p-numerical semigroups of Pythagorean triples

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Abstract

We give an explicit formula for the p-Frobenius number of primitive Pythagorean triples, that is the largest positive integer that can only be represented in p ways by combining the three integers of the Pythagorean triple. When $p = 0$, it is the original Frobenius number in the famous Diophantine problem of Frobenius. We also obtain closed forms for the number of positive integers, and the largest positive integer that can be represented in only p ways by combining the three integers of the Pythagorean triple. Our generalization is natural in terms of the Apéry set; a detailed analysis is needed, and the results are not trivial. Our method has an advantage in terms of visually grasping the elements of the Apéry set, and is useful to determine other related constants. In addition, as an application of our method, we can determine the p-Frobenius number of other triples such as those associated to the sides of integer-sided triangles with an angle of 60 degrees. This corresponds to the Diophantine equation $x^2 + y^2 - xy = z^2$; in principle, the method works for more general Diophantine equations also whose solutions can be similarly parameterized.

Keywords: Frobenius problem, Pythagorean triples, Apéry set, Diophantine equations

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1 Introduction

For integer $k \ge 2$, consider a set of positive integers $A = \{a_1, \dots, a_k\}$ with $\gcd(A) = \gcd(a_1, \dots, a_k) = 1$. To find the number of non-negative integral representations $x_1, x_2, \dots, x_n$, denoted by $d(n; A) = d(n; a_1, a_2, \dots, a_k)$, to $a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$ for a given positive integer n is one of the most important and interesting topics. This number is often called the *denumerant* and is equal to the coefficient of x^n in $1/(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_k})$ ([28]). Sylvester [27] and Cayley [4] showed that $d(n; a_1, a_2, \dots, a_k)$ can be expressed as the sum of a polynomial in n of degree $k-1$ and a periodic function of period $a_1a_2\cdots a_k$. For two variables, a formula for $d(n; a_1, a_2)$ is obtained in [31]. For three variables in the pairwise coprime case $d(n; a_1, a_2, a_3)$, in [9], the periodic function part is expressed in terms of trigonometric functions.

For a non-negative integer p, define S_p and G_p by

$$S_p(A) = \{n \in \mathbb{N}_0 | d(n; A) > p\}$$

and

$$G_p(A) = \{n \in \mathbb{N}_0 | d(n; A) \le p\}$$

respectively, satisfying $S_p \cup G_p = \mathbb{N}_0$, which is the set of non-negative integers. The set S_p is called *p-numerical semigroup* because $S(A) = S_0(A)$ is a usual numerical semigroup. G_p is the set of *p-gaps*. Define $g_p(A)$, $n_p(A)$ and $s_p(A)$ by

$$g_p(A) = \max_{n \in G_p(A)} n, \quad n_p(A) = \sum_{n \in G_p(A)} 1, \quad s_p(A) = \sum_{n \in G_p(A)} n\,,$$

respectively, and are called the *p-Frobenius number*, the *p-Sylvester number* (or *p-genus*) and the *p-Sylvester sum*, respectively. When $p = 0$, $g(A) = g_0(A)$, $n(A) = n_0(A)$ and $s(A) = s_0(A)$ are the original Frobenius number, Sylvester number (or genus) and Sylvester sum, respectively. To find such values is one of the crucial matters in the Diophantine problem of Frobenius. More detail descriptions of the p-numerical semigroups and their symmetric properties can be found in [18].

The Frobenius problem (also known as the Coin Exchange Problem or Postage Stamp Problem or Chicken McNugget Problem) has a long history and is one of the popular problems that has attracted the attention of experts as well as amateurs. For two variables $A = \{a, b\}$, it is known that

$$g(a, b) = (a-1)(b-1) - 1 \quad \text{and} \quad n(a, b) = \frac{(a-1)(b-1)}{2}$$

([28, 29]). For three or more variables, the Frobenius number cannot be given by any set of closed formulas which can be reduced to a finite set of certain polynomials ([5]). For three variables, various algorithms have been devised for finding the Frobenius number. For example, in [23], the Frobenius number is uniquely determined by six positive integers that are the solution to a system of three polynomial equations. In [7], a general algorithm is given by using 3×3 matrix. Nevertheless, explicit closed formulas have been found only for some special cases, including arithmetic, geometric, Mersenne, repunits and triangular (see [21, 24, 25] and references therein). We are interested in finding explicit closed forms, which is one of the most crucial matters in Frobenius problem. Our method has an advantage in terms of visually grasping the elements of the Apéry set, and is more useful to get more related values, including genus (Sylvester number), Sylvester sum [32], weighted power Sylvester sum [10, 19, 20] and so on.

We are interested in finding a closed or explicit form for the p-Frobenius number, which is more difficult when $p > 0$. For three or more variables, no concrete examples had been found until recently, when the first author succeeded in giving the p-Frobenius number as a closed-form expression for the triangular number triplet ([11]), for repunits ([12]), Fibonacci triplet ([16]), Jacobsthal triplets ([15, 14]) and arithmetic triplets ([17]).

It is well-known that the primitive Pythagorean triple (x, y, z) has the unique expression:

$$x = s^2 - t^2, \quad y = 2st, \quad z = s^2 + t^2\,,$$

where s and t are positive integers having different parity with $s > t$ and $\gcd(s, t) = 1$ (e.g., [30, Theorem 2.13]). In this paper, we give an explicit formula for the p-Frobenius number of primitive Pythagorean triples. As an application, we can also solve the analogous problem for triples of integers that form a triangle contains an angle of 60 degrees.

Theorem 1. *When $s < (\sqrt{2}+1)t$, for a nonnegative integer p with $p \le \lfloor t/(s-t) \rfloor$, we have*

$$\begin{aligned}&g_p(s^2 - t^2, 2st, s^2 + t^2)\\&= s\big((s+t)(s+t-2) - 2t^2\big) + p(s-t)(s^2+t^2)\,.\end{aligned}$$

When $s > (\sqrt{2}+1)t$, for a nonnegative integer p with $p \le \lfloor (s-t)/t \rfloor$, we have

$$g_p(s^2 - t^2, 2st, s^2 + t^2)$$

$$= s\big((s+t)(s+t-2)-2t^2\big)+pt(s^2+t^2)\,.$$

Remark. If $p=0$ in Theorem 1, Theorem 2.1 in [8] is recovered as a special case. We also give an explicit formula for the p-Sylvester number (p-genus) of primitive Pythagorean triples (Theorem 3 below). However, the result for $p=0$ has not been discovered yet.

For integer-sided triangles with an angle of 60 degrees, the analogue of Theorem 1 is the following result we prove.

Theorem 2. *Let s and t be positive integers having different parity with $s>t$, $\gcd(s,t)=1$ and $3\nmid s$. When $s<3t$, for a nonnegative integer p with $p\le\lfloor(2t)/(s-t)\rfloor$, we have*

$$\begin{aligned}&g_p(s^2-3t^2+2st,4st,s^2+3t^2)\\&=(s-t-1)(s^2+3t^2)+\big((p+1)s-(p-1)t-1\big)(4st)-(s^2-3t^2+2st)\,.\end{aligned}$$

When $s>3t$, for a nonnegative integer p with $p\le\lfloor(s-t)/(2t)\rfloor$, we have

$$\begin{aligned}&g_p(s^2-3t^2+2st,4st,s^2+3t^2)\\&=(2t-1)(s^2+3t^2)+\big(s+(2p+1)t-1\big)(s^2-3t^2+2st)-4st\,.\end{aligned}$$

Our method can be applied to obtain closed formulae for constants such as the p-Sylvester (power) sum [13, 32], and the p-Sylvester weighted sum [19, 20].

2 Preliminaries

For a positive integer p and a set of positive integers $A=\{a_1,a_2,\dots,a_k\}$ with $\gcd(A)=1$, denote by $\mathrm{R}_p(A)$ the set of all nonnegative integers whose representations in terms of $a_2,\dots,a_k$ with nonnegative integral coefficients have at least p ways. Note that when $p=0$, $\mathrm{R}_1\cup\mathrm{NR}(A)=\mathbb{N}\cup\{0\}$ (the set of nonnegative integers). We introduce the Apéry set (see [1]) below in order to obtain the formulas for $g_p(A)$, $n_p(A)$ and $s_p(A)$. Without loss of generality, we assume that $a_1=\min(A)$.

Definition 1. *Let p be a nonnegative integer. For a set of positive integers $A=\{a_1,a_2,\dots,a_\kappa\}$ with $\gcd(A)=1$ and $a_1=\min(A)$ we denote by*

$$\mathrm{Ap}_p(A)=\mathrm{Ap}_p(a_1,a_2,\dots,a_\kappa)=\{m_0^{(p)},m_1^{(p)},\dots,m_{a_1-1}^{(p)}\}\,,$$

the p-Apéry set of A, where each positive integer $m_i^{(p)}$ $(0 \le i \le a_1 - 1)$ *satisfies the conditions:*

$$\text{(i)}\, m_i^{(p)} \equiv i \pmod{a_1}, \quad \text{(ii)}\, m_i^{(p)} \in S_p(A), \quad \text{(iii)}\, m_i^{(p)} - a_1 \notin S_p(A)$$

Note that $m_0^{(0)}$ *is defined to be* 0.

It follows that for each p,

$$\mathrm{Ap}_p(A) \equiv \{0, 1, \dots, a_1 - 1\} \pmod{a_1}.$$

When $k \ge 3$, it is hard to find any explicit form of $g_p(A)$ as well as $n_p(A)$ and $s_p(A)$. Nevertheless, the following convenient formulas are known (For a more general case, see [13]). Though finding $m_j^{(p)}$ is enough hard in general, we can obtain it for some special sequences $(a_1, a_2, \dots, a_k)$.

Lemma 1. *Let* k *and* p *be integers with* $k \ge 2$ *and* $p \ge 0$*. Assume that* $\gcd(a_1, a_2, \dots, a_k) = 1$*. We have*

$$g_p(a_1, a_2, \dots, a_k) = \left(\max_{0 \le j \le a_1 - 1} m_j^{(p)}\right) - a_1, \tag{1}$$

$$n_p(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1-1} m_j^{(p)} - \frac{a_1 - 1}{2}, \tag{2}$$

$$s_p(a_1, a_2, \dots, a_k) = \frac{1}{2a_1} \sum_{j=0}^{a_1-1} \left(m_j^{(p)}\right)^2 - \frac{1}{2} \sum_{j=0}^{a_1-1} m_j^{(p)} + \frac{a_1^2 - 1}{12}. \tag{3}$$

Remark. When $p = 0$, the formulas (1), (2) and (3) reduce to the formulas by Brauer and Shockley [2], Selmer [26], and Tripathi [32], respectively:

$$g(a_1, a_2, \dots, a_k) = \left(\max_{1 \le j \le a_1 - 1} m_j\right) - a_1,$$

$$n(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1-1} m_j - \frac{a_1 - 1}{2},$$

$$s(a_1, a_2, \dots, a_k) = \frac{1}{2a_1} \sum_{j=0}^{a_1-1} (m_j)^2 - \frac{1}{2} \sum_{j=0}^{a_1-1} m_j + \frac{a_1^2 - 1}{12},$$

where $m_j = m_j^{(0)}$ $(1 \le j \le a_1 - 1)$ with $m_0 = m_0^{(0)} = 0$. More general formulas using Bernoulli numbers can be seen in [10].

3 Proof of the main theorem

3.1 The case where $s^2 - t^2$ is shortest

Let $s < (\sqrt{2}+1)t$, that is $s^2 - t^2 < 2st$. For simplicity, put

$$r_{i,j} := i(2st) + j(s^2 + t^2)$$

or just (i, j) when we tabulate these values.

First, consider the case $p = 0$. We shall show that the $(s^2 - t^2)$ elements in $\mathrm{Ap}_0(A)$ with $A = \{s^2 - t^2, 2st, s^2 + t^2\}$ are arranged as in Table 1.

$(0,0)$	$\cdots$	$(s-t-1,0)$	$(s-t,0)$	$\cdots\ \cdots$	$(s-1,0)$
$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(0,s-t-1)$	$\cdots$	$(s-t-1,s-t-1)$	$(s-t,s-t-1)$	$\cdots\ \cdots$	$(s-1,s-t-1)$
$(0,s-t)$	$\cdots$	$(s-t-1,s-t)$			
$\vdots$		$\vdots$			
$\vdots$		$\vdots$			
$(0,s-1)$	$\cdots$	$(s-t-1,s-1)$			

Table 1: $\mathrm{Ap}_0(s^2 - t^2, 2st, s^2 + t^2)$ when $s < (\sqrt{2}+1)t$

Since $\gcd(2t^2, s^2 - t^2) = 1$, it is enough to show that

$$\mathrm{Ap}_0(A) :\equiv \{j | 0 \le j \le s^2 - t^2 - 1\} \equiv \{2jt^2 | 0 \le j \le s^2 - t^2 - 1\} \pmod{s^2 - t^2}.$$

Since $r_{s-t+i,s-t+j} \equiv r_{i,j} \pmod{s^2 - t^2}$ and $r_{s-t+i,s-t+j} > r_{i,j}$ $(i, j \ge 0)$, any element of the form $r_{s-t+i,s-t+j}$ $(i, j \ge 0)$ is not in $\mathrm{Ap}_0(A)$. Since $r_{i,s+j} \equiv r_{t+i,j} \pmod{s^2 - t^2}$ and $r_{i,s+j} > r_{t+i,j}$ $(i, j \ge 0)$, any element of the form $r_{i,s+j}$ $(i, j \ge 0)$ is not in $\mathrm{Ap}_0(A)$. Since $r_{s+i,j} \equiv r_{i,t+j} \pmod{s^2 - t^2}$ and $r_{s+i,j} > r_{i,t+j}$ $(i, j \ge 0)$, any element of the form $r_{s+i,j}$ $(i, j \ge 0)$ is not in $\mathrm{Ap}_0(A)$. (See also Table 2 in these situations.) Therefore, only $s^2 - t^2$ elements in the area represented in Table 1 remain as candidates for the elements of $\mathrm{Ap}_0(A)$.

Now, all the elements $2jt^2 \pmod{s^2 - t^2}$ $(0 \le j \le s^2 - t^2 - 1)$ are arranged inside of the area represented in Table 1 as follows. First,

$$r_{0,j} \equiv 2jt^2 \pmod{s^2 - t^2} \quad (0 \le j \le s - 1)$$

and

$$r_{t,j} \equiv 2(s+j)t^2 \pmod{s^2 - t^2} \quad (0 \le j \le s - t - 1).$$

If $t \le s - t - 1$, this continues for $s - t \le j \le s - 1$. Then, by $r_{t,s} \equiv r_{2t,0} \pmod{s^2 - t^2}$, one moves to the column of $r_{2t,j}$ $(j \ge 0)$. If $t \ge s - t$, by $r_{t,s-t} \equiv r_{2t-s,0} \pmod{s^2 - t^2}$, one moves to the column of $r_{2t-s,j}$ $(j \ge 0)$.

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In general, assume that $r_{y,0} \equiv 2ht^2 \pmod{s^2-t^2}$ $(0 \le y \le s-1)$ for some non-negative integer h. Then for $j \ge 0$, $r_{y,j} \equiv 2(h+j)t^2 \pmod{s^2-t^2}$. If $y \ge s-t$, then by $r_{y,s-t} \equiv r_{y-s+t,0} \pmod{s^2-t^2}$ one moves to the column of $r_{y-s+t,j}$ $(j \ge 0)$ after $r_{y,j}$ $(0 \le j \le s-t-1)$. If $y \le s-t-1$, then by $r_{y,s} \equiv r_{y+t,0} \pmod{s^2-t^2}$ one moves to the column of $r_{y+t,j}$ $(j \ge 0)$ after $r_{y,j}$ $(0 \le j \le s-1)$. Since $\gcd(2t^2, s^2-t^2) = 1$, any of two element of the form $2jt^2 \pmod{s^2-t^2}$ $(0 \le j \le s^2-t^2-1)$ inside of the area in Table 1 is not overlapped.

Now we are on stage where we can determine the Frobenius number by using Lemma 1 (1). It is clear that the candidates to take the largest value in $\mathrm{Ap}_0(A)$ are at $(s-t-1, s-1)$ or $(s-1, s-t-1)$. Since $s^2+t^2 > 2st$, we have $r_{s-t-1,s-1} > r_{s-1,s-t-1}$. Hence,

$$
\begin{aligned}
&g_0(s^2-t^2, 2st, s^2+t^2)\\
&= (s-t-1)(2st) + (s-1)(s^2+t^2) - (s^2-t^2)\\
&= s\big((s+t)(s+t-2) - 2t^2\big)\,.
\end{aligned}
$$

3.1.1 $\quad p = 1$

All elements of $\mathrm{Ap}_1(A)$ are arranged in the form of shifting elements of $\mathrm{Ap}_0(A)$ whose residues are equal. Table 2 is as follows. That is, the $(s-t) \times (s-t)$ area at the lower left of $\mathrm{Ap}_0(A)$ is shifted to the upper right of $\mathrm{Ap}_1(A)$, and the $(s-t) \times (s-t)$ area at the upper right of $\mathrm{Ap}_0(A)$ is shifted to the lower left of $\mathrm{Ap}_1(A)$. Most of the other parts of $\mathrm{Ap}_0(A)$ shift in the lower right oblique direction as it is.

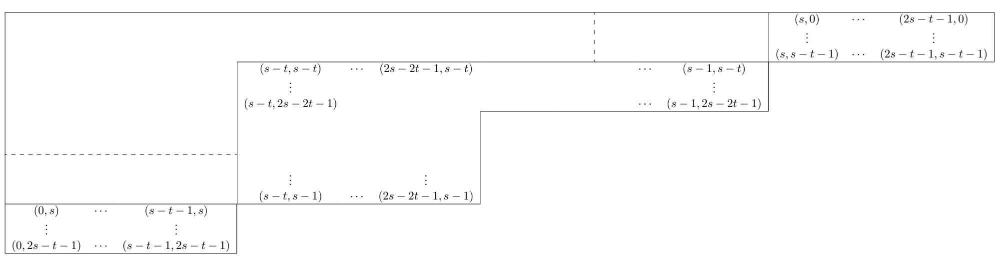

Table 2: $\mathrm{Ap}_1(s^2-t^2, 2st, s^2+t^2)$ when $s < (\sqrt{2}+1)t$

As checked in the case where $p = 0$, we have found that the set of all elements in these three areas is congruent to $\{0, 1, \dots, s^2-t^2-1\} \pmod{s^2-t^2}$. It is left to show that each element has at least two different representations. For the $(s-t) \times (s-t)$ area at the bottom left of Table 2, we have for

$0 \le y \le s-t-1$ and $0 \le z \le s-t-1$

$$0(s^2-t^2)+y(2st)+(s+z)(s^2+t^2) \\ = s(s^2-t^2)+(y+t)(2st)+z(s^2+t^2)\,.$$

For the $(s-t)\times(s-t)$ area at the top right of Table 2, we have for $0 \le y \le s-t-1$ and $0 \le z \le s-t-1$

$$0(s^2-t^2)+(s+y)(2st)+z(s^2+t^2) \\ = t(s^2-t^2)+y(2st)+(t+z)(s^2+t^2)\,.$$

For the middle area of $\mathrm{Ap}_1(A)$, we have for $0 \le y \le s-1$ and $0 \le z \le s-1$

$$0(s^2-t^2)+(s-t+y)(2st)+(s-t+z)(s^2+t^2) \\ = (s+t)(s^2-t^2)+y(2st)+z(s^2+t^2)\,.$$

In fact, for the elements in the area where $y \ge s-t$ and $z \ge s-t$, there are more than two representations belonging to $\mathrm{Ap}_p(A)$ ($p \ge 2$).

There are four candidates to take the largest value in $\mathrm{Ap}_1(A)$ and we can easily find that

$$r_{s-t-1,2s-t-1} > r_{s-1,2s-2t-1} > r_{2s-2t-1,s-1} > r_{2s-t-1,s-t-1}.$$

Hence, by Lemma 1 (1)

$$\begin{aligned} & g_1(s^2-t^2,2st,s^2+t^2) \\ & = (s-t-1)(2st)+(2s-t-1)(s^2+t^2)-(s^2-t^2) \\ & = s\big((s+t)(s+t-2)-2t^2\big)+(s-t)(s^2+t^2)\,. \end{aligned}$$

3.1.2 $p \ge 2$

When $p \ge 2$, it continues until $p \le \lfloor t/(s-t)\rfloor$, that the area of $\mathrm{Ap}_1(A)$ moves to the area of $\mathrm{Ap}_2(A)$, which moves to the area of $\mathrm{Ap}_3(A)$, and so on, in the correspondence relation modulo (s^2-t^2). Table 3 shows the areas of the $\mathrm{Ap}_p(A)$ ($p = 0,1,2,3$) for the case where $3 \le \lfloor t/(s-t)\rfloor < 4$. In Table 3, the area of $\mathrm{Ap}_0(A)$ is marked as 0 (including 0_a and 0_b); that of $\mathrm{Ap}_1(A)$ is marked as 1 (including 1_c and 1_d) with 1_a and 1_b; that of $\mathrm{Ap}_2(A)$ is marked as 2 (including 2_e and 2_f) with 2_a, 2_b, 2_c and 2_d; that of $\mathrm{Ap}_3(A)$ is marked as 3 with 3_a, 3_b, 3_c, 3_d, 3_e and 3_f. The areas having the same residue modulo (s^2-t^2) are determined as

$$0_a \Rightarrow 1_a \Rightarrow 2_a \Rightarrow 3_a\,,$$

$$
\begin{aligned}
0_b \Rightarrow 1_b \Rightarrow & 2_b \Rightarrow 3_b\,, \\
1_c \Rightarrow & 2_c \Rightarrow 3_c\,, \\
1_d \Rightarrow & 2_d \Rightarrow 3_d\,, \\
& 2_e \Rightarrow 3_e\,, \\
& 2_f \Rightarrow 3_f\,,
\end{aligned}
$$

and the main parts are as

$$
\begin{aligned}
&0 \text{ (excluding } 0_a \text{ and } 0_b) \Rightarrow 1 \text{ (including } 1_a \text{ and } 1_b)\,, \\
&1 \text{ (excluding } 1_c \text{ and } 1_d) \Rightarrow 2 \text{ (including } 2_e \text{ and } 2_f)\,, \\
&2 \text{ (excluding } 2_e \text{ and } 2_f) \Rightarrow 3\,.
\end{aligned}
$$

That is, the elements of the area of the lower left stair portions in $\mathrm{Ap}_p(A)$ correspond to the elements of the area of the upper right stair portion in $\mathrm{Ap}_{p+1}(A)$, and are aligned from the upper right row to the lower left. The elements of the area of the upper right stair portion in $\mathrm{Ap}_p(A)$ correspond to the elements of the area of the lower left stair portion in $\mathrm{Ap}_{p+1}(A)$, respectively, and line up in the upper right direction from the lowest left column. The elements of the area of $\mathrm{Ap}_p(A)$ in the center portion, except for the $(s-t)\times(s-t)$ area in the lower left and the $(s-t)\times(s-t)$ area in the upper right, correspond to the elements of the area of $\mathrm{Ap}_{p+1}(A)$ in the lower right diagonal direction.

More generally and more precisely, for $1 \le l \le p$, each element of the l-th $(s-t)\times(s-t)$ block from the left in the area of the lower left stair portions in $\mathrm{Ap}_p(A)$ is expressed by

$$
\begin{aligned}
&\big((l-1)s-(l-1)t+i, (p-l+1)s-(p-l)t+j\big) \\
&\qquad\qquad (0 \le i \le s-t-1,\ 0 \le j \le s-t-1)\,, \qquad (4)
\end{aligned}
$$

and for $1 \le l' \le p$, each element of the l'-th $(s-t)\times(s-t)$ block from the right in the area of the upper right stair portions in $\mathrm{Ap}_{p'}(A)$ is expressed by

$$
\begin{aligned}
&\big((p'-l'+1)s-(p'-l')t+i, (l'-1)s-(l'-1)t+j\big) \\
&\qquad\qquad (0 \le i \le s-t-1,\ 0 \le j \le s-t-1)\,. \qquad (5)
\end{aligned}
$$

Then we have the congruent relation for $p'=p+1$ and $l'=p'-l+1=p-l+2$

$$
\big((l-1)s-(l-1)t+i\big)(2st)+\big((p-l+1)s-(p-l)t+j\big)(s^2+t^2)
$$

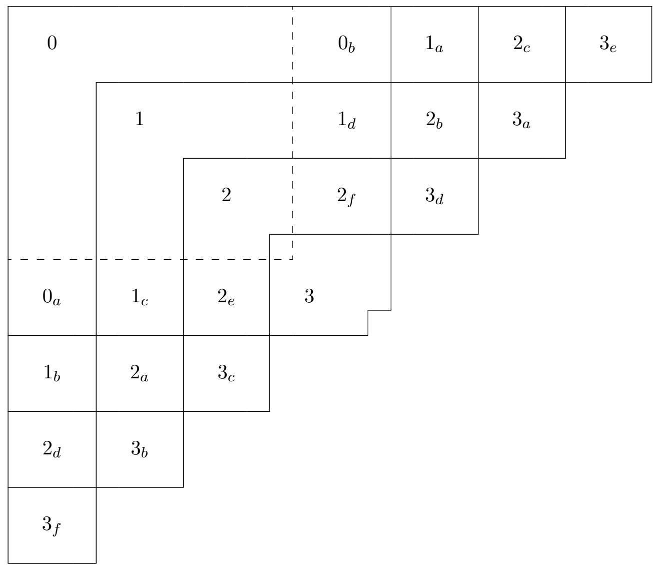

Table 3: $\mathrm{Ap}_p(s^2-t^2, 2st, s^2+t^2)$ $(p=0,1,2,3)$ when $s<(\sqrt{2}+1)t$

$$\equiv \bigl((p'-l'+1)s-(p'-l')t+i\bigr)(2st)+\bigl((l'-1)s-(l'-1)t+j\bigr)(s^2+t^2) \pmod{s^2-t^2},$$

as well as for $p=p'+1$ and $l=p-l'+1=p'-l'+2$.

For simplicity, denote by (x,y,z) the value of $x(s^2-t^2)+y(2st)+z(s^2+t^2)$. Each element of the leftmost $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p\ge 1)$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\bigl(0,0,ps-(p-1)t\bigr)\\ &=\bigl(js+(j-1)t, jt-(j-1)s, (p-j)s-(p-j)t\bigr)\\ &\qquad (j=1,2,\dots,p)\,.\end{aligned}$$

Note that $ps\le(p+1)t$ since $p\le\lfloor t/(s-t)\rfloor$.

Each element of the second from the left $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p\ge 2)$ has exactly $(p+1)$ representations, because

$$\begin{aligned}&\bigl(0,s-t,(p-1)s-(p-2)t\bigr)=\bigl(s+t,0,(p-2)s-(p-3)t\bigr)\\ &=\bigl(js+(j-1)t,(j-1)t-(j-2)s,(p-j-1)s-(p-j-1)t\bigr)\\ &\qquad (j=1,2,\dots,p-1)\,.\end{aligned}$$

Each element of the third from the left $(s-t)\times(s-t)$ area of $\mathrm{Ap}_p(A)$ $(p\ge 3)$ has exactly $(p+1)$ representations, because

$$\bigl(0,2s-2t,(p-2)s-(p-3)t\bigr)=\bigl(s+t,s-t,(p-3)s-(p-4)t\bigr)$$

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$$= \big(2s+2t, 0, (p-4)s-(p-5)t\big)$$
$$= \big(js+(j-1)t, (j-2)t-(j-3)s, (p-j-2)s-(p-j-2)t\big)$$
$$(j=1,2,\dots,p-2)\,.$$

In general, each element of the l-th ($1 \le l \le \lfloor t/(s-t) \rfloor$) from the left $(s-t) \times (s-t)$ area of $\mathrm{Ap}_p(A)$ ($p \ge l$) has exactly $(p+1)$ representations, because

$$\big(0, (l-1)s-(l-1)t, (p-l+1)s-(p-l)t\big)$$
$$= \big(i(s+t), (l-i-1)(s-t), (p-l-i+1)s-(p-l-i)t\big)$$
$$(i=1,2,\dots,l-1)$$
$$= \big(js+(j-1)t, (j-l+1)t-(j-l)s, (p-l-j+1)(s-t)\big)$$
$$(j=1,2,\dots,p-l+1)\,.$$

Similarly, each element of the l'-th ($1 \le l' \le \lfloor t/(s-t) \rfloor$) from the top right $(s-t) \times (s-t)$ area of $\mathrm{Ap}_p(A)$ ($p \ge l'$) has exactly $(p+1)$ representations, because

$$\big(0, (p-l'+1)s-(p-l')t, (l'-1)s-(l'-1)t\big)$$
$$= \big(i(s+t), (p-l'-i+1)s-(p-l'-i)t, (l'-i-1)(s-t)\big)$$
$$(i=1,2,\dots,l'-1)$$
$$= \big((j-1)s+jt, (p-l'-j+1)(s-t), (j-l'+1)t-(j-l')s\big)$$
$$(j=1,2,\dots,p-l'+1)\,.$$

Concerning the central portion of $\mathrm{Ap}_p(A)$, it is easy to see that each element is expressed by

$$\big(0, p(s-t)+i, p(s-t)+j\big) \quad (0 \le i \le s-t-1, 0 \le j \le pt-(p-1)s-1; \; s-t \le i \le pt-(p-1)s-1, 0 \le j \le s-t-1)\,, \tag{6}$$

and all elements have exactly $(p+1)$ representations, because

$$\big(0, p(s-t), p(s-t)\big) = \big(j(s+t), (p-j)(s-t), (p-j)(s-t)\big)$$
$$(j=1,2,\dots,p)\,.$$

Finally, the candidates to take the largest value in $\mathrm{Ap}_p(A)$ are clearly scattered in the lower right corners:

$$\big(0, l(s-t)-1, (p+2-l)s-(p+1-l)t-1\big) \quad (l=1,2,\dots,p),$$

$$\big(0,(p+1)(s-t)-1,s-1\big),\quad \big(0,s-1,(p+1)(s-t)-1\big),$$
$$\big(0,(p+2-l')s-(p+1-l')t-1,l'(s-t)-1\big)\quad (l'=1,2,\dots,p)\,.$$

By comparing these values, we can find that $(0,s-t-1,(p+1)s-pt-1)$ is the largest. Hence, by Lemma 1 (1)

$$\begin{aligned}
&g_p(s^2-t^2,2st,s^2+t^2)\\
&=(s-t-1)(2st)+\big((p+1)s-pt-1\big)(s^2+t^2)-(s^2-t^2)\\
&=s\big((s+t)(s+t-2)-2t^2\big)+p(s-t)(s^2+t^2)\,.
\end{aligned}$$

In addition, Theorem 1 does not hold for $p>\lfloor t/(s-t)\rfloor$. As can be seen from the example in Table 3, the elements of the central area of $\mathrm{Ap}_4(A)$ corresponding to the elements of the central area of $\mathrm{Ap}_3(A)$ are not all left, and there will be elements corresponding to another location. Due to the deviation, the place where the maximum value is taken also changes from $(0,s-t-1,(p+1)s-pt-1)$ in $\mathrm{Ap}_p(A)$ for $p>\lfloor t/(s-t)\rfloor$. In the case of the example in Table 4, for $p=4$, the elements in the area of the stair part on both sides still regularly move to the opposite side, but in the main central part, some surplus elements moves to the lower left ($3_i\Rightarrow 4_i$) and some to the upper-right ($3_k\Rightarrow 4_k$). In this case, in general, $(0,2s-2t-1,(p+1)s-pt-1)$ takes the largest value. It is as shown in Table 4. At $p=5$, the place where the largest value is taken becomes more complicated, since the corresponding residue part is further displaced.

In the table, ⓝ denotes the position of the largest element in $\mathrm{Ap}_n(A)$. Note that the area 3_h (and so, 4_h) does not exist if $t/(s-t)$ is an integer.

3.2 The case where $2st$ is shortest

Let $s>(\sqrt{2}+1)t$, that is $s^2-t^2>2st$. For simplicity, put

$$\gamma_{x,z}:=x(s^2-t^2)+z(s^2+t^2)$$

or just (x,z). First, consider the case $p=0$. All the $2st$ elements in $\mathrm{Ap}_0(A)$ are arranged as in Table 5.

Since $r_{t+i,t+j}\equiv r_{i,j}\pmod{2st}$ and $r_{t+i,t+j}>r_{i,j}$ $(i,j\ge 0)$, any element of the form $r_{t+i,t+j}$ $(i,j\ge 0)$ is not in $\mathrm{Ap}_0(A)$. Since $r_{i,s+j}\equiv r_{s+i,j}\pmod{2st}$ and $r_{i,s+j}>r_{s+i,j}$ $(i,j\ge 0)$, any element of the form $r_{i,s+j}$ $(i,j\ge 0)$ is not in $\mathrm{Ap}_0(A)$. Since $r_{s+t+i,j}\equiv r_{i,s-t+j}\pmod{2st}$ and $r_{s+t+i,j}>r_{i,s-t+j}$ $(i,j\ge 0)$, any element of the form $r_{s+t+i,j}$ $(i,j\ge 0)$ is not in $\mathrm{Ap}_0(A)$. Therefore, only $2st$ elements in the area represented in Table 5 remain as candidates for the elements of $\mathrm{Ap}_0(A)$.

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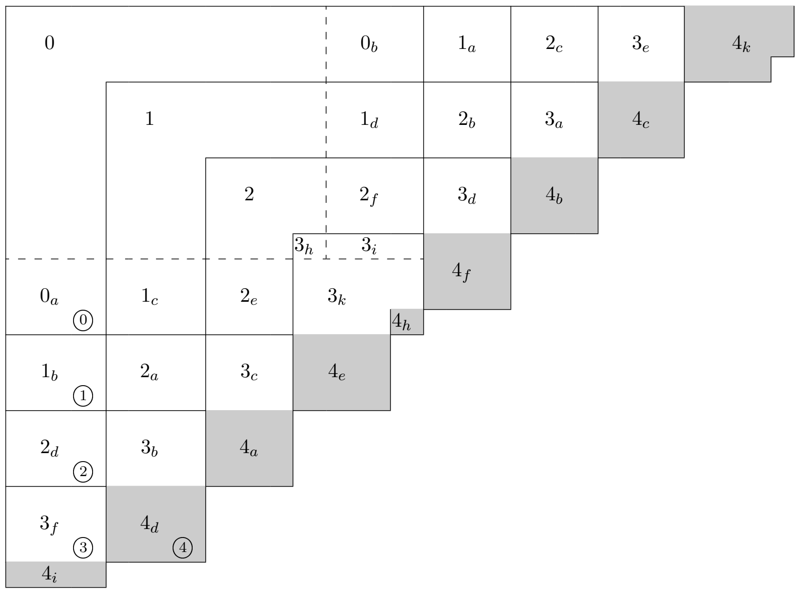

Table 4: $\mathrm{Ap}_p(s^2-t^2, 2st, s^2+t^2)$ $(p=4)$ when $s < (\sqrt{2}+1)t$

$(0,0)$	$\cdots$	$(t-1,0)$	$(t,0)$	$\cdots$ $\cdots$	$(s+t-1,0)$
$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(0,t-1)$	$\cdots$	$(t-1,t-1)$	$(t,t-1)$	$\cdots$ $\cdots$	$(s+t-1,t-1)$
$(0,t)$	$\cdots$	$(t-1,t)$			
$\vdots$		$\vdots$			
$\vdots$		$\vdots$			
$(0,s-1)$	$\cdots$	$(t-1,s-1)$			

Table 5: $\mathrm{Ap}_0(s^2-t^2, 2st, s^2+t^2)$ when $s > (\sqrt{2}+1)t$

It is similar to the case where $s < (\sqrt{2}+1)t$ to find that any of two elements in this area is not congruent modulo $(2st)$.

Since $s > t$, we have $\gamma_{s+t-1,t-1} > \gamma_{t-1,s-1}$. Hence,

$$\begin{aligned}
& g_0(s^2-t^2, 2st, s^2+t^2) \\
& = (s+t-1)(s^2-t^2) + (t-1)(s^2+t^2) - (2st) \\
& = s\bigl((s+t)(s+t-2) - 2t^2\bigr) .
\end{aligned}$$

When $p \ge 1$, the situation is somewhat similar to that of the case where $s < (\sqrt{2}+1)t$, but the role of $2st$ and s^2-t^2 is interchanged. Therefore, the calculation is not so similar.

Table 6 shows the case where $3 < \lfloor (s-t)/t \rfloor < 4$. The numbers $0, 1, 2, 3$ indicate the area of $\mathrm{Ap}_p(A)$ for $p = 0, 1, 2, 3$.

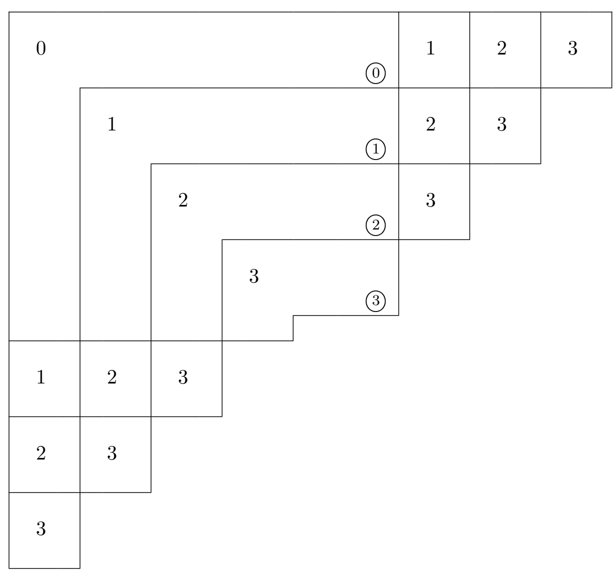

Table 6: $\mathrm{Ap}_p(s^2-t^2, 2st, s^2+t^2)$ $(p = 0, 1, 2, 3)$ when $s < (\sqrt{2}+1)t$

For simplicity, denote $\gamma_{x,z} = x(s^2-t^2) + z(s^2+t^2)$ by (x, z). More generally and more precisely, for $1 \le l \le p$, each element of the l-th $t \times t$ block from the left in the area of the lower left stair portions in $\mathrm{Ap}_p(A)$ is expressed by

$$((l-1)t+i, s+(p-l)t+j)(0 \le i \le t-1,\ 0 \le j \le t-1)\,, \qquad (7)$$

and for $1 \le l' \le p$, each element of the l'-th $t \times t$ block from the right in the area of the upper right stair portions in $\mathrm{Ap}_{p'}(A)$ is expressed by

$$(s+(p'-l'+1)t+i,(l'-1)t+j)(0 \le i \le t-1,\ 0 \le j \le t-1)\,. \tag{8}$$

Concerning the central portion of $\mathrm{Ap}_p(A)$, each element is expressed by

$$\bigl(pt+i, pt+j\bigr) \quad \begin{array}{l}(0 \le i \le t-1, 0 \le j \le s-pt-1;\\ \quad t \le i \le s-(p-1)t-1, 0 \le j \le t-1)\,.\end{array} \tag{9}$$

All the lower right elements of the $(t \times t)$ square areas and the central area are candidates for the largest value of $\mathrm{Ap}_p(A)$. And by comparison, we can see that the position at $\bigl(s+t-1,(p+1)t-1\bigr)$ takes the largest value, which is at the right-bottom of the central area, and in Figure 6, the position is shown by ⓟ $(p = 0, 1, 2, 3)$. Hence, by Lemma 1 (1)

$$\begin{aligned}&g_p(s^2-t^2, 2st, s^2+t^2)\\ &= (s+t-1)(s^2-t^2) + \bigl((p+1)t-1\bigr)(s^2+t^2) - 2st\\ &= s\bigl((s+t)(s+t-2) - 2t^2\bigr) + pt(s^2+t^2)\,.\end{aligned}$$

4 p-genus

We can use Table 1 to obtain an explicit form of genus (Sylvester number). First, let $s < (\sqrt{2}+1)t$. For a non-negative integer p, by the representation of each element in (4), (5) and (6), we have

$$\begin{aligned}&\sum_{w \in \mathrm{Ap}_p(A)} w\\ &= \sum_{l=1}^{p} \sum_{i=0}^{s-t-1} \sum_{j=0}^{s-t-1} \bigl(((l-1)s-(l-1)t+i)(2st)\\ &\qquad + ((p-l+1)s-(p-l)t+j)(s^2+t^2)\bigr)\\ &\quad + \sum_{l=1}^{p} \sum_{i=0}^{s-t-1} \sum_{j=0}^{s-t-1} \bigl(((p-l+1)s-(p-l)t+i)(2st)\\ &\qquad + ((l-1)s-(l-1)t+j)(s^2+t^2)\bigr)\\ &\quad + \sum_{i=0}^{s-t-1} \sum_{j=0}^{pt-(p-1)s-1} \bigl((p(s-t)+i)(2st) + (p(s-t)+j)(s^2+t^2)\bigr)\end{aligned}$$

$$+\sum_{i=s-t}^{pt-(p-1)s-1}\sum_{j=0}^{s-t-1}\bigl((p(s-t)+i)(2st)+(p(s-t)+j)(s^2+t^2)\bigr)$$
$$=\frac{(s+t)^2(s^3-s^2-2st^2+t^3+t^2)}{2}-\frac{p^2(s-t)^3(s+t)^2}{2}$$
$$+\frac{p(s+3t)(s^2-t^2)^2}{2}\,.$$

by Lemma 1 (2) we have

$$n_p(s^2-t^2,2st,s^2+t^2)$$
$$=\frac{1}{s^2-t^2}\Biggl(\frac{(s+t)^2(s^3-s^2-2st^2+t^3+t^2)}{2}$$
$$-\frac{p^2(s-t)^3(s+t)^2}{2}+\frac{p(s+3t)(s^2-t^2)^2}{2})\Biggr)-\frac{s^2-t^2-1}{2}$$
$$=\frac{s^3+2s^2(t-1)-2st-t^3+1}{2}-\frac{p}{2}(s^2-t^2)\bigl(p(s-t)-(s+3t)\bigr)\,.$$

Next, let $s>(\sqrt{2}+1)t$. For a non-negative integer p, by the representation of each element in (7), (8) and (9), we have

$$\sum_{w\in\mathrm{Ap}_p(A)}w$$
$$=\sum_{l=1}^{p}\sum_{i=0}^{t-1}\sum_{j=0}^{t-1}\bigl(((l-1)t+i)(s^2-t^2)$$
$$+(s+(p-l)t+j)(s^2+t^2)\bigr)$$
$$+\sum_{l=1}^{p}\sum_{i=0}^{t-1}\sum_{j=0}^{t-1}\bigl((s+(p-l+1)t+i)(s^2-t^2)$$
$$+((l-1)t+j)(s^2+t^2)\bigr)$$
$$+\sum_{i=0}^{t-1}\sum_{j=0}^{s-pt-1}\bigl((pt+i)(s^2-t^2)+(pt+j)(s^2+t^2)\bigr)$$
$$+\sum_{i=t}^{s-(p-1)t-1}\sum_{j=0}^{t-1}\bigl((pt+i)(s^2-t^2)+(pt+j)(s^2+t^2)\bigr)$$
$$=st\bigl(s^3+2s^2(t-1)-t^3\bigr)-p^2s^2t^3+ps^2t^2(4s-t)\,.$$

by Lemma 1 (2) we have

$$n_p(s^2-t^2,2st,s^2+t^2)$$

$$= \frac{1}{2st}\left(st\big(s^3+2s^2(t-1)-t^3\big)-p^2s^2t^3+ps^2t^2(4s-t)\big)\right)$$
$$-\frac{2st-1}{2}$$
$$= \frac{s^3+2s^2(t-1)-2st-t^3+1}{2}+\frac{pst}{2}\big(4s-(p+1)t\big)\,.$$

Theorem 3. *When $s < (\sqrt{2}+1)t$, for a non-negative integer p with $p \le \lfloor t/(s-t)\rfloor$, we have*

$$n_p(s^2-t^2, 2st, s^2+t^2)$$
$$= \frac{s^3+2s^2(t-1)-2st-t^3+1}{2}-\frac{p}{2}(s^2-t^2)\big(p(s-t)-(s+3t)\big)\,.$$

When $s > (\sqrt{2}+1)t$, for a non-negative integer p with $p \le \lfloor (s-t)/t\rfloor$, we have

$$n_p(s^2-t^2, 2st, s^2+t^2)$$
$$= \frac{s^3+2s^2(t-1)-2st-t^3+1}{2}+\frac{pst}{2}\big(4s-(p+1)t\big)\,.$$

5 Sylvester power sum and weighted sum

Our method, having an advantage in terms of visually grasping the elements of the Apéry set, is also useful to get Sylvester μ-th power sum

$$s_p^{(\mu)}(A) := \sum_{n\in G_p(A)} n^\mu$$

and weighted power sum with weight λ

$$s_{\lambda,p}^{(\mu)}(A) := \sum_{n\in G_p(A)} \lambda^n n^\mu \quad (\lambda \ne 1)\,,$$

where $A = \{a_1, \dots, a_k\}$ with $\gcd(A) = 1$ and $a_1 := \min(A)$.

In [13, Theorem 1, Theorem 2], both values are given explicitly by using Bernoulli numbers B_n, defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^\infty B_n \frac{x^n}{n!}$$

and Eulerian numbers $\left\langle {n \atop m} \right\rangle$, appearing in the generating function

$$\sum_{k=0}^\infty k^n x^k = \frac{1}{(1-x)^{n+1}}\sum_{m=0}^{n-1}\left\langle {n \atop m} \right\rangle x^{m+1}\quad (n\ge 1)\,,$$

respectively.

Lemma 2. *For integers k, p and μ with $k\ge 2$, $p\ge 0$ and $\mu\ge 1$, we have*

$$\begin{aligned}&s_p\mu(A)\\&=\frac{1}{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu+1}{\kappa}B_\kappa a_1^{\kappa-1}\sum_{i=0}^{a_1-1}\bigl(m_i^{(p)}\bigr)^{\mu+1-\kappa}+\frac{B_{\mu+1}}{\mu+1}(a_1^{\mu+1}-1)\,,\end{aligned}$$

Lemma 3. *For $\lambda^{a_1}\ne 1$ and a positive integer μ, we have*

$$\begin{aligned}&s_{\lambda,p}^{(\mu)}(A)\\&=\sum_{n=0}^{\mu}\frac{(-a_1)^n}{(\lambda^{a_1}-1)^{n+1}}\binom{\mu}{n}\sum_{j=0}^{n}\left\langle {n \atop n-j} \right\rangle\lambda^{ja_1}\sum_{i=0}^{a_1-1}\bigl(m_i^{(p)}\bigr)^{\mu-n}\lambda^{m_i^{(p)}}\\&\quad+\frac{(-1)^{\mu+1}}{(\lambda-1)^{\mu+1}}\sum_{j=0}^{\mu}\left\langle {\mu \atop \mu-j} \right\rangle\lambda^j\,.\end{aligned}$$

What we need is for an non-negative integer ν to obtain

$$\sum_{w\in\mathrm{Ap}_p(A)}w^\nu\quad\text{or}\quad\sum_{w\in\mathrm{Ap}_p(A)}w^\nu\lambda^w\,.$$

When $s<(\sqrt{2}+1)t$, by

$$\bigl(y(2st)+z(s^2+t^2)\bigr)^\nu=\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\bigl(y(2st)\bigr)^{\nu-\kappa}\bigl(z(s^2+t^2)\bigr)^\kappa\,,$$

we have

$$\begin{aligned}&\sum_{w\in\mathrm{Ap}_p(A)}w^\nu\\&=\sum_{l=1}^{p}\sum_{i=0}^{s-t-1}\sum_{j=0}^{s-t-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\bigl(((l-1)s-(l-1)t+i)(2st)\bigr)^{\nu-\kappa}\end{aligned}$$

$$\times\left(((p-l+1)s-(p-l)t+j)(s^2+t^2)\right)^\kappa$$
$$+\sum_{l=1}^{p}\sum_{i=0}^{s-t-1}\sum_{j=0}^{s-t-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\left(((p-l+1)s-(p-l)t+i)(2st)\right)^{\nu-\kappa}$$
$$\times\left(((l-1)s-(l-1)t+j)(s^2+t^2)\right)^\kappa$$
$$+\sum_{i=0}^{s-t-1}\sum_{j=0}^{pt-(p-1)s-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\left((p(s-t)+i)(2st)\right)^{\nu-\kappa}$$
$$\times\left((2st)+(p(s-t)+j)(s^2+t^2)\right)^\kappa$$
$$+\sum_{i=s-t}^{pt-(p-1)s-1}\sum_{j=0}^{s-t-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\left((p(s-t)+i)(2st)\right)^{\nu-\kappa}$$
$$\times\left((p(s-t)+j)(s^2+t^2)\right)^\kappa .$$

When $s>(\sqrt{2}+1)t$, we have

$$\sum_{w\in\mathrm{Ap}_p(A)}w^\nu$$
$$=\sum_{l=1}^{p}\sum_{i=0}^{t-1}\sum_{j=0}^{t-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\left(((l-1)t+i)(s^2-t^2)\right)^{\nu-\kappa}$$
$$\times\left((s+(p-l)t+j)(s^2+t^2)\right)^\kappa$$
$$+\sum_{l=1}^{p}\sum_{i=0}^{t-1}\sum_{j=0}^{t-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\left((s+(p-l+1)t+i)(s^2-t^2)\right)^{\nu-\kappa}$$
$$\times\left(((l-1)t+j)(s^2+t^2)\right)^\kappa$$
$$+\sum_{i=0}^{t-1}\sum_{j=0}^{s-pt-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\left((pt+i)(s^2-t^2)\right)^{\nu-\kappa}\left((pt+j)(s^2+t^2)\right)^\kappa$$
$$+\sum_{i=t}^{s-(p-1)t-1}\sum_{j=0}^{t-1}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\left((pt+i)(s^2-t^2)\right)^{\nu-\kappa}\left((pt+j)(s^2+t^2)\right)^\kappa .$$

By substituting each of the above identities into the formula in Lemma 2 we obtain a general explicit formula. It is similar about the weighted sums.

Though the above general expression cannot be further simplified, for a specific ν, we can get a more explicit form. For example, when $s>(\sqrt{2}+1)t$, for $\nu = 2$, we have

$$\sum_{w\in\mathrm{Ap}_p(A)} w^2$$
$$= \frac{st}{3}\bigl(2s^6+6s^5(t-1)+s^4(8t^2-12t+5)-6s^3t^3-2s^2(2t-3)t^3$$
$$+t^4(2t^2-1)\bigr)-\frac{8p^3s^4t^4}{3}+2pst^3\bigl(3s^4-s^3(2t-1)+t^4\bigr)$$
$$+\frac{2pst^2}{3}\bigl(6s^5+3s^4(3t-4)-s^3(2t-3)t-6s^2t^3+3t^5\bigr)\,.$$

Together with the form where $\nu = 1$:

$$\sum_{w\in\mathrm{Ap}_p(A)} w = st\bigl(s^3+2s^2(t-1)-t^3\bigr)+ps^2t^2(4s-(p+1)t)\,,$$

by using Lemma 2 we obtain an explicit form of the Sylvester sum $s_p(A) = s_p^{(1)}(A)$. It is similar when $s < (\sqrt{2}+1)t$.

Proposition 1. *When $s < (\sqrt{2}+1)t$, for a non-negative integer p with $p \le \lfloor t/(s-t)\rfloor$, we have*

$$s_p(s^2-t^2, 2st, s^2+t^2)$$
$$= \frac{1}{12}\bigl(2s^6+6s^5(t-1)+s^4(8t^2-18t+5)-6s^3t(t^2+2t-2)$$
$$-2s^2t^2(2t^2-3t-2)+6st^4+2t^6-t^4-1\bigr)$$
$$-\frac{p^3}{3}(s^2-t^2)^3+\frac{p^2}{2}\bigl(s^5+6s^4t^2-2s^3t^2(4t+1)+st^4+2t^6\bigr)$$
$$+\frac{p}{6}\bigl(2s^6+3s^5(t-1)+6s^4t(3t-2)-6s^3t^2(4t+1)+12s^2t^3-3st^4(t-3)+4t^6\bigr)\,.$$

When $s > (\sqrt{2}+1)t$, for a non-negative integer p with $p \le \lfloor (s-t)/t\rfloor$, we have

$$s_p(s^2-t^2, 2st, s^2+t^2)$$
$$= \frac{1}{12}\bigl(2s^6+6s^5(t-1)+s^4(8t^2-18t+5)-6s^3t(t^2+2t-2)$$
$$-2s^2t^2(2t^2-3t-2)+6st^4+2t^6-t^4-1\bigr)$$
$$-\frac{2}{3}p^3s^3t^3+\frac{p^2t^2}{2}\bigl(3s^4-s^3(2t-1)+s^2t+t^4\bigr)$$
$$+\frac{pt}{6}\bigl(6s^5+3s^4(3t-4)-s^3t(2t+9)-3s^2t^2(2t-1)+3t^5\bigr)\,.$$

6 Application to triples associated to integer-sided triangles with a 60 degree angle

In the above sections, we considered triples (x, y, z) satisfying the Diophantine equation $x^2 + y^2 = z^2$. This method is applicable to triples satisfying another Diophantine equation. Instead of a right triangle, consider three sides of a triangle that has an angle of 60 degrees. Namely, consider the triples $(x, y, z) = (s^2 - 3t^2 + 2st, 4st, s^2 + 3t^2)$, satisfying $x^2 + y^2 - xy = z^2$. For primitivity, we need the additional condition, $3 \nmid s$; then $\gcd(s^2 - 3t^2 + 2st, 4st, s^2 + 3t^2) = 1$.

Note that if $s = 3t$, then $x = y = z$. Thus, there are two cases to consider.

1. If $s < 3t$, then $x = s^2 - 3t^2 + 2st$ is the shortest side.

2. If $s > 3t$, then $y = 4st$ is the shortest side.

In the case of [1], we have and $x < z < y$. Then all the elements of the 0-Apéry set are given as in Table 7.

$(0,0)$	$\cdots$	$(s-t-1,0)$	$(s-t,0)$	$\cdots\ \cdots$	$(s+t-1,0)$
$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(0,s-t-1)$	$\cdots$	$(s-t-1,s-t-1)$	$(s-t,s-t-1)$	$\cdots\ \cdots$	$(s+t-1,s-t-1)$
$(0,s-t)$	$\cdots$	$(s-t-1,s-t)$			
$\vdots$		$\vdots$			
$\vdots$		$\vdots$			
$(0,s+t-1)$	$\cdots$	$(s-t-1,s+t-1)$			

Table 7: $\mathrm{Ap}_0(s^2 - 3t^2 + 2st, 4st, s^2 + 3t^2)$ when $s < 3t$

Since $s < 3t$, the largest value is at $(s - t - 1, s + t - 1)$. Hence, by $x < z < y$, we have

$$g_0(s^2 - 3t^2 + 2st, 4st, s^2 + 3t^2)$$
$$= (s - t - 1)(s^2 + 3t^2) + (s + t - 1)(4st) - (s^2 - 3t^2 + 2st)\,.$$

By applying a similar method, as long as $p \le \lfloor (2t)/(s - t) \rfloor$, the largest value of $\mathrm{Ap}_p(A)$ is at $\bigl(s - t - 1, s + t - 1 + p(s - t)\bigr)$. Thus,

$$g_p(s^2 - 3t^2 + 2st, 4st, s^2 + 3t^2)$$
$$= (s - t - 1)(s^2 + 3t^2) + \bigl((p + 1)s - (p - 1)t - 1\bigr)(4st) - (s^2 - 3t^2 + 2st)\,.$$

In the case of [2], we have $y < z < x$. Then all the elements of the 0-Apéry set are given as in Table 8.

$(0,0)$	$\cdots$	$(2t-1,0)$	$(2t,0)$	$\cdots\ \cdots$	$(s+t-1,0)$
$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(0,2t-1)$	$\cdots$	$(s-t-1,s-t-1)$	$(s-t,s-t-1)$	$\cdots\ \cdots$	$(s+t-1,2t-1)$
$(0,2t)$	$\cdots$	$(2t-1,2t)$			
$\vdots$		$\vdots$			
$\vdots$		$\vdots$			
$(0,s+t-1)$	$\cdots$	$(2t-1,s+t-1)$			

Table 8: $\mathrm{Ap}_0(s^2-3t^2+2st, 4st, s^2+3t^2)$ when $s > 3t$

Since $s > 3t$, the largest value is at $(2t-1, s+t-1)$. Hence, by $y < z < x$, we have

$$g_0(s^2-3t^2+2st, 4st, s^2+3t^2) = (2t-1)(s^2+3t^2) + (s+t-1)(s^2-3t^2+2st) - 4st\,.$$

By applying a similar method, as long as $p \le \lfloor (s-t)/(2t) \rfloor$, the largest value of $\mathrm{Ap}_p(A)$ is at $\bigl(2t-1, s+t-1+p(2t)\bigr)$. Thus,

$$g_p(s^2-3t^2+2st, 4st, s^2+3t^2) = (2t-1)(s^2+3t^2) + \bigl(s+(2p+1)t-1\bigr)(s^2-3t^2+2st) - 4st\,.$$

Theorem 4. *Let s and t be positive integers having different parity with $s > t$, $\gcd(s,t) = 1$ and $3 \nmid s$. When $s < 3t$, for a nonnegative integer p with $p \le \lfloor (2t)/(s-t) \rfloor$, we have*

$$g_p(s^2-3t^2+2st, 4st, s^2+3t^2) = (s-t-1)(s^2+3t^2) + \bigl((p+1)s-(p-1)t-1\bigr)(4st) - (s^2-3t^2+2st)\,.$$

When $s > 3t$, for a nonnegative integer p with $p \le \lfloor (s-t)/(2t) \rfloor$, we have

$$g_p(s^2-3t^2+2st, 4st, s^2+3t^2) = (2t-1)(s^2+3t^2) + \bigl(s+(2p+1)t-1\bigr)(s^2-3t^2+2st) - 4st\,.$$

More generally, when we consider Diophantine equations in three variables which have infinitely many triples of solutions given by two-dimensional parameters as above, it would be interesting to analyse whether the same method as above can be applied. They will be discussed in subsequent works.

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7 Some remarks

It is very difficult to completely determine the p-Frobenius and the p-Sylvester numbers for all non-negative integers p, as seen in [11, 12, 15, 14, 16, 17] too. The reason is that if p is greater than a certain value, the regularity of the p-Apéry set is broken. This is the principal reason why we only proved in Theorem 1 partial results for the p-Frobenius and the p-Sylvester numbers of primitive Pythagorean triples for p bounded by a certain constant. For example, for $\langle s^2-t^2, 2st, s^2+t^2\rangle$ if $s=4m+1$, $t=2m$ (with $m\ge 2$), then the results are given only for $p=0$; and if $s=10m+1$, $t=4m$ (with $m\ge 2$), then the results are given only for $p=0,1$.

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