The Spectral Geometry of Biregular Graphs: a Quantum Graph Approach

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Abstract

The spectral theory of the Laplace differential operator for biregular quantum graphs is developed. Trees are studied in detail. Generating functions for closed nonbacktracking walks appear when resolvents for trees are related to resolvents for biregular graphs they cover. The relationship between resolvent traces for finite graphs and walk generating functions is especially productive. A detailed description of the rational extension of nonbacktracking walk generating functions is presented.

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1 Introduction

Let $\mathcal{G}$ denote a locally finite connected simple graph with a countable vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Edges are treated as intervals of length 1. As a consequence of this identification one may use the Hilbert space $L^2(\mathcal{G}) = \oplus_{e\in\mathcal{E}} L^2(e)$ and a standard construction to equip $\mathcal{G}$ with a self-adjoint differential operator $\mathcal{L}$ which acts by $-D^2$ on its domain.

Quantum graph methods are used here to study the spectral geometry of biregular graphs; these are bipartite graphs whose vertex degrees are the same for all vertices in one of the two vertex classes. The main results use the spectral data of a discrete Laplacian Δ to compute the generating function for counts of nonbacktracking closed walks on finite biregular graphs.

For finite graphs, previously established results link the spectra of a discrete Laplacian Δ to the spectrum of the differential operator $\mathcal{L}$. Biregular graphs have the biregular tree $\mathcal{T}$ as a universal covering space. The spectral theory of the differential operator $\mathcal{L}$ is developed in detail for the biregular tree $\mathcal{T}$, including a detailed description of the spectrum and resolvent $R_{\mathcal{T}}(\lambda) = (\mathcal{L} - \lambda I)^{-1}$. Covering space theory allows the resolvents $R_{\mathcal{G}}(\lambda) = (\mathcal{L} - \lambda I)^{-1}$ for graphs $\mathcal{G}$ to be constructed using the tree resolvents.

When $\mathcal{G}$ is finite, the resolvent $R_{\mathcal{G}}(\lambda)$ is a trace class operator. Two expressions for the trace of the resolvent are developed. The first is a conventional series based on eigenvalues. The second expression describes the trace as a product of a term coming from $\mathcal{T}$ and a term involving a generating function for closed nonbacktracking walks in $\mathcal{G}$. By connecting these two descriptions one obtains a formula for the generating function. This formula is combined with complex variable methods to show that the walk generating functions extend as rational functions, with considerable information about pole numbers and locations.

There is, of course, a large literature on spectral theory for combinatorial graphs; [12] is a modern reference, and [24] discusses infinite graphs. The linkage between the spectra of quantum graphs with equal edge lengths and discrete graph operators, originally developed by [3] and [11] for $\mathcal{L}$, was extended to Schrödinger operators on possibly infinite quantum graphs in [25]. The methods used in this work for biregular graphs go beyond the location of the spectrum to provide a detailed description of the resolvent, useful for computing functions of $\mathcal{L}$, and the related multiplier function theory.

There has been considerable work on the spectral geometry of finite regular graphs, when each vertex has the same degree. Connections between

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the spectral theory of the discrete Laplacian and walk generating functions were previously explored in [7]. A more recent and extensive treatment of related material is [31]. Related work from a quantum graph viewpoint is in [8]. Analyses for biregular graphs or irregular graphs have also used the nonbacktracking adjacency operator. References include [1], [2] and [18].

Many years ago Bob Brooks suggested that the techniques of [8] could be used in the biregular case. His suggestion is gratefully acknowledged. Thanks too to the referee who made beneficial suggestions.

This work starts with a discussion of quantum graphs and the differential operator $\mathcal{L}$. For the finite graphs studied here, the eigenvalues of $\mathcal{L}$ can be computed from the eigenvalues of a conventional discrete graph Laplacian Δ together with some additional combinatorial data. The regular behavior of the eigenvalue sequence of $\mathcal{L}$ allows for a compact description of the trace of $R_{\mathcal{G}}(\lambda)$ with explicit dependence on the eigenvalues of Δ.

The next section introduces biregular graphs. Important examples for this work are the complete bipartite graphs whose resolvent trace is computed. Conveniently, from the quantum graph perspective regular graphs can be converted to biregular graphs with minimal effect on the spectrum, allowing this work to subsume some of the previous analysis of regular graphs.

The spectral theory for a biregular tree $\mathcal{T}$ is then developed. Given an edge e in $\mathcal{T}$, there are special solutions of the eigenvalue equation $-D^2y = \lambda y$ which are functions of the distance from e. The propagation of these solutions depends on multipliers $\mu^{\pm}(\lambda)$ whose behavior largely determines the spectrum of $\mathcal{L}$ for the tree. The functions $\mu^{\pm}(\lambda)$ have a central role in the construction of the resolvents $R_{\mathcal{T}}(\lambda)$; their properties as analytic functions are discussed.

The final section opens with a discussion of closed nonbacktracking walks in finite graphs and their generating functions. The generating functions for complete bipartite graphs are computed. This is followed by material on covering spaces and the construction of resolvents $R_{\mathcal{G}}(\lambda)$ from $R_{\mathcal{T}}(\lambda)$. When $\mathcal{G}$ is a finite biregular graph the resolvent traces are closely related to walk generating functions, which are described in detail.

2 Differential operators on graphs

Background material on quantum graphs [4, 21], tailored to our purposes, will be used. In this work edges have length one, and are usually identified

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with $[0,1]$. By using the identification of edges with intervals, function spaces and differential operators may be defined. Let $L^2(e) = L^2[0,1]$ be the usual Lebesgue space. $L^2(\mathcal{G})$ denotes the Hilbert space $\oplus_{e \in \mathcal{E}} L^2(e)$. For $f \in L^2(\mathcal{G})$, the function $f_e : [0,1] \to \mathbb{C}$ is the restriction of f to the edge e. The inner product is

$$\langle f, g \rangle = \int_{\mathcal{G}} f \overline{g} = \sum_{e \in \mathcal{E}} \int_0^1 f_e(x) \overline{g_e(x)} \, dx.$$

A formal second derivative operator $-D^2$ acts componentwise on functions $f \in L^2(\mathcal{G})$ in its domain $\mathcal{D}$. Functions $f \in \mathcal{D}$ are continuous on $\mathcal{G}$, and continuously differentiable on each edge, with f_e' absolutely continuous for each edge e, and $f'' \in L^2(\mathcal{G})$. At each vertex v the function f must satisfy

$$\sum_{e \sim v} \partial_\nu f_e(v) = 0, \tag{2.1}$$

where $e \sim v$ indicates an edge e incident on the vertex v. Here ∂_ν denotes the derivative computed in 'outward pointing' local coordinates which identify each edge e incident on v with a copy of $[0,1]$ and identify v with 0. The operator $\mathcal{L} : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ acting by $-D^2$ with domain $\mathcal{D}$ is self-adjoint [4]. The spectrum is a subset of $[0,\infty)$.

If the vertex set $\mathcal{V}$ is finite then $\mathcal{L}$ has discrete spectrum, with eigenvalues $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \dots$, listed with multiplicity, and an orthonormal basis of eigenfunctions ϕ_n with eigenvalues λ_n [4, p. 67]. The resolvent $R_{\mathcal{G}}(\lambda) = (\mathcal{L} - \lambda I)^{-1}$ is trace class [26, p. 206-212] or [17, p. 521-525] with summable eigenvalue sequence $1/(\lambda_n - \lambda)$ for λ in the resolvent set ρ, the complement of the spectrum. The resolvent can be written as an integral operator with kernel

$$R_{\mathcal{G}}(\lambda, x, y) = \sum_n \frac{\phi_n(x)\overline{\phi_n(y)}}{\lambda_n - \lambda}, \quad x, y \in \mathcal{G}.$$

Since $\{\phi_n\}$ is an orthonormal sequence the trace is

$$\mathrm{tr} R_{\mathcal{G}}(\lambda) = \sum_n \frac{1}{\lambda_n - \lambda} = \int_{\mathcal{G}} R(\lambda, x, x) dx. \tag{2.2}$$

The next proposition records some simple facts.

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Proposition 2.1. *Suppose $\mathcal{G}$ is finite. If $\Im(\lambda) \ne 0$ then $\Im(\lambda)\Im(\mathrm{tr} R_{\mathcal{G}}(\lambda)) > 0$. ($\mathrm{tr} R_{\mathcal{G}}(\lambda)$ is a Herglotz-Nevanlinna function.) The function $\mathrm{tr} R_{\mathcal{G}}(\lambda)$ has exactly one root between distinct eigenvalues λ_n and λ_{n+1}, and no other roots.*

Proof. Since $\lambda_n \in \mathbb{R}$,

$$\Im(\frac{1}{\lambda_n - \lambda}) = \frac{1}{2i}[\frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_n - \overline{\lambda}}] = \frac{1}{2i}\frac{\lambda - \overline{\lambda}}{|\lambda_n - \lambda|^2},$$

leading to the first claim, which also shows that any roots of $\mathrm{tr} R_{\mathcal{G}}(\lambda)$ must be real. If $\lambda < 0$ then $\mathrm{tr} R_{\mathcal{G}}(\lambda)$ is a sum of positive terms, and there is a pole at $\lambda = 0$. Finally, for $\lambda \in \mathbb{R}$, $\lim_{\lambda \downarrow \lambda_n} = -\infty$, $\lim_{\lambda \uparrow \lambda_n} = +\infty$ and

$$\frac{d}{d\lambda} \sum_n \frac{1}{\lambda_n - \lambda} = \sum_n \frac{1}{(\lambda_n - \lambda)^2},$$

so $\mathrm{tr} R_{\mathcal{G}}(\lambda)$ is increasing between the eigenvalues. □

2.1 Trace formula 1

When $\mathcal{G}$ is finite, a strong link between the eigenvalues of $\mathcal{L}$ and a discrete Laplacian Δ leads to a formula for $\mathrm{tr} R_{\mathcal{G}}(\lambda)$. Given a vertex $v \in \mathcal{G}$, let $w_1, \dots, w_{deg(v)}$ be the vertices adjacent to v. On the vertex space $\mathbb{V}$ consisting of functions $f : \mathcal{V} \to \mathbb{C}$, one has the adjacency operator

$$Af(v) = \sum_{i=1}^{deg(v)} f(w_i),$$

and the degree operator

$$T_{deg} f(v) = deg(v) f(v).$$

The operator Δ defined by

$$\Delta f(v) = f(v) - T_{deg}^{-1} A f(v) \tag{2.3}$$

is similar (in the sense of matrix conjugation) [12, pp. 3-7] to the symmetric discrete Laplacian $I - T_{deg}^{-1/2} A T_{deg}^{-1/2}$; the eigenvalues are real and nonnegative. In case $\mathcal{G}$ is biregular the matrix $T_{deg}^{-1/2} A T_{deg}^{-1/2}$ is a constant multiple of A.

Suppose $y(x)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda = \omega^2$. If the edge from v to w_i is identified with $[0,1]$, then

$$y(x) = y(v)\cos(\omega x) + \frac{y(w_i) - y(v)\cos(\omega)}{\sin(\omega)}\sin(\omega x), \quad \sin(\omega) \ne 0,$$

and y may be recovered from its values at $0, 1$ except when $\sin(\omega) = 0$. The condition (2.1) gives

$$f(v)\cos(\omega) = \frac{1}{\deg(v)}\sum_j f(w_j).$$

The argument can be reversed, leading to the following well known result [3, 11] or [4, p. 90-92].

Proposition 2.2. *If $\lambda_k \notin \{n^2\pi^2 \mid n = 0,1,2,\dots\}$, then λ_k is an eigenvalue of $\mathcal{L}$ if and only if $\nu = 1 - \cos(\sqrt{\lambda_k})$ is an eigenvalue of Δ. In this case λ_k and ν have the same geometric multiplicity.*

Henceforth, $\mathcal{G}$ is assumed to be bipartite with $N_{\mathcal{V}}$ vertices and $N_{\mathcal{E}}$ edges. Additional spectral information is available; results similar to the next pair of propositions can be found in [9, 19]. Different aspects of bipartite quantum graphs are treated in [20]. Let $E(\lambda)$ denote the eigenspace of $\mathcal{L}$ with eigenvalue λ. First a standard observation: if $y \in E(0)$ then

$$0 = \int_{\mathcal{G}} y\mathcal{L}y = \int_{\mathcal{G}} (y')^2,$$

so y must be constant on each edge. Since y is continuous on $\mathcal{G}$, $E(0)$ consists of the functions constant on $\mathcal{G}$.

The edge space $\mathbb{E}$ of $\mathcal{G}$ consists of functions $f : \mathcal{E} \to \mathbb{C}$ which are constant on each edge. Pick a spanning tree $\mathcal{T}$ for $\mathcal{G}$, letting $e_1, \dots, e_M$, with $M = N_{\mathcal{E}} - N_{\mathcal{V}} + 1$, be the edges of $\mathcal{G}$ not in $\mathcal{T}$. For each $m = 1, \dots, M$ there is [6, p. 53] a cycle C_m containing e_m together with a path in $\mathcal{T}$ connecting the vertices of e_m. Let $Z_0(\mathcal{G})$ be the subspace of $\mathbb{E}$ generated by the independent functions with the value 1 on C_m and zero on the complement.

Proposition 2.3. *If $\mathcal{G}$ is finite and bipartite, then*

$$\dim E(n^2\pi^2) = N_{\mathcal{E}} - N_{\mathcal{V}} + 2, \quad n = 1, 2, \dots.$$

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Proof. First note that if y is an eigenfunction of $\mathcal{L}$ with eigenvalue $n^2\pi^2$ that vanishes at any vertex, then it vanishes at all vertices. Let $E_0(n^2\pi^2) \subset E(n^2\pi^2)$ denote those eigenfunctions of $\mathcal{L}$ with eigenvalue $n^2\pi^2$ vanishing at the vertices.

Since $\mathcal{G}$ is bipartite all cycles C_m are even. Suppose C_m has distinct vertices $v_0, \dots, v_{2k-1}$, edges $\{v_{j-1}, v_j\}$ for $j = 1, \dots, 2k-1$ and $\{v_{2k-1}, v_0\}$. Pick coordinates x identifying $\{v_{j-1}, v_j\}$ with $[j-1, j]$ and $\{v_{2k-1}, v_0\}$ with $[2k-1, 2k]$. The functions f_m with values $\sin(n\pi x)$ on the edges of C_m and 0 on all other edges are independent eigenfunctions of $\mathcal{L}$.

Now suppose that $y \in E_0(n^2\pi^2)$. By subtracting a linear combination $\sum a_m f_m$ we may assume that y vanishes on the edges e_m. If v is a boundary vertex v of the spanning tree $\mathcal{T}$ with v in edge e of $\mathcal{T}$, then $y = 0$ on all edges of $\mathcal{G}$ incident on v except possibly e. But the vertex conditions (2.1) force y to vanish on e as well, and thus on all of $\mathcal{T}$. So $\dim E_0(n^2\pi^2) = \dim Z_0(\mathcal{G}) = N_{\mathcal{E}} - N_{\mathcal{V}} + 1$.

Finally, identify each edge with $[0, 1]$ so that the R vertices are 0 and the B vertices are 1. If $\lambda = n^2\pi^2$, an eigenfunction vanishing at no vertices is given by $\cos(n\pi x)$ on each edge. □

Proposition 2.4. *If $\mathcal{G}$ is finite and bipartite, then the eigenvalues λ of $\mathcal{L}$ satisfy*

(i) there are precisely $N_{\mathcal{V}} - 2$ eigenvalues with $0 < \lambda < \pi^2$, counted with multiplicity.

(ii) there are precisely $N_{\mathcal{V}} - 2$ eigenvalues with $\pi^2 < \lambda < (2\pi)^2$, counted with multiplicity, and these have the form $\lambda = (2\pi - \mu)^2$ for μ an eigenvalue of $\mathcal{L}$ with $0 < \mu < \pi^2$; λ and μ have the same multiplicity.

Proof. Since $\mathcal{G}$ is bipartite [12, p. 7], Δ has eigenvalues $\nu_0 = 0, \nu_{N_{\mathcal{V}}-1} = 2$, both with multiplicity 1. All eigenvalues ν of Δ satisfy $0 \le \nu \le 2$. Thus there are $N_{\mathcal{V}} - 2$ eigenvalues ν of Δ with $0 < \nu < 2$. These correspond to eigenvalues λ of $\mathcal{L}$ with $0 < \sqrt{\lambda} < \pi$. Since $1 - \cos(\sqrt{\lambda})$ is symmetric about $\sqrt{\lambda} = \pi$, conclusion (ii) holds.

□

Let α_k^2, $k = 1, \dots, 2N_{\mathcal{V}} - 4$ be the eigenvalues of $\mathcal{L}$, with multiplicity, satisfying $0 < \alpha_k^2 < (2\pi)^2$ and $\alpha_k^2 \ne \pi^2$. The description of the eigenvalues of $\mathcal{L}$ implies that the trace of $R_{\mathcal{G}}(\lambda)$ can be rewritten as

$$\operatorname{tr}(R_{\mathcal{G}}(\lambda)) = -\frac{1}{\lambda} + \sum_{k=1}^{2N_{\mathcal{V}}-4} \sum_{m=-\infty}^{\infty} \frac{1}{[\alpha_k + 2m\pi]^2 - \lambda} + (N_{\mathcal{E}} - N_{\mathcal{V}} + 2) \sum_{m=1}^{\infty} \frac{1}{[m\pi]^2 - \lambda}$$

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A standard residue calculation will be sketched to evaluate these sums; more details are in [22, p.149]. The function $\pi \cot(\pi z)/[(\alpha + 2\pi z)^2 - \lambda]$ has simple poles with residues $1/[(\alpha + 2\pi m)^2 - \lambda]$ at the integers m, and simple poles at $z = (-\alpha \pm \sqrt{\lambda})/(2\pi)$. Integrate over the contours γ_N which bound the squares with vertices $(N + 1/2)(\pm 1 \pm i)$ to get

$$0 = \lim_{N \to \infty} \frac{1}{2\pi i} \int_{\gamma_N} \frac{\pi \cot(\pi z)}{(\alpha + 2\pi z)^2 - \lambda} \, dz = \sum_{m=-\infty}^{\infty} \frac{1}{(\alpha + 2\pi m)^2 - \lambda}$$

$$+ \frac{1}{4\sqrt{\lambda}} \cot(\frac{-\alpha + \sqrt{\lambda}}{2}) - \frac{1}{4\sqrt{\lambda}} \cot(\frac{-\alpha - \sqrt{\lambda}}{2}).$$

Trigonometric identies give

$$\sum_{m=-\infty}^{\infty} \frac{1}{[\alpha + 2\pi m]^2 - \lambda} = \frac{\sin(\sqrt{\lambda})}{2\sqrt{\lambda}} \frac{1}{\cos(\sqrt{\lambda}) - \cos(\alpha)} \tag{2.4}$$

Similarly,

$$\sum_{m=1}^{\infty} \frac{1}{[m\pi]^2 - \lambda} = \frac{1}{2\lambda} - \frac{\cot(\sqrt{\lambda})}{2\sqrt{\lambda}},$$

and so

$$\operatorname{tr}(R_{\mathcal{G}}(\lambda)) = \frac{N_{\mathcal{E}} - N_{\mathcal{V}}}{2\lambda} + \frac{\sin(\sqrt{\lambda})}{2\sqrt{\lambda}} \sum_{k=1}^{2N_{\mathcal{V}}-4} \frac{1}{\cos(\sqrt{\lambda}) - \cos(\alpha_k)} \tag{2.5}$$

$$-(N_{\mathcal{E}} - N_{\mathcal{V}} + 2) \frac{\cot(\sqrt{\lambda})}{2\sqrt{\lambda}}$$

3 Biregular graphs

A graph $\mathcal{G}$ is bipartite if the vertex set $\mathcal{V}$ can be partitioned into two subsets, $\mathcal{V}_R$ and $\mathcal{V}_B$, and all edges e of the edgeset $\mathcal{E}$ have the form $e = \{r, b\}$ with $r \in \mathcal{V}_R$ and $b \in \mathcal{V}_B$. $\mathcal{G}$ is biregular if the degree of each vertex depends only on the vertex class, R or B. A simple example is the complete bipartite graph $K(m_B, m_R)$ with m_B vertices in $\mathcal{V}_B$ and m_R vertices in $\mathcal{V}_R$. The edge set consists of all $\{v, w\}$ with $v \in \mathcal{V}_B$ and $w \in \mathcal{V}_R$. Another important example is the biregular tree $\mathcal{T}$, with vertices in $\mathcal{V}_B$ having degree $\delta_B + 1 > 1$, and vertices in $\mathcal{V}_R$ have degree $\delta_R + 1 > 1$.

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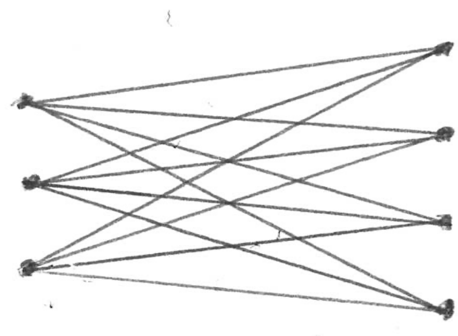

Figure 1: A complete bipartite graph, $K(3,4)$

Here are two ways of generating biregular graphs from a regular graph Γ. Suppose $\mathcal{V}_1$ and $\mathcal{V}_2$ are two copies of the vertex set of Γ. The bipartite cover $\widetilde{\Gamma}$ of Γ has vertex set the disjoint union of $\mathcal{V}_1$ and $\mathcal{V}_2$. If $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$, then there is an edge $\{v_1, v_2\}$ in $\widetilde{\Gamma}$ if and only if the corresponding vertices v_1, v_2 in Γ are adjacent. $\widetilde{\Gamma}$ is connected if and only if Γ is connected and not bipartite. The second method works well for quantum graphs. For each edge e of Γ, simply add a vertex v to the middle of e, using the continuity and derivative condition of (2.1). The vertex v will have degree 2, but the domain of $\mathcal{L}$ is not affected [4, p. 14]. When all edges are required to have length 1, the edges of the original graph effectively have length 2. Rescaling of edge lengths for a finite graph by a factor 2 converts eigenvalues of $\mathcal{L}$ from λ_n to $\lambda_n/4$. In this fashion the spectral theory of regular quantum graphs is subsumed into the spectral theory of biregular quantum graphs.

Computations for the complete bipartite graphs will be useful later. To compute the Δ spectrum for $K(m_B, m_R)$, index the R vertices as $v_1, \dots, v_{m_R}$ and the B vertices as $v_{m_R+1}, \dots, v_{m_B+m_R}$. The matrix $T_{deg}^{-1}A$ of (2.3) is

$$T_{deg}^{-1}A = \begin{pmatrix} 0_{m_R,m_R} & C_1 \\ C_2 & 0_{m_B,m_B}, \end{pmatrix}$$

where C_1 is an $m_R \times m_B$ block with all entries $1/m_B$ and C_2 is an $m_B \times m_R$ block with all entries $1/m_R$. Since

$$(T_{deg}^{-1}A)^2 = \begin{pmatrix} C_1C_2 & 0_{m_R,m_B} \\ 0_{m_B,m_R} & C_2C_1 \end{pmatrix},$$

and $\mathrm{tr}T_{deg}^{-1}A = 0$ one finds that 0 is an eigenvalue of $T_{deg}^{-1}A$ with multiplicity $m_R + m_B - 2$, while the other eigenvalues are ± 1. Thus the eigenvalues of $\Delta = I - T_{deg}^{-1}A$ are 1 with multiplicity $m_R + m_B - 2$, 0 and 2.

By Proposition 2.2 the eigenvalues $\lambda \ne n^2\pi^2$ of the corresponding differential operator $\mathcal{L}$ are $(\frac{\pi}{2} + n\pi)^2$ for $n = 0, 1, 2, \dots$, each with multiplicity $m_R + m_B - 2$. By Proposition 2.3 the multiplicity of $\lambda = n^2\pi^2$ for $n = 1, 2, 3, \dots$ is $m_R m_B - (m_R + m_B) + 2$. Finally $\lambda = 0$ is a eigenvalue with multiplicity 1. In particular, $\cos(\alpha_k) = 0$. Letting $\mathrm{tr}R_{CB}(\lambda)$ denote the trace of the resolvent for $\mathcal{L}$ on a complete bipartite graph, (2.5) gives

$$\mathrm{tr}R_{CB}(\lambda) = \frac{m_R m_B - (m_R + m_B)}{2\lambda} + (\delta_R + \delta_B)\frac{\tan(\sqrt{\lambda})}{\sqrt{\lambda}} \tag{3.1}$$

$$-(m_R m_B - (m_R + m_B) + 2)\frac{\cot(\sqrt{\lambda})}{2\sqrt{\lambda}}.$$

4 Biregular trees

4.1 Multipliers

Now consider the case of the biregular tree $\mathcal{T}$ whose edges all have length 1. Vertices in $\mathcal{V}_B$ have degree $d_B = \delta_B + 1 > 1$, and vertices in $\mathcal{V}_R$ have degree $d_R = \delta_R + 1 > 1$.

Fix a vertex $r_0 \in \mathcal{V}_R$, identified with $0 \in \mathbb{R}$, and an initial edge $e_0 = \{r_0, b_0\}$, which is identified with the interval $[0, 1]$. Two classes of rays, extending from the edge e_0, will be used: one whose vertex sequence starts r_0, b_0, the other starting with b_0, r_0. The rays rooted at r_0 are subgraphs of $\mathcal{T}$ with vertex set $\{r_0, b_0, r_1, b_1, \dots\}$, indexed by nonnegative integers n, with edges $\{r_n, b_n\}$ and $\{b_n, r_{n+1}\}$. The rays rooted at b_0 will have a vertex sets $\{b_0, r_0, b_{-1}, r_{-1} \dots\}$ indexed by nonpositive integers n, with edges $\{b_n, r_n\}$ and $\{r_n, b_{n-1}\}$. As metric graphs, a ray rooted at r_0 may be identified with $[0, \infty)$, and a ray rooted at b_0 may be identified with $(-\infty, 1]$. With these identifications, even integers $2n$ correspond to vertices r_n and odd integers $2n + 1$ correspond to vertices b_n. Let $\mathcal{T}_0$ (resp. $\mathcal{T}_1$) denote the subtree of $\mathcal{T}$ obtained as the union of the rays rooted at r_0 (resp. b_0).

Solutions $y(x, \lambda)$ of

$$-D^2 y = \lambda y \tag{4.1}$$

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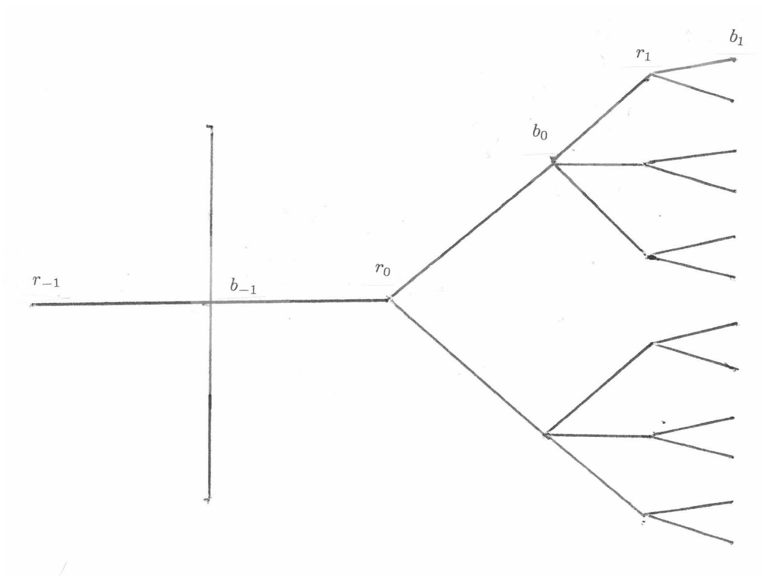

Figure 2: Part of a biregular tree with vertex degrees $3, 4$

on $[0,1]$ will be extended radially to $\mathcal{T}_0$ (resp. $\mathcal{T}_1$), with values depending only on $x \in [0,\infty)$ (resp. $x \in (-\infty, 1]$). Similar solutions are commonly used for spectral theoretic studies of radial trees; a variety of issues are treated in [10, 13, 28, 29, 30].

These extended solutions will be continuous at each vertex, with (2.1) satisfied. Such solutions $y(x,\lambda)$ may be continued from edge e_0 along the indicated rays by using the following jump conditions at integers x. The continuity requirement is $y(x^-,\lambda) = y(x^+,\lambda)$. For rays rooted at r_0 the derivative conditions are satisfied by taking

$$\begin{aligned} y'(x^+,\lambda) &= y'(x^-,\lambda)/\delta_R, \quad x > 0, \text{even}, \\ y'(x^+,\lambda) &= y'(x^-,\lambda)/\delta_B, \quad x > 0, \text{odd}, \end{aligned} \tag{4.2}$$

while for rays rooted at b_0 the conditions are

$$\begin{aligned} y'(x^-,\lambda) &= y'(x^+,\lambda)/\delta_R, \quad x < 1, \text{even}, \\ y'(x^-,\lambda) &= y'(x^+,\lambda)/\delta_B, \quad x < 1, \text{odd}. \end{aligned} \tag{4.3}$$

The propagation of initial data $y(v,\lambda), y'(v,\lambda)$ for the extended solutions is described by transition (or transfer) matrices. When initial data is decomposed using the eigenvectors of the transition matrix, the eigenvalues act as multipliers.

Let $\omega = \sqrt{\lambda}$, with $\omega \ge 0$ for $\lambda \ge 0$ and the square root taken continuously for $\lambda \in \mathbb{C} \setminus i(-\infty, 0)$. On the initial edge e_0 the equation (4.1) has a solution basis $\cos(\omega x), \sin(\omega x)/\omega$. At $x = 0^+$ this basis satisfies

$$\begin{pmatrix} \cos(\omega 0^+) & \sin(\omega 0^+)/\omega \\ -\omega \sin(\omega 0^+) & \cos(\omega 0^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.4}$$

The abbreviations

$$c(\lambda) = \cos(\omega), \quad c'(\lambda) = -\omega \sin(\omega)$$

$$s(\lambda) = \sin(\omega)/\omega, \quad s'(\lambda) = \cos(\omega)$$

will be used.

Suppose a solution y of (4.1) satisfies $y(0^+, \lambda) = a$ and $y'(0^+, \lambda) = b$. At $x = 1^-$ the solution will have values given by

$$\begin{pmatrix} y(1^-, \lambda) \\ y'(1^-, \lambda) \end{pmatrix} = M_0(\lambda) \begin{pmatrix} a \\ b \end{pmatrix}, \quad M_0(\lambda) = \begin{pmatrix} c(\lambda) & s(\lambda) \\ c'(\lambda) & s'(\lambda) \end{pmatrix}.$$

By (4.2), the initial data at $x = 1^+$ is then

$$\begin{pmatrix} y(1^+, \lambda) \\ y'(1^+, \lambda) \end{pmatrix} = J_{0,B} \begin{pmatrix} y(1^-, \lambda) \\ y'(1^-, \lambda) \end{pmatrix}, \quad J_{0,B} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_B \end{pmatrix}.$$

Initial data propagation from $x = 1^+$ to $x = 2^-$ is again given by multiplication by $M_0(\lambda)$, while

$$\begin{pmatrix} y(2^+, \lambda) \\ y'(2^+, \lambda) \end{pmatrix} = J_{0,R} \begin{pmatrix} y(2^-, \lambda) \\ y'(2^-, \lambda) \end{pmatrix}, \quad J_{0,R} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_R \end{pmatrix}.$$

Continuing in this fashion, the propagation of initial data from $x = 2n^+$ to $x = (2n+2)^+$ for $n \ge 0$, is given by

$$T_0(\lambda) = J_{0,R} M_0 J_{0,B} M_0$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_R \end{pmatrix} \begin{pmatrix} c(\lambda) & s(\lambda) \\ c'(\lambda) & s'(\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_B \end{pmatrix} \begin{pmatrix} c(\lambda) & s(\lambda) \\ c'(\lambda) & s'(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} c^2 + sc'/\delta_B & cs + ss'/\delta_B \\ c'c/\delta_R + s'c'/(\delta_R \delta_B) & sc'/\delta_R + (s')^2/(\delta_R \delta_B) \end{pmatrix}$$

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Note that

$$\det(T_0(\lambda)) = \det(J_{0,R} M_0 J_{0,B} M_0) = \frac{1}{\delta_B \delta_R}, \tag{4.5}$$

$$trT_0(\lambda) = \cos^2(\omega)(1 + \frac{1}{\delta_B \delta_R}) - \sin^2(\omega)(\frac{1}{\delta_R} + \frac{1}{\delta_B}).$$

Next, treat the behavior of similar solutions which start at $x = 1^-$ and extend to decreasing values of x. Still using the basis $\cos(\omega x), \sin(\omega x)/\omega$, the matrix taking initial data from $x = 1^-$ to $x = 0^+$ is

$$M_1(\lambda) = M_0^{-1}(\lambda) = \begin{pmatrix} s'(\lambda) & -s(\lambda) \\ -c'(\lambda) & c(\lambda) \end{pmatrix}.$$

From (4.3) the jump in initial data from $x = 0^+$ to $x = 0^-$ is given by

$$J_{1,R} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_R \end{pmatrix}.$$

Similarly, the jump from $x = -1^+$ to $x = -1^-$ is given by

$$J_{1,B} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_B \end{pmatrix}.$$

More generally, the propagation of initial data from $x = (2n+1)^-$ to $x = (2n-1)^-$ for $n \le 0$, is given by

$$T_1(\lambda) = J_{1,B} M_1 J_{1,R} M_1$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_B \end{pmatrix} \begin{pmatrix} s'(\lambda) & -s(\lambda) \\ -c'(\lambda) & c(\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta_R \end{pmatrix} \begin{pmatrix} s'(\lambda) & -s(\lambda) \\ -c'(\lambda) & c(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} (s')^2 + sc'/\delta_R & -s's - sc/\delta_R \\ -c's'/\delta_B - cc'/(\delta_B \delta_R) & c's/\delta_B + c^2/(\delta_B \delta_R) \end{pmatrix}$$

Again $\det(T_1(\lambda)) = \frac{1}{\delta_B \delta_R}$ and $\mathrm{tr}(T_1(\lambda)) = \mathrm{tr}(T_0(\lambda))$ Since the transition matrices $T_j(\lambda)$ have the same determinants and traces, they have the same eigenvalues

$$\mu^{\pm}(\lambda) = \mathrm{tr}T_j/2 \pm \sqrt{\mathrm{tr}(T_j)^2/4 - \det(T_j)}, \tag{4.6}$$

Again, take the square root nonnegative when the argument is nonnegative, and continuous if the argument is in $\mathbb{C} \setminus i(-\infty, 0)$.

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Suppose the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has eigenvalues $\mu^\pm$. Then

$$\begin{pmatrix} a-\mu^\pm & b \\ c & d-\mu^\pm \end{pmatrix} \begin{pmatrix} b \\ \mu^\pm - a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so the vectors $[b, \mu^\pm - a]$ are eigenvectors. Using $s'(\lambda) = c(\lambda)$ these eigenvectors for $T_j(\lambda)$ are

$$E_0^\pm(\lambda) = \begin{pmatrix} sc + sc/\delta_B \\ \mu^\pm - c^2 - sc'/\delta_B \end{pmatrix}, \quad E_1^\pm(\lambda) = \begin{pmatrix} -sc - sc/\delta_R \\ \mu^\pm - c^2 - sc'/\delta_R \end{pmatrix} \tag{4.7}$$

The eigenvectors $E_0^\pm(\lambda), E_1^\pm(\lambda)$ are nonzero except for a discrete set of $\lambda \in \mathbb{R}$.

When λ is real the matrices $T_j(\lambda)$ are real-valued. Then if $\mathrm{tr}(T_j)^2/4 - \det(T_j)$ is nonpositive the eigenvalues are conjugate pairs, with

$$|\mu^\pm(\lambda)| = \frac{1}{\sqrt{\delta_B \delta_R}}, \quad \frac{-2}{\sqrt{\delta_B \delta_R}} \le \mathrm{tr}(T_j) \le \frac{2}{\sqrt{\delta_B \delta_R}}. \tag{4.8}$$

If $\mathrm{tr}(T_j)^2/4 - \det(T_j)$ is nonnegative, the eigenvalues are real. Since $\mu^+\mu^- = 1/(\delta_B \delta_R)$, the eigenvalues have the same sign.

Lemma 4.1. *The eigenvalues $\mu^\pm(\lambda)$ satisfy*

$$\lim_{\lambda \to -\infty} e^{-2|\mathrm{Im}\sqrt{\lambda}|} \mu^+(\lambda) = \frac{\delta_R + 1}{2\delta_R} \frac{\delta_B + 1}{2\delta_B}, \quad \lim_{\lambda \to -\infty} e^{2|\mathrm{Im}\sqrt{\lambda}|} \mu^-(\lambda) = \frac{2}{\delta_R + 1} \frac{2}{\delta_B + 1} \tag{4.9}$$

Proof. For $\lambda < 0$ let $\omega = \sqrt{\lambda}$ as before. In exponential form

$$c(\lambda) = (e^{i\omega} + e^{-i\omega})/2 = (e^{-|\omega|} + e^{|\omega|})/2, \quad c'(\lambda) = -\omega(e^{i\omega} - e^{-i\omega})/2i,$$

$$s(\lambda) = (e^{i\omega} - e^{-i\omega})/(2i\omega) = (e^{-|\omega|} - e^{|\omega|})/(2i\omega),$$

Since $\mathrm{tr}T_0(\lambda) = \mathrm{tr}T_1(\lambda)$, equation (4.5) gives

$$\lim_{\lambda \to -\infty} e^{-2|\mathrm{Im}\sqrt{\lambda}|} \mathrm{tr}\,(T_j) = \frac{1}{4}[1 + \frac{1}{\delta_R} + \frac{1}{\delta_B} + \frac{1}{\delta_R \delta_B}] = \frac{\delta_R + 1}{2\delta_R} \frac{\delta_B + 1}{2\delta_B}$$

(4.6) first gives the estimate for $\mu^+(\lambda)$, with the estimate for $\mu^-(\lambda)$ following from $\mu^+\mu^- = \frac{1}{\delta_R \delta_B}$. □

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Lemma 4.2. *If $|\mu^\pm(\lambda)| = 1/\sqrt{\delta_B\delta_R}$, then λ is in the essential spectrum of $\mathcal{L}$; consequently λ is real.*

Proof. The argument follows that of [8, Lem 3.2]. The transition matrix $T_0(\lambda)$ has an eigenvector, with eigenvalue $\mu^-(\lambda)$, which we take as the initial data at $x = 0^+$ for a nontrivial radial solution of (4.1) on $\mathcal{T}_0$. This solution satisfies $y(x+2,\lambda) = \mu^-(\lambda)y(x,\lambda)$ along each ray in $\mathcal{T}_0$ based at r_0. The self adjoint conditions (2.1) hold for y except possibly at r_0. These extended solutions decay as $x \to \infty$. For $n = 1,2,3,\dots$ let $\mathcal{T}_{2n+3}$ denote the subtree of $\mathcal{T}_0$ with $0 \le x \le (2n+3)$.

A modification of $y(x,\lambda)$ will provide approximate eigenfunctions of $\mathcal{L}$. For $0 < x < 1$ fix a C^2 function $\eta(x)$ such that $\eta(x) = 0$ for $0 < x < 1/4$ and $\eta(x) = 1$ for $3/4 < x < 1$. For $n = 1,2,3,\dots$ define radial functions ϕ_n on $\mathcal{T}$ by

$$\phi_n(x) = \begin{Bmatrix} \eta(x), & 0 < x < 1, \\ \eta(2n+3-x), & 2n+2 < x < 2n+3, \\ 1, & 1 \le x \le 2n+2, \\ 0, & \text{otherwise.} \end{Bmatrix}$$

If $y_n(x,\lambda) = \phi_n(x)y(x,\lambda)$ on $\mathcal{T}_0$, and $y_n(x,\lambda)$ is then extended by 0 to $\mathcal{T}$, the resulting function is in the domain of $\mathcal{L}$. Since $y_n(x,\lambda) = y(x,\lambda)$ for $1 \le x \le 2n$,

$$\int_{\mathcal{T}} |y_n|^2 \ge \sum_{k=1}^{n} (\delta_B\delta_R)^k \int_{0\le x\le 2} |(\mu^-)^k y|^2 = n \int_{0\le x\le 2} |y|^2.$$

Also, for some constants C_1, C_2,

$$\int_{\mathcal{T}} |[-D^2 + q - \lambda]y_n|^2 \le C_1[1 + (\delta_B\delta_R)^n|\mu^-|^{2n}] \int_{0\le x\le 1} [|y| + |y'|]^2 \le C_2,$$

so $y_n/\|y_n\|$ serves as an approximate eigenfunction of norm 1, with

$$\|(\mathcal{L} - \lambda)\frac{y_n}{\|y_n\|_2}\|_2 \to 0, \quad n \to \infty.$$

λ is thus in the spectrum of the self adjoint operator $\mathcal{L}$.

Recall that the spectrum of a self-adjoint operator has a two part partition: the discrete spectrum, which is the set of isolated eigenvalues of finite multiplicity, and the essential spectrum. By shifting the initial point

r_0, the construction of the functions y_n can be easily modified to produce a sequence ψ_n with pairwise disjoint supports such that $\|\psi_n\|_2 = 1$ and $\|(\mathcal{L} - \lambda I)\psi_n\|_2 \to 0$. By Weyl's criterion [26, p. 237] λ is in the essential spectrum of $\mathcal{L}$.

□

By Lemma 4.2 the set

$$\sigma_1 = \{\lambda \in \mathbb{C} \mid |\mu^\pm(\lambda)| = |\delta_R \delta_B|^{-1/2}\} = \{\lambda \in \mathbb{R} \mid \mathrm{tr}(T_j)^2 - 4\det(T_j) \le 0\}$$

is contained in the half line $[0, \infty)$. The next result describes the main features of $\mu^\pm(\lambda)$.

Theorem 4.3. *The eigenvalues $\mu^\pm(\lambda)$ may be chosen to be single valued analytic functions in the complement of σ_1 with the asymptotic behaviour given in Lemma 4.1. On this domain $|\mu^+(\lambda)| > 1/\sqrt{\delta_R\delta_B}$ and $|\mu^-(\lambda)| < 1/\sqrt{\delta_R\delta_B}$. These functions $\mu^+(\lambda)$ and $\mu^-(\lambda)$ have continuous extensions to the real axis from the upper and lower half planes which are analytic except on the discrete set where $\mathrm{tr}(T_j)^2 = 4/(\delta_R\delta_B)$. If $\nu \in \sigma_1$ then*

$$\lim_{\epsilon \to 0^+} \mu^\pm(\nu + i\epsilon) - \mu^\pm(\nu - i\epsilon) = 2i\,\mathrm{Im}\,(\mu^\pm(\nu))\,. \tag{4.10}$$

Proof. Since $\mathrm{tr}(T_j)$ and $\det(T_j)$ are entire functions of λ, the eigenvalues $\mu^\pm(\lambda)$ (and eigenvectors (4.7)) will be analytic in any simply connected domain with $\mathrm{tr}(T_j)^2 - 4/(\delta_R\delta_B) \ne 0$. Use $U = \mathbb{C} \setminus [0, \infty)$ as such a domain.

The eigenvalues satisfy $\mu^+\mu^- = 1/(\delta_R\delta_B)$, so $|\mu^+|$ and $|\mu^-|$ and are distinct in $\mathbb{C} \setminus \sigma_1$. Since $|\mu^+(\lambda)| > 1/\sqrt{\delta_R\delta_B}$ and $|\mu^-(\lambda)| < 1/\sqrt{\delta_R\delta_B}$ in U, the extensions across $\mathbb{R} \setminus \sigma_1$ are single valued.

To obtain the continuous extension to the real axis, note that the set of points where the eigenvalues coalesce, or $\mathrm{tr}(T_j)^2 - 4/(\delta_R\delta_B) = 0$, is the zero set of an entire function, which has isolated (real) zeroes r_i. The analytic functions $\mu^\pm(\lambda)$, given by (4.6), thus have an analytic continuation from either half plane to the real axis with these points r_i omitted. At these points

$$\lim_{\lambda \to r_i} \mu^\pm(\lambda) = \mathrm{tr}\,(T_j)/2$$

independent of the branch of the square root.

We have observed in Lemma 4.2 that if $|\mu^\pm(\nu)| = 1/\sqrt{\delta_R\delta_B}$, then $\nu \in \mathbb{R}$. If $\mathrm{tr}\,(T_j)^2 = \det(T_j) = 4/(\delta_R\delta_B)$, then both sides of (4.10) are 0. Suppose

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instead that $\operatorname{tr}(T_j)^2/4 - \det(T_j) < 0$, so that the eigenvalues $\mu^{\pm}(\nu)$ are a nonreal conjugate pair. Since the eigenvalues are distinct, they extend analytically across the real axis. There are two possibilities: either (i) (4.10) holds, in which case the extension of $\mu^{\pm}(\nu+i\epsilon)$ is $\mu^{\mp}(\nu-i\epsilon)$, or (ii) $\mu^{\pm}(\nu+i\epsilon)$ extends to $\mu^{\pm}(\nu - i\epsilon)$. The second case will be excluded because $|\mu^+| > 1/\sqrt{\delta_R\delta_B}$ in the complement of σ_1. If (ii) held, then μ^+ would be an analytic function of λ in a neighborhood of ν satisfying $|\mu^+(\nu)| = 1/\sqrt{\delta_R\delta_B}$, and $|\mu^+(\lambda)| \ge 1/\sqrt{\delta_R\delta_B}$. But this violates the open mapping theorem [14, p. 162], so (i) must hold. The treatment of μ^- is the same. □

Define additional solutions of (4.1) on $(0,1)$ by

$$V(x,\lambda) = sc(1+\frac{1}{\delta_B})\cos(\omega x) + [\mu^- - c^2 - \frac{sc'}{\delta_B}]\sin(\omega x)/\omega,$$

$$U(x,\lambda) = -sc(1+\frac{1}{\delta_R})\cos(\omega(1-x)) - [\mu^- - c^2 - \frac{sc'}{\delta_R}]\sin(\omega(1-x))/\omega.$$

$V(x,\lambda)$ (resp. $U(x,\lambda)$) has initial data at $x=0^+$ (resp. $x=1^-$) given by the eigenvector E_0^- (resp. E_1^-).

Theorem 4.4. *If $\lambda \in \mathbb{C}\setminus\sigma_1$ then $V(x,\lambda)$ may be extended to a solution of (4.1) on $\mathcal{T}_0$ which satisfies the vertex conditions* (4.2)*, and so* (2.1)*, for $x > 0$, and is square integrable on $\mathcal{T}_0$. The analogous statements hold for $U(x,\lambda)$ on $\mathcal{T}_1$.*

Proof. If $\lambda \in \mathbb{C}\setminus\sigma_1$ then by Theorem 4.3 the eigenvalue $\mu^-(\lambda)$ is well defined. Extend $V(x,\lambda)$ as a solution to (4.1) on $x > 1$ using the conditions (4.2). For integers $n > 0$, the initial data at $x = 2n^+$ is given by

$$\begin{pmatrix} V(2n^+,\lambda) \\ V'(2n^+,\lambda) \end{pmatrix} = T_0^n(\lambda)\begin{pmatrix} V(0^+,\lambda) \\ V'(0^+,\lambda) \end{pmatrix} = (\mu^-)^n \begin{pmatrix} V(0^+,\lambda) \\ V'(0^+,\lambda) \end{pmatrix}.$$

Since $|\mu^-(\lambda)| < 1/\sqrt{\delta_R\delta_B}$ the computation

$$\int_{\mathcal{T}_\mathcal{R}} |y|^2 = \sum_{k=0}^{\infty} (\delta_B\delta_R)^k \int_{0\le x\le 2} |(\mu^-)^k y|^2 = \sum_{k=0}^{\infty} \left(\delta_B\delta_R(\mu^-)^2\right)^k \int_{0\le x\le 2} |y|^2,$$

establishes the square integrability. □

4.2 The resolvent and spectrum for trees

The resolvent $(\mathcal{L} - \lambda I)^{-1}$ is defined in the resolvent set ρ, the complement of the spectrum of $\mathcal{L}$. The solutions U and V of Theorem 4.4 can be used to construct the resolvent of $\mathcal{L}$ on $\mathcal{T}$. The Wronskian $W(\lambda) = W(U,V) = U(x,\lambda)V'(x,\lambda) - U'(x,\lambda)V(x,\lambda)$, defined for $x \in (0,1)$, is independent of x. Evaluation at $x = 1^-$ gives

$$W(\lambda) = \Big[sc(1+\frac{1}{\delta_B})c' + (\mu^- - c^2 - \frac{sc'}{\delta_B})c\Big]sc(1+\frac{1}{\delta_R}) \tag{4.11}$$

$$+s\Big[(1+\frac{1}{\delta_B})c^2 + (\mu^- - c^2 - \frac{sc'}{\delta_B})\Big]\Big[(\mu^- - c^2 - \frac{sc'}{\delta_R})\Big]$$

The identity $sc' - c^2 = sc' - cs' = -1$ leads to a simplification,

$$W(\lambda) = s\Big[c^2(1+\frac{1}{\delta_B}) + (\mu^- - c^2 - \frac{sc'}{\delta_B})\Big]\Big[c^2(1+\frac{1}{\delta_R}) + (\mu^- - c^2 - \frac{sc'}{\delta_R})\Big]$$

$$-sc^2(1+\frac{1}{\delta_B})(1+\frac{1}{\delta_R})$$

$$= s\Big[\frac{1}{\delta_B} + \mu^-\Big]\Big[\frac{1}{\delta_R} + \mu^-\Big] - sc^2(1+\frac{1}{\delta_B})(1+\frac{1}{\delta_R}).$$

$\mathcal{L}$ will have eigenvalues whenever $\delta_R\delta_B > 1$. Define

$$\sigma_2 = \{\lambda \in \mathbb{C} \setminus \sigma_1 \mid W(\lambda) = 0\}.$$

Since $s(\lambda)$ is a factor of $W(\lambda)$, the Wronskian $W(\lambda)$ vanishes whenever $s(\lambda) = 0$, that is if $\sqrt{\lambda} = n\pi$ for $n = 1,2,3.\dots$. Notice that if $s(\lambda) = 0$, then $c^2(\lambda) = 1$. Then $tr(T_j(\lambda))^2/4 - \det T_j(\lambda) = (1 - 1/(\delta_B\delta_R))/4$, which is strictly positive if $\delta_R\delta_B > 1$, and $\lambda \notin \sigma_1$ by (4.6). Whenever $\delta_R\delta_B > 1$, the set σ_2 is thus not empty. The next result further describes σ_2.

Lemma 4.5. *σ_2 is a discrete subset of $\mathbb{R} \setminus \sigma_1$. If $\delta_R\delta_B > 1$ and $s(\lambda) = 0$ then $\lambda \in \sigma_2$. If $\lambda \in \sigma_2$ and U and V are not identically zero, which is the case if $s(\lambda) = 0$, then λ is an eigenvalue of $\mathcal{L}$.*

Proof. Having initial data $E_0^-(\lambda)$ and $E_1^-(\lambda)$, the solutions $V(x,\lambda)$ and $U(x,\lambda)$ are not identically 0 except for a discrete set of $\lambda \in \mathbb{R}$. In particular, if $s(\lambda) = 0$ then $|\mu^-| < 1/\sqrt{\delta_B\delta_R}$, and the initial data for $V(x,\lambda)$ and $U(x,\lambda)$ is nonzero by (4.7).

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The vanishing of $W(\lambda)$ means that V and U are linearly dependent on $(0,1)$. If $U(x,\lambda) = bV(x,\lambda)$ for some constant b and $\lambda \in \mathbb{C} \setminus \sigma_1$, the function

$$w(x,\lambda) = \begin{Bmatrix} bV(x,\lambda), x \in \mathcal{T}_0 \\ U(x,\lambda), x \in \mathcal{T}_1 \end{Bmatrix}$$

is an nontrivial eigenfunction of $\mathcal{L}$. Since $\mathcal{L}$ is self-adjoint, $\lambda \in \mathbb{R}$ and $W(\lambda) \ne 0$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The discreteness of σ_2 follows since $W(\lambda)$ is analytic in $\mathbb{C} \setminus \sigma_1$. □

Proposition 4.6. *Suppose $Eig(\lambda)$ is the eigenspace for an eigenvalue λ of $\mathcal{L}$ on $L^2(\mathcal{T})$. Then* $\dim(Eig(\lambda)) = \infty$.

Proof. Let $g_1 \in Eig(\lambda)$ have norm 1. For any $\epsilon > 0$ there is a finite set F_1 of edges of $\mathcal{T}$ such that $\int_{F_1} |g_1|^2 \ge 1 - \epsilon$. If β is an automorphism of $\mathcal{T}$ then $g_\beta = g \circ \beta \in Eig(\lambda)$. For any finite N there are automorphisms β_n for $n = 1, \dots, N$ such that the corresponding sets $F_n = \beta_n F_1$ for the eigenfunctions $g_n = g \circ \beta_n$ are pairwise disjoint.

Suppose $g_m = \sum_{j \ne m} c_j g_j$. Consider the $N \times N$ Gram matrix A with entries $A_{k,n} = \langle g_k, g_n \rangle$. The determinant is unchanged if the $m-th$ row is replaced by column entries

$$\langle g_m, g_n \rangle - \langle \sum_{j \ne m} c_j g_j, g_n \rangle = 0.$$

That is, $\det(A) = 0$ if the functions g_n are linearly dependent [16, p. 441]. But $\lim_{\epsilon \downarrow 0} A = 1$, so $\dim(Eig(\lambda)) \ge N$.

□

For $\lambda \in \mathbb{C} \setminus (\sigma_1 \cup \sigma_2)$ define the kernel

$$R_e(x,t,\lambda) = \begin{Bmatrix} -U(x,\lambda)V(t,\lambda)/W, & -\infty < x \le t, 0 \le t \le 1\,, \\ -U(t,\lambda)V(x,\lambda)/W, & 0 \le t \le 1, t \le x < \infty\,. \end{Bmatrix}. \tag{4.12}$$

Suppose $f_e : [0,1] \to \mathbb{C}$ has support in $(0,1)$. A simple computation [5, p. 309] shows that the function

$$h_e(x) = \int_0^1 R_e(x,t,\lambda) f_e(t)\, dt$$

satisfies $[-D^2 - \lambda] h_e = f_e$. By Theorem 4.4 h_e is square integrable on $\mathcal{T}$ and satisfies the boundary conditions (2.1) at each vertex of $\mathcal{T}$. Thus h_e is in the domain of $\mathcal{L}$. Since integration of f_e against the meromorphic kernel $R_e(x,t,\lambda)$ agrees with the resolvent $R(\lambda)f_e$ as long as $W(\lambda) \ne 0$, they must agree for all $\lambda \in \rho$.

For $f \in L^2(\mathcal{T})$, let f_e denote the restriction of f to the edge e with the edge orientation described for (4.12).

Theorem 4.7. *For $\lambda \in \mathbb{C} \setminus (\sigma_1 \cup \sigma_2)$,*

$$R(\lambda)f = \sum_e \int_0^1 R_e(x,t,\lambda) f_e(t)\, dt, \tag{4.13}$$

the sum converging in $L^2(\mathcal{T})$.

Proof. The formula (4.13) agrees with the resolvent of $\mathcal{L}$ in the resolvent set, which includes $\mathbb{C}\backslash\mathbb{R}$. Let $[a,b]$ be a compact interval contained in $\mathbb{R}\backslash\{\sigma_1 \cup \sigma_2\}$. If P denotes the family of spectral projections for $\mathcal{L}$, then [27, p. 237,264] for any $f \in L^2(\mathcal{T})$

$$\frac{1}{2}[P_{[a,b]} + P_{(a,b)}]f = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b [R(\lambda + i\epsilon) - R(\lambda - i\epsilon)]f \ d\lambda\,. \tag{4.14}$$

By the observations above, the right hand side of (4.14) vanishes on the dense set of f supported on finitely many edges. This means that $[P_{[a,b]} + P_{(a,b)}]f = 0$ for all $f \in L^2(\mathcal{T})$, and $[a,b]$ is in the resolvent set ρ. Thus $\mathbb{C} \setminus \{\sigma_1 \cup \sigma_2\} \subset \rho$. □

5 Closed walks and resolvent traces

In this section the analysis of the resolvent of the biregular tree $R_{\mathcal{T}}(\lambda)$ is extended to the resolvent $R_{\mathcal{G}}(\lambda)$ of a general biregular graph $\mathcal{G}$. $\mathcal{T}$ and $\mathcal{G}$ are linked via the theory of covering spaces; [23] and [15] are sources for the theory and its application to graphs. A feature of the approach using covering spaces is that generating functions for closed nonbacktracking walks in $\mathcal{G}$ arise naturally.

5.1 Walks

A combinatorial walk of length l in $\mathcal{G}$ (or $\mathcal{T}$) starting at the vertex v_1 and ending at v_{l+1} is a sequence $v_1, e_1, v_2, e_2, \dots, e_l, v_{l+1}$ with adjacent vertices

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v_i, v_{i+1} joined by edges e_i. A walk is closed if $v_1 = v_{l+1}$, and nonbacktracking if consecutive undirected edges are distinct. A closed walk may be nonbacktracking if the edge $\{v_1, v_2\}$ is the same as $\{v_1, v_{l+1}\}$; in the terminology of [31] tails are allowed.

If η_l denotes the number of nonbacktracking closed walks of length $l \ge 1$ in $\mathcal{G}$, then $p_{\mathcal{G}}(z) = N_{\mathcal{E}} + \sum_{l=1}^{\infty} \eta_l z^l$ will be the associated generating function. When $\mathcal{G}$ is finite with adjacency matrix A, an elementary result in graph spectral theory gives the number $\widetilde{\eta}_l$ of closed walks of length l (with backtracking allowed) as the trace of the $l - th$ power of A. The comparison $\eta_l \le \widetilde{\eta}_l$ can be used to show that $p_{\mathcal{G}}(z)$ converges to an analytic function in a neighborhood of $z = 0$.

Using an identification of the edges with intervals of unit length, there is an edgepath $\gamma : [0, l] \to \mathcal{G}$ which traverses the edges e_i of a walk in order at unit speed. Any loop in $\mathcal{G}$ with basepoint v_1 is homotopic [15, p. 86] to such an edgepath. By adding vertices of degree 2 (without changing the lengths of the original edges), closed edgepaths may be assumed to start and end at edge midpoints.

It will be useful to determine the walk generating function $p_{\mathcal{G}}(z)$ for the complete bipartite graphs $K(m_B, m_R)$ with m_B vertices in $\mathcal{V}_B$ and m_R vertices in $\mathcal{V}_R$. Closed walks in bipartite graphs have even length, so assume $l = 2k$ is even. Start by counting the number of nonbacktracking walks of length $2k - 1$ starting at $v \in \mathcal{V}_R$ and ending in $\mathcal{V}_B$. The vertex v may be followed by any of m_B vertices in $\mathcal{V}_B$. To avoid backtracking, there are then $m_R - 1$ available vertices in $\mathcal{V}_R$, then $m_B - 1$ available vertices in $\mathcal{V}_B$, and so on. After $2k - 1$ steps the number of such walks is $m_B(m_R - 1)^{k-1}(m_B - 1)^{k-1}$. The last step must return to v, but to avoid backtracking the walks which returned to v after $2k - 2$ steps are omitted. The count C_k of closed nonbacktracking walks of length $2k$ starting at v satisfies $C_k = m_B(m_R - 1)^{k-1}(m_B - 1)^{k-1} - C_{k-1}$. Multiplying by the number of starting R vertices and doing the same for starts in $\mathcal{V}_B$ shows that the total number W_k of closed nonbacktracking walks of length $2k$ with designated starting vertex satisfies

$$W_k = 2m_R m_B (m_R - 1)^{k-1}(m_B - 1)^{k-1} - W_{k-1}, \quad k \ge 2, \quad W_1 = 0.$$

That is, for $k \ge 2$,

$$W_k = 2m_R m_B (-1)^{1-k} \sum_{j=1}^{k-1} (-1)^j (m_R - 1)^j (m_B - 1)^j$$

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$$= \frac{2m_R m_B(m_R-1)(m_B-1)}{1+(m_R-1)(m_B-1)}[(m_R-1)^{k-1}(m_B-1)^{k-1}+(-1)^k]$$

Adding the number of edges for order zero term gives the generating function

$$P_{CG}(z) = \frac{m_R m_B}{2} + \frac{2m_R m_B(m_R-1)(m_B-1)}{1+(m_R-1)(m_B-1)} \times \tag{5.1}$$

$$\times \Big[\sum_{k=2}^{\infty}(m_R-1)^{k-1}(m_B-1)^{k-1}z^k + \sum_{k=2}^{\infty}(-1)^k z^k\Big]$$

$$= \frac{m_R m_B}{2} + \frac{2m_R m_B(m_R-1)(m_B-1)}{1+(m_R-1)(m_B-1)}\Big[\frac{(m_R-1)(m_B-1)z^2}{1-(m_R-1)(m_B-1)z} + \frac{z^2}{1+z}\Big].$$

W_k is a count of walks of length $2k$, so the closed nonbacktracking walk generating function is

$$p_{CG}(z) = P_{CG}(z^2) = \tag{5.2}$$

$$\frac{m_R m_B}{2} + \frac{2m_R m_B(m_R-1)(m_B-1)}{1+(m_R-1)(m_B-1)}\Big[\frac{(m_R-1)(m_B-1)z^4}{1-(m_R-1)(m_B-1)z^2} + \frac{z^4}{1+z^2}\Big].$$

Suppose $\{r, b\}$ is an edge of $K(m_B, m_R)$, or any bipartite graph, with midpoint w. The closed nonbacktracking walks have one walk starting at b and followed by r, and one starting at r and followed by b. Without changing the count, these walks can be viewed as starting at w followed by b or r. This alternate view will be useful later.

5.2 Coverings

Each (connected, simple) biregular graph $\mathcal{G}$ has a universal covering space $(\mathcal{T}, p)$, where as before $\mathcal{T}$ is the biregular tree. The continuous map $p : \mathcal{T} \to \mathcal{G}$ is such that for each $x \in \mathcal{G}$ there is an open neighborhood N containing x such that $p^{-1}(N)$ is a union of pairwise disjoint sets, each homeomorphic to N. The R, B labeling of tree vertices can be chosen so that p preserves vertex class.

Pick basepoints $\xi_0 \in \mathcal{G}$ and $\widetilde{\xi}_0 \in p^{-1}(\xi_0)$. If $\widetilde{\gamma}$ is a nonbacktracking edgepath in $\mathcal{T}$ starting at $\widetilde{\xi}_0$ and ending at $\widetilde{\xi}_1 \in p^{-1}(\xi_0)$, then $\gamma = p(\widetilde{\gamma})$ will be a closed nonbacktracking edgepath in $\mathcal{G}$. Every closed nonbacktracking edgepath arises in this way [15, p. 86].

Suppose that ξ_0 is a point in the interior of the edge $e_0 \in \mathcal{G}$, and that $\widetilde{\xi_0} \in p^{-1}(\xi_0)$. Let $\widetilde{e}_0$ be the edge of $\mathcal{T}$ containing $\widetilde{\xi_0}$. Then given any function $f \in L^2(e_0)$, there is a corresponding function $\widetilde{f} \in L^2(\mathcal{T})$ such that

$$\widetilde{f}(\widetilde{\xi}) = \left\{ \begin{array}{ll} f(p(\widetilde{\xi})), & \widetilde{\xi} \in \widetilde{e}_0 \,, \\ 0, & \widetilde{\xi} \notin \widetilde{e}_0 \end{array} \right\}.$$

Proposition 5.1. *Suppose that $f \in L^2(\mathcal{G})$ has support on an edge e_0. There is a $C > 0$ such that if $|Im(\sqrt{\lambda})| > C$ then for $\xi \in \mathcal{G}$*

$$[R_{\mathcal{G}}(\lambda)f](\xi) = \sum_{\widetilde{\xi} \in p^{-1}(\xi)} [R_{\mathcal{T}}(\lambda)\widetilde{f}](\widetilde{\xi}).$$

The sum and its first two derivatives converge uniformly for $\xi \in \mathcal{G}$.

Proof. The proof has two parts: a formal verification and a proof that the sum converges. Consider the two sums

$$H(\xi,\lambda) = \sum_{\widetilde{\xi} \in p^{-1}(\xi)} [R_{\mathcal{T}}(\lambda)\widetilde{f}](\widetilde{\xi}), \quad h(\xi,\lambda) = \sum_{\widetilde{\xi} \notin \widetilde{e}_0} [R_{\mathcal{T}}(\lambda)\widetilde{f}](\widetilde{\xi}).$$

Note that

$$(-D^2 - \lambda)H(\xi,\lambda) = \left\{ \begin{array}{ll} f(\xi), & \xi \in e_0, \\ 0, & \xi \notin e_0. \end{array} \right.$$

Since the vertex conditions are satisfied in the tree, they are still satisfied when we sum over vertices in the tree.

To check on the convergence of $H(\xi,\lambda)$ it suffices to check $h(\xi,\lambda)$. Since the local homeomorphism of edges extends to edge pairs $\{r_n, b_n\}, \{b_n, r_{n+1}\}$, uniform convergence of the series for the $k-th$ derivative, $k = 0,1,2$ of $h(\xi,\lambda)$ is implied by

$$\sum_{n=0}^{\infty} (\delta_B \delta_R |\mu^-|)^n < \infty,$$

or $\delta_B \delta_R |\mu^-(\lambda)| < 1$. For C sufficiently large, this holds for $|\operatorname{Im}(\sqrt{\lambda})| > C$ by Lemma 4.1.

If $\mathcal{G}$ is not a finite graph we must still check that $h \in L^2(\mathcal{G})$. Let $e_a \in \mathcal{G}$ denote an edge pair as above. It will suffice to consider the summands of h with $\widetilde{\xi} \in \mathcal{T}_0$. This contribution to the $L^2(\mathcal{G})$ norm is dominated by a sum

$$\sum_{e_a \in \mathcal{G}} \int_0^2 |V(x,\lambda)|^2 | \sum_{\widetilde{e_a} \in p^{-1}(e_a)} (\mu^-)^{k(m)/2}|^2,$$

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where the sum is taken over distinct edge pairs e_a, and $k(m)$ (which may be taken even) is the distance from the tree root r_0 to $\widetilde{e_a}$.

This sum converges with

$$\sum_{e_a} |S_{e_a}|^2, \quad S_{e_a} = \sum_{\widetilde{e_a} \in p^{-1}(e_a)} (\mu^-)^{k(m)}.$$

If the largest magnitude of a term in S_{e_a} is $|\mu^-|^n$, then

$$|S_{e_a}| \le \sum_{j=n}^{\infty} (\delta_R \delta_B |\mu^-|)^j \le \frac{\delta_R \delta_B |\mu^-|^n}{1 - \delta_R \delta_B |\mu^-|}.$$

There are at most $(\delta_R \delta_B)^n$ sums S_{e_a} with a leading term whose magnitude is as large as $|\mu^-|^n$. This count gives the bound

$$\sum_{e_a} |S_{e_a}|^2 \le \sum_n (\delta_R \delta_B)^n \Big(\frac{(\delta_R \delta_B)^n |\mu^-|^n}{1 - \delta_R \delta_B |\mu^-|} \Big)^2 \le \frac{1}{[1 - \delta_R \delta_B |\mu^-|]^2} \sum_n (\delta_R \delta_B)^{3n} |\mu^-|^{2n}.$$

Thus h is square integrable if $(\delta_R \delta_B)^3 |\mu^-|^2 < 1$.

After modifying C, it follows that $H(\xi, \lambda)$ is in the domain of $\mathcal{L}$, with $(-D^2 - \lambda) H(\xi, \lambda) = f$.

□

The diagonal of the resolvent kernel and the trace of the resolvent of a finite biregular graph $\mathcal{G}$ contain information about closed walks in $\mathcal{G}$. Let e be an edge of the biregular graph $\mathcal{G}$ with midpoint g and vertices r and b of e identified with $[0,1]$ as above. Define $P_e(z) = 1 + \sum_{l>0} \eta_l z^{l/2}$ with coefficients η_l counting closed nonbacktracking walks of length l starting at g. Recall that $\eta_l = 0$ if l is odd.

Theorem 5.2. *There is a $C > 0$ such that for $|\operatorname{Im}(\sqrt{\lambda})| > C$ and $t \in e$ the diagonal of the resolvent may be written as*

$$R_{\mathcal{G}}(t, t, \lambda) = P_e(\mu^-(\lambda)) \frac{U(t, \lambda) V(t, \lambda)}{-W(\lambda)}, \quad t \in e. \tag{5.3}$$

If $\mathcal{G}$ is a finite biregular graph with $N_{\mathcal{E}}$ edges, then

$$\operatorname{tr} R_{\mathcal{G}}(\lambda) = \frac{P_{\mathcal{G}}(\mu^-(\lambda))}{-W(\lambda)} \int_0^1 U(t, \lambda) V(t, \lambda) \, dt, \tag{5.4}$$

Here $P_{\mathcal{G}}(z) = \sum_e P_e(z) = N_{\mathcal{E}} + \sum_{l>0} \eta_l z^{l/2}$, where η_l counts closed nonbacktracking walks of length l, with one of the $N_{\mathcal{E}}$ basepoints at the midpoint of an edge.

Proof. Fix $\widetilde{g} \in \mathcal{T}$, identifying the edge containing g with $[0,1]$ and let the function $x : \mathcal{T} \to \mathbb{R}$ be as described above.

Suppose that $\xi \in e$, where f is supported. Split the resolvent sum into three parts,

$$[R_{\mathcal{G}}(\lambda)f](\xi) = [R^0_{\mathcal{T}}(\lambda)\widetilde{f}](\xi) + [R^+_{\mathcal{T}}(\lambda)\widetilde{f}](\xi) + [R^-_{\mathcal{T}}(\lambda)\widetilde{f}](\xi), \tag{5.5}$$

where the $0, +, -$ terms are the resolvent sums of Theorem 5.1 coming respectively from $\widetilde{\xi} \in \widetilde{e}$, $x(\widetilde{\xi}) > 1$ and $x(\widetilde{\xi}) < 0$. In the following sums, $e_m \in p^{-1}(e)$ means that there is a closed edgepath γ starting at g which lifts to a path $\widetilde{\gamma}$ based at $\widetilde{g}$ and ending in e_m. Let $l(m) = 2k$ be the length of $\widetilde{\gamma}$. The three terms are given by integration against kernels, with $0 \le t \le 1$. $R^0_{\mathcal{T}}(x,t,\lambda)$ is given by (4.12) with $0 \le x \le 1$, while

$$R^+(x,t,\lambda) = \frac{U(t,\lambda)V(x,\lambda)}{-W(\lambda)} \sum_{x(e_m)>1} [\mu^-(\lambda)]^{l(m)/2}, \quad e_m \in p^{-1}(e), \tag{5.6}$$

and

$$R^-(x,t,\lambda) = \frac{V(t,\lambda)U(x,\lambda)}{-W(\lambda)} \sum_{x(e_m)<0} [\mu^-(\lambda)]^{l(m)/2}, \quad e_m \in p^{-1}(e). \tag{5.7}$$

That is, a closed nonbacktracking walk of length l contributes powers $\mu^-(\lambda)^{l/2}$.

Introduce the functions

$$P_e^+(z) = \sum_{x(e_m)>1} z^{l(m)/2} = \sum_{l>0} \eta_l^+ z^{l/2}, \quad P_e^-(z) = \sum_{x(e_m)<0} z^{l(m)/2} = \sum_{l>0} \eta_l^- z^{l/2},$$

and $P_e(z) = 1 + P_e^+(z) + P_e^-(z)$. The coefficients η_l^+, η_l^-, count closed edgepaths in $\mathcal{G}$ based at the midpoint of e, and whose lift from the midpoint of $\tilde{e}$ to the midpoint of $\tilde{e}$ has length l, with $x(\tilde{e})$ respectively greater than 1 or less than 0. Setting $x = t$ and including the three terms in (5.5) gives the description of the diagonal of the resolvent in the statement of the theorem. □

For finite biregular graphs $\mathcal{G}$, (5.4) expresses $\mathrm{tr} R_{\mathcal{G}}(\lambda)$ as a product, with the factor

$$\frac{1}{W(\lambda)} \int_0^1 U(t,\lambda)V(t,\lambda)\, dt$$

independent of $\mathcal{G}$. Dividing the general case of (5.4) by the example of the complete bipartite graph with the same degrees gives the corollary

$$\frac{P_{\mathcal{G}}(\mu^-(\lambda))}{P_{CB}(\mu^-(\lambda))} = \frac{\mathrm{tr} R_{\mathcal{G}}(\lambda)}{\mathrm{tr} R_{CB}(\lambda)}. \tag{5.8}$$

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5.3 Structure of $P_{\mathcal{G}}(z)$

The effectiveness of (5.8) is based on our knowledge of three of the four functions which appear. The resolvent trace $\mathrm{tr}R_{\mathcal{G}}(\lambda)$ is computed in (2.5) from the eigenvalues of the discrete Laplacian Δ. The special case $\mathrm{tr}R_{CB}(\lambda)$ is then presented in (3.1). For the complete bipartite graph $m_B = \delta_R + 1$ and $m_R = \delta_B + 1$, so (5.2) becomes

$$P_{CG}(\mu^-(\lambda)) = \frac{(\delta_B+1)(\delta_R+1)}{2} \tag{5.9}$$

$$+\frac{2(\delta_B+1)(\delta_R+1)\delta_B\delta_R}{1+\delta_B\delta_R}\Big[\frac{\delta_B\delta_R\mu^-(\lambda)^2}{1-\delta_B\delta_R\mu^-(\lambda)}+\frac{\mu^-(\lambda)^2}{1+\mu^-(\lambda)}\Big]$$

The tree based term in (5.4) satisfies

$$-\frac{1}{W(\lambda)}\int_0^1 U(t,\lambda)V(t,\lambda)\ dt = \mathrm{tr}R_{CB}(\lambda)/P_{CB}(\mu^-(\lambda)).$$

From (5.1) it is easy to verify that the generating function $P_{CG}(z)$ in series form converges to an analytic function for $|z| < \frac{1}{\delta_B\delta_R}$, and that it extends to a rational function for $z \in \mathbb{C}$. Currently, the function $P_{\mathcal{G}}(\mu^-(\lambda))$ is only defined on the range of $\mu^-(\lambda)$. As described in Theorem 4.3, $|\mu^-(\lambda)| < 1/\sqrt{\delta_B\delta_R}$. Since $P_{CG}(z)$ has a rational extension, (5.8) can be used to extend $P_{\mathcal{G}}(z)$ by taking

$$z = \begin{Bmatrix}\mu^-(\lambda), & |z| < 1/\sqrt{\delta_R\delta_B},\\ \mu^+(\lambda), & |z| > 1/\sqrt{\delta_R\delta_B}.\end{Bmatrix} \tag{5.10}$$

The trace formula

$$\mu^+(\lambda) + \mu^-(\lambda) = \mu^- + 1/(\delta_R\delta_B\mu^-) = \mathrm{tr}(T_j(\lambda)) \tag{5.11}$$

leads to consideration of the mapping $\phi(z) = z + 1/(cz)$, where $z \in \mathbb{C} \setminus \{0\}$ and $c > 0$. This discussion follows [22, p. 89], where additional information is available. If $w = z + 1/(cz)$, then $2z = w \pm \sqrt{w^2 - 4/c}$, and $\phi(z)$ is two to one unless $w = \pm 2c^{-1/2}$. Since $w \pm \sqrt{w^2 - 4/c} \neq 0$ for $w \in \mathbb{C}$, $\phi(z)$ is also surjective.

In addition, $\phi(z) = \phi(1/(cz))$, so ϕ is injective for $|z| < c^{-1/2}$ and $|z| > c^{-1/2}$. The image of the circle $z = c^{-1/2}e^{i\theta}$ is the interval $[-2c^{-1/2}, 2c^{-1/2}]$. Thus $\phi : \{0 < |z| < c^{-1/2}\} \to \mathbb{C} \setminus [-2c^{-1/2}, 2c^{-1/2}]$ and $\phi : \{|z| > c^{-1/2}\} \to \mathbb{C} \setminus [-2c^{-1/2}, 2c^{-1/2}]$ are both conformal maps.

To further extend the definition of z in (5.10), observe that $\mathcal{C} = \{(\lambda, \mu^\pm(\lambda)) \in \mathbb{C}^2\}$ is a Riemann surface presented as a two-sheeted branched cover of $\mathbb{C}$. As long as the eigenvalues of $T_j(\lambda)$ are distinct, $\mu(\lambda)$ is a locally defined analytic function. The proof of Theorem 4.3 shows that $\mathcal{C}$ is connected. The definition of z in (5.10) may be extended as the value of μ as a function from $\mathcal{C}$ to $\mathbb{C}$.

Lemma 5.3. *The extended map $z = \mu$ taking $\mathcal{C}$ to $\mathbb{C} \setminus \{0\}$ is surjective, with the locally defined $\mu(\lambda)$ satisfying $\mu'(\lambda) \ne 0$ unless* $\cos(\omega)\sin(\omega)/\omega = 0$.

Proof. From (4.5),

$$\mathrm{tr} T_j(\lambda) = \cos^2(\omega)(1 + \frac{1}{\delta_B})(1 + \frac{1}{\delta_R}) - (\frac{1}{\delta_R} + \frac{1}{\delta_B}). \tag{5.12}$$

One checks easily that the range of $\mathrm{tr}T_j(\lambda)$ is $\mathbb{C}$.

The comments above about the mapping $\phi(z)$ mean that all pairs $\{\mu^-, \mu^+ = 1/(\delta_R\delta_B\mu^-)\}$ with $\mu^- \ne 0$ occur for some values of λ. With $|\mu^-| < 1/\sqrt{\delta_R\delta_B}$ and $|\mu^+| > 1/\sqrt{\delta_R\delta_B}$ the individual values of $\mu^-(\lambda)$ and $\mu^+(\lambda)$ are well-defined and analytic. The values of $\mu^-(\lambda)$ and $\mu^+(\lambda)$ respectively cover the sets $\{0 < |z| < 1/\sqrt{\delta_R\delta_B}\}$ and $\{|z| > 1/\sqrt{\delta_R\delta_B}\}$, showing that the range of μ includes $\mathbb{C} \setminus \{0, |z| = 1/\sqrt{\delta_R\delta_B}\}$.

It remains to consider $|\mu^\pm(\lambda)| = 1/\sqrt{\delta_R\delta_B}$, in which case $\lambda \in [0,\infty)$ by Lemma 4.2. As $\cos^2(\omega)$ decreases from 1 to 0 in (5.12), $\mathrm{tr}T_j(\lambda)$ decreases from $1 + \frac{1}{\delta_B}\frac{1}{\delta_R}$ to $-(\frac{1}{\delta_R} + \frac{1}{\delta_B})$. From $(1/\sqrt{\delta_R} - 1/\sqrt{\delta_B})^2 \ge 0$ follows $1/\delta_R + 1/\delta_B \ge 2/\sqrt{\delta_r\delta_B}$ and from $(1 - 1/\delta_R)(1 - 1/\delta_B) \ge 0$ follows $1 + \frac{1}{\delta_B}\frac{1}{\delta_R} \ge (\frac{1}{\delta_R} + \frac{1}{\delta_B})$. Thus

$$-(\frac{1}{\delta_R} + \frac{1}{\delta_B}) \le -2/\sqrt{\delta_R\delta_B} < 2/\sqrt{\delta_R\delta_B} \le 1 + \frac{1}{\delta_B}\frac{1}{\delta_R}.$$

The behavior is similar as $\cos^2(\omega)$ increases from 0 to 1. From (4.6) and (4.8) the eigenvalues $\mu^\pm(\lambda)$ traverse the upper and lower semicircles $|\mu^\pm(\lambda)| = 1/\sqrt{\delta_R\delta_B}$ once in each of these segments where $|\mathrm{tr}T_j(\lambda)| \le 2/\sqrt{\delta_R\delta_B}$, finishing the surjectivity of μ. By Theorem 4.3 continuation of $\mu^\pm(\lambda)$ across an interval with $|\mathrm{tr}T_j(\lambda)| < 2/\sqrt{\delta_R\delta_B}$ is analytic as $\mu^\pm(\lambda)$ switches to the $\mu^\mp(\lambda)$ sheet.

Computing the derivative of $\mathrm{tr}T_j(\lambda)$ in two ways gives

$$\frac{d\mathrm{tr}T_j(\lambda)}{d\lambda} = -\cos(\omega)\frac{\sin(\omega)}{\omega}(1 + \frac{1}{\delta_R})(1 + \frac{1}{\delta_B}),$$

and

$$\frac{d}{d\lambda}(\mu^- + \frac{1}{\delta_R\delta_B\mu^-}) = (\mu^-)'(1 - \frac{1}{\delta_R\delta_B(\mu^-)^2})$$

so zeros of the derivative of $\mu^-(\lambda)$ are constrained by

$$\mu^-(\lambda)'(1 - \frac{1}{\delta_R\delta_B(\mu^-(\lambda))^2}) = -\cos(\omega)\frac{\sin(\omega)}{\omega}(1+\frac{1}{\delta_R})(1+\frac{1}{\delta_B}). \tag{5.13}$$

From $\mu^+(\lambda)\mu^-(\lambda) = 1/(\delta_R\delta_B)$,

$$\mu^+(\lambda)' = -\mu^-(\lambda)'\mu^+(\lambda)/\mu^-(\lambda),$$

so the derivatives of μ^+ and μ^- vanish together.

□

As noted, $\mu(\lambda)$ is a locally defined analytic function on the connected set $\{\mu^+(\lambda) \neq \mu^-(\lambda)\}$, so locally (5.8) extends to

$$\frac{P_{\mathcal{G}}(\mu(\lambda))}{P_{CB}(\mu(\lambda))} = \frac{\mathrm{tr}R_{\mathcal{G}}(\lambda)}{\mathrm{tr}R_{CB}(\lambda)}. \tag{5.14}$$

or globally

$$\frac{P_{\mathcal{G}}(\mu)}{P_{CB}(\mu)} = \frac{\mathrm{tr}R_{\mathcal{G}}(\lambda)}{\mathrm{tr}R_{CB}(\lambda)}. \tag{5.15}$$

The left side of (5.14) depends on $\mu(\lambda)$, so that is true of the right side as well, implying that $P_{\mathcal{G}}(z)$ is well defined. The right side of (5.14) depends on λ, so $P_{\mathcal{G}}(\mu^+(\lambda))/P_{CB}(\mu^+(\lambda)) = P_{\mathcal{G}}(\mu^-(\lambda))/P_{CB}(\mu^-(\lambda))$.

Theorem 5.4. *If $\mathcal{G}$ is a finite biregular graph then the generating function $P_{\mathcal{G}}(z)$ extends analytically to*

$$\{|z| < 1/\sqrt{\delta_R\delta_B}\} \setminus \{z \in \mathbb{R} \mid 1/(\delta_R\delta_B) \leq |z| < 1/\sqrt{\delta_R\delta_B}\} \tag{5.16}$$

and

$$\{|z| > 1/\sqrt{\delta_R\delta_B}\} \setminus \{z \in \mathbb{R} \mid 1/\sqrt{\delta_R\delta_B} < |z| \leq 1\},$$

except for poles at $z = \pm i$.

Proof. The resolvent traces in (5.14) are meromorphic in $\mathbb{C}$, with poles in $[0,\infty)$. As noted in Proposition 2.1, $\mathrm{tr}R_{CB}(\lambda) = 0$ implies $\lambda \geq 0$. Thus $Q(\lambda) = \mathrm{tr}R_{\mathcal{G}}(\lambda)/\mathrm{tr}R_{CB}(\lambda)$ is meromorphic in $\mathbb{C}$, with all poles in $[0,\infty)$. By Theorem 4.3 $\mu^-(\lambda)$ and $\mu^+(\lambda)$ are analytic in the complement of $\sigma_1 \subset [0,\infty)$. Since P_{CG} is rational, $P_{\mathcal{G}}(\mu^-(\lambda))$ and $P_{\mathcal{G}}(\mu^+(\lambda))$ are meromorphic in $\mathbb{C} \setminus \sigma_1$.

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By (5.13), $\mu^-(\lambda)'$ is not zero unless $\cos(\omega)\sin(\omega)/\omega = 0$; these roots only occur for $\lambda \ge 0$. If $\mu^-(\lambda_0)' \ne 0$, then $\mu^+(\lambda_0)' \ne 0$, in which case both $\mu^+(\lambda)$ and $\mu^-(\lambda)$ are conformal maps in a neighborhood of λ_0. If $Q(\lambda)$ is analytic in a neighborhood of λ_0 and if $P_{CB}(\Psi(\lambda))$ is also analytic in a neighborhood of $\mu^\pm(\lambda_0)$, then (5.14) provides a definition of $P_{\mathcal{G}}(z) = P_{\mathcal{G}}(\Psi(\lambda))$ as an analytic function in some neighborhood of $z_0 = \mu^\pm(\lambda_0)$.

$Q(\lambda)$ is analytic in $\mathbb{C} \setminus [0,\infty)$. As noted for (4.8), $\mathrm{tr}(T_j)(\lambda)$ is real when λ is real. If $\lambda \in \mathbb{R}$ and $\mathrm{tr}(T_j)^2/4 - \det(T_j) \le 0$ then $|\mu^\pm(\lambda)| = 1/\sqrt{\delta_B\delta_R}$. If $\lambda \ge 0$ and $\mathrm{tr}(T_j)^2/4 - \det(T_j) \ge 0$ then $\mu^\pm(\lambda) \in \mathbb{R}$ and $|\mathrm{tr}(T_j)(\lambda)| \le 1 + \frac{1}{\delta_R}\frac{1}{\delta_B}$ by (5.12). By (4.6)

$$\mu^+ \le \frac{1}{2}(1 + \frac{1}{\delta_R}\frac{1}{\delta_B}) + \sqrt{(1 - \frac{1}{\delta_R}\frac{1}{\delta_B})^2/4} = 1,$$

so $\mu^-(\lambda) \ge 1/(\delta_R\delta_B)$ for $\lambda \ge 0$. Finally, the rational function $P_{CB}(z)$ has poles at $z = \pm i, \pm 1/(\delta_R\delta_B)$. Thus $P_{\mathcal{G}}(z)$ is analytic in the sets in (5.16). □

By (5.12) the functions $\mathrm{tr}T_j(\lambda)$, and so the (unordered pair of) eigenvalues $\mu^\pm(\lambda)$, are periodic in ω with period π. Thus any singularities in $P_{\mathcal{G}}(\mu)$ must already appear in the strip $0 \le \Re(\omega) \le \pi$. The function $Q(\lambda) = \mathrm{tr}R_{\mathcal{G}}(\lambda)/\mathrm{tr}R_{CB}(\lambda)$ only has singularities in $[0,\infty)$, meaning that singular points for $P_{\mathcal{G}}(z)$ coming from $Q(\lambda)$ will appear for $0 \le \lambda \le \pi^2$, where $Q(\lambda)$ has a finite set of poles.

By (2.5) the pole locations for $\mathrm{tr}R_{\mathcal{G}}(\lambda)$ with $0 \le \lambda \le \pi^2$ are 0, π^2 and the points with $\cos(\sqrt{\lambda}) = 1 - \nu_k$ for $k = 1,\dots,N_{\mathcal{V}} - 2$. For the complete bipartite graph $\mathcal{L}$ has eigenvalues at $0, (\pi/2)^2, \pi^2$ in the interval $[0,\pi^2]$. By Proposition 2.1 the function $\mathrm{tr}R_{CG}(\lambda)$ has two roots ξ_i lying between the consecutive eigenvalue pairs. Thus in the strip $0 \le \Re(\lambda) \le \pi^2$ the function $Q(\lambda)$ has at most $N_{\mathcal{V}} + 2$ poles, all lying in $[0,\pi^2]$.

Theorem 5.5. *If $\mathcal{G}$ is a finite biregular graph then the generating function $P_{\mathcal{G}}(z)$ extends to a rational function. The poles are all located in the set*

$$\{z = \pm i\} \cup \{|z| = 1/\sqrt{\delta_R\delta_B}\} \cup \{z \in \mathbb{R} \mid 1/(\delta_R\delta_B) \le |z| < 1\}.$$

Proof. The function $P_{\mathcal{G}}(z)$ is analytic except possibly at a finite collection of points for which:

(i) λ is a pole of $Q(\lambda)$,

(ii) $\mu^\pm(\lambda)' = 0$,

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(iii) $\mu^-(\lambda) = \mu^+(\lambda)$,

(iv) z is a pole of $P_{CG}(z)$.

By (5.14) the isolated singular points for $z \in \mathbb{C}$ are not essential singularities, so [14, p. 105-109] $P_{\mathcal{G}}(z)$ is meromorphic in $\mathbb{C}$ with a finite set of poles, which are limited to the sets (i) and (iv). In the strip $0 \le \Re(\omega) \le \pi$, note that $|\mathrm{tr} T_j(\lambda)| \to \infty$, and so $|\mu^+(\lambda)| \to \infty$ as $|\Im(\omega)| \to \infty$. From (2.5) the resolvent traces have limit zero as $|\Im(\lambda)| \to \infty$ in this strip. Moreover $P_{CB}(z)$ has a pole at ∞. Thus $P_{\mathcal{G}}(z)$ is a meromorphic function with finitely many poles in $\mathbb{C} \cup \infty$, so is rational [14, p. 141].

□

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