu}(u)$$
$$=\langle \kappa^\mu(f)\xi\,,\,\xi\rangle.$$

□

The hypotheses of amenability and σ-compactness of G are not needed for the following proposition.

Proposition 5.5. *Assume that μ has full support. Then $\|\operatorname{Ind}\rho^{G\ltimes X}(f)\| \le \|\rho^G(f)\|$ for all $f \in C_c(G)$.*

Proof. Note that the representation $\operatorname{Ind}\rho^{G\ltimes X}$ is equivalent with $\operatorname{Ind} L$ of Theorem 5.2. Indeed, if $u \in G^{(0)}$, $V(u) : \mathcal{L}(u) \to \mathcal{K}(u)$ defined via

$$V(u)(\xi)(x)(g) = \xi(x)(g, g^{-1}x) \text{ where } r(g) = \omega(x) = u$$

is a unitary. Moreover V intertwines $\operatorname{Ind}\rho^{G\ltimes X}$ and $\operatorname{Ind} L$ in the sense of Remark 3.10.

Therefore $\operatorname{Ind}\rho^{G\ltimes X} \cong \kappa^\mu \otimes \rho^G$ and by Corollary 4.9 we have $\kappa^\mu \otimes \rho^G$ equivalent to $(\dim \kappa_\mu) \otimes \rho^G$. It follows that for all $f \in C_c(G)$ we have

$$\|\kappa^\mu(f)\| \le \|(\kappa^\mu \otimes \rho^G)(f)\| = \|((\dim \kappa^\mu) \otimes \rho^G)(f)\| = \|\rho^G(f)\|.$$

□

6. The Renault-Deaconu groupoid

Let X be a locally compact Hausdorff space and let $T : X \to X$ be a local homeomorphism. Assume that T is positively expansive and exact (see [18, Page 2069]). Then the Renault-Deaconu groupoid $G(X,T)$ associated to T was described in Example 2.2.

Remark 6.1. A probability measure μ on $X = G^{(0)}$ defines a state ϕ_μ on $C^*(G(X,T))$ such that $\phi_\mu(f) = \int_{G^{(0)}} f|_{G^{(0)}}d\mu$ for $f \in C_c(G(X,T))$. It is known that ϕ_μ is a KMS state for the $\mathbb{R}$-action given by $\alpha_t(f)(\gamma) = e^{itc(\gamma)}f(\gamma)$ at inverse temperature β iff μ is quasi-invariant for $G(X,T)$ with Radon-Nikodym derivative $D_\mu = e^{-\beta c}$, see [18, Theorem 3.5]. Here $c : G(X,T) \to \mathbb{Z},\ c(x,k,y) = k$.

If $\psi : X \to (0,\infty)$ is continuous, then there is a continuous cocycle $D_\psi : G(X,T) \to (0,\infty)$ given by

$$D_\psi(x, m-n, y) = \frac{\psi(x)\psi(Tx)\cdots\psi(T^{m-1}x)}{\psi(y)\psi(Ty)\cdots\psi(T^{n-1}y)}.$$

The transfer operator $\mathcal{L}_\psi : C(X) \to C(X)$ is given by

$$(\mathcal{L}_\psi f)(x) = \sum_{Ty=x} \psi(y)f(y). \tag{6.0.1}$$

We recall the following result, see Proposition 3.4.1 in [31].

Proposition 6.2. *If μ is a probability measure on X, then μ is quasi-invariant for $G(X,T)$ with Radon-Nikodym derivative $D_\mu = \dfrac{dr^*\mu}{ds^*\mu}$ if and only if $\mathcal{L}_\psi^*\mu = \mu$, where $\mathcal{L}_\psi^*$ is the dual operator acting on the space of finite measures on X.*

Example 6.3. Assume that (X,d) is a metric space and $T : X \to X$ is a local homeomorphism such that $\displaystyle\lim_{y\to x} \frac{d(Tx,Ty)}{d(x,y)} = \varphi(x) > 0$ for all $x \in X$. Let $\psi(x) = \varphi(x)^{-s}$, where s is the Hausdorff dimension of (X,d), Then the normalized Hausdorff measure μ of d is quasi-invariant for $G(X,T)$ ([14, Theorem 3.4]).

In particular, for $0 < r_j < 1, j = 1, ..., k$ and $X = \{1, 2..., k\}^{\mathbb{N}}$ with metric d such that $\operatorname{diam}(Z(x_0x_1\cdots x_n)) = r_{x_0}r_{x_1}\cdots r_{x_n}$, the Hausdorff dimension s is the unique solution of the equation $r_1^s + r_2^s + \cdots + r_k^s = 1$ and μ is given by $\mu(Z(x_0x_1\cdots x_n)) = r_{x_0}^s r_{x_1}^s \cdots r_{x_n}^s$. The one-sided shift $T : X \to X$ gives $\displaystyle\lim_{y\to x} \frac{d(Tx,Ty)}{d(x,y)} = \varphi(x) = \frac{1}{r_{x_0}}$ and (μ, s) is such that $\dfrac{dT^*\mu}{d\mu} = \varphi^s$.

Suppose $G = G(X,T)$ acts on the left on the space Y via $\omega : Y \to X$. Define

$$\tilde{T} : Y \to Y,\ \tilde{T}(z) = (T(\omega(z)), -1, \omega(z)) \cdot z.$$

Then $\tilde{T}$ is a local homeomorphism such that $\omega \circ \tilde{T} = T \circ \omega$. Moreover, the action groupoid $G \ltimes Y$ is isomorphic to the groupoid $\tilde{G} = G(Y, \tilde{T})$ via the map

$$\Psi : \tilde{G} \to G * Y,\ \Psi((z, m-n, y)) = (\omega(z), m-n, \omega(y)), y),$$

for $z, y \in Y$, see [15, Theorem 2.2].

In particular, we can construct quasi-invariant measures on Y as quasi-invariant measures on $G \ltimes Y \cong G(Y, \tilde{T})$.

Example 6.4. Let E be a locally finite directed graph which has no sources. Let $E^* := \displaystyle\bigcup_{k\geq 0} E^k$ be the space of finite paths, where

$$E^k = \{e_1e_2\cdots e_k : e_i \in E^1,\ r(e_{i+1}) = s(e_i)\},$$

and let E^∞ be the infinite path space with the topology given by $Z(\alpha) = \{\alpha x : x \in E^\infty\}$ for $\alpha \in E^*$. We assume that E^∞ is a totally disconnected space, homeomorphic to the Cantor set. On $X = E^\infty$, consider the shift $T : X \to X,\ \ T(x)_i = x_{i+1}$ which is a local homeomorphism. The groupoid $G(X,T)$ is called the graph groupoid and its C^*-algebra is denoted by $C^*(E)$.

Recall that for $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta)$, we denote

$$Z(\alpha, \beta) = \{\gamma \in G(X,T) \mid \gamma = (\alpha x, |\alpha| - |\beta|, \beta y)\},$$

which are compact open bisections. The indicator functions $\{1_{Z(v,v)} \mid v \in E^0\}$ and $\{1_{Z(e,s(e))} \mid e \in E^1\}$ generate $C^*(E)$, see [16, Proposition 4.1] (where the range and source maps are reversed).

If $G(X,T)$ acts on its unit space X by $(x,k,y) \cdot y = x$, let μ be the Markov measure on X determined by a map $p : E \to (0,\infty)$ satisfying $\sum\limits_{r(e)=v} p(e) = 1$ for every $v \in E^0$ and a map $\mu_0 : E^0 \to (0,\infty)$ satisfying $\sum\limits_{v \in E^0} \mu_0(v) = 1$ such that

$$\mu(Z(e_1 e_2 \cdots e_n)) = \mu_0(r(e_1))p(e_1)p(e_2)\cdots p(e_n).$$

Then μ is quasi-invariant for $G(X,T)$ and $D(ex,1,x) = \frac{1}{p(e)}$.

The Koopman representation κ^μ of $G(X,T)$ associated to μ acts on $L^2(X,\mu)$ by rank 1 operators, since $L^2(\omega^{-1}(s(g)), \mu_{s(g)})$ reduces to $\mathbb{C}$.

We now determine the operators $\kappa^\mu(f) \in \mathcal{B}(L^2(X,\mu))$ for the above indicator functions. We have for $\xi \in L^2(X,\mu)$

$$\kappa^\mu(1_{Z(v,v)})\xi(x) = \sum_{r(\gamma)=x} 1_{Z(v,v)}(\gamma)\kappa^\mu_\gamma(\xi)(x) = \sum_y 1_{Z(v,v)}((x,k,y))\xi((y,k,x)\cdot x) =$$

$$= \sum_y 1_{Z(v,v)}((x,k,y))\xi(y) = \begin{cases} \xi(x) \text{ if } r(x) = v \\ 0 \text{ if } r(x) \neq v, \end{cases}$$

$$\kappa^\mu(1_{Z(e,s(e))})\xi(x) = \frac{1}{\sqrt{p(e)}} \sum_{r(\gamma)=x} 1_{Z(e,s(e))}(\gamma)\kappa^\mu_\gamma(\xi)(x) =$$

$$= \frac{1}{\sqrt{p(e)}} \sum_y 1_{Z(e,s(e))}((x,k,y))\xi(y) = \begin{cases} \frac{1}{\sqrt{p(e)}}\xi(z) \text{ if } x = ez \\ 0 \text{ if } x \neq ez. \end{cases}$$

Denote by $P_v = \kappa^\mu(1_{Z(v,v)})$ and $S_e = \kappa^\mu(1_{Z(e,s(e))})$. Since $L^2(X,\mu)$ decomposes as $\bigoplus\limits_{v \in E^0} L^2(vX,\mu)$, we note that P_v acts as identity on $L^2(vX,\mu)$ and is 0 otherwise. It follows that $P_v^* = P_v = P_v^2$ and $\sum\limits_{v \in E^0} P_v = I$. Also, S_e takes $L^2(s(e)X,\mu)$ to $L^2(eX,\mu)$ and

$$S_e^* S_e = P_{s(e)}, \quad \sum_{r(e)=v} S_e S_e^* = P_v.$$

Since $\{P_v, S_e\}$ satisfy the same relations as $\{1_{Z(v,v)}\}$ and $\{1_{Z(e,s(e))}\}$ for $v \in E^0, e \in E^1$, it follows that $C^*(\kappa^\mu)$ is a quotient of $C^*(E)$. Since $\mu_0(v) \neq 0$ for all $v \in E^0$ and $p(e) \neq 0$ for all $e \in E^1$, it follows that the partial isometries S_e are all non-zero. Using the same proof as Theorem 3.7 of [17] it follows that κ^μ is faithful and, thus, $C^*(E)$ is isomorphic with $C^*(\kappa^\mu)$.

Remark 6.5. By allowing μ_0 and p to take zero values at specific vertices and edges, one can recover the ideal structure of $C^*(E)$ from the resulting Koopman representations.

Assume, for simplicity, that E satisfies condition (K): every vertex $v \in E^0$ either has no loop based at v or at least two loops based at v ([17, Section 6] where the notation for r and s is reversed compared to ours). Recall also that a subset H of E^0 is called hereditary if whenever $e \in E^1$ and $s(e) \in H$, then $r(e) \in H$. The set H is called saturated if whenever $r(s^{-1}(v)) \subset H$, then $v \in H$. It is known that there is an isomorphism between the lattice of saturated hereditary subsets of E^0 and the lattice of ideals of $C^*(E)$ ([16, Theorem 6.6]) given via $H \mapsto I(H)$, where

$$I(H) = \overline{\text{span}}\{1_{Z(\alpha,\beta)} : \alpha, \beta \text{ finite paths with } s(\alpha) = s(\beta) \in H\}.$$

Let H be a saturated hereditary set and let $\mu_0 : E^0 \to [0,\infty)$ and $p : E^1 \to [0,\infty)$ be defined such that $\sum_{v\in E^0} \mu_0(v) = 1$, $\sum_{r(e)=v} p(e) = 1$ for all $v \in E^0$, $\mu_0(v) = 0$ for all $v \in H$ and $p(e) = 0$ for all $e \in s^{-1}(H)$. Then (μ_0, p) defines a quasi-invariant measure μ on X as above and one can easily check, using computations like in the previous example, that $\ker \kappa^\mu = I(H)$.

Example 6.6. In this example we follow the notation of [15]: we let $W = \{1, \dots, N\}$ for some integer $N \geq 2$, W^n is the set of words of length n over the alphabet W, and $W^* = \bigcup_{n\geq 0} W^n$ is the set of finite words over W. We let $X = W^\infty$ be the set of infinite words (sequences) with elements in W and $T : X \to X$ be the shift map: $T(x_1x_2x_3\cdots) = (x_2x_3\cdots)$. As in the previous example, the topology on X is given by the clopen cylinders $Z(w) = \{wx : x \in X\}$ for all $w \in W^*$. Then $G = G(X,T)$ is the Cuntz groupoid ([25, Section III.2]) and $C^*(G)$ is isomorphic with the Cuntz algebra $\mathcal{O}_N$. Let (Y, d) be a complete metric space and let $(F_1, \dots, F_N)$ be an iterated function system on Y ([13]). That is, each F_i is a strict contraction on Y. We assume further that each F_i is a homeomorphism. There is a unique compact invariant set K ([13, Theorem 3.1.3]) such that $K = \bigcup_{i=1}^N F_i(K)$. Assume that the iterated function system is totally disconnected: $F_i(K) \bigcap F_j(K) = \emptyset$ if $i \neq j$. In this case K is a totally disconnected set.

We recall next the construction of a "fractafold" bundle $\mathfrak{F}$ on which G acts and an invariant measure on $\mathfrak{F}$ as given in [15, Section 3]. For $w \in W^n$ we write $F_w^{-1}(A) = F_{w_1}^{-1} \circ \cdots \circ F_{w_n}^{-1}(A)$ and $F_w(A) = F_{w_n} \circ \cdots \circ F_{w_1}(A)$. For $x \in X$ or $x \in W^*$ we write $x(n) := x_1 \cdots x_n$ and set $\mathfrak{F}_n(x) = F_{x(n)}^{-1}(K)$. Then $\mathfrak{F}_n(x) \subset \mathfrak{F}_{n+1}(x)$ and the infinite blow-up of K at x is $\mathfrak{F}(x) = \bigcup_{n\geq 0} \mathfrak{F}_n(x)$ endowed with the inductive limit topology (see [34, Section 5.4] for a short introduction to blow-ups). The *fractafold bundle* $\mathfrak{F}$ is defined as the increasing union of $\mathfrak{F}_n := \bigsqcup_{w\in W^*} Z(w)\times \mathfrak{F}_n(w)$ endowed with the inductive limit topology. Then $\mathfrak{F}$ is a Hausdorff space and the map $\omega : \mathfrak{F}\to X$, $\omega(x,t) = x$ is continuous, open and surjective. Under the assumption that the iterated function system is totally disconnected, $\mathfrak{F}$ is locally compact. The groupoid $G(X,T)$ acts on $\mathfrak{F}$ via

$$(x,m-n,y)(y,t) = (x, F^{-1}_{x(m)}(F_{y(n)}(t))).$$

There is a unique invariant probability measure μ on K ([13, Theorem 4.4.1]) such that $\mu(A) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\mu(F_i^{-1}(A))$ for all Borel subsets A of K. One can extend μ to an infinite measure μ_x on $\mathfrak{F}_x$ via $\mu_x(A) = N^n\mu(F_{x(n)}(A))$ if $A\in \mathfrak{F}_n(x)$. Consider the measure ν on X generated by weights $\{1/N,\dots,1/N\}$. That is $\nu(Z(w)) = (1/N)^n$ for all $w\in W^n$ and $n\ge 0$. Then there is a unique *G-invariant* measure μ_∞ on $\mathfrak{F}$ such that $\mu_\infty(U\times A) = \nu(T^n(U))\cdot\mu(F_{w(n)}(A))$ for all $n\ge 0$, $w\in W^n$, and $U\times A\subset Z(w)\times\mathfrak{F}_n(w)$ ([15, Proposition 3.11]). Note that the measure $\tilde{\mu}$ in the decomposition of μ_∞ equals ν and is quasi-invariant for G.

The Koopman representation κ of G on μ_∞ extends to a representation of $\mathcal{O}_N$ that acts on $L^2(\mathfrak{F},\mu_\infty)$ via

$$\kappa(f)\xi(x,t) = \sum_{(x,m-n,y)\in G} f(x,m-n,y)\xi(y, F^{-1}_{y(n)}(F_{x(m)}(t)))$$

for all $(x,t)\in L$, $\xi\in L^2(\mathfrak{F},\mu_\infty)$, and $f\in C_c(G)$. In particular, if $S_i = 1_{Z(i,\emptyset)}$ are the Cuntz isometries generating $C^*(G)\cong\mathcal{O}_N$, where

$$Z(i,\emptyset) := \{(ix,1,x) : x\in X\},$$

then

$$\kappa(S_i)\xi(x,t) = \begin{cases}\xi(T(x),F_i(t)) & \text{if } x\in Z(i)\\ 0 & \text{otherwise}\end{cases}$$

for all $i = 1,\dots,N$. We note that $\kappa(S_i)\neq 0$ for all $i = 1,\dots N$. To see this, let $\xi\in L^2(\mathfrak{F},\mu_\infty)$ be defined via

$$\xi(x,t) = 1_{\mathfrak{F}_0}(x,t) = \begin{cases}1 & \text{if } x\in X \text{ and } t\in K\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\xi\in L^2(L,\mu_\infty)$ since $\mu_\infty(\mathfrak{F}_0) = 1$. Then $\kappa(S_i)(\xi) = \xi_i$ where

$$\xi_i(x,t) = 1_{Z(i)\times F_i^{-1}(K)} = \begin{cases}1 & \text{if } x\in Z(i) \text{ and } t\in F_i^{-1}(K)\\ 0 & \text{otherwise,}\end{cases}$$

for all $i = 1,\dots,N$. Since

$$\int_{\mathfrak{F}}\xi_i(x,t)\,d\mu_\infty(x,y) = \mu_\infty(Z(i)\times F_i^{-1}(K)) = \mu(K) = 1$$

by the definition of μ_∞, we get $\kappa(S_i)\neq 0$. Therefore $C^*(\kappa)\cong\mathcal{O}_N$.

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One can build other G-invariant measures on $\mathfrak{F}$ by considering invariant measures for K using non-equal strictly positive weights $\{p_1, \dots, p_N\}$ such that $\sum_{i=1}^N p_i = 1$. There is a unique measure on K that satisfies $\mu(A) = \sum_{i=1}^N p_i \mu(F_i^{-1}(A))$ for all Borel subsets A of K. Also, one can define a measure ν on X based on the weights via $\nu(Z(w)) = p_{w_1} \cdot \cdots p_{w_n}$ for all $w \in W^*$. Then one can prove that the measure μ_∞ defined as above, $\mu_\infty(U \times A) = \nu(T^n(U)) \cdot \mu(F_{w(n)}(A))$ for all $n \geq 0$, $w \in W^n$, and $U \times A \subset Z(w) \times L_n(w)$, is a G-invariant measure. Under our assumption that $p_i > 0$ for all $i = 1, \dots, N$, a similar analysis proves that $C^*(\kappa) \cong \mathcal{O}_N$.

Example 6.7. Consider again the Cuntz groupoid as defined in the previous example: $X = \{1, \dots, N\}^{\mathbb{N}}$ and $T : X \to X$ is the shift. We show that if Y is any left G-space and if μ is any G-invariant measure on Y with *full support*, then $C^*(\kappa) \cong \mathcal{O}_N$. This example generalizes easily to the case of finite graphs that satisfies the (K)-condition or, equivalently, Cuntz-Krieger algebras that satisfy condition (II).

Let Y be a locally compact Hausdorff left G-space with anchor map $\omega : Y \to X = G^{(0)}$ and assume that μ is a G-invariant measure on Y. Recall that T lifts to a local homeomorphism $\tilde{T} : Y \to Y$ defined via $\tilde{T}(z) = (T(\omega(z)), -1, \omega(z)) \cdot z$ for all $z \in Y$ and $G \ltimes Y \cong G(Y, \tilde{T})$. Therefore there is $\psi : Y \to \mathbb{R}_+^*$ such that μ is invariant for the dual of the transfer operator $\mathcal{L}_\psi$ defined as in (6.0.1). Let $S_i = 1_{Z(i,\emptyset)}$, $i = 1, \dots, N$, be the Cuntz isometries that generate $C^*(G)$. Then

$$\kappa(S_i)\xi(z) = \sum_{(\omega(z), m-n, x) \in G} S_i(\omega(z), m-n, x)\xi((x, n-m, \omega(z)) \cdot z) \cdots \Delta_\mu((x, n-m, \omega(z)), z)^{1/2}$$

$$= \begin{cases} \xi((T(\omega(z)), -1, \omega(z)) \cdot z)\Delta_\mu((T(\omega(z)), -1, \omega(z)), z)^{1/2} & \text{if } z \in \omega^{-1}(Z(i)) \\ 0 & \text{otherwise} \end{cases}$$

which, by the identification of $G \ltimes Y$ with $G(Y, \tilde{T})$

$$= \begin{cases} \xi(\tilde{T}(z))\Delta_\mu(\tilde{T}(z), -1, z)^{1/2} & \text{if } z \in \omega^{-1}(Z(i)) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \xi(\tilde{T}(z))\psi(z)^{-1/2} & \text{if } z \in \omega^{-1}(Z(i)) \\ 0 & \text{otherwise.} \end{cases}$$

Since ψ is strictly positive and μ has full support, it follows that $\kappa(S_i) \neq 0$ for all $i = 1, \dots, N$ and, thus, $C^*(\kappa) \cong \mathcal{O}_N$.

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