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ON THE WELL-POSEDNESS OF THE 2D EULER-BÉNARD SYSTEM WITH NONLINEAR THERMAL DIFFUSIVITY.

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ABSTRACT. This study presents the global existence of a unique solution within the Yudovich class for the Bénard system with nonlinear thermal diffusivity.

1. Introduction

The inviscid Bénard model is one of the systems that play an important role in the study and understanding of some of the terms and phenomena related to fluid dynamics and hydrodynamics. This model is represented by the following system:

$$(1)\quad \begin{cases} \theta_t + v\cdot\nabla\theta - \operatorname{div}(\kappa(\theta)\nabla\theta) = \mu v_2 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ v_t + v\cdot\nabla v + \nabla P = \theta\vec{e}_2 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \operatorname{div} v = 0, \\ (v(0,x),\theta(0,x)) = (v_0(x),\theta_0(x)), \end{cases}$$

here $v=(v_1,v_2)$ represents the velocity field which is assumed to be free-divergence, θ and P denotes the temperature and the pression of the fluid respectively, the term $\theta\vec{e}_2$ referred as the buoyancy force and the term μv_2 describes the Rayleigh-Bénard convection in the gravity direction. In what follows κ is a function stands for the thermal diffusivity. By neglecting the temperature θ in the system (1), then we will get the Euler-equations

$$(2)\quad \begin{cases} v_t + v\cdot\nabla v + \nabla P = 0, & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \operatorname{div} v = 0, \\ v(0,x) = v_0(x). \end{cases}$$

This equation have been extensively studied by many mathematicians in various function spaces. For example, in the $d-$dimensional case ($d>2$), Kato and Ponce [18] showed that the equation (2) is locally well-posed, whenever the initial velocity belongs to $W^{s,p}$, with $\frac{d}{p}+1<s$, thereafter Chemin [8] extends this result to the Hölder class $\mathscr{C}^s$, with $1<s$, and then by Chae [6] in the Besov framework. It is worth noting that until now, in 3 dimensional, the global theory for this equations still

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an open problem. The concept of vorticity ($w \triangleq \nabla \times v$) is very important to study and understand some phenomena related to the system (2). For instance, Beale, Kato and Majda [2] proved the following blow up criterion:

$$\text{The lifespan } T_* \text{ is finite if and only if } \int_0^{T_*} \|w(\tau)\|_{L^\infty} d\tau = +\infty.$$

In the 2−dimensional case, the vorticity obeys the following transport equations

$$w_t + v \cdot \nabla w = 0,$$

then as an immediate application for the (BKM)-criterion[1], we can find the global existence of Kato's solutions. Yudovich in his famous paper [37] showed the existence/uniqueness of weak solution to the system (2) with initial vorticity belongs to $L^p \cap L^\infty$, where $p > 1$. The Yudovich's result extended by Serfati [29], where he just needs from the vorticity to be bounded. By back to the system (1), which can be considered as a generalization for the Euler equations, which can be divided into several cases:

- CASE 1: (isotropic case) κ is a positive constant:

Chae [7] proved the global regularity for the classical Boussinesq system $(1)_{\kappa>0,\mu=0}$, this result extended by Hmidi and Keraani [16] to the critical Besov spaces. For other related results in the critical Besov spaces see [14, 15, 23, 24, 26]. Concerning the Yudovich framework Danchin and Paicu [9] showed that the systems $(1)_{\kappa>0,\mu=0}$ and $(1)_{\kappa>0,\mu>0}$ admit a unique global solution belongs to $L^2 \cap L^\infty$ (see also [32, 22]).

- CASE 2: (anisotropic case) κ is a positive constant with a dissipation in one direction $(\partial^2_{x_i})_{i=1,2}$ instead of the full Laplacian operator:

Danchin and Paicu [10] proved the global existence of the anisotropic Boussinesq system, Adhikari et al. investigated the anisotropic Boussinesq system in different situations in [1]. In the reference [21], Ma and Zhang established the global weak solution for the 2D Bénard system with vertical dissipation in the first component of the velocity field and horizontal thermal diffusivity. They also provided global regularity criteria for this weak solution. Additionally, they demonstrated the global existence and regularity criteria of the weak solution for the Bénard system in other cases involving partial viscosity and thermal diffusivity. More recentely, the author [25] established the global existence of weak solution for the Bénard system with variable viscosity. The existence uniqueness and stability of such a system were explored by many authors, the reader can consult the following references [4, 5, 33, 12].

- CASE 3: κ is a positive constant and the operator $-\Delta$ is replaced by the fractional dissipation $(-\Delta)^s$[2]:

The system $(1)_{\kappa>0,\mu=0}$ explored by many authors, where Hmidi et al. [17] proved the global existence for the inviscid Boussinesq system $(1)_{\kappa>0,\mu=0}$ with critical dissipation corresponding to the case $s = 1/2$, the method of [17] helps to cancel the effect of the rough term of the Lipschitz norm of the temperature. More recently, Melkemi et al. [27] proved the global existence of Yudovich's solution for Boussinesq system $(1)_{\kappa>0,\mu=0}$ with general source term and critical dissipation. Based on the technique used in [17], Wu and Xue [34] proved the global existence of the Yudovich solution for the

[1](BKM) is an abbreviation for Beale, Kato and Majda criterion

[2]For $s \in (0,1]$ the operator $(-\Delta)^s$ is defined via its Fourier transform $\mathscr{F}\big((-\Delta)^s g\big)(\xi) \triangleq |\xi|^s \mathscr{F}(g)(\xi)$

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inviscid Bénard $(1)_{\kappa>0,\mu>0}$ system with subcritical dissipation, later Xu and Xue [31] investigated the Yudovich type solution for the inviscid Boussinesq system with critical dissipation. More recently, Z. Ye [36] was able to show the global existence of the Yudovich solution for the Bénard equation with critical dissipation.

- CASE 4: κ is a function depends on temperature:

Díaz and Galiano [11] proved the existence and uniqueness of solutions for the system $(1)_{\mu=0}$ on domains, recently Li and Xu [19] proved the global regularity for the system $(1)_{\mu=0}$, when $(v_0,\theta_0)\in\Big(H^s(\mathbb{R}^2)\Big)^2$, and they have assumed that the coefficient κ is a smooth function such that $\frac{1}{C_0}\le\kappa(\cdot)\le C_0$. Recently, Li [20], established the global regularity for the system $(1)_{\mu>0}$

More recently, M. Paicu and N. Zhu [28] investigated and studied the Yudovich type solution for the system $(1)_{\mu=0}$, where they have assumed that the nonlinear thermal diffisuvity coefficient is sufficiently close to some positive constant in L^∞. More precisely, they assumed that, the function κ obeys the following conditions:

- Boundedness of the function κ :
$$C_0^{-1}\le\kappa(x)\le C_0,\quad\text{for all } x\in\mathbb{R};$$
- boundedness of the first derivative of κ :
$$\kappa'(x)\le C_0;$$
- smallness condition: there exists a constant ε such that

$$\|\kappa(\theta)-1\|_{L^\infty}\le\varepsilon. \tag{3}$$

The key idea of [28] is based on taking advantage of the heat kernel semi-group. From Duhamel's formula, the authors were able to get the high regularities for the temperature via using the fact that the operator
$$\mathscr{A}(f)\triangleq\int_0^t\nabla^2e^{(t-\tau)\Delta}f_x(\tau)d\tau$$
is continuous from $L^r\big((0,t);L^p\big)$ into itself and the fact that the nonlinear thermal diffisuvity coefficient is sufficiently close to some positive constant, thus the authors get the uniform bound for the vorticity in $L^2\cap L^\infty$, which is enough to obtain the global existence of solution. Then, Y. Maafa and M. Zerguine [38] used the same technique of [28] to study the vortex patch problem of the system $(1)_{\mu=0}$. In 2022, Z. He and X. Liao [13], studied the global regularity for the Boussinesq system with temperature-dependent thermal and viscosity diffusions in general Sobolev spaces. More recently, Ye [35] dropped the boundedness of the first derivative of the function κ, and the smallness condition (3).

Main Result

In the current paper, we are mainly interested with (CASE 4) corresponding to the case when κ is a function depends on temperature and μ is a positive constant. To be more precise, we study the system (1), where the initial vorticity belongs to the Yudovich class and we assume that the function κ is a smooth function that obeys the following conditions:
$$\kappa_*\le\kappa(z)\le\kappa^*\ \text{and}\ |\kappa'(z)|\le C_0\ \text{for all}\ z\in\mathbb{R}.$$

and we have the following result:

Theorem 1.1. *Let* $v_0 \in L^2(\mathbb{R}^2)$ *such that* $\text{div} v_0 \triangleq 0$ *and* $w_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *We assume that* $\mu \geq 0$[3] *and* $\theta_0 \in H^2(\mathbb{R}^2)$. *Then, the system* (1) *has a* ***unique global solution*** (v, w, θ) *such that*

$$v \in L^\infty([0,T]; L^2(\mathbb{R}^2)), \quad w \in L^\infty([0,T]; L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)),$$

and

$$\theta \in L^\infty([0,T]; H^2(\mathbb{R}^2) \cap \dot{W}^{1,\infty}(\mathbb{R}^2)), \quad \Delta(\kappa(\theta)\nabla\theta)) \in L^2([0,T], L^2(\mathbb{R}^2)).$$

Let us say a few words about the strategy of our proof, the main difficulty lies in the presence of the nonlinear thermal diffisuvity, which makes the control of $\|\theta\|_{L^1_T \text{Lip}}$ seems tough. First, we start by establishing L^2-estimate of velocity, vorticity and temperature, then we move to deal with the $L^\infty-$norm of vorticity. In [30], Sun and Zhang observed that the primitive of the function κ, which is denoted by $\Phi(t,x) := \int_0^\theta \kappa(s) ds$ satisfies the following system:

$$\begin{cases} \Phi_t + v \cdot \nabla\Phi - \kappa(\theta)\Delta\Phi = \kappa(\theta) v_2 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \Phi(0,x) = \Phi_0(x). \end{cases} \tag{4}$$

So, we perform the $\dot{H}^1-$estimates of Φ, then we can get that Φ belongs to $L^2([0,T], \dot{H}^2)$ therefore by using an interpolation theorem combined with the fact that $\partial_i \Phi = \kappa(\theta)\partial_i\theta$ and κ is bounded function we get the following estimate $\|\nabla\theta\|_{L^1_T L^4} < +\infty$, after this we move to performing $\dot{H}^2-$estimates for the function Φ, which leads us to control $\|\nabla\theta\|_{L^1_T L^\infty}$ thus we obtain the $L^\infty-$norm of the vorticity, hence the global Yudovich-solutions. The uniqueness part will be treated by adopting the Yudovich method. This paper is arranged as follows, where in the next section, we perform some a priori estimate for the velocity, tempurature and vorticity. In the last section we prove the uniqueness of solution for the system (1).

Notations: As usual, throughout the paper, we agree with the following notations, where we denote by C any positive constant which changes from one line to another and we shall use the notation $X \lesssim Y$ instead of the notation, $\exists C_0 > 0$ such that $X \leq CY$ and C_0 is a positive constant depending on the initial data. For every $p \in [1,\infty]$, $\|\cdot\|_{L^p}$ denotes the L^p norms. For $T > 0$, we denote by C_T any constant depends on the initial datum and the time T.

Proof of Theorem 1.1

In what follows, we present the proof of Theorem 1.1, we start first with some a priori estimates for the velocity, temperature, thermal diffusivity, and vorticity.

2. A priori estimate

This section is devoted to perform some a priori estimate of the temperature, velocity and vorticity.

[3] For the clarity of the computations, we assume that $\mu = 1$

L^2-estimates for velocity and temperature:

The basic L^2-estimates of velocity and temperature are given as follows:

Proposition 2.1. *We assume that* (θ, v) *is a smooth solution for* (1) *such that* $\theta_0, v_0 \in L^2$*, then*

$$\|v(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\kappa_* \|\nabla\theta\|_{L_t^2 L^2} \leq C_T, \tag{5}$$

Proof. By taking the L^2 inner product of the temperature and velocity equations and integrating by parts over the spatial variable, we obtain

$$\frac{d}{dt}\big(\|v(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2\big) + \kappa_* \int_{\mathbb{R}^2} |\nabla\theta(t,x)|^2 dx \leq 2\int_{\mathbb{R}^2} \theta v_2 dx.$$

From the Hölder and Young inequalities, and the Gronwall inequality we obtain the required estimate. □

L^2-estimate of the vorticity:

Proposition 2.2. *Let v be a smooth divergence free field with* $w \triangleq \operatorname{curl} v$*. We assume that* (w, θ) *a smooth solution for* (1)*, then we have*

$$\|w\|_{L^2} \leq C_T.$$

Proof. First we observe that the vorticity w obeys the following equation

$$w_t + v \cdot \nabla w = \partial_1 \theta. \tag{6}$$

The classical L^2 estimate, ensures that

$$\|w(t)\|_{L^2} \leq \|w_0\|_{L^2} + \int_0^t \|\nabla\theta(\tau)\|_{L^2} d\tau.$$

Hölder inequality and (5) leads to

$$\|w(t)\|_{L^2} \leq \|w_0\|_{L^2} + C_T \|\theta_0\|_{L^2}^2 \leq C_T.$$

□

In order to control the $L^\infty-$bound of vorticity we need to get more information about the temperature, first, we observe that the function $\Phi \equiv \int_0^\theta \kappa(s) ds$ satisfies the following parabolic equation:

$$\begin{cases} \Phi_t + v \cdot \nabla\Phi - \kappa(\theta)\Delta\Phi = \kappa(\theta) v_2 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \Phi(0,x) = \Phi_0(x). \end{cases} \tag{7}$$

Global bound in $\dot{H}^1$:

Proposition 2.3. *Let v be a smooth divergence free field. We assume that Φ a smooth solution for* (7)*, then we have:*

(8) $$\|\Phi\|_{\dot{H}^1} + \|\Phi\|_{L^2_T \dot{H}^2} \leq C_T,$$

(9) $$\|\nabla\theta\|_{L^1_t L^4} \leq C_T.$$

Proof. We start by proving (8). Multiplying (7) by $-\Delta\Phi$ and integrating over $\mathbb{R}^2$, we obtain

$$\frac{d}{dt}\|\nabla\Phi(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} \kappa(\theta)|\Delta\Phi(t,x)|^2 dx = \int_{\mathbb{R}^2} v\cdot\nabla\Phi\Delta\Phi dx - \int_{\mathbb{R}^2} \kappa(\theta) v_2 \Delta\Phi,$$

We know that

$$\kappa_* \leq \kappa(\theta) \leq \kappa^*,$$

and

$$\kappa_*\|\Delta\Phi(t)\|_{L^2} \leq \int_{\mathbb{R}^2} \kappa(\theta)|\Delta\Phi(t,x)|^2 dx$$

then, from the above inequalities and after integrating by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\nabla\Phi(t)\|_{L^2}^2 + \kappa_*\|\Delta\Phi(t)\|_{L^2}^2 &\leq \int_{\mathbb{R}^2} |D_j v_i||D_i\Phi||D_j\Phi| dx + \kappa^* \int_{\mathbb{R}^2} |\nabla v_2||\nabla\Phi| dx \\
&\leq \|w(t)\|_{L^2}\|\nabla\Phi\|_{L^4}^2 + \kappa^*\|w(t)\|_{L^2}\|\nabla\Phi(t)\|_{L^2} \\
&\leq \|w(t)\|_{L^2}\|\nabla\Phi(t)\|_{L^2}\|\Delta\Phi(t)\|_{L^2} + \kappa^*\|w(t)\|_{L^2}\|\nabla\Phi(t)\|_{L^2} \\
&\leq \kappa^*\|w(t)\|_{L^2}\|\nabla\Phi(t)\|_{L^2}(1 + \|\Delta\Phi(t)\|_{L^2}) \\
&\leq \frac{\kappa_*}{2}\|\Delta\Phi\|_{L^2}^2 + \frac{\kappa_*}{2} + C_\kappa\|w(t)\|_{L^2}^2\|\nabla\Phi\|_{L^2}^2,
\end{aligned}$$

we point out that we have used in the above inequality the Young, Ladyzhenskaya inequalities and the Calderón-Zygmund inequality. Integrating with respect to time and by using Gronwall Lemma, we obtain

$$\|\Phi(t)\|_{\dot{H}^1} + \frac{\kappa_*}{2}\int_0^t \|\Delta\Phi(\tau)\|_{L^2}^2 d\tau \leq C_T.$$

□

For (9), we apply again the Ladyzhenskaya inequality, we obtain

$$\kappa_*\|\nabla\theta\|_{L^4} \leq \|\nabla\Phi\|_{L^4} \leq \|\Phi\|_{L^2}^{1/2}\|\Delta\Phi\|_{L^2}^{1/2}.$$

Consequently, we find out

$$\|\nabla\theta\|_{L^1_t L^4} \leq C_T.$$

Global bound in $\dot{H}^2$:

Proposition 2.4. *Let v be a smooth divergence free field. We assume that Φ a smooth solution for (7), then we have:*

$$\|\Phi(t)\|_{\dot{H}^2}^2 + \kappa_* \int_0^T \|\Delta\nabla\Phi(\tau)\|_{L^2}^2 d\tau \lesssim C_T. \tag{10}$$

Proof. We multiply (7) by $\Delta^2\Phi$ and integrating with respect to space variable, we obtain after using Hölder inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\Delta\Phi|^2dx + \kappa_*\int_{\mathbb{R}^2}|\Delta\nabla\Phi|^2dx &\leq \int_{\mathbb{R}^2}|\nabla v||\nabla\Phi||\Delta\nabla\Phi|dx + \int_{\mathbb{R}^2}|v||\Delta\Phi||\Delta\nabla\Phi|dx\\ &+ \int_{\mathbb{R}^2}|\kappa'(\theta)||\nabla\theta||\Delta\nabla\Phi|dx + |\int_{\mathbb{R}^2}\nabla v_2\nabla\Delta\Phi dx|\\ &\leq \|\nabla v\|_{L^4}\|\nabla\Phi\|_{L^4}\|\Delta\nabla\Phi\|_{L^2} + \|v\|_{L^\infty}\|\Delta\Phi\|_{L^2}\|\Delta\nabla\Phi\|_{L^2}\\ &+ \kappa^*\|\nabla\theta\|_{L^2}\|\Delta\nabla\Phi\|_{L^2} + \|\nabla v\|_{L^2}\|\Delta\nabla\Phi\|_{L^2},\end{aligned}$$

by using Young's inequality, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\Delta\Phi|^2dx + \kappa_*\int_{\mathbb{R}^2}|\Delta\nabla\Phi|^2dx &\leq C_\kappa\|\nabla v\|_{L^4}^2\|\nabla\Phi\|_{L^4}^2 + \frac{\kappa_*}{8}\|\Delta\nabla\Phi\|_{L^2}^2 + C_\kappa\|v\|_{L^\infty}^2\|\Delta\Phi\|_{L^2}^2\\ &+ \frac{\kappa_*}{8}\|\Delta\nabla\Phi\|_{L^2}^2 + C_\kappa\|\nabla\theta\|_{L^2}^2 + \frac{\kappa_*}{8}\|\Delta\nabla\Phi\|_{L^2}^2\\ &+ C_\kappa\|\nabla v\|_{L^2}^2 + \frac{\kappa_*}{8}\|\Delta\nabla\Phi\|_{L^2}^2.\end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\Delta\Phi|^2dx + \frac{\kappa_*}{2}\int_{\mathbb{R}^2}|\Delta\nabla\Phi|^2dx \lesssim C_T + \|v\|_{L^\infty}^2\|\Delta\Phi\|_{L^2}^2.$$

Integrating with respect to time, we infer that

$$\|\Phi(t)\|_{\dot{H}^2}^2 + \kappa_*\int_0^T \|\Delta\nabla\Phi(\tau)\|_{L^2}^2 d\tau \lesssim C_T + \int_0^T \|v(\tau)\|_{L^\infty}^2\|\Phi(\tau)\|_{\dot{H}^2}^2 d\tau.$$

In order to close the above inequality, we need to bound $\|v(\tau)\|_{L^\infty}$, in fact, from the interpolation inequality, we have

$$\begin{aligned}\|v(\tau)\|_{L^\infty}^2 &\leq \|v(\tau)\|_{L^2}^{2/3}\|\nabla v(\tau)\|_{L^4}^{4/3}\\ &\leq \|v(\tau)\|_{L^2}^{2/3}\|w(\tau)\|_{L^4}^{4/3}.\end{aligned}$$

According to Gronwall's Lemma, we find out the desired result. □

Global bound of the vorticity in L^∞:

Proposition 2.5. *Let v be a smooth divergence free field with $w \triangleq \operatorname{curl} v$. We assume that (w,θ) be a smooth solution for (1), then we have*

$$\|w\|_{L^\infty} \le C_T.$$

Proof. From the maximum principle of (6), we have

$$\|w(t)\|_{L^\infty} \le \|w_0\|_{L^\infty} + \|\nabla\theta\|_{L^1_T L^\infty}.$$

We have,

$$\kappa_* \|\nabla\theta(t)\|_{L^1_T L^\infty} \le \|\nabla\Phi\|_{L^1_T L^\infty}. \tag{11}$$

On one hand, from interpolation inequality, we obtain

$$\begin{aligned} \|\nabla\Phi\|_{L^\infty} &\le C\|\nabla\Phi\|_{L^2}^{1/3}\|\Delta\Phi\|_{L^4}^{2/3} \\ &\lesssim C_T\|\Delta\Phi\|_{L^4}^{2/3}. \end{aligned}$$

On the other hand, we have, from Ladyzhenskaya's inequality:

$$\|\Delta\Phi(t)\|_{L^4}^{2/3} \lesssim \|\Delta\Phi(t)\|_{L^2}^{1/3}\|\Delta\nabla\Phi(t)\|_{L^2}^{1/3}.$$

Then from Young's inequality and (10), we arrive to

$$\|\Delta\Phi(t)\|_{L^1_T L^4}^{2/3} \lesssim C_T.$$

Hence,

$$\|\nabla\theta(t)\|_{L^1_T L^\infty} \le \frac{1}{\kappa_*}\|\nabla\Phi\|_{L^1_T L^\infty} \lesssim C_T,$$

from this we conclude that

$$\|w(t)\|_{L^\infty} \le C_T.$$

□

With these results we can obtain the existence of global solutions for the system (1) and it will be done in similar way in [28], thus we omit the details.

3. Uniqueness

This part is concerned with the uniqueness of the solution for the system (1). For the proof we adopt the Yudovich method. We assume that the system (1) admits to solutions (v_1,θ_1,p_1) and (v_2,θ_2,p_2) with same initial datum and we set $\bar v = v_1 - v_2$, $\bar\theta = \theta_1 - \theta_2$ and $\bar p = p_1 - p_2$.

$$\begin{cases} \bar\theta_t + v_2\cdot\nabla\bar\theta - \operatorname{div}(\kappa(\theta_2)\nabla\bar\theta) = -\bar v\cdot\nabla v_1 + \operatorname{div}((\kappa(\theta_2)-\kappa(\theta_1))\nabla\theta_1) + \bar v^2 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \bar v_t + v_2\cdot\nabla\bar v + \nabla\bar p = \bar\theta\vec e_2 - \bar v\cdot\nabla v_1 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \end{cases}$$

From the classical L^2-estimate we find out, for $r\in[2,\infty)$,

$$\frac{1}{2}\frac{d}{dt}(\|\bar{v}(t)\|_{L^2}^2)\leq\|\nabla v_1\|_{L^r}\|\bar{v}\|_{L^{2r'}}^2+\|\bar{v}\|_{L^2}\|\bar{\theta}\|_{L^2}$$
$$\lesssim r\|\nabla v_1\|_{\mathbb{L}}\|\bar{v}\|_{L^\infty}^{2/r}\|\bar{v}\|_{L^2}^{2/r'}+\|\bar{v}\|_{L^2}\|\bar{\theta}\|_{L^2},$$

where $\|\nabla f\|_{\mathbb{L}}\triangleq\sup_{r\in[2,\infty)}\frac{\|\nabla f\|_{L^r}}{r}$.
In a similar way, we estimate the temperature term $\bar{\theta}$, where the standard L^2-estimate gives

$$\frac{1}{2}\frac{d}{dt}(\|\bar{\theta}(t)\|_{L^2}^2)+\kappa_*\|\nabla\bar{\theta}\|_{L^2}^2\leq\|\nabla\theta_1\|_{L^\infty}\|\bar{\theta}\|_{L^2}\|\bar{v}\|_{L^2}+\|\kappa(\theta_2)-\kappa(\theta_1)\|_{L^4}\|\nabla\theta_1\|_{L^4}\|\nabla\bar{\theta}\|_{L^2}$$
$$+\|\bar{v}\|_{L^2}\|\theta\|_{L^2}$$

Combining Young's inequality with an interpolation theorem, we get

$$\frac{1}{2}\frac{d}{dt}(\|\bar{\theta}(t)\|_{L^2}^2)+\kappa_*\|\nabla\bar{\theta}\|_{L^2}^2\leq(1+\|\nabla\theta_1\|_{L^\infty})\|\bar{\theta}\|_{L^2}\|\bar{v}\|_{L^2}+\|\bar{\theta}\|_{L^2}^{1/2}\|\nabla\bar{\theta}_1\|_{L^4}\|\nabla\bar{\theta}\|_{L^2}^{3/2}$$
$$\lesssim(1+\|\nabla\theta_1\|_{L^\infty})\|\bar{\theta}\|_{L^2}\|\bar{v}\|_{L^2}+\|\bar{\theta}\|_{L^2}^2\|\nabla\bar{\theta}_1\|_{L^4}^{4/3}+\frac{\kappa_*}{2}\|\nabla\bar{\theta}\|_{L^2}^2.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}(\|\bar{\theta}(t)\|_{L^2}^2)+\frac{\kappa_*}{2}\|\nabla\bar{\theta}\|_{L^2}^2\leq(1+\|\nabla\theta_1\|_{L^\infty})\|\bar{\theta}\|_{L^2}\|\bar{v}\|_{L^2}+\|\bar{\theta}\|_{L^2}^2\|\nabla\bar{\theta}_1\|_{L^4}^{4/3}.$$

Let δ be a small parameter and by setting

$$\mathbb{Y}_\delta(t)\triangleq\Big(\|\bar{v}\|_{L^2}^2+\|\bar{\theta}\|_{L^2}^2+\delta\Big)^{\frac{1}{2}}.$$

According to the previous estimates, we find

$$\frac{d}{dt}\mathbb{Y}_\delta(t)\lesssim r\|\nabla v_1\|_{\mathbb{L}}\|\bar{v}\|_{L^\infty}^{2/r}\mathbb{Y}_\delta^{1-2/r}(t)+C\Big(1+\|\nabla\theta_1\|_{L^\infty}+\|\nabla\theta_1\|_{L^4}^{4/3}\Big)\mathbb{Y}_\delta(t).$$

Setting $\Upsilon_\delta(t)\triangleq\exp\Big(-\int_0^t(1+\|\nabla\theta_1(\tau)\|_{L^\infty}+\|\nabla\theta_1(\tau)\|_{L^4}^{4/3})d\tau\Big)\mathbb{Y}_\delta(t)$.
From the above estimates, we have

$$\frac{2}{r}\Upsilon_\delta^{\frac{2}{r}-1}(t)\frac{d}{dt}\Upsilon_\delta(t)\lesssim\|\nabla v_1\|_{\mathbb{L}}\|\bar{v}\|_{L^\infty}^{2/r}exp\big(-\frac{2}{r}\int_0^t(1+\|\nabla\theta_1(\tau)\|_{L^\infty}+\|\nabla\theta_1(\tau)\|_{L^4}^{4/3})d\tau\big),$$

after an integration with respect to time, we infer that

$$\Upsilon_\delta(t)\leq\Big(\delta^{1/r}+C\int_0^t\|\nabla v_1\|_{\mathbb{L}}\|\bar{v}\|_{L^\infty}^{2/r}\Big)^{r/2}.$$

Since $\|\nabla v_1\|_{\mathbb{L}}$ is locally bounded, then we can find $T_0 > 0$ such that

$$\int_0^{T_0} \|\nabla v_1(\tau)\|_{\mathbb{L}} d\tau < \frac{1}{2}.$$

Thus by letting $r \to +\infty$ we deduce that $(\bar{v}, \bar{\theta})$ equals to 0 on $[0, T_0]$ and by using a bootstrap argument we can deduce that $(\bar{v}, \bar{\theta})$ equals to 0 on $[0, T]$, for all $T > 0$, which finishes the proof of the uniqueness part.

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