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CHARACTERIZING SOME GENERALIZATIONS OF LINDELÖF FRAMES

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ABSTRACT. We study frames with the property that whenever two closed sublocales with a compact intersection cover the frame, then at least one of the closed sublocales in the cover is Lindelöf. Frames with this characteristic will be called *LJ*-frames. One advantage of studying *LJ*-frames is that it provides grounds to explore some interactions between connectedness, compactness, and the Lindelöf property. The class of *LJ*-frames contains all connected frames without points; so not every *LJ*-frame is spatial. The latter is also supported by the fact that *LJ*-frames are a generalization of Lindelöf frames, and Lindelöf frames need not be spatial; the Booleanization of the frame of open sets of the real line is one example of non-spatial Lindelöf frames. We show that almost Lindelöf Boolean frames are *LJ*-frames. Localic counterparts of some results that are available in spaces are provided herein. Moreover, we characterize *LJ*-frames via their remainders in compact regular frames; the latter has not been explored in spaces.

1. Introduction

In this paper, we study and characterize some classes of frames which are generalizations of the Lindelöf ones. For details and motivation, let us recall some jargon from the literature. Throughout, "space" means "topological space." An *LJ-space* is a space X with the property that if $X = A \cup B$, where A and B are closed in X with $A \cap B$ compact, then either A or B is Lindelöf (see paper by Gao in [15]). This is a generalization of the notion of a *J-space* that Michael introduced in [21][1] which has the same definition modulo replacing "Lindelöf" with "compact" (also studied by Mthethwa and Taherifar in [23, 24]).

We study *LJ*-spaces in a larger terrain of pointfree topology. An *LJ*-frame is the pointfree topology counterpart of an *LJ*-space, which we defined in the abstract. A space X is an *LJ*-space if and only if the frame of open sets of X is an *LJ*-frame. One of the nice characteristics of the pointfree counterpart of *J*-spaces (called *J-frames* by Mthethwa in [22]), at least from a pointfree topology practitioner viewpoint, is that a frame with no points is a *J*-frame precisely when it is connected. Since every *J*-frame is an *LJ*-frame, then all connected frames with no points are examples of non-spatial *LJ*-frames; a technique for constructing pointless connected frames is provided in this paper.

Here is a synopsis of this paper: §2 consists of all the basic vocabulary that we use throughout the paper. In §3, we show, among other things, that if a frame is almost Lindelöf (à la Dube [10])

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[1]The letter "*J*" is motivated by the fact that these spaces have a characterization that resembles the statement of the Jordan curve theorem, see [21, Proposition 3.3 (b)].

and Boolean, then it is an *LJ*-frame. We construct a non-spatial example of a *J*-frame. Taking a cue from Gao [15], we establish classes of frames called the strong *LJ*-frames, and the semi-strong *LJ*-frames. The latter classes of frames satisfy conditions which are stronger than the *LJ*-frame property; we provide differentiating examples. Perhaps, the interesting aspect of strong *LJ*-frames and the semi-strong *LJ*-frames is that the interplay between connectedness, compactness, and the Lindelöf property is witnessed: connected components of semi-strong *LJ*-frames and strong *LJ*-frame inherit the *LJ*-semi-strongness and the *LJ*-strongness from the ambient frame. There exists a *J*-frame with a connected closed sublocale that does not inherit this property. In §4, we characterize *LJ*-classes of frames via remainders in compactifications; and these results do not appear in the classical topology literature.

2. Preliminaries

2.1. *Frames.* Recall that a *frame* L is a complete lattice which satisfies the join-infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

while a *coframe* L is a complete lattice satisfying the meet-infinite distributive law:

$$x \vee \bigwedge S = \bigwedge \{x \vee s \mid s \in S\}$$

for every $x \in L$ and every $S \subseteq L$. Henceforth, we will write L and M for frames. The top and bottom elements of L will be denoted by 1 and 0, respectively. For any $a, b \in L$, we say that a is *rather below* b, written as $a \prec b$, if $a \wedge c = 0$ and $c \vee b = 1$, for some $c \in L$. The *pseudocomplement* of $a \in L$ is defined by $a^* = \bigvee\{x \in L \mid x \wedge a = 0\}$. A frame L is called *regular* if $a = \bigvee\{x \in L \mid x \prec a\}$, for all $a \in L$. A *Boolean frame* is a frame L such that $a^{**} = a$ for all $a \in L$. A subset $A \subseteq L$ is a cover of L if $1 = \bigvee A$. A frame L is *compact (Lindelöf)* if for any cover $A \subseteq L$ there exist a finite (countable) $S_0 \subseteq S$ such that $1 = \bigvee S_0$. An element $c \in L$ is *connected* if whenever $c = a \vee b$ and $a \wedge b = 0$, then either $a = 0$ or $b = 0$. A frame L is *connected* if its top element is connected, and L is locally connected if every element can be expressed as a join of connected elements in L. A *spatial frame* is a frame that is isomorphic to a frame, $\mathfrak{O}(X)$, of open sets of some topological space X. An element $p \neq 1$ in a lattice L is *meet-irreducible* if for any $a, b \in L$, with $a \wedge b \leq p$, either $a \leq p$ or $b \leq p$. By a *point* in a frame L, we mean an element $p \in L$ which is meet-irreducible.

2.2. *The sublocale lattice.* In a frame L, there is a binary operation $\to$, called the *Heyting operation*, such that for any $a, b, c \in L$, one has $c \leq a \to b \iff c \wedge a \leq b$. A *sublocale* of a frame L is a subset S of L such that S is closed under arbitrary meets and for each $x \in L$ and each $s \in S$, $x \to s \in S$. The sets $\mathfrak{c}_L(a) = {\uparrow}a = \{x \in L \mid a \leq x\}$ and $\mathfrak{o}_L(a) = \{a \to x \mid x \in L\}$ are sublocales of L, called the *closed* and the *open* sublocales associated with $a \in L$. The lattice $S(L)$ of all sublocales of a frame L is, in general, a non-complemented coframe under inclusion (see [25, Theorem III.3.2.1]) where the meets are precisely the intersections and the joins are as follows:

$$\bigvee_{i \in I} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i \in I} S_i\}$$

for any $\{S_i\}_{i \in I} \subseteq S(L)$. The bottom element of $S(L)$ is $\mathsf{O} := \{1\}$ (we call this the *void* sublocale) and L is its top element. Throughout the paper, joins of sublocales of L will be taken in the coframe $S(L)$. We shall say that a collection of sublocales $\{S_i \mid i \in I\} \subseteq S(L)$ is a cover of a sublocale S if $S \subseteq \bigvee_i S_i$. If all the S_i are closed (or open), we will say that $\{S_i \mid i \in I\}$ is a *closed* (or an *open*) *cover* of S. Being a coframe, we have the co-Heyting operation $\smallsetminus$ on $S(L)$, satisfying $A \smallsetminus B \subseteq C \Longleftrightarrow A \subseteq B \vee C$ for any $A, B, C \in S(L)$. For any $S, R \in S(L)$, we have the following formula by Isbell [16]:

$$R \smallsetminus S = \bigvee \{T \in S(L) \mid T \subseteq R \text{ and } T \cap S = \mathrm{O}\}.$$

The *supplement*, $L \smallsetminus S$, of a sublocale S in a frame L is given by:

$$L \smallsetminus S = \bigvee \{T \in S(L) \mid T \cap S = \mathrm{O}\} = \bigcap \{R \in S(L) \mid R \vee S = L\}.$$

For the proof of the second equality in the formula above, see [27, Lemma 1.1 (3)]. Note that $S \vee (L \smallsetminus S) = L$, but in general, the equation $S \cap (L \smallsetminus S) = \mathrm{O}$ does not hold. We say S is *complemented in* L if $S \cap (L \smallsetminus S) = \mathrm{O}$; that is, $L \smallsetminus S$ is the complement of S in the lattice $S(L)$. We shall often use the following fact (for example, see [22, Lemma 3.1]) for free:

Lemma 2.1. *If $S, T \in \mathsf{S}(L)$ with $S \subseteq T$ and S or T is complemented, then $L \smallsetminus T \subseteq L \smallsetminus S$.*

Since the content of [13, Remarks 3.3 (a) & (b)] prevails in this paper, we paraphrase it below for the coframe $S(L)$:

Lemma 2.2. *Let S be a complemented sublocale of L with complement $L \smallsetminus S$. Then, for any $T \in \mathsf{S}(L)$, we have:*

(1) *$S \smallsetminus T = S \cap (L \smallsetminus T)$ and $T \smallsetminus S = T \cap (L \smallsetminus S)$.*
(2) *$S \smallsetminus (R \smallsetminus T) = (S \cap T) \vee (S \smallsetminus R)$ whenever T is complemented. In particular, if T is complemented, then $S \smallsetminus (L \smallsetminus T) = S \cap T$.*

The *closure*, $\overline{S}$, and the *interior*, $\mathrm{int}_L(S)$, of a sublocale S of L are given by:

$$\overline{S} = \{x \in L \mid x \geq \bigwedge S\} \quad \text{and} \quad \mathrm{int}_L(S) = \bigvee \{\mathfrak{o}_L(x) \mid \mathfrak{o}_L(x) \subseteq S\}.$$

A sublocale S of L is *dense* if $\overline{S} = L$. The *frontier* of $S \in S(L)$ is the sublocale given by $\mathrm{Fr}_L(S) = \overline{S} \cap \overline{L \smallsetminus S}$. We speak of *disjoint* sublocales S and T if $S \cap T = \mathrm{O}$. We say that a sublocale S is *connected* if the top element of S is connected in S. Here is an equivalent formulation of connectedness:

Lemma 2.3. *The following conditions are equivalent:*

(1) *A non-void sublocale S of a frame L is connected.*
(2) *Whenever $a, b \in L$ and $S \subseteq \mathfrak{o}_L(a) \vee \mathfrak{o}_L(b)$ with $S \cap \mathfrak{o}_L(a) \cap \mathfrak{o}_L(b) = \mathrm{O}$, then either $S \cap \mathfrak{o}_L(a) = \mathrm{O}$ or $S \cap \mathfrak{o}_L(b) = \mathrm{O}$.*
(3) *Whenever $a, b \in L$ and $S \subseteq \mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)$ with $S \cap \mathfrak{c}_L(a) \cap \mathfrak{c}_L(b) = \mathrm{O}$, then either $S \cap \mathfrak{c}_L(a) = \mathrm{O}$ or $S \cap \mathfrak{c}_L(b) = \mathrm{O}$.*

2.3. *Definitions and examples of LJ-classes of spaces.* Recall that a space X is *Lindelöf* if every open cover of X can be reduced to a countable one. Now, let us recall some formal definitions from [15]:

Definition 2.4. A space X is called:

(1) a *strong LJ-space* if every compact $K \subseteq X$ is contained in a closed Lindelöf $A \subseteq X$ such that $X \smallsetminus A$ is connected.

(2) a *semi-strong LJ-space* if every compact $K \subseteq X$ is contained in a closed Lindelöf $A \subseteq X$ such that $A \cup C = X$ for some connected $C \subseteq X \setminus K$.

(3) an *LJ-space* if whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is Lindelöf.

Collectively, the we shall call the spaces above the *LJ-classes* of spaces.

From [15, Theorem 1], one has the implications: Lindelöf $\Longrightarrow$ strong *LJ*-space $\Longrightarrow$ semi-strong *LJ*-space $\Longrightarrow$ *LJ*-space. These implications are, in general, not reversible:

Example 2.5. The long line $Z := [0, \omega_1) \times [0, 1)$ with the order topology generated by the lexicographical order is a regular Hausdorff space (for example, see Steen and Seebach Jr. [29]).

(1) The ordinal space $[0, \omega_1)$ is a regular Hausdorff space [29], and it is an *LJ*-space which is not a semi-strong *LJ*-space, by [15, Proposition 4 (1)].

(2) In [15, Example 5 (1)], a regular Hausdorff space Y (being a regular Hausdorff space $Z \times \mathbb{R}^+$, where $\mathbb{R}^+$ is a set of non-negative real numbers with the standard topology) that is a semi-strong *LJ*-space but not a strong *LJ*-space is given.

(3) It is shown in [15, Proposition 3] that Z is a strong *LJ*-space, and of course, Z is not Lindelöf (see [29]).

For more on *LJ-classes* of spaces, we refer the reader to [15].

3. *LJ*-classes of frames: properties and characterisations

Note that a sublocale S of a frame L is Lindelöf if and only if whenever S covered by a collection $\mathscr{U}$ of open sublocales of L, then it is covered by countably many members of $\mathscr{U}$.

Definition 3.1. A frame L is called:

(1) a *strong LJ-frame* if every compact sublocale $K \subseteq L$ is contained in a closed Lindelöf sublocale $S \subseteq L$ such that $L \setminus S$ is connected.

(2) a *semi-strong LJ-frame* if for every compact sublocale $K \subseteq L$, there exists a closed Lindelöf sublocale $S \subseteq L$ containing K and a connected sublocale $C \subseteq L \setminus K$ with $S \vee C = L$.

(3) an *LJ-frame* if whenever $L = S \vee T$ for closed sublocales S, T of L with $S \cap T$ compact, then S or T is Lindelöf.

Collectively, we call the frames defined in Definition 3.1 the *LJ-classes* of frames.

Remark 3.2. Recall that a property $\mathscr{P}$ in pointfree topology is a *conservative* extension of the same property in spaces given that a space X has property $\mathscr{P}$ in classical topology if and only if the frame $\mathfrak{O}X$ has property $\mathscr{P}$ in pointfree topology:

(1) It is not difficult to verify that Definition 3.1 (3) is a conservative pointfree extension of the corresponding notion in spaces if the T_D axiom is assumed; the same is true for the J-frame notion.

(2) Definition 3.1 (1) is conservative when regularity and the T_D axiom is assumed; that is, a T_D-space X is a regular strong *LJ*-space if and only if $\mathfrak{O}X$ is a regular strong *LJ*-frame.

(3) For a T_D-space X, it is true that if X is a regular semi-strong *LJ*-space then $\mathfrak{O}X$ is a regular semi-strong *LJ*-frame. We do not know whether the converse of the latter holds true for any T_D-space, the glitch here is that a connected sublocale of $\mathfrak{O}X$ is not expected to be induced, unless if it is a component; in which case it would be closed (see [25, XIII.2.5.3]) and therefore, induced. Recall

that *scattered spaces* are characterized by the property that every non-empty closed set contains an isolated point. A frame L is said to be *scattered* if $S(L)$ is a frame. A frame L is scattered if and only if $S(L)$ is a Boolean algebra, see [5]. Scatteredness is conservative since, if X is a T_D-space, then X is scattered if and only if each sublocale of $S(\mathfrak{O}X)$ is complemented, by [4, Theorem 2.4.2]. Hence, if X is a T_D-space, then X is a scattered regular semi-strong LJ-space if and only if $\mathfrak{O}X$ is a regular scattered semi-strong LJ-frame.

Remark 3.3. An equivalent formulation of Definition 3.1 is provided below using elements:

(1)′ A frame L is an LJ-frame if for any $a,b \in L$ such that $a \wedge b = 0$ and $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)$ is compact, either $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(b)$ is Lindelöf. We shall use this formulation in Proposition 3.6.

(2)′ A regular frame L is a *strong LJ-frame* if and only if for every $a \in L$ with $\mathfrak{c}_L(a)$ compact, there exists a connected element $b \leq a$ in L such that $\mathfrak{c}_L(b)$ is Lindelöf.

(3)′ A regular frame L is a *semi-strong LJ-frame* if and only if whenever $k \in L$ and $\mathfrak{c}_L(k)$ is compact, then there exists $l \leq k$ with $\mathfrak{c}_L(l)$ Lindelöf and a connected sublocale $C \subseteq \mathfrak{o}_L(k)$ such that $\mathfrak{c}_L(l) \vee C = L$.

Verily, (2)′ and (3)′ are true for all frames with the property that compact sublocales are closed (e.g., the strongly Hausdorff frames).

The following characterization of pointless J-frames was proved in [22, Proposition 4.9] for regular frames; we provide a proof which does not assume regularity:

Proposition 3.4. *Let L be a frame with no points. Then L is a J-frame if and only if it is connected.*

Proof. ($\Longrightarrow$) Let L be a J-frame with no points. Suppose L is not connected. Then $L = S \vee T$ for some non-trivial closed sublocales S and T of L such that $S \cap T = \mathrm{O}$. Observe that $S \cap T$ is compact, and L is a J-frame, so we may assume S is compact. But every nontrivial compact locale has at least one point, by [18, Lemma III.1.9]; so S has at least one point. By [25, Lemma VI.3.1.1], $\mathrm{Pt}(S) = S \cap \mathrm{Pt}(L)$, so L has at least one point, which is a contradiction.

($\Longleftarrow$) Suppose L is connected. Let $L = S \vee T$, where S,T are closed sublocales of L such that $S \cap T$ is compact. If $S \cap T \neq \mathrm{O}$, then $S \cap T$ has at least one point by [18, Lemma III.1.9], so L has at least one point by [25, Lemma VI.3.1.1], which is a contradiction. Thus, $S \cap T = \mathrm{O}$. Since L is connected, then $S = \mathrm{O}$ or $T = \mathrm{O}$. Hence, S or T is compact. □

Example 3.5. Recall that the *Booleanization* of a frame L is the frame whose underlying set is $\{a \in L : a = a^{**}\} = \{b^* : b \in L\}$:

(i) A Boolean algebra is a distributive lattice in which every element is complemented, and a frame that is a Boolean algebra is called a *Boolean frame*. Let B be a pointless Boolean frame (for example, the Booleanization of the frame of open sets of real numbers). Then B is a disconnected frame with no points. Thus, B is a non-spatial frame that is not a J-frame, by Proposition 3.4.

(ii) A connected frame with no points is a non-spatial J-frame; and whence a non-spatial LJ-frame. It would be pleasant to know whether or not non-spatial examples of LJ-frame which are not J-frames exists.

For the next result, we consider *almost Lindelöf frames*; these were characterized by Dube in [10, Proposition 4.10] as those frames with the property that whenever $a,b \in \mathrm{Coz}\,L$ and $a \vee b = 1$, then $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(b)$ is Lindelöf.

Proposition 3.6. *Let L be a Boolean frame. If L is almost Lindelöf, then L is an LJ-frame.*

Proof. Suppose that L is an almost Lindelöf frame. To show that L is an LJ-frame, let $a,b \in L$ be such that $a \wedge b = 0$ and $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)$ is compact. Since L is a Boolean frame, then $a \vee a^* = 1$ and $a, a^* \in L = \mathrm{Coz}\, L$. But L is almost Lindelöf, so $\mathfrak{c}_L(a)$ or $\mathfrak{c}_L(a^*)$ is Lindelöf. If $\mathfrak{c}_L(a)$ is Lindelöf, then we are done. If $\mathfrak{c}_L(a^*)$ is Lindelöf, then $\Big(\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)\Big) \vee \mathfrak{c}_L(a^*)$ is Lindelöf. Now, note that $b = b \wedge a^*$, so $\Big(\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b)\Big) \vee \mathfrak{c}_L(a^*) = L \cap \mathfrak{c}_L(b \wedge a^*) = \mathfrak{c}_L(b)$. Hence, $\mathfrak{c}_L(b)$ is Lindelöf. □

It is worth emphasizing again that there are non-spatial LJ-frames, which is essentially the reason for the importance of Proposition 3.6. In the proof of the result below (and for the remainder of this paper), we use for free the fact that the Lindelöf property is inherited by closed sublocales.

Theorem 3.7. *Let L be a frame. Consider the following conditions:*

(1) *L is Lindelöf.*
(2) *L is a strong LJ-frame.*
(3) *L is a semi-strong LJ-frame.*
(4) *L is an LJ-frame.*

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$.

Proof. $(1) \Longrightarrow (2)$ Note that $L = \mathfrak{c}_L(0)$, so for any compact sublocale K of L, one has that $K \subseteq \mathfrak{c}_L(0)$, $\mathfrak{c}_L(0)$ is Lindelöf, and $L \smallsetminus \mathfrak{c}_L(0) = \mathfrak{o}_L(0) = \mathsf{O}$ is connected. Therefore, L is a strong LJ-frame.

$(2) \Longrightarrow (3)$ Let K be a compact sublocale of L. So, there exists a closed Lindelöf sublocale S of L containing K such that $L \smallsetminus S$ is connected. Since S is complemented in L, then $K \subseteq S$ implies $L \smallsetminus S \subseteq L \smallsetminus K$. We have $(L \smallsetminus S) \vee S = L$, so L is a semi-strong LJ-frame.

$(3) \Longrightarrow (4)$ Let $L = S \vee T$ where S,T are closed sublocales of L with $S \cap T$ compact. Use the LJ-strongness of L to find a closed Lindelöf sublocale A of L with $S \cap T \subseteq A$, and a connected sublocale $C \subseteq L \smallsetminus (S \cap T)$ such that $A \vee C = L$. From $L = S \vee T$ we get $C = (S \cap C) \vee (T \cap C)$. Since $S \cap T$ is complemented in L, $C \subseteq L \smallsetminus (S \cap T)$ implies that $C \cap (S \cap T) = \mathsf{O}$. But C is connected, so $S \cap C = \mathsf{O}$ or $T \cap C = \mathsf{O}$. If $S \cap C = \mathsf{O}$, then $A \vee C = L \Longrightarrow S \cap A = S \Longrightarrow S \subseteq A \Longrightarrow S$ is Lindelöf. Similarly, if $T \cap C = \mathsf{O}$ then T is Lindelöf. So, L is an LJ-frame. □

Example 3.8. Recall that regularity and Lindelöfness are conservative properties.

(1) Since any ordinal space is a scattered, Hausdorff, and regular (see [29]), then $[0,\omega_1)$ has all these properties; in particular, it is a scattered regular T_D-space. Thus, by Example 2.5 (1) and Remark 3.2 (3), we have that $\mathfrak{O}([0,\omega_1))$ is a regular LJ-frame that is not a semi-strong LJ-frame.
(2) Since each Hausdorff space is a T_D-space, then for the regular Hausdorff space Y in Example 2.5 (2), $\mathfrak{O}Y$ is a regular semi-strong LJ-frame, by Remark 3.2 (3). Moreover, $\mathfrak{O}Y$ is a regular frame that is not a strong LJ-frame, by conservativeness of the LJ-strongness for regular spaces (Remark 3.2 (2)).
(3) The frame $\mathfrak{O}Z$, for Z in Example 2.5 (3), is an example of a regular non-Lindelöf strong LJ-frame.

Characterizations of strong and semi-strong LJ-frames in terms of open covers of certain sublocales are what we present next:

Proposition 3.9. *The following conditions are equivalent on a regular frame L:*

(1) *L is a semi-strong LJ-frame if and only if whenever K is a compact sublocale of L and $\mathscr{W}$ is a cover of $L \smallsetminus K$ by disjoint open sublocales of L, then there is $W \in \mathscr{W}$ and a connected $C \subseteq W$ such that $\overline{L \smallsetminus C}$ is Lindelöf.*

(2) *L is a strong LJ-frame if and only if whenever K is a compact sublocale of L and $\mathscr{W}$ is a cover of $L \smallsetminus K$ by disjoint open sublocales of L, then there is $W \in \mathscr{W}$ and an open connected $C \subseteq W$ such that $L \smallsetminus C$ is Lindelöf.*

Proof. (1) For the forward implication, suppose K is a compact sublocale of L with $L \smallsetminus K \subseteq \bigvee \mathscr{W}$ for a collection $\mathscr{W}$ of disjoint open sublocales of L. Using the *LJ*-semi-strongness of L, we find a connected $C \subseteq L \smallsetminus K$ and a closed Lindelöf sublocale S such that $K \subseteq S$ with $S \vee C = L$. Suppose $C \neq C \cap W$ for all $W \in \mathscr{W}$. Fix W_0 in $\mathscr{W}$, and let $W_1 = \bigvee\{W \in \mathscr{W} \mid W \neq W_0\}$. Then we have

$$C = (C \cap W_0) \vee (C \cap W_1) \quad \text{with} \quad (C \cap W_0) \cap (C \cap W_1) = \mathsf{O}.$$

The latter is not possible since C is connected. So, $C \subseteq W$ for some $W \in \mathscr{W}$. Furthermore, since $S \vee C = L$, and S is complemented in L, then $L \smallsetminus S \subseteq C$. Taking supplements and using the complementedness of $L \smallsetminus S$, we get $L \smallsetminus C \subseteq S$. Thus, $\overline{L \smallsetminus C} \subseteq S$, since S is closed. Therefore, $\overline{L \smallsetminus C}$ is Lindelöf.

Conversely, if K is a compact sublocale of L, then by regularity of L, K is closed. So, $\mathscr{W} = \{L \smallsetminus K, \mathsf{O}\}$ is a disjoint open cover of $L \smallsetminus K$. By the hypothesis, there is a connected sublocale $C \subseteq L \smallsetminus K$ such that $\overline{L \smallsetminus C}$ is Lindelöf. Moreover, $C \subseteq L \smallsetminus K \Longrightarrow K \subseteq L \smallsetminus C \subseteq \overline{L \smallsetminus C}$ and $\overline{L \smallsetminus C} \vee C = L$; so L is a semi-strong *LJ*-frame.

(2) For the forward direction, proceed as in (1), but use the *LJ*-strongness of L to find a closed Lindelöf sublocale S of L such that $K \subseteq S$ and $L \smallsetminus S$ is connected. Put $C := L \smallsetminus S$. Then C is the open connected sublocale with the property that $C \subseteq W$ for some $W \in \mathscr{W}$ (with the same proof as in (1)), and $L \smallsetminus C = S$ is Lindelöf.

For the reverse implication, proceed as in (1) again, but here, find an open connected $C \subseteq L \smallsetminus K$ such that $L \smallsetminus C$ is Lindelöf. Complementedness of K and C implies that $K = L \smallsetminus (L \smallsetminus K) \subseteq L \smallsetminus C$ and $L \smallsetminus (L \smallsetminus C) = C$ is connected. ☐

In (1) $\Longleftrightarrow$ (4) of Theorem 3.12, we prove the *LJ*-frame counterpart of Proposition 3.9. We shall need the following lemmas:

Lemma 3.10. *If B is a closed non-Lindelöf sublocale of L and a sublocale $C \subseteq B$ is Lindelöf, then there is a closed non-Lindelöf sublocale $D \subseteq B$ such that $D \cap C = \mathsf{O}$.*

Proof. Let B be a closed non-Lindelöf sublocale of L and take a collection $\mathscr{U}$ of open sublocales of L such that $B \subseteq \bigvee\{U \mid U \in \mathscr{U}\}$ and $\mathscr{U}$ has no countable subcover of B. Use Lindelöfness of C to find a countable $\mathscr{V} \subseteq \mathscr{U}$ such that $C \subseteq \bigvee\{V \mid V \in \mathscr{V}\} := S$. Put $D := B \smallsetminus S$, and have $D \subseteq B$. Since S is complemented in L (being open), then $D \cap C = (B \smallsetminus S) \cap C = \big((L \smallsetminus S) \cap B\big) \cap C = \mathsf{O}$; where the second equality follows by Lemma 2.2 (1). ☐

Lemma 3.11. *Let L be an LJ-frame. If $\mathscr{W}$ is a collection of disjoint open sublocales of L that covers $L \smallsetminus K$ with K compact and $K \subseteq U$ for some open $U \in \mathsf{S}(L)$, then*

$$\mathscr{Y} = \{W \in \mathscr{W} \mid W \nsubseteq U\}$$

is countable.

Proof. Let $\mathscr{W}, K$, and U be as postulated and suppose that $\mathscr{Y}$ is uncountable. Then $\mathscr{W}$ is an uncountable set and we can partition $\mathscr{W}$ as $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$, where $\mathscr{W}_1 \cap \mathscr{W}_2 = \emptyset$, with both $\mathscr{W}_1 \cap \mathscr{Y} = \{W \in \mathscr{W}_1 \mid W \nsubseteq U\}$ and $\mathscr{W}_2 \cap \mathscr{Y} = \{W \in \mathscr{W}_2 \mid W \nsubseteq U\}$ uncountable. Let $V_1 = \bigvee \mathscr{W}_1 = \bigvee\{W \in \mathscr{W} \mid W \in \mathscr{W}_1\}$ and $V_2 = \bigvee \mathscr{W}_2 = \bigvee\{W \in \mathscr{W} \mid W \in \mathscr{W}_2\}$. Then $L \smallsetminus K \subseteq \bigvee\{W \mid W \in \mathscr{W}\} = V_1 \vee V_2$ and $V_1 \cap V_2 = \mathsf{O}$; the latter because the members of $\mathscr{W}$ are disjoint open sublocales, and any sublocale distributes over the join of open ones. Moreover, since each V_i is a join of open sublocales of L, then V_1 and V_2 are open sublocales of L. Since V_1 and V_2 are complemented in L, then $L \smallsetminus K \subseteq V_1 \vee V_2 \Longrightarrow (L \smallsetminus V_1) \cap (L \smallsetminus V_2) = L \smallsetminus (L \smallsetminus K) \subseteq K$. Since K is a compact sublocale of L and $(L \smallsetminus V_1) \cap (L \smallsetminus V_2)$ is closed, then $(L \smallsetminus V_1) \cap (L \smallsetminus V_2)$ is compact. Now, $V_1 \cap V_2 = \mathsf{O} \Longrightarrow (L \smallsetminus V_1) \vee (L \smallsetminus V_2) = L$. Therefore, $L \smallsetminus V_1$ or $L \smallsetminus V_2$ is Lindelöf, since L is an *LJ*-frame. Suppose $L \smallsetminus V_2$ is Lindelöf. Therefore, $\overline{V_1}$ is Lindelöf because $V_1 \cap V_2 = \mathsf{O} \Longrightarrow V_1 \subseteq L \smallsetminus V_2 \Longrightarrow \overline{V_1} \subseteq L \smallsetminus V_2$. Put $C := \overline{V_1} \smallsetminus U$. Since U is complemented, then $C = (L \smallsetminus U) \cap \overline{V_1} \subseteq \overline{V_1}$, so C is Lindelöf. We now show that $C \subseteq \bigvee(\mathscr{W}_1 \cap \mathscr{Y})$. First, use the fact that $(L \smallsetminus V_1) \cap (L \smallsetminus V_2) \subseteq K$ and $\overline{V_1} \subseteq L \smallsetminus V_2$ to get $(L \smallsetminus V_1) \cap \overline{V_1} \subseteq K$. Intersecting with $L \smallsetminus U$ on both sides of the previous containment, and using the fact that $K \subseteq U$ with U complemented, gives us $(L \smallsetminus U) \cap (L \smallsetminus V_1) \cap \overline{V_1} \subseteq (L \smallsetminus U) \cap K = \mathsf{O}$. Let $A = \bigvee\{W \in \mathscr{W}_1 \mid W \subseteq U\}$ and $B = \bigvee\{W \in \mathscr{W}_1 \mid W \nsubseteq U\}$. Then $V = A \vee B$, and we have $(L \smallsetminus U) \cap (L \smallsetminus (A \vee B)) \cap \overline{V_1} = \mathsf{O}$. That is, $(L \smallsetminus U) \cap (L \smallsetminus A) \cap (L \smallsetminus B) \cap \overline{V_1} = \mathsf{O}$, since A and B are complemented. But $A \subseteq U$, so $L \smallsetminus U \subseteq L \smallsetminus A$. This implies that $(L \smallsetminus U) \cap (L \smallsetminus B) \cap \overline{V_1} = \mathsf{O}$. It follows that $(L \smallsetminus U) \cap \overline{V_1} \subseteq L \smallsetminus (L \smallsetminus B) = B = \bigvee\{W \in \mathscr{W}_1 \mid W \nsubseteq U\}$. That is, $C \subseteq \bigvee(\mathscr{W}_1 \cap \mathscr{Y})$. So, there exists a countable cover $\mathscr{B} \subseteq \mathscr{W}_1 \cap \mathscr{Y}$ of C. But $\mathscr{W}_1 \cap \mathscr{Y}$ is infinite and uncountable, so we can find $W_0 \in \mathscr{W}_1 \cap \mathscr{Y}$ such that $W_0 \notin \mathscr{B}$. Now, $W_0 \nsubseteq U$ and $W_0 \cap W = \mathsf{O}$ for all $W \in \mathscr{B}$. Observe that $W_0 \subseteq \bigvee \mathscr{W}_1 = V_1 \subseteq \overline{V_1}$ and $W_0 \cap C \subseteq W_0 \cap \bigvee \mathscr{B} = \bigvee\{W_0 \cap W \mid W \in \mathscr{B}\} = \mathsf{O}$. From the latter, we get $W_0 \subseteq L \smallsetminus C = L \smallsetminus \big((L \smallsetminus U) \cap \overline{V_1}\big) = U \vee (L \smallsetminus \overline{V_1})$. Therefore $W_0 \subseteq \overline{V_1} \cap \big(U \vee L \smallsetminus \overline{V_1}\big) = \overline{V_1} \cap U \subseteq U$. This is a contradiction. So, $\mathscr{Y}$ is countable. □

Theorem 3.12. *The following conditions are equivalent on a frame L:*

(1) *L is an LJ-frame.*
(2) *For any sublocale S of L with $\mathrm{Fr}_L(S)$ compact, either $\overline{S}$ or $\overline{L \smallsetminus S}$ is Lindelöf.*
(3) *If S and T are disjoint closed sublocales of L with $\mathrm{Fr}_L(S)$ or $\mathrm{Fr}_L(T)$ compact, then S or T is Lindelöf.*
(4) *If K is a compact sublocale of L and $\mathscr{W}$ is a collection of disjoint open sublocales of L that covers $L \smallsetminus K$, then $L \smallsetminus W$ is Lindelöf for some $W \in \mathscr{W}$.*

Proof. (1) $\Longrightarrow$ (2) Take $S \in \mathcal{S}(L)$ such that $\mathrm{Fr}_L(S) = \overline{S} \cap \overline{L \smallsetminus S}$ is compact. Observe that $L = S \vee (L \smallsetminus S) \subseteq \overline{S} \vee \overline{L \smallsetminus S}$, therefore $L = \overline{S} \vee \overline{L \smallsetminus S}$. But L is an *LJ*-frame, so $\overline{S}$ or $\overline{L \smallsetminus S}$ is Lindelöf.

(2) $\Longrightarrow$ (3) Let S and T be disjoint closed sublocales of L with $\mathrm{Fr}_L(S)$ or $\mathrm{Fr}_L(T)$ compact. Suppose $\mathrm{Fr}_L(S)$ is compact, then $\overline{S}$ or $\overline{L \smallsetminus S}$ is Lindelöf, by the hypothesis. But S is closed, so $\overline{S} = S$. Thus, S or $\overline{L \smallsetminus S}$ is Lindelöf. If S is Lindelöf, then we are done. Suppose $\overline{L \smallsetminus S}$ is Lindelöf and notice that $S \cap T = \mathsf{O} \Longrightarrow T \subseteq L \smallsetminus S \subseteq \overline{L \smallsetminus S}$. Since T is closed and $\overline{L \smallsetminus S}$ is Lindelöf, then T is Lindelöf.

(3) $\Longrightarrow$ (1) Let $L = S \vee T$ where S, T are closed sublocales of L with $S \cap T$ compact. Suppose that T is not Lindelöf, we show that S is Lindelöf. Since $S \cap T \subseteq T$ and $S \cap T$ is Lindelöf (being compact), use Lemma 3.10 to find a closed non-Lindelöf sublocale $D \subseteq T$ such that $D \cap (S \cap T) = \mathsf{O}$. Therefore $S \cap D = \mathsf{O}$. Notice that $L = S \vee T$ and the complementedness of S implies that $L \smallsetminus S \subseteq T$, and since $\mathrm{Fr}_L(S) = \overline{S} \cap \overline{L \smallsetminus S} \subseteq S \cap T$, then $\mathrm{Fr}_L(S)$ is compact. So, S or D is Lindelöf, by the hypothesis. But D is not Lindelöf, so S is Lindelöf.

$(4) \Longrightarrow (1)$ Let $L = S \vee T$ where S, T are closed sublocales of L with $S \cap T$ compact. Put $K := S \cap T$. Then $L \smallsetminus K = (L \smallsetminus S) \vee (L \smallsetminus T)$ and $(L \smallsetminus S) \cap (L \smallsetminus T) = \mathsf{O}$. By (4), either $L \smallsetminus (L \smallsetminus S)$ or $L \smallsetminus (L \smallsetminus T)$ is Lindelöf; that is, S or T is Lindelöf.

$(1) \Longrightarrow (4)$ Let $\mathscr{W}$ be a collection of disjoint open sublocales of L that covers $L \smallsetminus K$ with K compact. We consider two cases:

Case 1: Suppose that there is a $W_0 \in \mathscr{W}$ such that $\overline{W_0}$ is non-Lindelöf. Put $W_1 := \bigvee\{W \in \mathscr{W} \mid W \neq W_0\}$. Observe that $L \smallsetminus K \subseteq W_0 \vee W_1$ and $W_0 \cap W_1 = \mathsf{O}$. The latter implies that $L = (L \smallsetminus W_0) \vee (L \smallsetminus W_1)$ and the former implies that $(L \smallsetminus W_0) \cap (L \smallsetminus W_1) \subseteq K$, so $(L \smallsetminus W_0) \cap (L \smallsetminus W_1)$ is compact. Since L is an *LJ*-frame, either $L \smallsetminus W_0$ or $L \smallsetminus W_1$ is Lindelöf. But $W_0 \cap W_1 = \mathsf{O} \Longrightarrow W_0 \subseteq L \smallsetminus W_1 \Longrightarrow \overline{W_0} \subseteq L \smallsetminus W_1$, so if $L \smallsetminus W_1$ is Lindelöf, then so is $\overline{W_0}$, and this is a contradiction. Thus, $L \smallsetminus W_0$ is Lindelöf. This proves that (4) holds true for this case.

Case 2: Suppose that $\overline{W}$ is Lindelöf for all $W \in \mathscr{W}$. We first show that L is Lindelöf. So let $\mathscr{V}$ be a collection of open sublocales of L that covers L. Use compactness of K to find a finite $\mathscr{V}_0 \subseteq \mathscr{V}$ such that $K \subseteq \bigvee \mathscr{V}_0 := U$. Then the set $\mathscr{Y} = \{W \in \mathscr{W} \mid W \nsubseteq U\}$ is countable, by Lemma 3.11. Write $\mathscr{Y} = \{W_n \mid n \in \mathbb{N}\}$. For each $W_n \in \mathscr{Y}$, we have that $\overline{W_n}$ is Lindelöf, and since $\overline{W_n} \subseteq \bigvee \mathscr{V}$, then $W_n \subseteq \overline{W_n} \subseteq \bigvee \mathscr{V}_n := V_n$ for a countable collection $\mathscr{V}_n \subseteq \mathscr{V}$. Now, $L = K \vee (L \smallsetminus K) \subseteq U \vee \left(\bigvee \mathscr{W}\right) = U \vee \left(\bigvee\{W \in \mathscr{W} \mid W \subseteq U\}\right) \vee \left(\bigvee\{W \in \mathscr{W} \mid W \nsubseteq U\}\right) = U \vee \bigvee\{W_n \mid n \in \mathbb{N}\} \subseteq (\bigvee \mathscr{V}_0) \vee \bigvee\{V_n \mid n \in \mathbb{N}\}$. Thus, L is Lindelöf, and whence, so are the closed sublocales of L. In particular, $L \smallsetminus W$ is Lindelöf for all $W \in \mathscr{W}$. □

Recall that a sublocale of a frame L is called a *component* if it is a maximal connected sublocale.

Theorem 3.13. *Let L be a regular frame. Consider the conditions:*

(1) *L is a strong LJ-frame.*

(2) *L is a semi-strong LJ-frame.*

(3) *L is an LJ-frame. If L is locally connected, then* $(1) \Longleftrightarrow (2) \Longleftrightarrow (3)$.

Proof. We have seen from Theorem 3.7 that, in general, $(1) \Longrightarrow (2) \Longrightarrow (3)$. So, let L be locally connected. We show that $(3) \Longrightarrow (1)$. Suppose L is an *LJ*-frame and let K be a compact sublocale of L. Let $\mathscr{C}$ be the collection of all distinct components of L. Then, by [25, XIII.2.5.2], elements of $\mathscr{C}$ are disjoint. Since L is a locally connected frame, then by [25, Proposition XIII.3.2], $L = \bigvee \mathscr{C}$ and each $C \in \mathscr{C}$ is open (and closed). One can deduce from [3, Corollary 1.4] that any open sublocale U of a locally connected frame L is a join of components of L that are contained in U. In particular, since K is closed (being a compact sublocale of a regular frame), we have $L \smallsetminus K = \bigvee\{C \in \mathscr{C} \mid C \subseteq L \smallsetminus K\}$. Therefore, by the equivalence of (1) and (4) in Theorem 3.12, there exists $C_0 \in \mathscr{C}$ such that $C_0 \subseteq L \smallsetminus K$ and $L \smallsetminus C_0$ is Lindelöf. Put $S := L \smallsetminus C_0$. Then S is a closed Lindelöf sublocale of L and $L \smallsetminus S = C_0$, so $L \smallsetminus S$ is connected. Finally, since K is complemented, the containment $C_0 \subseteq L \smallsetminus K$ implies that $K \subseteq L \smallsetminus C_0 = S$. □

It is shown in [25, Proposition XIII.2.4] that any binary join of connected sublocales having a non-trivial intersection is again connected. A component of any frame is always a closed sublocale, by [25, XIII.2.5.3].

Proposition 3.14.

(1) *If L is a strong LJ-frame, so is every component of L.*
(2) *If L is a semi-strong LJ-frame, so is every component of L.*

Proof.
(1) Suppose C is a component of L and let K be a compact sublocale of C. By the LJ-strongness of L, there is a closed Lindelöf sublocale $T \subseteq L$ such that $K \subseteq T$ and $L \smallsetminus T$ is connected. If $C \subseteq T$, we are done because this implies C is Lindelöf, and therefore, a strong LJ-frame by (1)$\Longrightarrow$(3) of Theorem 3.7. If $C \nsubseteq T$, then $C \cap (L \smallsetminus T) \neq \mathsf{O}$ since T is closed. Now, C and $L \smallsetminus T$ are connected, and therefore, so is $C \vee (L \smallsetminus T)$. But C is a component so $C = C \vee (L \smallsetminus T)$. That is, $L \smallsetminus T \subseteq C$. Consider the closed sublocale $S := T \cap C$ of C. Clearly, $K \subseteq S$, and S is Lindelöf since it is also a closed sublocale of the Lindelöf frame T. It is only left to prove that the supplement of S in C, $C \smallsetminus S$, is connected. This may be done by showing that $C \smallsetminus S = L \smallsetminus T$. First, recall from [13, Remark 4.2 (b)] that supplements in the sublocale are calculated the same as in the parent frame; in particular, $C \smallsetminus S = C \cap \big(L \smallsetminus (T \cap C)\big) = C \cap \big((L \smallsetminus T) \vee (L \smallsetminus C)\big) = C \cap (L \smallsetminus T) = L \smallsetminus T$. In the previous calculation, we use the fact that S and C are closed (in particular, complemented) in L and apply Lemma 2.2 (1) and [13, Proposition 3.2 (5)].
(2) Suppose C is a component of L and let K be a compact sublocale of C. The LJ-semi-strongness of L implies that there is a closed Lindelöf sublocale $T \subseteq L$ such that $K \subseteq T$ and a connected sublocale $D \subseteq L \smallsetminus K$ with $T \vee D = L$. If $C \subseteq T$, then C is Lindelöf, and therefore a semi-strong LJ-frame by (1)$\Longrightarrow$(2) of Theorem 3.7. If $C \nsubseteq T$, then $C \cap (L \smallsetminus T) \neq \mathsf{O}$. Put $S := C \cap T$. Now, $T \vee D = L \Longrightarrow L \smallsetminus T \subseteq D$, and $K \subseteq T \Longrightarrow L \smallsetminus T \subseteq L \smallsetminus K$, so $L \smallsetminus T \subseteq D \cap (L \smallsetminus K) = D$. Thus, $C \cap (L \smallsetminus T) \subseteq C \cap D$. So, $C \cap D \neq \mathsf{O}$. It follows that $D \vee C$ is connected, and since C is a component, then $D \vee C = C$. That is, $D \subseteq C$. Observe that K is a sublocale of C that is contained in T, so S is a closed Lindelöf sublocale of C with $K \subseteq S$. Lastly, $D \subseteq C \cap (L \smallsetminus K) = C \smallsetminus K$ (where the last equality follows by Lemma 2.2 (1)), and $S \vee D = (C \cap T) \vee D = (C \vee D) \cap (T \vee D) = C \cap L = C$. ☐

Example 3.15.
(1) All the closed subspaces of the J-space $[0, \omega_1)$ are also J-spaces, see [15, Proposition 4]. Therefore, the closed sublocales of the J-frame $\mathfrak{O}([0, \omega_1))$ are induced by closed subsets of $[0, \omega_1)$, so they are all J-frames. We do not know whether or not Proposition 3.14 holds true for LJ-frames, but the J-space analog of it is not true: in [21, Example 9.2], a locally compact Hausdorff J-space X with a connected component C which is not a J-space is constructed. In fact, C is isomorphic to $\mathbb{R}$, and $\mathbb{R}$ is not a J-space by [21, Proposition 2.2]. Since C is a T_D-space, then $\mathfrak{O}C$ is not a J-frame, by conservativeness of the J-frame property for T_D-spaces. It is shown in [7, Proposition 3.6] that connected subsets induce connected sublocales in a T_D-space, so the J-frame $\mathfrak{O}X$ has a connected closed sublocale $\widetilde{C}$ which is not a J-frame because $\widetilde{C} = \mathfrak{c}_{\mathfrak{O}X}(X \smallsetminus C) \cong \mathfrak{O}C$. We may not conclude here that $\widetilde{C}$ is a component of $\mathfrak{O}X$ since connected sublocales of $\mathfrak{O}X$ may not be induced.
(2) In general, Proposition 3.14 is not true for arbitrary closed sublocales:
(a) The long line Z is a regular strong LJ-space having a closed subspace $A := [0, \omega_1) \times \{0\}$ which is homeomorphic to $[0, \omega_1)$ and this is not a strong LJ-space, see [15, Proposition 4]. Therefore, A is regular and $\mathfrak{O}A$ is not a strong LJ-frame, by conservativeness of the LJ-strongness property for regular spaces. Since $\widetilde{A} \cong \mathfrak{O}A$, then $\widetilde{A}$ is a closed sublocale of the strong LJ-frame $\mathfrak{O}Z$ which is not a strong LJ-frame.

(b) In [15, Example 5], the author constructed a subspace Y of the regular Hausdorff $Z\times\mathbb{R}^+$, where Z is the long line and $\mathbb{R}^+$ is the set of nonnegative real numbers with the usual topology. This Y is a regular Hausdorff space which is semi-strong LJ-space and it has closed subspace F which is not even an LJ-space. Since the open set lattice of a regular T_D-space which is a semi-strong LJ-space is a semi-strong LJ-frame, then $\mathfrak{O}Y$ is a regular semi-strong LJ-frame having the closed sublocale $\widetilde{F}$ that is not an LJ-frame (hence $\widetilde{F}$ is not a semi-strong LJ-frame).

We now study the nature of binary closed covers of a frame that makes it to belong in an LJ-class.

Theorem 3.16. *Let L be a regular frame. If $\{S,T\}$ is a closed cover of L with $S\cap T$ compact, then the following conditions are equivalent:*
(1) *L is a strong LJ-frame.*
(2) *One of S and T is Lindelöf and the other is a strong LJ-frame.*

Proof. (1) $\Longrightarrow$ (2)
If L is a strong LJ-frame, then it is an LJ-frame by (2) $\Longrightarrow$ (4) of Theorem 3.7. So we may assume that T is Lindelöf. We show that S is a strong LJ-frame, so let K_1 be a compact sublocale of S. We have a compact sublocale $K=K_1\vee(S\cap T)$ of L; so $K\subseteq A$ for some closed Lindelöf sublocale A of L such that $L\smallsetminus A$ is connected. Put $A_1:=A\cap S$, $C_1:=(L\smallsetminus A)\cap S$ and $C_2:=(L\smallsetminus A)\cap T$. Hence, A_1 is Lindelöf, and $K_1\subseteq A_1$. Since $L\smallsetminus A=C_1\vee C_2$, then $C_1=\mathsf{O}$ or $C_2=\mathsf{O}$, by connectedness of $L\smallsetminus A$. If $C_2=\mathsf{O}$, then, by Lemma 2.2 (1), we have $S\smallsetminus A_1=S\cap(L\smallsetminus A_1)=S\cap((L\smallsetminus A)\vee(L\smallsetminus S))=S\cap C_1=C_1=L\smallsetminus A$. Thus, $S\smallsetminus A_1$ is connected. Therefore, S is a strong LJ-frame. Suppose, $C_1=\mathsf{O}$, then $L=A\vee(L\smallsetminus A)\Longrightarrow S=(S\cap A)\vee(S\cap(L\smallsetminus A))=S\cap A=A_1$, so S is Lindelöf. Thus, S is a strong LJ-frame by (1) $\Longrightarrow$ (2) of Theorem 3.7.

(2) $\Longrightarrow$ (1) Suppose T is Lindelöf and S is a strong LJ-frame. Take a compact sublocale K of L. Regularity of L implies that K is closed, so $K\cap S$ is a compact sublocale (being a closed sublocale of K) contained in S. Put $K_1:=(K\cap S)\vee(T\cap S)$. We have that K_1 is a compact sublocale of S; find a closed Lindelöf sublocale A_1 of S containing K_1 such that $S\smallsetminus A_1$ is connected. Now, put $A:=A_1\vee T$. Then A is a closed Lindelöf sublocale of L. Now, observe that $K=(K\cap S)\vee(K\cap T)\subseteq K_1\vee T\subseteq A_1\vee T=A$. It only remains to show that $L\smallsetminus A$ is connected, which we show by arguing that $L\smallsetminus A=S\smallsetminus A_1$. To see this, first note that on the one hand $L\smallsetminus T\subseteq S$, so $L\smallsetminus A=(L\smallsetminus A_1)\cap(L\smallsetminus T)\subseteq(L\smallsetminus A_1)\cap S=S\smallsetminus A_1$. On the other hand, since $T\cap S\subseteq K_1\subseteq A_1\subseteq S$, then $(S\smallsetminus A_1)\cap A=((L\smallsetminus A_1)\cap S)\cap(A_1\vee T)=(L\smallsetminus A_1)\cap(T\cap S)\subseteq(L\smallsetminus A_1)\cap A_1=\mathsf{O}$, so $S\smallsetminus A_1\subseteq L\smallsetminus A$. Thus, $L\smallsetminus A=S\smallsetminus A_1$, and whence, $L\smallsetminus A$ is connected. □

Next, we show that the result above holds for semi-strong LJ-frames:

Theorem 3.17. *Let L be a regular frame. If $\{S,T\}$ is a closed cover of L with $S\cap T$ compact, then the following conditions are equivalent:*
(1) *L is a semi-strong LJ-frame.*
(2) *One of S and T is Lindelöf and the other is a semi-strong LJ-frame.*

Proof. (1) $\Longrightarrow$ (2) We proceed as in the proof of Theorem 3.16, but use the LJ-semi-strongness of L, there is a closed Lindelöf sublocale A of L containing K and a connected $C\subseteq L\smallsetminus K$ such that $A\vee C=L$. Put $A_1:=A\cap S$, $C_1:=C\cap S$ and $C_2:=C\cap T$. So A_1 is Lindelöf and $K_1\subseteq A_1$. Since $C=C_1\vee C_2$, it follows by connectedness of C that $C_1 = \mathrm{O}$ or $C_2 = \mathrm{O}$. If $C_2 = \mathrm{O}$, then $C = C_1 = C \cap S$, and so $C \subseteq S$. A sublocale of a regular locale is regular, by [18, Proposition III.1.2 (i)]. So, S is regular, and whence K_1 is a closed (and therefore, a complemented) sublocale of S, by [18, Proposition III.1.2 (iii)]. It follows that $C \subseteq (L \smallsetminus K) \cap S = S \smallsetminus K \subseteq S \smallsetminus K_1$. Moreover, $C \vee A_1 = S$, so the *LJ*-semi-strongness of S follows. If $C_1 = \mathrm{O}$, then from $A \vee C = L$, one has $S = A_1$, so S is Lindelöf. Thus, S is a semi-strong *LJ*-frame by (1) $\Longrightarrow$ (2) of Theorem 3.7.

(2) $\Longrightarrow$ (1) Proceed as in the proof of Theorem 3.16 but with T Lindelöf and S semi-strong, and $K_1 := K \cap S$. We have $K_1 \subseteq K$, so K_1 is a compact sublocale of S; find a closed Lindelöf sublocale A_1 of S containing K_1 and a connected $C \subseteq S \smallsetminus K_1$ such that $A_1 \vee C = S$. Let $A = A_1 \vee T$. Then A is a closed Lindelöf sublocale of L. Observe that $K \subseteq A$. It is now not difficult to check that $C \subseteq L \smallsetminus K$ and $A \vee C = L$. □

We now show that the *LJ*-frame analog of Theorem 3.16 and Theorem 3.17 holds true without the assumption of regularity on the ambient frame.

Theorem 3.18. *If $\{S,T\}$ is a closed cover of a frame L with $S \cap T$ compact, then the following conditions are equivalent:*

(1) *L is an LJ-frame.*

(2) *One of S and T is Lindelöf and the other is an LJ-frame.*

Proof. (1) $\Longrightarrow$ (2) By definition, S or T is Lindelöf. Assume that T is Lindelöf. We shall show that S is an *LJ*-frame. Suppose $S = A \vee B$ for some closed sublocales A, B of S such that $A \cap B$ is compact. Since S is a closed sublocale of L, then A, B are closed sublocales of L. Now, $L = A \vee (B \vee T)$ and $A \cap (B \vee T) = (A \cap B) \vee (A \cap T) \subseteq (A \cap B) \vee (S \cap T)$, so $A \cap (B \vee T)$ is compact, being a closed sublocale contained in a compact one. Thus, A or $B \vee T$ is Lindelöf, since L is an *LJ*-frame. If A is Lindelöf, then we are done. If $B \vee T$ is Lindelöf, then B inherits the Lindelöf property from $B \vee T$ since B is closed.

(2) $\Longrightarrow$ (1) Without loss of generality, suppose that T is Lindelöf and that S is an *LJ*-frame. We shall show that L is an *LJ*-frame. Suppose $L = A \vee B$ for some closed sublocales A, B of L such that $A \cap B$ is compact. Now, let:

$$A_1 = A \cap S \quad \text{and} \quad B_1 = B \cap S,$$
$$A_2 = A \cap T \quad \text{and} \quad B_2 = B \cap T.$$

Then, $A_1 \vee B_1 = (A \cap S) \vee (B \cap S) = (A \vee B) \cap S = L \cap S = S$ and $A_1 \cap B_1$ is compact since it is a closed sublocale contained in $A \cap B$. Thus, A_1 or B_1 is Lindelöf. Assume B_1 is Lindelöf, and note that B_2 is Lindelöf since it is a closed sublocale contained in T. Hence, B is Lindelöf since $B_1 \vee B_2 = B$. □

We have seen in Proposition 3.14 that components always inherit the *LJ*-strongness (and the *LJ*-semi-strongness) from the parent frame if it has this property. In general, closed sublocales need not behave this way, even for *LJ*-frames (see Example 3.15). For closed sublocales with a compact frontier, one has:

Corollary 3.19. *Let S be a closed sublocale of a frame L with $\mathrm{Fr}_L(S)$ compact.*

(1) *If L is a regular strong LJ-frame, so is S.*

(2) *If L is a regular semi-strong LJ-frame, so is S.*

(3) *If L is an LJ-frame, so is S.*

Proof. We show that (3) is true. Note that $L = S \vee \overline{L \smallsetminus S}$ and $\mathrm{Fr}_L(S) = S \cap \overline{L \smallsetminus S}$ is compact. Since L is an LJ-frame, then one of S and $\overline{L \smallsetminus S}$ is Lindelöf and the other is an LJ-frame, by Theorem 3.18. In particular, if $\overline{L \smallsetminus S}$ is Lindelöf, then S is an LJ-frame. If S is Lindelöf, then we are done, by (1) $\Longrightarrow$ (4) of Theorem 3.7. Part (1) and (2) follows by a similar argument; but in these case, one must accordingly apply Theorem 3.16 and (1) $\Longrightarrow$ (2) of Theorem 3.7 for part (1), and Theorem 3.17 together with (1) $\Longrightarrow$ (3) of Theorem 3.7 for part (2). □

Corollary 3.20. *Let $\{S, T\}$ be a closed cover of a frame L with T Lindelöf.*

(1) *If L is regular and S is a strong LJ-frame with $\mathrm{Fr}_L(S)$ compact, so is L.*
(2) *If L is regular and S is a semi-strong LJ-frame, so is L.*
(3) *If S is an LJ-frame, so is L.*

Proof. Note that part (2) follows by the proof of (2) $\Longrightarrow$ (1) of Theorem 3.17 and part (3) is a consequence of the proof of (2) $\Longrightarrow$ (1) of Theorem 3.18. However, part (1) does not follow by the proof of (2) $\Longrightarrow$ (1) Theorem 3.16 because this proof uses the compactness of $S \cap T$; a condition that we do not have here. But compactness of $\mathrm{Fr}_L(S)$ ought to be enough to rectify this. First, note that $L = S \vee \overline{L \smallsetminus S}$ and $S \cap \overline{L \smallsetminus S}$ is compact. Now, $L = S \vee T$ and the complementedness of S implies that $L \smallsetminus S \subseteq T$. Thus, $\overline{L \smallsetminus S} \subseteq T$. It follows that $\overline{L \smallsetminus S}$ is Lindelöf. But S is a strong LJ-frame, so by (2) $\Longrightarrow$ (1) Theorem 3.16, L is a strong LJ-frame.

□

Corollary 3.21. *Suppose $L = S \vee U$ where U is an open sublocale of a frame L with $\overline{U}$ compact and S is any sublocale of L.*

(1) *If S is a regular strong LJ-frame, so is L.*
(2) *If L is regular and S is a semi-strong LJ-frame, so is L.*
(3) *If S is an LJ-frame, so is L.*

Proof. We show that (3) is true. Let $A = L \smallsetminus U$. Then A is a closed sublocale of S. Furthermore, $\mathrm{Fr}_L(A) = A \cap \overline{L \smallsetminus A} = A \cap \overline{U} \subseteq \overline{U}$. Therefore $\mathrm{Fr}_L(A)$ is compact and it is contained in S. But S is an LJ-frame, so A is an LJ-frame by Corollary 3.19 (3). Notice that $L = A \vee \overline{U}$, and $A \cap \overline{U}$ is compact (being a closed sublocale of L contained in $\overline{U}$). It now follows by Theorem 3.18 that L is an LJ-frame. The proof for (1) (respectively, (2)) follows similarly, but instead it applies Corollary 3.19 (1) and Theorem 3.16 (respectively, Corollary 3.19 (2) and Theorem 3.17). □

Theorem 3.22. *Let $\{S, T\}$ be a closed cover of a frame L with $S \cap T$ non-Lindelöf.*

(1) *If L is regular and S and T are semi-strong LJ-frames, so is L.*
(2) *If S and T are LJ-frames, so is L.*

Proof. (1) To show that L is a semi-strong LJ-frame, let K be a compact sublocale of a regular L. Put $K_1 := K \cap S$ and $K_2 := K \cap T$. Then K_1 and K_2 are compact. Since K_1 is compact and $K_1 \subseteq S$, there is a closed Lindelöf sublocale A_1 of S containing K_1 and connected $C_1 \subseteq S \smallsetminus K_1$ such that $C_1 \vee A_1 = S$. Similarly, there is a closed Lindelöf sublocale A_2 of T containing K_2 and connected $C_2 \subseteq T \smallsetminus K_2$ with $C_2 \vee A_2 = T$. Now, put $A := A_1 \vee A_2$ and $C := C_1 \vee C_2$. It is now clear that A is a closed Lindelöf sublocale containing K such that $C \vee A = L$. Also, $C = C_1 \vee C_2 \subseteq (S \smallsetminus K_1) \vee (T \smallsetminus K_2) = (S \cap (L \smallsetminus K_1)) \vee (T \cap (L \smallsetminus K_2)) = (S \cap (L \smallsetminus (K \cap S))) \vee (T \cap (L \smallsetminus (K \cap T))) = (S \cap (L \smallsetminus K)) \vee (T \cap$

$(L \smallsetminus K)) = (S \vee T) \cap (L \smallsetminus K) = L \smallsetminus K$. We now argue that C is connected. We do this by showing that $C_1 \cap C_2 \neq \mathrm{O}$. Note that $C_1 \vee A_1 = S$ and $C_2 \vee A_2 = T$ implies that $S \smallsetminus A_1 \subseteq C_1$ and $T \smallsetminus A_2 \subseteq C_2$. Therefore, $(S \smallsetminus A_1) \cap (T \smallsetminus A_2) \subseteq C_1 \cap C_2$. We show that $(S \smallsetminus A_1) \cap (T \smallsetminus A_2) \neq \mathrm{O}$. Suppose not, and note that $(S \smallsetminus A_1) \cap (T \smallsetminus A_2) = \mathrm{O} \Longrightarrow (S \cap (L \smallsetminus A_1)) \cap (T \cap (L \smallsetminus A_2)) = \mathrm{O} \Longrightarrow (S \cap T) \cap (L \smallsetminus A) = \mathrm{O} \Longrightarrow (S \cap T) \subseteq A \Longrightarrow S \cap T$ is Lindelöf, which is a contradiction. Thus, $(S \smallsetminus A_1) \cap (T \smallsetminus A_2) \neq \mathrm{O}$, and this establishes the connectedness of C.

(2) Suppose that S and T are *LJ*-frames. To show that L is an *LJ*-frame, let $L = A \vee B$ and suppose that $A \cap B$ is compact. Put:

$$A_1 := A \cap S \quad \text{and} \quad B_1 := B \cap S,$$

$$A_2 := A \cap T \quad \text{and} \quad B_2 := B \cap T.$$

Note that $A_1 \vee B_1 = S$ and $A_2 \vee B_2 = T$. Since $A_1 \cap B_1 \subseteq A \cap B$ and $A_2 \cap B_2 \subseteq A \cap B$, then $A_1 \cap B_1$ and $A_2 \cap B_2$ are compact. But S is an *LJ*-frame, so either A_1 or B_1 is Lindelöf. Similarly, since T is an *LJ*-frame, either A_2 or B_2 is Lindelöf. We show that if B_1 is Lindelöf, then B is Lindelöf. To this end, suppose that B_1 is Lindelöf. Observe that $S \cap T = (A_1 \vee B_1) \cap (A_2 \vee B_2) = (A_1 \cap A_2) \vee (B_1 \cap A_2) \vee (A_1 \cap B_2) \vee (B_1 \cap B_2) \subseteq A_2 \vee (A \cap B) \vee B_1$. On the one hand, $S \cap T$ is non-Lindelöf, by the hypothesis. On the other hand, $A \cap B$ is compact, and therefore Lindelöf. But, B_1 is also Lindelöf. Putting all of this together, we get that A_2 cannot be Lindelöf, otherwise $S \cap T$ would be Lindelöf. It follows that B_2 must be Lindelöf. We now have that B_1 and B_2 are Lindelöf, so $B_1 \vee B_2 = B$ is Lindelöf. By a similar argument, we can show that if A_1 is Lindelöf, so is A. Therefore, L is an *LJ*-frame. □

Remark 3.23. The strong *LJ*-frame analog of Theorem 3.22 is not true. This can be observed from the fact that in spaces, see [15, Example 5], one has a regular Hausdorff semi-strong *LJ*-space Y which is not a strong *LJ*-space that has a closed cover $\{A, B\}$ with $A \cap B$ non-Lindelöf, but A and B are strong *LJ*-spaces. Therefore, Theorem 3.22 does not hold in frames, by conservativeness of the *LJ*-strongness for regular spaces. Moreover, the semi-strong *LJ*-space Y (so Y is an *LJ*-space) in [15, Example 5] has a closed cover $\{Y, F\}$ with $Y \cap F = F$ non-Lindelöf, but F is not an *LJ*-space (an therefore not a semi-strong *LJ*-space). So, $\mathfrak{O}Y$ is a semi-strong *LJ*-frame (hence, an *LJ*-frame) with $\mathfrak{O}Y = \widetilde{Y} \vee \widetilde{F}$ and $\widetilde{Y \cap F} = \widetilde{F}$ non-Lindelöf such that $\widetilde{F}$ is not an *LJ*-frame. Thus, the converses of (1) and (2) of Theorem 3.22 are not true.

4. *LJ*-frames and remainders

This section contains results that do not appear in topological spaces. Here, we characterize the classes of *LJ*-frames using their remainders from compact regular extension. A compact (Lindelöf) regular frame M containing a frame L as a dense sublocale is called a *compactification (Lindelöfication)* of L. The notion of a remainder of a frame in its compactification was introduced by Baboolal in [2] using congruences. An equivalent sublocalic definition of a remainder of a frame in its compactification is provided below:

Definition 4.1. [13] Let M be an compactification of L. The *remainder* of L in M is the sublocale $M \smallsetminus L$ of M.

Theorem 4.2. *Let M be a compactification of a frame L. The following conditions are equivalent:*

(1) *L is an LJ-frame.*
(2) *If $M \smallsetminus L \subseteq U$, for an open sublocale U of M such that $\{W_1, W_2\}$ is a disjoint open cover of $U \cap L$, then $L \smallsetminus W_1$ or $L \smallsetminus W_2$ is Lindelöf.*

Proof. (1)$\Longrightarrow$(2) Suppose L is an *LJ*-frame and let $M \smallsetminus L \subseteq U$ where U is an open sublocale of M such that $U \cap L = W_1 \vee W_2$, $W_1 \cap W_2 = \mathrm{O}$, and each W_i is an open sublocale of $U \cap L$ (therefore each W_i is open in L). From $M \smallsetminus L \subseteq U$, one has that $M \smallsetminus U \subseteq L$. Put $K := M \smallsetminus U$. Then K is a closed sublocale of a compact frame M. Since K is contained in L (in particular, K is complemented in L), then K is a compact sublocale of L. Note that U is complemented (being open) in M, therefore $L \smallsetminus K = L \smallsetminus (M \smallsetminus U) = U \cap L$, by Lemma 2.2 (1). That is, $L \smallsetminus K = W_1 \vee W_2$. It follows that $K = (L \smallsetminus W_1) \cap (L \smallsetminus W_2)$. Therefore, $(L \smallsetminus W_1) \cap (L \smallsetminus W_2)$ is compact. From $W_1 \cap W_2 = \mathrm{O}$ and the fact that each W_i is complemented in L, we get that $L = (L \smallsetminus W_1) \vee (L \smallsetminus W_2)$. Since L is an *LJ*-frame, then $L \smallsetminus W_1$ or $L \smallsetminus W_2$ is Lindelöf.

(2)$\Longrightarrow$(1) Let $L = A \vee B$ where A and B are closed sublocales of L with $A \cap B$ compact. We show that A or B is Lindelöf. Since M is regular, then $A \cap B$ is a closed sublocale of M. Let $U = M \smallsetminus (A \cap B)$. Then U is open in M. The complementedness of $A \cap B$ in M and the fact that $A \cap B \subseteq L$ implies that $M \smallsetminus L \subseteq M \smallsetminus (A \cap B) = U$. Note that

$$\begin{aligned} U \cap L &= (A \vee B) \cap (M \smallsetminus (A \cap B)) \\ &= (A \cap (M \smallsetminus (A \cap B))) \vee (B \cap (M \smallsetminus (A \cap B))). \end{aligned}$$

We claim that $A \cap (M \smallsetminus (A \cap B)) = L \smallsetminus B$. Indeed, since $A \cap B$ is complemented in M, then $(A \cap B) \cap (M \smallsetminus (A \cap B)) = \mathrm{O}$. Hence, $A \cap (M \smallsetminus (A \cap B)) \subseteq L \smallsetminus B$. For the reverse inclusion, first note that since $L = A \vee B$ and B is complemented in L, we must have $L \smallsetminus B = (A \cap (L \smallsetminus B)) \vee (B \cap (L \smallsetminus B)) = A \cap (L \smallsetminus B)$. That is, $L \smallsetminus B \subseteq A$. Now, observe that $A \cap B \subseteq B$ implies that $L \smallsetminus B \subseteq L \smallsetminus (A \cap B) \subseteq M \smallsetminus (A \cap B)$; the last containment is true since the supplements in L are calculated the same way as in M, by [13, Remark 4.2 (b)]. It follows that $L \smallsetminus B \subseteq A \cap (M \smallsetminus (A \cap B))$. A similar argument can be used to show that $B \cap (M \smallsetminus (A \cap B)) = L \smallsetminus A$. Hence,

$$U \cap L = (L \smallsetminus B) \vee (L \smallsetminus A).$$

Note that $(L \smallsetminus B)$ and $(L \smallsetminus A)$ are open sublocales of L which are both contained in the open sublocale U of M, so they are open in $U \cap L$. Moreover, $(L \smallsetminus B) \cap (L \smallsetminus A) = L \smallsetminus (A \vee B) = L \smallsetminus L = \mathrm{O}$. So, by the hypothesis, either $L \smallsetminus (L \smallsetminus A)$ or $L \smallsetminus (L \smallsetminus B)$ is Lindelöf. By the complementedness of A and B in L, the latter means A or B is Lindelöf; showing that L is an *LJ*-frame. □

We now present the strong *LJ*-frame analog of Theorem 4.2:

Theorem 4.3. *Let M be a compactification of a frame L. Then the following conditions are equivalent:*
(1) *L is a strong LJ-frame.*
(2) *If $M \smallsetminus L \subseteq U$ and U is an open sublocale of M, then there exists a closed Lindelöf sublocale T of L such that $L \smallsetminus T$ is connected and $L \smallsetminus T \subseteq U$.*

Proof. (1)$\Longrightarrow$(2) Suppose that $M \smallsetminus L \subseteq U$ for some open sublocale U of M. Here too, $M \smallsetminus U$so it is a compact sublocale of L. But L is a strong *LJ*-frame, so there exists a closed Lindelöf sublocale T of L such that $M \smallsetminus U \subseteq T$ and $L \smallsetminus T$ is connected. From $M \smallsetminus U \subseteq T$ we get $M = U \vee T$, by [25, VI.4.5], where the join is taken in $S(M)$. Intersecting with $L \smallsetminus T$ on both side of the previous equation and using the fact that T is complemented in L, we get $L \smallsetminus T \subseteq U$.

(2)$\Longrightarrow$(1) Let K be a compact sublocale of L. Then K is a closed sublocale of M, by regularity of M. Since $K \subseteq L$, then the complementedness of K in M implies that $M \smallsetminus L \subseteq M \smallsetminus K$. Now, $M \smallsetminus K$ is an open sublocale of M, so by the hypothesis, then there exists a closed Lindelöf sublocale T of L such that $L \smallsetminus T$ is connected and $L \smallsetminus T \subseteq M \smallsetminus K$. Using the complementedness of K in M again, we get $K = M \smallsetminus (M \smallsetminus K) \subseteq M \smallsetminus (L \smallsetminus T) \subseteq T$; the last containment follows from the fact that if $S \in S(M)$ and $S \cap (L \smallsetminus T) = \mathrm{O}$, then $S \subseteq T$ since T is complemented in L. □

The semi-strong *LJ*-frame version of Theorem 4.2 and Theorem 4.3 is presented below:

Theorem 4.4. *Let M be a compactification of a frame L. Then the following conditions are equivalent:*

(1) *L is a semi-strong LJ-frame.*

(2) *If $M \smallsetminus L \subseteq \mathrm{int}_M(S)$ where S is a sublocale of M, then there exists a closed Lindelöf sublocale T of L and a connected sublocale C of L such that $M \smallsetminus \mathrm{int}_M(S) \subseteq T$ and $L \smallsetminus T \subseteq C \subseteq \mathrm{int}_M(S)$.*

Proof. (1)$\Longrightarrow$(2) Suppose that $M \smallsetminus L \subseteq \mathrm{int}_M(S)$ for some sublocale S of M. Now, being a join of open sublocales of M, $\mathrm{int}_M(S)$ is open in M. So, $M \smallsetminus \mathrm{int}_M(S)$ is a closed sublocale of M, whence, $M \smallsetminus \mathrm{int}_M(S)$ inherits compactness from M. Since $M \smallsetminus L \subseteq \mathrm{int}_M(S)$, then the complementedness of $\mathrm{int}_M(S)$ in M implies that $M \smallsetminus \mathrm{int}_M(S) \subseteq L$. Using the *LJ*-semi-strongness of L, we get a closed Lindelöf sublocale T of L and a connected sublocale C of L such that $M \smallsetminus \mathrm{int}_M(S) \subseteq T$, $C \subseteq L \smallsetminus (M \smallsetminus \mathrm{int}_M(S))$, and $T \vee C = L$. First, the latter and the complementedness of T in L implies that $L \smallsetminus T \subseteq C$. Furthermore, $L \smallsetminus (M \smallsetminus \mathrm{int}_M(S)) = L \cap \mathrm{int}_M(S)$, by Lemma 2.2 (2), since $\mathrm{int}_M(S)$ is complemented in M. Thus, $C \subseteq L \cap \mathrm{int}_M(S) \subseteq \mathrm{int}_M(S)$. Therefore, $L \smallsetminus T \subseteq C \subseteq \mathrm{int}_M(S)$.

(2)$\Longrightarrow$(1) Let K be a compact sublocale of L. So, $M \smallsetminus K$ is an open sublocale of M, and whence $\mathrm{int}_M(M \smallsetminus K) = M \smallsetminus K$. Since $K \subseteq L$, then the complementedness of K in M implies that $M \smallsetminus L \subseteq M \smallsetminus K$. Therefore, by the hypothesis, there exists a closed Lindelöf sublocale T of L and a connected sublocale C of L such that $M \smallsetminus (M \smallsetminus K) \subseteq T$ and $L \smallsetminus T \subseteq C \subseteq M \smallsetminus K$. So, $K \subseteq T$. Now, by applying [25, VI.4.5] to $L \smallsetminus T \subseteq C$ we get that $L \subseteq T \vee C$, where the join is taken in $S(L)$. That is, $T \vee C = L$. From $C \subseteq M \smallsetminus K$, and the fact that $C \subseteq L$, one has $C \subseteq (M \smallsetminus K) \cap L = L \smallsetminus K$ by the complementedness of K in M and Lemma 2.2 (1). □

The spatial versions of Theorem 4.2, Theorem 4.3, and Theorem 4.4 are presented below, and they do not appear in the classical topology literature. Let us remind the reader that a compact Hausdorff space is regular, and that Hausdorffness and regularity are inherited by subspaces, so X is regular and Hausdorff (and therefore a T_D-space) in the result below:

Corollary 4.5. *Let Y be a compact Hausdorff space containing a space X:*

(1) *X is a strong LJ-space if and only if whenever $Y \smallsetminus X \subseteq U$ and U is an open subset of Y, then there exists a closed Lindelöf subset T of X such that $X \smallsetminus T$ is connected and $X \smallsetminus T \subseteq U$.*

(2) *X is a scattered semi-strong LJ-space if and only if whenever $Y \smallsetminus X \subseteq \mathrm{int}_Y(S)$ where S is a subset of Y, then there exists a closed Lindelöf subset T of X and a connected subset C of X such that $Y \smallsetminus \mathrm{int}_Y(S) \subseteq T$ and $X \smallsetminus T \subseteq C \subseteq \mathrm{int}_Y(S)$.*

(3) *X is an LJ-space if and only if whenever $Y \smallsetminus X \subseteq U$, and U is an open subset of Y such that $\{W_1, W_2\}$ is a disjoint open cover of $U \cap X$, then $X \smallsetminus W_1$ or $X \smallsetminus W_2$ is Lindelöf.*

Remark 4.6. We close the paper with the following remarks:

(i) It is worth noting that the density of L in M is not a requirement for the proofs of Theorem 4.2, Theorem 4.3, and Theorem 4.4. Thus, these results hold true for any compact regular frame M containing L. Note that in Theorem 4.2, Theorem 4.3, and Theorem 4.4, compactness of M is vital for the proof of (1)$\Longrightarrow$(2); so one does not expect this implication to be true for a non-compact M. However, (2)$\Longrightarrow$(1) holds for any regular frame M, in all cases. Madden and Vermeer ([20]) constructed a Lindelöfication for completely regular frames, the regular Lindelöf reflection λL for a completely regular frame L. So, in particular, the implications (2)$\Longrightarrow$(1) of Theorem 4.2, Theorem 4.3, and Theorem 4.4 are true for the Lindelöfication λL, where L is completely regular. It is important to mention here that the latter is novel since the mentioned Lindelöf reflection does not exist in spaces.

(ii) A sublocale is *relatively connected* in the ambient frame if it is not contained in a join of two disjoint nontrivial open sublocale of the parent frame. It was shown in [22, Theorem 5.5] that a completely regular frame L is a J-frame if and only if the remainder $\beta L\smallsetminus L$ is relatively connected in βL, where βL is the Stone-Čech compactification of L. The proof of the latter is heavily dependent on the fact that βL is a *perfect compactification* (in accord with Baboolal [2]). Now, by [11, Lemma 2.4], λL is a perfect Lindelöfication. It is, therefore, natural to wonder if a completely regular LJ-frame L can be characterized using the remainder $\lambda L\smallsetminus L$ in a similar (or perhaps different) way that J-frames were characterized via $\beta L\smallsetminus L$.

(iii) As pointed out in Example 3.5 (ii), we do not know whether or not non-spatial examples of LJ-frames that are not J-frames exist. This, together with part (ii) of this remark, alerts for a general study of the non-spatiality of J-frames, LJ-frames, and other related classes of frames. We pursue this direction elsewhere.

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