THE FIRST VARIATIONAL PROBLEMS OF THE TOTAL MASS OF LOG-CONCAVE FUNCTIONS UNDER THE GAUGE TRANSFORM

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Abstract. This paper is originated from Colesanti and Fragalà studying the surface area measure of a log-concave function. On the class of log-concave functions, we study the first variation of the total mass functional under the gauge transform $\mathcal{J}$ and gauge addition, which is associated with the dual Minkowski theory of convex bodies in a natural way.

1. Introduction and main results

The research object of this paper is log-concave function f on $\mathbb{R}^n$, where there exists a convex function φ such that $f = e^{-\varphi}$.

In recent years, the target of log-concave functions is taking essentially any result from convex geometric analysis and trying to adapt it to functional settings, see [1],[3],[5],[10],[12],[13]. In 2013, Colesanti-Fragalà in [6] gave the concept of the total mass of the log-concave function and solved the first variational problem for the addition associated with the Legendre transform $\mathcal{L}$ (see [3]). As well, the corresponding Minkowski problems are studied later by Cordero-Erausquin and Klartag in [7] and Rotem in [14]. In 2020, based on Colesanti-Fragalà's work, Fang-Xing-Ye [9] generalized the Minkowski problem on log-concave functions to L_p case, which leads to the corresponding first variational problem and the L_p-Minkowski problem.

In 2011, Artstein-Avidan and Milman [4] discovered a new order reversing operator polarity transform $\mathcal{A}$, which gave the functional form for duality of noncompact convex set. In addition, they also composed the Legendre transform and polarity transform, which resulted in a new order preserving transform the gauge transform $\mathcal{J}$.

When one considers the class $C_{vx_0}(\mathbb{R}^n)$ of non-negative lower-semicontinuous convex functions $\varphi : \mathbb{R}^n \to [0, \infty]$ which take the value 0 at 0, the gauge transform could be written as

$$(\mathcal{J}\varphi)(x) = (\mathcal{A}\mathcal{L}\varphi)(x) = \inf\{r > 0 : \varphi(x/r) \leq 1/r\}. \tag{1}$$

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In 2021, Florentin and Segal [8] discussed some inequalities of a new addition derived from the gauge transform. For consistency, we still use "$\boxtimes$" as the gauge addition between convex functions as follows,

$$\varphi \boxtimes \psi = \mathcal{J}^{-1}(\mathcal{J}\varphi + \mathcal{J}\psi), \tag{2}$$

where $\varphi, \psi \in C_{vx_0}(\mathbb{R}^n)$. Because of the reflexivity of $\mathcal{J}$ (see Proposition 3.3 for details), the above equation can also be expressed as

$$\varphi \boxtimes \psi = \mathcal{J}(\mathcal{J}\varphi + \mathcal{J}\psi).$$

In the same paper, when one gives the epi-graph of convex function φ

$$epi(\varphi) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : \varphi(x) \le z, z \in \mathbb{R}\},$$

they also gave the geometric meanings of this new addition,

$$epi(\varphi \boxtimes \psi) = (epi(\varphi)^\circ + epi(\psi)^\circ)^\circ, \tag{3}$$

where it could be interpreted geometrically as the duality of dual sum of two noncompact convex sets.

Note that this gauge addition could be transferred to the log-concave functions by

$$\alpha.f \boxplus \beta.g = e^{-\varphi.\alpha \boxtimes \psi.\beta} = e^{-\mathcal{J}(\alpha\mathcal{J}\varphi + \beta\mathcal{J}\psi)}, \tag{4}$$

where $f = e^{-\varphi}, g = e^{-\psi}$, $\alpha, \beta \ge 0$.

Before discussing the first variational problem of log-concave functions, the most important thing is to identify a good notion of "area measure" for log-concave functions. To that aim, reference [6], we set

$$J(f) = \int_{\mathbb{R}^n} f dx,$$

where dx denotes integration with respect to the Lebesgue measure in $\mathbb{R}^n$.

Inspired by the dual Minkowski theory of convex bodies [11] and the geometric meanings of gauge addition, it is natural to study the first variation of the total mass of log-concave functions with respect to gauge addition, by

$$\begin{aligned} \widetilde{\delta} J(f, g) &= \lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{f \boxplus t.g - f}{t} dx \\ &= \lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{e^{-\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)(x)} - e^{-\varphi(x)}}{t} dx. \end{aligned} \tag{5}$$

In general, it seems impossible to find a explicit expression for the first variation $\delta\widetilde{J}(f, g)$. Therefore, our integral representation formulae are settled in the similar class by Colesanti-Fragalà in [6]. Here, we still use $\mathcal{A}'$ of log-concave functions $f = e^{-\varphi}$ such that φ belongs to

$$\mathcal{L}' := \{\varphi \in C_{vx_0}(\mathbb{R}^n) : \mathrm{dom}(\varphi) = \mathbb{R}^n, \varphi \in \mathcal{C}^2_+(\mathbb{R}^n \backslash \{o\}), \lim_{||x|| \to +\infty} \frac{\varphi(x)}{||x||} = +\infty\},$$

therefore, $\mathcal{A}'$ is defined as

$$\mathcal{A}' = \{f = e^{-\varphi} : \varphi \in \mathcal{L}'\}.$$

Now, we introduce the concept of controlled perturbation and discuss the case $f = g$,

$$\widetilde{\delta} J(f,f) = -[(n + \log J(f))J(f) + Ent(f)], \tag{6}$$

where $f \in \mathcal{A}'$, $J(f) > 0$, and

$$Ent(f) = \int_{\mathbb{R}^n} f \log f dx - J(f) \log J(f)$$

introduced in [6]. Furthermore, if the first variation $\widetilde{\delta} J(f,g)$ satisfies the controlled perturbation and dominated convergence theorems, in Section 4 we obtain the explicit expression of the first variation by

$$\widetilde{\delta} J(f,g) = \int_{\mathbb{R}^n} e^{-\varphi(x)}[\varphi(x) - < x, \nabla\varphi(x) >]\varphi(x)\mathcal{J}\psi(\frac{x}{\varphi(x)})dx, \tag{7}$$

where $f, g \in \mathcal{A}'$, $f = e^{-\varphi}, g = e^{-\psi}$.

Finally, we prove that the first variation $\widetilde{\delta} J(f,g)$ satisfies the dominated convergence theorem, i.e

$$\widetilde{\delta} J(f,g) = \lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{e^{-\varphi_t(x)} - e^{-\varphi(x)}}{t} dx = \int_{\mathbb{R}^n} \lim_{t \to 0^+} \frac{e^{-\varphi_t(x)} - e^{-\varphi(x)}}{t} dx,$$

this also proves that the first variation of the total mass of log-concave functions under gauge addition exists.

2. Definition and preliminaries

This section provides preliminaries and notations required for log-concave functions and convex bodies. More details can be found in [16].

We work in the n-dimensional Euclidean space $\mathbb{R}^n$. In space $\mathbb{R}^n$, $< x, y >$ represents the inner product between x and y, and $||x||$ represents the Euclidean norm, for every $x, y \in \mathbb{R}^n$.

In addition, this paper use "o" denote the origin, with $B_n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ to be the unit ball and $S^n = \{x \in \mathbb{R}^n : ||x|| = 1\}$ to be the unit sphere in $\mathbb{R}^n$.

In this paper, dom(φ)=$\{x \in \mathbb{R}^n : \varphi(x) < +\infty\}$, and R($\varphi$) represents the range of the function φ. $\mathcal{H}^m(x)$ is the m-dimensional Hausdorff measure, and the abbreviation for the Lebesgue integral measure $d\mathcal{H}^n(x)$ is represented by dx.

A set $K \in \mathbb{R}^n$ is said to be a *convex body* if K is a compact convex set with nonempty interior.

Definition 2.1. *The set K is called a convex set on $\mathbb{R}^n$ if K satisfies the following conditions*

$$\lambda x + (1 - \lambda)y \in K, \quad \forall x, y \in K,$$

where for every $\lambda \in [0, 1]$.

The family of convex bodies is denoted by $\mathcal{K}^n$, with $\mathcal{K}_o^n$ to be the collection of convex bodies containing "o" in their interiors. And $V_n(K) = \mathcal{H}^n(K)$ said to be n-dimensional volume of a convex body K, where $K \in \mathcal{K}^n$.

In addition, the polar body of a convex body is introduced as follows.

Definition 2.2. *For every $K \in \mathcal{K}^n$, the polar body of K is defined as*

$$K^o := \{y \in \mathbb{R}^n :< x, y >\leq 1, \forall x \in K\}. \tag{8}$$

The polar body of a convex body K can be seen as a duality relationship with a convex body K, which plays a crucial role in Minkowski's duality theory.

Definition 2.3. *For every $K \in \mathcal{K}^n$, h_K represents the support function of the convex body K, defined as*

$$h_K(x) := \sup_{y\in K} < x, y >, \quad \forall x \in \mathbb{R}^n. \tag{9}$$

Support function is one of the most important tools to study convex sets, it plays a role in connecting convex bodies and convex functions. Obviously, the support function is a positively homogeneous function, i.e. for $\forall \alpha > 0$,

$$h_K(\alpha x) = \alpha h_K(x).$$

Using the Legendre transform (see Definition 2.10), we can also get, for $\forall K \in \mathcal{K}^n$,

$$\mathcal{L}h_K(x) = \begin{cases} 0, & x \in K \\ +\infty, & x \notin K, \end{cases}$$

thus, $K = \mathrm{dom}(\mathcal{L}h_K)$. In addition, it can be seen from [16, Corollary 1.7.3] that when $y = \nabla h_K(x) \in \partial K$, formula (9) can fetch the supremum, i.e

$$h_K(x) =< \nabla h_K(x), x > . \tag{10}$$

According to formula (9), ∂K and h_K are one-to-one correspondents.

Definition 2.4. *For every $K \in \mathcal{K}^n$, the radial function of K is represented by $p_K(\cdot)$, defined by:*

$$p_K(x) := \max\{\lambda \geq 0 : \lambda x \in K\}, \tag{11}$$

for every $x \in \mathbb{R}^n$.

In addition, the reciprocal of the radius function is the well-known Minkowski function, which is represented by $||\cdot||_K$.

For every $K \in \mathcal{K}_o^n$, the following statement can be further obtained.

Proposition 2.5. *If $o \in int(K)$, then*

$$p_K(x) = \frac{1}{h_{K^o}(x)}. \tag{12}$$

Using the definition of support function and polar body, the proof of proposition 2.5 is obvious.

In this paper, I_K^∞ and $\mathcal{X}_\mathcal{K}$ are used to represent two different type of characteristic functions as follows,

Definition 2.6.

$$I_K^\infty := \begin{cases} 0, & \textit{if } x \in K \\ +\infty, & \textit{if } x \notin K, \end{cases}$$

$$\mathcal{X}_K := \begin{cases} 1, & \textit{if } x \in K \\ 0, & \textit{if } x \notin K. \end{cases}$$

Thus, we introduce the Minkowski addition.

Definition 2.7. *For every $K, L \in \mathcal{K}^n$, their Minkowski addition is defined by:*

$$K + L = \{x + y : x \in K, y \in L\}.$$

The geometric significance of Minkowski addition is also obvious and can be combined with support functions.

Lemma 2.8. *For every $K, L \in \mathcal{K}^n$, $\alpha, \beta \geq 0$, we have*

$$h_{\alpha K + \beta L} = \alpha h_K + \beta h_L.$$

The proof can be found in reference [16].

Now, we give more definitions on convex functions.

Definition 2.9. $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ *is convex function on $\mathbb{R}^n$, if for every $x, y \in \mathbb{R}^n$, φ satisfies*

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y),$$

where $\lambda \in [0, 1]$.

In this paper, $\mathcal{C}$ denotes the set of all convex functions from $\mathbb{R}^n$ to $\mathbb{R} \bigcup \{+\infty\}$. Through the convexity of φ, we can know that $\mathrm{dom}(\varphi)$ is a convex set, φ is called *proper*, if $\mathrm{dom}(\varphi) \neq \emptyset$. According to reference [9], we say $\varphi \in C_+^2(E)$, if φ is second-order differentiable on $E \subseteq \mathrm{int}(\mathrm{dom}(\varphi))$ and its Hessian matrix is positive definite.

Since this article is concerned with convex functions defined on $\mathbb{R}^n$, same as in reference [4], $C_{vx}(\mathbb{R}^n)$ denotes the set of lower semi-concontinuous convex functions on $\mathbb{R}^n$, i.e

$$C_{vx}(\mathbb{R}^n) = \{\varphi : \varphi \in \mathcal{C} \text{ and lower semi-continuous}\},$$

with a special subclass

$$C_{vx_0}(\mathbb{R}^n) = \{\varphi \in C_{vx} : \varphi(0) = 0 \text{ and } \varphi \text{ is nonnegative}\}.$$

Now, we introduce the definitions and some properties of Legendre transform and Polarity transform, and gauge transform.

Definition 2.10. *For every* $\varphi \in C_{vx}(\mathbb{R}^n)$*, we have*

$$(\mathcal{L}\varphi)(x) = \sup_{y\in\mathbb{R}^n}\{< x, y > -\varphi(y)\}. \tag{13}$$

Legendre transform [3] is a classical mathematical tool, which transforms the function on vector space into the function on dual space, and has good properties such as homogeneity, order reversing and reflexivity. In addition, Legendre transform has good geometric properties.

Proposition 2.11. *If the convex function* φ *is differentiable at the point* y*, then* $\mathcal{L}\varphi(x)$ *takes* sup *at* $x = \nabla\varphi(y)$*, i.e*

$$\mathcal{L}(\nabla\varphi(y)) =< y, \nabla\varphi(y) > -\varphi(y).$$

From proposition 2.11, we can draw the following inference.

Corollary 2.12. *Let* φ *be differentiable at the point* y*, then when* $x = \nabla\varphi(y)$*,* $y = \nabla\mathcal{L}\varphi(x)$*.*

If the Legendre transform is applied to the support function h_K, we have

$$\mathcal{L}h_K = I_K^\infty.$$

Similar to the Legendre transform, Polarity transform in [2] is also order reversing, and also has good properties such as homogeneity, reflexivity, defined as

Definition 2.13. *For every* $\varphi \in C_{vx_0}(\mathbb{R}^n)$*,*

$$(\mathcal{A}\varphi)(x) := \begin{cases} \sup_{\{y\in\mathbb{R}^n : f(y)>0\}} \frac{<x,y>-1}{f(y)}, & \text{if } x \in \{f^{-1}(0)\}^o \\ +\infty, & \text{if } x \notin \{f^{-1}(0)\}^o. \end{cases}$$

Combined with the definition of the characteristic function, we have

$$\mathcal{A}I_K^\infty = I_{K^o}^\infty.$$

3. Gauge transform and gauge addition

3.1. **The gauge transform and its properties.** In 2011, Artstein-Avidan and Milman gave the definition of gauge transform ($\mathcal{J}$) in [4, Corollary 6].

Definition 3.1. $\mathcal{J} : C_{vx_0}(\mathbb{R}^n) \to C_{vx_0}(\mathbb{R}^n)$,

$$\mathcal{J} = \mathcal{A}\mathcal{L} = \mathcal{L}\mathcal{A}.$$

Through the first definition and calculation of gauge transform, another expression of $\mathcal{J}$ can be obtained.

Proposition 3.2. *For every* $\varphi \in C_{vx_0}(\mathbb{R}^n)$*, then*

$$(\mathcal{J}\varphi)(x) = \inf\{r > 0 : \varphi(x/r) \leq 1/r\}.$$

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First of all, the $\mathcal{J}$ operate is closed in $C_{vx_0}(\mathbb{R}^n)$, since $\mathcal{A}$ and $\mathcal{L}$ are closed in $C_{vx_0}(\mathbb{R}^n)$, namely:

$$\mathcal{J}\varphi \in C_{vx_0}(\mathbb{R}^n),$$

for every $\varphi \in C_{vx_0}(\mathbb{R}^n)$.

Through the proof in [4], we can directly give some basic properties of $\mathcal{J}$, such as order preserving, reflexivity, homogeneity, etc.

Proposition 3.3. *For every $\varphi, \psi \in C_{vx_0}(\mathbb{R}^n)$, $\alpha > 0$, $\mathcal{J}$ has the following properties.*

1 order preserving: if for every $x \in \mathbb{R}$, $\varphi(x) \geq \psi(x)$, then

$$\mathcal{J}\varphi \geq \mathcal{J}\psi.$$

2 reflexivity:

$$\mathcal{J}\mathcal{J}\varphi = \varphi.$$

3 homogeneity:

$$\mathcal{J}(\alpha\varphi)(x) = (1/\alpha)(\mathcal{J}\varphi)(\alpha x). \tag{14}$$

As can be seen from [4], if the definition of support function is introduced, the following propositions are presented.

Proposition 3.4. *For every $K \in \mathcal{K}^n$, then*

$$\mathcal{J}h_K = \mathcal{A}\mathcal{L}h_K = \mathcal{A}I_K^{\infty} = I_{K^o}^{\infty}. \tag{15}$$

It can be seen from [2] that if the definition of the above atlas is introduced, some meanings of $\mathcal{J}$ in geometry it can be understood as the following Lemma.

Lemma 3.5. *For every $\varphi \in C_{vx_0}(\mathbb{R}^n)$, then*

$$F(epi(\varphi)) = epi(\mathcal{J}\varphi), \tag{16}$$

where $F(x, z) = (\frac{x}{z}, \frac{1}{z})$.

3.2. **Gauge addition.** It is now possible to define a new addition derived from the $\mathcal{J}$, called gauge addition in this paper, with the mathematical notation "$\boxplus$" to be consistent with [8].

Definition 3.6. *For every $\varphi, \psi \in C_{vx_0}(\mathbb{R}^n)$ and $f = e^{-\varphi}, g = e^{-\psi}$,*

$$f \boxplus g = e^{-\varphi \boxtimes \psi} = e^{-\mathcal{J}(\mathcal{J}\varphi + \mathcal{J}\psi)}. \tag{17}$$

For every $\alpha, \beta \geq 0$,

$$\alpha.f \boxplus \beta.g = e^{-\varphi.\alpha \boxtimes \psi.\beta} = e^{-\mathcal{J}(\alpha\mathcal{J}\varphi + \beta\mathcal{J}\psi)}. \tag{18}$$

However, it is not enough to consider only functions in $C_{vx_0}(\mathbb{R}^n)$. For every $\varphi \in C_{vx_0}(\mathbb{R}^n)$, the continuity and differentiability of φ on $\mathrm{dom}(\varphi)$ cannot be guaranteed. Therefore,we can consider a subclass of $C_{vx_0}(\mathbb{R}^n)$, be denoted as $\mathcal{L}'$.

$$\mathcal{L}' := \{\varphi \in C_{vx_0}(\mathbb{R}^n) : \mathrm{dom}(\varphi) = \mathbb{R}^n, \varphi \in \mathcal{C}_+^2(\mathbb{R}^n \backslash \{o\}), \lim_{||x|| \to +\infty} \frac{\varphi(x)}{||x||} = +\infty\}.$$

Remark 3.7. *By* [4],[6],[9] *we can know* $\mathcal{A}$ *and* $\mathcal{L}$ *is closed in* $\mathcal{L}'$*, therefore* $\mathcal{J}$ *is closed in* $\mathcal{L}'$*.*

Remark 3.8. *"." of the above equation are defined as the product of the function in* $\mathcal{L}'$ *and the constant coefficient in* $\mathbb{R}^+$*.*

Remark 3.9. *If* $x = o$*, then, for every* $\alpha, \beta \geq 0$*, we have* $\varphi.\alpha \boxtimes \psi.\beta(o) = 0$*.*

It can be calculated directly.

Remark 3.10. *If* $\alpha = 1$*,* $\beta = t > 0$*, then*

$$f \boxplus t.g = e^{-\varphi \boxtimes \psi.t} = e^{-\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)} =: e^{-\varphi_t}, \tag{19}$$

where $\varphi_0(x) = \varphi(x)$*.*

Remark 3.11. *If* $\alpha = 1 - \lambda$*,* $\beta = \lambda$*,* $\lambda \in [0, 1]$*, then*

$$(1-\lambda).f \boxplus \lambda.g = e^{-\mathcal{J}((1-\lambda)\mathcal{J}\varphi + \lambda\mathcal{J}\psi)} =: e^{-\varphi \boxtimes_\lambda \psi} = f \boxplus_\lambda g. \tag{20}$$

We can see from [8] that there is a geometric expression for "$\boxplus_\lambda$".

Example 3.12. *For every* $\varphi, \psi \in C_{vx_0}(\mathbb{R}^n)$*,* $\lambda \in [0, 1]$*, then*

$$epi(\varphi \boxtimes_\lambda \psi) = ((1-\lambda)epi(\varphi)^\circ + \lambda epi(\psi)^\circ)^\circ. \tag{21}$$

Since the function φ_t plays an important role in solving the first variational problem, we now explore some potential properties of the function φ_t.

3.3. **Monotonicity of φ_t.** For the function $t \mapsto \varphi_t$, the monotonicity of φ_t on $[0, +\infty)$ can be obtained by using the isotonic property of $\mathcal{J}$, that is, the following lemma.

Lemma 3.13. *For every* $\varphi, \psi \in \mathcal{L}'$*,* $\varphi_t = \varphi \boxtimes \psi.t$*, then, for every* $t \in [0, 1]$*,* $x \in \mathbb{R}^n$*, we have*

$$\varphi(x) \leq \varphi_t(x) \leq \varphi_1(x). \tag{22}$$

Proof. By the definition of $\psi(x)$, we can know, for every $x \in \mathbb{R}^n$, $\psi(x) \geq 0$, therefore, $\mathcal{J}\psi(x) \geq 0$, then

$$\mathcal{J}\varphi(x) + t\mathcal{J}\psi(x) \geq \mathcal{J}\varphi(x),$$
$$\mathcal{J}\varphi(x) + \mathcal{J}\psi(x) \geq \mathcal{J}\varphi(x) + t\mathcal{J}\psi(x).$$

By applying the $\mathcal{J}$ on both sides of the above formula at the same time, and combining with the order preservation of the $\mathcal{J}$operator, we can get

$$\varphi_t(x) \geq \varphi(x),$$
$$\varphi_1(x) \geq \varphi_t(x),$$

To sum up, we can get $\varphi_1(x) \geq \varphi_t(x) \geq \varphi(x)$. □

It is worth noting that the following inferences can be drawn directly from the above lemma.

Corollary 3.14. *For every* $f, g \in \mathcal{A}'$*,* $t \in (0, 1)$*, let* $f_t = f \boxplus t.g$*, then*

$$f \geq f_t \geq f_1.$$

3.4. **Continuity of φ_t.** For every $\varphi \in \mathcal{L}'$, φ is a continuous function on $\mathbb{R}^n$. Then, from the closure of "⊠" in $\mathcal{L}'$, for every $t \geq 0$, $\varphi_t(x)$ is continuous on $\mathbb{R}^n$.

Of course, to find the first variation, the first thing to consider is the continuity of the function $\varphi_t(x)$ at $t = 0$. Therefore, for every $x \in \mathbb{R}^n$, it becomes particularly important to solve the existence problem of the limit $\lim_{t\to 0^+} \varphi_t(x)$.

Before introducing the next theorem, in order to guarantee the existence of the limit $\lim_{t\to 0^+} \varphi_t(x)$, it is necessary to place some restrictions on the functions φ and ψ, which requires the introduction of the concept of controlled perturbation.

Definition 3.15. *Let $\varphi, \psi \in \mathcal{L}'$. We say ψ is a controllable disturbance of φ, if there exists $C > 0$ such that for every $x \in \mathbb{R}^n$ one has*

$$\mathcal{J}\psi(x) \leq C\mathcal{J}\varphi(x). \tag{23}$$

The controllable perturbation is introduced mainly to avoid the maximum growth order of the function $\mathcal{J}\psi$ being higher than the maximum growth order of the function $\mathcal{J}\varphi$. As a result, the limit $\lim_{t\to 0^+} \varphi_t$ may not converge to φ. Therefore, taking the function ψ here is a controllable disturbance of the function φ, on this premise, it can be proved that the following theorem is true.

Theorem 3.16. *If ψ is a controllable disturbance of φ, then, for every $x \in \mathbb{R}^n$, the following limit exists, that is*

$$\lim_{t\to 0^+} \varphi_t(x) = \varphi(x). \tag{24}$$

Remark 3.17. *From* [6, Definition 4.4]*, it can be seen that if the function φ, ψ satisfies the allowable disturbance, it must satisfy the controllable disturbance.*

Proof of Theorem 3.16. From the definition of the $\mathcal{J}$, we can know

$$\varphi_t(x) = \mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)(x) = \inf\{r > 0 : (\mathcal{J}\varphi + t\mathcal{J}\psi)(x/r) \leq 1/r\}.$$

Since for every $t \geq 0$, $x \in \mathrm{dom}(\varphi)$, we have $(\mathcal{J}\varphi + t\mathcal{J}\psi)(x) \geq \mathcal{J}\varphi(x)$, therefore, it is known by the order preservation of $\mathcal{J}$, $\varphi_t(x) \geq \varphi(x)$.Take the limit off both sides and have $\underline{\lim}_{t\to 0^+} \varphi_t(x) \geq \varphi(x)$. So we just need to prove $\lim_{t\to 0^+} \varphi_t(x) \leq \varphi(x)$. From the definition of controlled disturbance, we can know

$$\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi) \leq \mathcal{J}(\mathcal{J}\varphi + tC\mathcal{J}\varphi) = \mathcal{J}((1 + tC)\mathcal{J}\varphi),$$

combining the homogeneity of $\mathcal{J}$, we can know

$$\varphi_t(x) \leq \mathcal{J}((1 + tC)\mathcal{J}\varphi)(x) = \frac{1}{1 + tC}\varphi((1 + tC)x),$$

so the upper limit on both sides is

$$\overline{\lim_{t\to 0^+}}\, \varphi_t(x) \leq \overline{\lim_{t\to 0^+}}\, \frac{1}{1 + tC}\varphi((1 + tC)x) = \varphi(x),$$

to sum up, for every $x \in \mathrm{dom}(\varphi)$, $\lim_{t\to 0^+} \varphi_t(x) = \varphi(x)$. $\square$

4. The first variational problem

With the concept of total mass on $\mathbb{R}^n$, combined with the definition of the new addition "$\boxplus$", we can give the first variational problem of the total mass.

Definition 4.1. *Let $f, g \in \mathcal{A}', f = e^{-\varphi}, g = e^{-\psi}$, define $\widetilde{\delta}J(f,g)$ by*

$$\begin{aligned}\widetilde{\delta}J(f,g) &= \lim_{t\to 0^+} \int_{\mathbb{R}^n} \frac{f \boxplus t.g - f}{t} dx \\ &= \lim_{t\to 0^+} \int_{\mathbb{R}^n} \frac{e^{-\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)(x)} - e^{-\varphi(x)}}{t} dx.\end{aligned}$$

whenever the limit exists.

It is said to be the first variational formula of the log-concave function derived from gauge addition. Next, we prove the existence of $\widetilde{\delta}J(f,g)$ and compute its explicit expression.

4.1. The existence and explicit expression for $\widetilde{\delta}\mathbf{J}(\mathbf{f},\mathbf{f})$.

Before considering $\widetilde{\delta}J(f,g)$, we consider the existence and explicit expression of $\widetilde{\delta}J(f,f)$ when $f = g$. By the definition of $\widetilde{\delta}J(f,g)$, we know that for every $f \in \mathcal{A}'$, there is

$$\widetilde{\delta}J(f,f) = \lim_{t\to 0^+} \int_{\mathbb{R}^n} \frac{e^{-\mathcal{J}((1+t)\mathcal{J}\varphi)(x)} - e^{-\varphi(x)}}{t} dx.$$

The definition of "*Ent*" in [6, Proposition 3.11] needs to be introduced here for the convenience of proof.

Definition 4.2. *For every $f \in \mathcal{A}'$, $f = e^{-\varphi}$, we have*

$$Ent(f) = \int_{\mathbb{R}^n} f \log f dx - J(f) \log J(f),$$

by [6]*, we can know $Ent(f) \in (-\infty, +\infty)$.*

Through simplification and calculation, the following theorem can be obtained.

Theorem 4.3. *For every $f \in \mathcal{A}'$, $J(f) > 0$, then*

$$\begin{aligned}\widetilde{\delta}J(f,f) &= -\int_{\mathbb{R}^n} f \log f dx - n \int f dx \\ &= -[(n + \log J(f))J(f) + Ent(f)].\end{aligned}$$

Proof. We can know from the homogeneity of $\mathcal{J}$,

$$\mathcal{J}((1+t)\mathcal{J}\varphi)(x) = \frac{1}{1+t}\varphi((1+t)x),$$

therefore

$$\widetilde{\delta}J(f,f) = \lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{e^{-\frac{1}{1+t}\varphi((1+t)x)} - e^{-\varphi(x)}}{t}dx$$
$$= \lim_{t\to 0^+}[\int_{\mathbb{R}^n}\frac{e^{-\frac{1}{1+t}\varphi((1+t)x)}}{t}dx - \int_{\mathbb{R}^n}\frac{e^{-\varphi(x)}}{t}dx]. \tag{25}$$

Let $y=(1+t)x$, then:

$$\lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{e^{-\frac{1}{1+t}\varphi((1+t)x)}}{t}dx = \lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{e^{-\frac{1}{1+t}\varphi(y)}}{t(1+t)^n}dy,$$

and

$$\widetilde{\delta}J(f,f) = \lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{e^{-\frac{1}{1+t}\varphi(x)} - (1+t)^n e^{-\varphi(x)}}{(1+t)^n t}dx.$$

We can simplify it further

$$\widetilde{\delta}J(f,f) = -\lim_{t\to 0^+}[\int_{\mathbb{R}^n}\frac{e^{-\varphi(x)} - e^{-\frac{1}{1+t}\varphi(x)}}{t}dx - [\frac{(1+t)^n-1}{t}]\int_{\mathbb{R}^n}e^{-\varphi(x)}dx].$$

For ease of calculation, the following marks can be made

$$A_1 = \int_{\mathbb{R}^n}\frac{e^{-\varphi(x)} - e^{-\frac{1}{1+t}\varphi(x)}}{t}dx, \tag{26}$$
$$A_2 = [\frac{(1+t)^n-1}{t}]\int_{\mathbb{R}^n}e^{-\varphi(x)}dx, \tag{27}$$

therefore

$$\widetilde{\delta}J(f,f) = \lim_{t\to 0^+}(-A_1 - A_2).$$

When $t\to 0^+$, apply L'Hospital's rule to t, we get

$$\lim_{t\to 0^+}A_2 = n\int_{\mathbb{R}^n}e^{-\varphi(x)}dx = nJ(f), \tag{28}$$

from the definition of the total mass function, $A_2 \in (0,+\infty)$.

Let's continue our discussion of the existence of A_1 by simplifying A_1 as follows

$$A_1 = \int_{\mathbb{R}^n}e^{-\frac{1}{1+t}\varphi(x)}(\frac{e^{-\frac{t}{1+t}\varphi(x)}-1}{t})dx$$
$$\geq \int_{\mathbb{R}^n}e^{-\frac{1}{1+t}\varphi(x)}(\frac{e^{-t\varphi(x)}-1}{t})dx,$$

let $h_t = e^{-t\varphi}$, apply Lagrange's mean value theorem, exists $\xi\in(0,t)$,

$$\frac{dh_t}{dt}|_{t=\xi} = \frac{e^{-t\varphi}-1}{t} = -\varphi e^{-\xi\varphi},$$

$$\begin{aligned}A_1 &\geq -\int_{\mathbb{R}^n} e^{-\frac{1}{1+t}\varphi(x)}\varphi(x)e^{-\xi\varphi(x)}dx\\&= -\int_{\mathbb{R}^n} e^{-(\frac{1-t}{2+2t}+\xi)\varphi(x)}\varphi(x)e^{-\varphi(x)/2}dx\\&\geq -\int_{\mathbb{R}^n} e^{-(\frac{1-t}{2+2t}+\xi)\varphi(x)}mdx, \qquad (29)\end{aligned}$$

the last inequality is true because for every $s \in \mathbb{R}^+$, $se^{-s/2} \leq m := 2/e$.

From the proof procedure of [6, Proposition 3.11], when $t \to 0^+$, for every $x \in \mathbb{R}^n$, $\varphi(x) \geq 0$, therefore, when $t \to 0^+$,

$$-\int_{\mathbb{R}^n} e^{-(\frac{1-t}{2+2t}+\xi)\varphi(x)}mdx \geq -\int_{\mathbb{R}^n} e^{-\frac{1}{6}\varphi(x)}mdx. \qquad (30)$$

From the definition of the total mass function, we can see that the right formula in (30) converges, that is

$$-\int_{\mathbb{R}^n} e^{-\frac{1}{6}\varphi(x)}mdx \in (-\infty, 0).$$

Therefore, by applying the dominated convergence theorem, we can get

$$\begin{aligned}\widetilde{\delta}J(f,f) &= \lim_{t\to 0^+}(-A_1 - A_2)\\&= \lim_{t\to 0^+}[-\int_{\mathbb{R}^n} \frac{e^{-\varphi(x)} - e^{-\frac{1}{1+t}\varphi(x)}}{t}dx - nJ(f)]\\&= -\int_{\mathbb{R}^n} \lim_{t\to 0^+} \frac{e^{-\varphi(x)} - e^{-\frac{1}{1+t}\varphi(x)}}{t}dx - nJ(f)\\&= \int_{\mathbb{R}^n} e^{-\varphi(x)}\varphi(x)dx - nJ(f)\\&= -\int_{\mathbb{R}^n} f\log f dx - nJ(f). \qquad (31)\end{aligned}$$

By the definition of A_1, we can know $A_1 \leq 0$, therefore, $-A_1 \geq 0$.

By [6, Proposition 3.11], we can know

$$\int_{\mathbb{R}^n} \varphi e^{-\varphi}dx = \int_{\mathbb{R}^n} \varphi e^{-\varphi/2}e^{-\varphi/2}dx \leq m\int_{\mathbb{R}^n} e^{-\varphi/2}dx < +\infty,$$

therefore, we have $-\int_{\mathbb{R}^n} f\log f dx \in (0, +\infty)$.

To sum up, the explicit expression of $\widetilde{\delta}J(f,f)$ can be obtained

$$\widetilde{\delta}J(f,f) = \lim_{t\to 0^+}(-A_1 - A_2) = -\int_{\mathbb{R}^n} f\log f dx - n\int f dx.$$

□

In addition, this section proves the existence of the limit function $\widetilde{\delta}J(f,f)$, namely, $\widetilde{\delta}J(f,f) \in (-\infty, 0)$.

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Here, by the definition of "*Ent*" in [6, Proposition 3.11], we can obtain

$$\widetilde{\delta}J(f,f) = -[(n + \log J(f))J(f) + Ent(f)],$$

where $Ent(f) \in (-\infty, +\infty)$.

4.2. **An explicit expression for $\widetilde{\delta}\mathbf{J}(\mathbf{f},\mathbf{g})$.** The existence of $\widetilde{\delta}J(f,f)$ shows that under certain conditions, the first variation of the total mass function of the log-concave function exists. Next, the existence and explicit expression of $\widetilde{\delta}J(f,g)$ are further proved.

From the definition of $\widetilde{\delta}J(f,g)$, if it satisfies the dominated convergence theorem, then the limit and integral are commutative and we can obtain

$$\begin{aligned}\widetilde{\delta}J(f,g) &= \lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{e^{-\varphi_t(x)} - e^{-\varphi(x)}}{t}dx\\ &= \int_{\mathbb{R}^n}\lim_{t\to 0^+}\frac{e^{-\varphi_t(x)} - e^{-\varphi(x)}}{t}dx\\ &= -\int_{\mathbb{R}^n}\lim_{t\to 0^+} e^{-\varphi_t}\frac{d\varphi_t}{dt}|_{t=0^+}dx,\end{aligned} \tag{32}$$

substitute $\varphi_t = \mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)$ into formula (32) to get

$$\widetilde{\delta}J(f,g) = -\int_{\mathbb{R}^n}\lim_{t\to 0^+} e^{-\mathcal{J}(\mathcal{J}\varphi+t\mathcal{J}\psi)}\frac{d\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)}{dt}|_{t=0^+}dx. \tag{33}$$

Through simplification and calculation, the following theorem can be obtained.

Theorem 4.4. *For every $f, g \in \mathcal{A}'$, $f = e^{-\varphi}$, $g = e^{-\psi}$, if ψ is controlled disturbance of φ and the limit $\widetilde{\delta}J(f,g)$ satisfies the dominated convergence theorem, then the first variational problem can be expressed explicitly as follows*

$$\widetilde{\delta}J(f,g) = \int_{\mathbb{R}^n} e^{-\varphi(x)}[\varphi(x) - < x, \nabla\varphi(x) >]\varphi(x)\mathcal{J}\psi(\frac{x}{\varphi(x)})dx. \tag{34}$$

Next, on the premise that Theorem 4.6 and controlled disturbance are true, we begin to prove that Theorem 4.4 is true. To simplify the calculation, we can first consider the right formula in the integral (33) and express it with I, denoted as

$$I = \frac{d\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)}{dt}|_{t=0^+}.$$

Definition 4.5. *We say that a function φ defined in $\mathbb{R}^n$ is monotonically increasing if*

$$\varphi(\lambda x) \textit{ is monotonically increasing about } \lambda,$$

where $\lambda \geq 0$, $x \in \mathbb{R}^n$.

For every $\varphi \in \mathcal{L}'$, since φ is convex function and $\varphi(0) = 0$, it is obvious that φ is monotonically increasing. Using the order preserving of $\mathcal{J}$, we also know $\mathcal{J}\varphi$ is monotonically increasing.

Proof. From the second definition of the $\mathcal{J}$, for any fixed $x \in \mathbb{R}^n$, every $t > 0$, we have

$$\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)(x) = \inf\{r > 0 : (\mathcal{J}\varphi + t\mathcal{J}\psi)(\frac{x}{r}) \leq \frac{1}{r}\}, \tag{35}$$

The monotonicity and convexity of the function $(\mathcal{J}\varphi + t\mathcal{J}\psi)$ imply that there exists $r_1(t) > 0$ for which the infimum is attained in equation (35):

$$r_1(t) = \varphi_t(x) = \mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)(x) = \frac{1}{(\mathcal{J}\varphi + t\mathcal{J}\psi)(\frac{x}{r_1(t)})}. \tag{36}$$

The derivative of both sides of equation (36) with respect to t gets

$$I = \frac{dr_1(t)}{dt}|_{t=0^+} = \frac{d\varphi_t(x)}{dt}|_{t=0^+} = \frac{d\mathcal{J}(\mathcal{J}\varphi + t\mathcal{J}\psi)(x)}{dt}|_{t=0^+} = r_1'(t)|_{t=0^+},$$

further simplification has

$$\begin{aligned} I &= \frac{< \nabla\mathcal{J}\varphi(\frac{x}{r_1}) + t\nabla\mathcal{J}\psi(\frac{x}{r_1}), x > \cdot r_1'(t) - \mathcal{J}\psi(\frac{x}{r_1}) \cdot r_1^2(t)}{[\mathcal{J}\varphi(\frac{x}{r_1}) + t\mathcal{J}\psi(\frac{x}{r_1})]^2 r_1^2(t)}|_{t=0^+} \\ &= \frac{\mathcal{J}\psi(\frac{x}{r_1}) \cdot r_1^2(t)}{< \nabla\mathcal{J}\varphi(\frac{x}{r_1}) + t\nabla\mathcal{J}\psi(\frac{x}{r_1}), x > -[\mathcal{J}\varphi(\frac{x}{r_1}) + t\mathcal{J}\psi(\frac{x}{r_1})]^2 r_1^2(t)}|_{t=0^+}. \end{aligned}$$

By Theorem 3.16, the existence of the limit of the function $\varphi_t(x)$ at $t \to 0^+$, we have

$$\lim_{t\to 0^+} r_1(t) = \lim_{t\to 0^+} \varphi_t(x) = \varphi(x),$$

thus, when $t = 0^+$, $r_1(t) = \varphi(x)$ is obtained by substituting into I,

$$I = \frac{\mathcal{J}\psi(\frac{x}{\varphi(x)}) \cdot \varphi^2(x)}{< \nabla\mathcal{J}\varphi(\frac{x}{\varphi(x)}), x > -1}. \tag{37}$$

Considering function $\mathcal{J}\varphi(\frac{x}{\varphi(x)})$, from the definition of the $\mathcal{J}$, we can know

$$\mathcal{J}\varphi(\frac{x}{\varphi(x)}) = \inf\{s > 0 : \varphi(\frac{x}{\varphi(x)s}) \leq \frac{1}{s}\}.$$

For any fixed $x \in \mathbb{R}^n$, by the monotonicity and convexity of the function φ, it exists unique $s_1 > 0$

$$s_1 = \frac{1}{\varphi(\frac{x}{\varphi(x)s_1})},$$

it can be solved by the above formula

$$s_1 = \frac{1}{\varphi(x)},$$

therefore

$$\mathcal{J}\varphi(\frac{x}{\varphi(x)}) = s_1 = \frac{1}{\varphi(x)}, \tag{38}$$

Using y^j, x^i to represent the j coordinate function of $\frac{x}{\varphi(x)}$ and the i coordinate function of x, and partial differentiation of x^i is performed on both sides of the above formula, we have

$$\frac{\partial \mathcal{J}\varphi(\frac{x}{\varphi(x)})}{\partial x^i} = \sum_j \frac{\partial \mathcal{J}\varphi}{\partial y^j}(\frac{x}{\varphi(x)}) \cdot \frac{\partial y^j(\frac{x}{\varphi(x)})}{\partial x^i}$$
$$= -\frac{1}{\varphi^2(x)} \cdot \frac{\partial \varphi(x)}{\partial x^i}, \tag{39}$$

from the definition of the coordinate function

$$y^j(\frac{x}{\varphi(x)}) = \frac{x^j}{\varphi(x)}.$$

Thereupon

$$\frac{\partial y^j(\frac{x}{\varphi(x)})}{\partial x^i} = \frac{\frac{\partial x^j}{\partial x^i} \cdot \varphi(x) - x^j \cdot \frac{\partial \varphi(x)}{\partial x^i}}{\varphi^2(x)}.$$

Putting the above formula into (39) has

$$\sum_j \frac{\partial \mathcal{J}\varphi}{\partial y^j}(\frac{x}{\varphi(x)}) \cdot \frac{\frac{\partial x^j}{\partial x^i} \cdot \varphi(x) - x^j \cdot \frac{\partial \varphi(x)}{\partial x^i}}{\varphi^2(x)} = -\frac{1}{\varphi^2(x)} \cdot \frac{\partial \varphi(x)}{\partial x^i},$$

this is what we get when we simplify it

$$[< \nabla \mathcal{J}\varphi(\frac{x}{\varphi(x)}), x > -1] \cdot \frac{\partial \varphi(x)}{\partial x^i} \cdot \frac{1}{\varphi(x)} = \frac{\partial \mathcal{J}\varphi}{\partial y^i}(\frac{x}{\varphi(x)}).$$

For ease of calculation, let $A_0 =< \nabla \mathcal{J}\varphi(\frac{x}{\varphi(x)}), x > -1$, since

$$\sum_i \frac{\partial \mathcal{J}\varphi}{\partial y^i}(\frac{x}{\varphi(x)}) \cdot x^i =< \nabla \mathcal{J}\varphi(\frac{x}{\varphi(x)}), x >,$$

therefore

$$\frac{A_0}{\varphi(x)} \cdot < \nabla \varphi(x), x >= A_0 + 1,$$

so we can figure out

$$A_0 = \frac{\varphi(x)}{< \nabla \varphi(x), x > -\varphi(x)}. \tag{40}$$

Substituting (40) into (37), we have

$$I = \mathcal{J}\psi(\frac{x}{\varphi(x)}) \cdot \varphi(x) \cdot [< \nabla \varphi(x), x > -\varphi(x)].$$

To sum up, an explicit expression of $\widetilde{\delta} J(f,g)$ can be found

$$\widetilde{\delta} J(f,g) = \int_{\mathbb{R}^n} e^{-\varphi(x)} \mathcal{J}\psi(\frac{x}{\varphi(x)}) \cdot \varphi(x) \cdot [\varphi(x) - < \nabla \varphi(x), x >] dx.$$

It is worth noting that the definition of the $\mathcal{L}$ is known by introducing it

$$\widetilde{\delta} J(f,g) = -\int_{\mathbb{R}^n} e^{-\varphi(x)} \mathcal{J}\psi(\frac{x}{\varphi(x)}) \cdot \varphi(x) \cdot [(\mathcal{L}\varphi)(\nabla \varphi(x))] dx.$$

□

4.3. Existence proof of $\widetilde{\delta}\mathbf{J}(\mathbf{f},\mathbf{g})$.

Theorem 4.6. *For every $f, g \in \mathcal{A}'$, $f = e^{-\varphi}, g = e^{-\psi}$, if ψ is controlled disturbance of φ, then we always get $\widetilde{\delta}J(f,g)$ to satisfy the dominated convergence theorem, i.e*

$$\begin{aligned}\widetilde{\delta}J(f,g) &= \lim_{t\to 0^+}\int_{\mathbb{R}^n}\frac{e^{-\varphi_t(x)}-e^{-\varphi(x)}}{t}dx\\ &= \int_{\mathbb{R}^n}\lim_{t\to 0^+}\frac{e^{-\varphi_t(x)}-e^{-\varphi(x)}}{t}dx.\end{aligned}$$

Proof. From Definition 3.15, if ψ is a controllable disturbance of φ, then there exists $C > 0$ such that for every $x \in \mathbb{R}^n$ we have

$$(\mathcal{J}\psi)(x) \le C(\mathcal{J}\varphi)(x),$$

under these conditions,

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{e^{-\varphi(x)}-e^{-\mathcal{J}(\mathcal{J}\varphi+t\mathcal{J}\psi)(x)}}{t}dx &\le \int_{\mathbb{R}^n}\frac{e^{-\varphi(x)}-e^{-\mathcal{J}(\mathcal{J}\varphi+tC\mathcal{J}\varphi)(x)}}{t}dx\\ &= \int_{\mathbb{R}^n}\frac{e^{-\varphi(x)}-e^{-\mathcal{J}((1+tC)\mathcal{J}\varphi)(x)}}{t}dx,\end{aligned}$$

using the homogeneity of the $\mathcal{J}$ again, we can get

$$\int_{\mathbb{R}^n}\frac{e^{-\varphi(x)}-e^{-\mathcal{J}((1+tC)\mathcal{J}\varphi)(x)}}{t}dx = \int_{\mathbb{R}^n}\frac{e^{-\varphi(x)}-e^{-\frac{1}{1+tC}\varphi((1+tC)x)}}{t}dx. \tag{41}$$

Let $y = (1+tC)x$ and plug in the second term to the right of (41)

$$\int_{\mathbb{R}^n}\frac{e^{-\frac{1}{1+tC}\varphi((1+tC)x)}}{t}dx = \int_{\mathbb{R}^n}\frac{1}{(1+tC)^n}\frac{e^{-\frac{1}{1+tC}\varphi(y)}}{t}dy, \tag{42}$$

then let $y = x$ substitute (42) and combine (41) to have

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{(1+tC)^n e^{-\varphi(x)}-e^{-\frac{1}{1+tC}\varphi(x)}}{t}dx\\ =&[\frac{(1+tC)^n-1}{t}]\int_{\mathbb{R}^n}e^{-\varphi(x)}dx + \int_{\mathbb{R}^n}\frac{e^{-\varphi(x)}-e^{-\frac{1}{1+tC}\varphi(x)}}{t}dx.\end{aligned} \tag{43}$$

Similar to the calculation of $\widetilde{\delta}J(f,f)$, (43) is represented by $C_1 + C_2$, where

$$C_1 = [\frac{(1+tC)^n-1}{t}]\int_{\mathbb{R}^n}e^{-\varphi(x)}dx,$$

when $t \to 0^+$, apply L 'Hospital's rule

$$C_1 = nC\int_{\mathbb{R}^n}e^{-\varphi(x)}dx \in (0,+\infty).$$

While

$$
\begin{aligned}
C_2 &= \int_{\mathbb{R}^n} \frac{e^{-\varphi(x)} - e^{-\frac{1}{1+tC}\varphi(x)}}{t} dx \\
&= \int_{\mathbb{R}^n} e^{-\frac{1}{1+tC}\varphi(x)} \cdot \frac{e^{-\frac{tC}{1+tC}\varphi(x)} - 1}{t} dx \\
&\geq \int_{\mathbb{R}^n} e^{-\frac{1}{1+tC}\varphi(x)} \cdot \frac{e^{-tC\varphi(x)} - 1}{t} dx. \qquad (44)
\end{aligned}
$$

Let $l_t = e^{-tC\varphi}$, apply Lagrange mean value theorem, exists $\xi \in (0, t)$

$$
\frac{dl_t}{dt}|_{t=\xi} = \frac{e^{-tC\varphi} - 1}{t} = -C\varphi \cdot e^{-C\xi\varphi},
$$

substitute into (44) to obtain

$$
\begin{aligned}
C_2 &\geq -\int_{\mathbb{R}^n} C\varphi \cdot e^{-\frac{1}{1+tC}\varphi(x)} \cdot e^{-C\xi\varphi} dx \\
&= -\int_{\mathbb{R}^n} Ce^{-\frac{1+C\xi+2C^2t\xi-Ct}{2+2Ct}\varphi(x)} \varphi(x) e^{-\varphi(x)/2} dx \\
&\geq -\int_{\mathbb{R}^n} Ce^{-\frac{1+C\xi+2C^2t\xi-Ct}{2+2Ct}\varphi(x)} m dx, \qquad (45)
\end{aligned}
$$

where m is the constant value defined earlier.

Therefore, when t is small enough, for example, let $t \leq \frac{1}{C}$ makes

$$
0 < \frac{1 + C\xi + 2C^2t\xi - Ct}{2 + 2Ct} \varphi(x) < 1,
$$

we can see that (45) is convergent, i.e

$$
-\int_{\mathbb{R}^n} Ce^{-\frac{1+C\xi+2C^2t\xi-Ct}{2+2Ct}\varphi(x)} m dx \in (-\infty, 0).
$$

So, applying the dominated convergence theorem gives us

$$
\begin{aligned}
\widetilde{\delta} J(f,g) &= \lim_{t\to 0^+} (1+tC)^{-n}(C_1 + C_2) \\
&= \int_{\mathbb{R}^n} \lim_{t\to 0^+} [nCe^{-\varphi(x)} + \frac{e^{-\varphi(x)} - e^{-\frac{1}{1+tC}\varphi(x)}}{t}] dx \\
&= C \int_{\mathbb{R}^n} [ne^{-\varphi(x)} - \varphi(x)e^{-\varphi(x)}] dx,
\end{aligned}
$$

because

$$
-\int_{\mathbb{R}^n} C\varphi(x)e^{-\varphi(x)} dx \in (-\infty, 0),
$$

therefore

$$
\widetilde{\delta} J(f,g) \in (-\infty, +\infty).
$$

In summary, $\widetilde{\delta} J(f,g)$ satisfies the dominated convergence theorem. □

Thus, under the condition that ψ is a controllable disturbance of φ, we prove the existence and explicit expression of the solution to the first variational problem of log-concave function derived from the $\mathcal{J}$. This provides a foundation for further study of Brunn-Minkowski theory on unbounded convex sets.

Conflict of interest

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

References

[1] D. Alonso-Gutiérrez, *A reverse Rogers-Shephard inequality for log-concave functions*, J. Geom. Anal., **374** (2019) 1719–1771.
[2] S. Artstein-Avidan, D. Florentin, adn A. Segal, *Functional Brunn-Minkowski inequalities induced by polarity*, Adv. Math., (1) **364** (2020) 107006.
[3] S. Artstein-Avidan and V. Milman, *The concept of duality in convex analysis, and the characterization of the Legendre transform*, Ann. Math., (2) **169** (2009) 661–674.
[4] S. Artstein-Avidan and V. Milman, *Hidden structures in the class of convex functions and a new duality transform*, J. Eur. Math. Soc., (4) **4** (2011) 975–1004.
[5] U. Caglar and E. Werner, *Divergence for s-concave and log concave functions*, Adv. Math., **257** (2014) 219–247.
[6] A. Colesanti and I. Fragalà, *The first variation of the total mass of log-concave functions and related inequalities*, Adv. Math., **24** (2013) 708–749.
[7] D. Cordero-Erausquin and B. Klartag, *Moment measures*, J. Funct. Anal. **268** (2015), 3834–3866.
[8] D. Florentin and A. Segal, *On the Linear Structures Induced by the Four Order Isomorphisms Acting on $Cvx_0(\mathbb{R}^n)$*, Journal of convex analysis, (3) **28** (2021) 751-760.
[9] N. Fang, S. Xing, and D. Ye, *Geometry of log-concave functions: the L_p Asplund sum and the L_p Minkowski problem*, Cal. Var. PDE, **61** (2020) 45.
[10] N. Fang and J. Zhou, *LYZ ellipsoid and Petty projection body for log-concave functions*, Adv. Math., **340** (2018) 914–959.
[11] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, *Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems*, Acta. Math., **216** (2016) 325–388.
[12] B. Klartag and V. Milman, *Geometry of log-concave functions and measures*, Geom. Dedicata, **112** (2005) 169–182.
[13] L. Rotem, *Support functions and mean width for α-concave functions*, Adv. Math., **243** (2013) 168–186.
[14] L. Rotem, *Surface area measures of log-concave functions*, J. d'Ana. Math., **147** (2022) 373–400.
[15] H. Minkowski, *Volumen und Oberfläche*, Math. Ann., (4) **57** (1903) 447–495.
[16] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Cambridge University Press, 1993.

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