FRACTIONAL TIKHONOV REGULARIZATION
METHOD FOR SIMULTANEOUS INVERSION OF THE
SOURCE TERM AND INITIAL DATA IN A
TIME-FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. This paper is concerned with the problem of identifying the space-dependent source term and initial value simultaneously for a time-fractional diffusion equation. The inverse problem is ill-posed, and the idea of decoupling it into two operator equations is applied. In order to solve this inverse problem, a fractional Tikhonov regularization method is proposed. Furthermore, the corresponding convergence estimates are presented by using the a-priori and a-posteriori parameter choice rules. Several numerical examples compared with the classical Tikhonov regularization are constructed for verifying the accuracy and efficiency of the proposed method.

1. Introduction.

In recent years, more and more researchers are concerned about the problem of the time-fractional diffusion. They can describe anomalous diffusion phenomena in place of classical diffusion processes. The time-fractional diffusion equation replaces the standard time derivative with the time-fractional derivative, and derives it, which can be applied in anomalous diffusion and mechanical fields [1–4, 13]. We know the time-fractional diffusion models have a better performance than integer-order diffusion models in simulating real super-diffusion and sub-diffusion processes [4, 17, 18], because the fractional differential operator have the non-local property. The direct problem of the time-fractional diffusion equation, namely the initial value problem and the

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initial boundary value problem, has been extensively studied in recent years, see [1, 6–9]. However, in some practical cases, boundary data or initial data, diffusion coefficients, or parts of the source term may not be given, and we hope to find them with additional measurement data, which will lead to some fractional diffusion inverse problems. But by reading [10], we know that most inverse problems are pathological, and we need to use a regular approach to solve them. The inverse problem of the fractional-order diffusion equation has also recently been studied. For the inverse source problem of the time fractional order diffusion equation, a large number of research results have been studied. However, the inverse problems for simultaneous inversion of multi-parameters in time-fractional diffusion equations have not been investigated widely. Here, we only list some references about this topic [19, 22–24].

In the paper, we consider the following time-fractional diffusion equation under homogeneous Dirichlet boundary condition:

$$\begin{cases}
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + Lu(x,t) = f(x), & (x,t) \in \Omega \times [0,T], \\
u(x,t) = 0, & (x,t) \in \partial \Omega \times [0,T], \\
u(x,0) = \phi(x), & x \in \Omega,
\end{cases}
$$

(1.1)

where the time-fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order $\alpha (0 < \alpha < 1)$, which defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial u}{\partial \eta} d\eta, \quad 0 < \alpha < 1.
$$

Let $\Omega$ be an open bounded domain in $\mathbb{R}^d (1 \leq d \leq 3)$ with sufficiently smooth boundary $\partial \Omega$. Here, $\alpha \in (0, 1)$, and $L$ is a symmetric uniformly elliptic operator that is defined by

$$L(u) = -\sum_{i=1}^d \frac{\partial}{\partial x_i} \left( \sum_{j=1}^d a_{i,j} \frac{\partial}{\partial x_j} u(x) \right) + c(x)u(x).
$$

(1.2)

By the properties of the $L$ operator, we know the coefficients in (1.2) satisfy

$$a_{i,j} = a_{j,i}, \quad 1 \leq i, j \leq d, \quad c(x) \geq 0, \forall x \in \Omega,
$$

(1.3)
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\[(1.4) \quad v \sum_{i=1}^{d} \xi_i^2 \leq \sum_{i,j=1}^{d} a_{i,j}(x) \xi_i \xi_j, \quad \forall x \in \Omega, \xi \in \mathbb{R}^d, v > 0.\]

If the source term \( f(x) \) and initial data \( \phi(x) \) are given, the problem (1.1) is called the direct problem. The inverse problem for (1.1) is little known. Inverse problems appear when there is no data given, such as the initial data, source term, boundary value, diffusion coefficient, et al. By adding some given data, we can obtain inverse problems.

In this paper, we reconstruct the source term \( f(x) \) and initial condition \( u(x,0) = \phi(x) \) from observable data \( g_i(\cdot) = u(\cdot, T_i), (i = 1, 2) \), and we assume that \( 0 < T_1 < T_2 \). Since the noisy measurements are inevitable, we denote the measurements with error as \( g_1^\delta(\cdot) \) and \( g_2^\delta(\cdot) \), which satisfy

\[ \| u(\cdot, T_1) - g_1^\delta(\cdot) \| \leq \delta, \quad \| u(\cdot, T_2) - g_2^\delta(\cdot) \| \leq \delta, \]

where \( \| \cdot \| \) is \( L^2 \) norm, and \( \delta > 0 \) is the noise level. For \( \alpha = 1 \), the above inverse problem is regarded as the simultaneous inversion of a classical parabolic equation [25].

The time-fractional diffusion equation (1.1) has received a lot of attention recently because there are many applications in various engineering fields. Mathematical theories of anomalous diffusion equations and associative numerical methods are often discussed, for example, [1–4, 11–13] and the references therein. The inverse problem of the time-fractional diffusion equation has also been extensively studied. For example, the backward problem is explored by quasi-reversibility method, optimization method, data regularization method, and spectral truncation method [14–16, 19, 26].

To the best knowledge of the authors, there are few problems of determining the source term and the initial value simultaneously in the fractional diffusion equations. In [19], Ruan et al. proposed a standard Tikhonov regularisation method to solve the inverse problem, and they gave the convergence rate for an a-priori and a-posteriori regularisation parameter choice rules, respectively. In [20], Saouli et al. used a modified Tikhonov regularization method to solve considered problem, and they obtained convergence estimates between the exact solutions and their regularized approximations. In [21], Wen et al.
used the Landweber iteration method to construct the solution of the problem of the corresponding conjugate operator equation.

In this paper, we use the fractional Tikhonov regularization method to solve the problem of simultaneously identifying the initial data and the source term for a time-fractional diffusion equation (1.1). In [30], Klann firstly proposed the fractional Tikhonov method in 2008. Compared with the standard Tikhonov method, its numerical fit is more better. In [31], Li used the fractional Tikhonov to consider the heat conduction inverse problem of the time diffusion equation. In [28], Wang et al. used the fractional Tikhonov method to treat time fractional diffusion equation. In [27], Xue et al. used the fractional Tikhonov method to identify the source of the temporal fractional diffusion equation. To overcome the difficulties caused by the ill-posedness, Yang et al. used the fractional Tikhonov regularization method for finding source terms in a time-fractional radial heat equation in [33]. In [29], Yang et al. used the fractional Tikhonov regularization methods for identifying the initial value problem for a time-fractional diffusion equation. In [32], Djennadi et al. deal with two types of inverse problems for diffusion equations involving Caputo fractional derivatives in time and fractional Sturm-Liouville operator for space using the fractional Tikhonov regularization method.

The rest of the paper is structured as follows: in Section 2, we state some preliminaries for our upcoming discussion. Conditional stability and ill-posedness for the simultaneously inverse problem are provided in Section 3. In Section 4, we apply the fractional Tikhonov regularization method and give convergence rates under both the a-priori and the a-posterior parameter choice rules. In Section 5, several numerical examples are constructed to verify accuracy and efficiency of the proposed method. Finally, we give a brief conclusion in Section 6.

2. Preliminaries.

In order to facilitate the subsequent proof and theoretical derivation, we give the following definition, lemmas and remark.

Definition 2.1. ([12]) For arbitrary constants $\alpha > 0$ and $\beta \in \mathbb{R}$, we will consider the behaviour of the Mittag-Leffler function defined as
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follows

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}, \quad z \in \mathbb{C}. \]

Lemma 2.2. ([12]) (a) For \( 0 < \alpha < 1 \) and \( \eta > 0 \),

\[ 0 \leq E_{\alpha,1}(-\eta) < 1, \quad \frac{d^n}{d\eta^n} E_{\alpha,1}(-\lambda \eta^\alpha) = -\lambda E_{\alpha,1}(-\lambda \eta^\alpha). \]

In addition, \( E_{\alpha,1}(-\eta) \) is fully monotonic. That is to say \((-1)^n d^n E_{\alpha,1}(-\eta)/d\eta^n \geq 0\), when \( \eta \to +\infty \), \( E_{\alpha,1}(-\eta) \) satisfies the following approximation relation:

\[ E_{\alpha,1}(-\eta) = \frac{1}{\eta^\alpha(1 - \alpha)} + \mathcal{O}(|\eta|^{-2}). \]

(b) For \( \lambda > 0 \), \( \alpha > 0 \) and positive integer \( m \in \mathbb{N} \),

\[ \frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha - m} E_{\alpha,\alpha - m + 1}(-\lambda t^\alpha), \quad t > 0. \]

Lemma 2.3. ([34]) For any \( \lambda_k \) satisfying \( \lambda_k > \lambda_1 > 0 \), there exist positive constants \( C \), \( \overline{C} \) and \( C_1 \) depending on \( \alpha \), \( T \), \( \lambda_1 \) such that

\[ \frac{C}{\lambda_k} \leq E_{\alpha,1}(-\lambda_k T^\alpha) \leq \frac{\overline{C}}{\lambda_k}, \]

\[ \frac{C_1}{\lambda_k T^\alpha} \leq E_{\alpha,1+\alpha}(-\lambda_k T^\alpha) \leq \frac{1}{\lambda_k T^\alpha}, \]

where \( C_1(\alpha, T, \lambda_1) = 1 - E_{\alpha,1}(-\lambda_1 T^\alpha) \).

Remark 2.4. In this paper, we need to use two moments \( T_1 \), \( T_2 \), according to Lemma 2.3, there exist positive constants \( \overline{C}_1 \), \( \overline{C}_1 \) and \( \overline{C}_2 \), \( \overline{C}_2 \) such that:

\[ \frac{C_1}{\lambda_k} \leq E_{\alpha,1}(-\lambda_k T_1^\alpha) \leq \frac{\overline{C}_1}{\lambda_k}, \]

\[ \frac{C_2}{\lambda_k} \leq E_{\alpha,1}(-\lambda_k T_2^\alpha) \leq \frac{\overline{C}_2}{\lambda_k}. \]

We now prove the following lemmas:
Lemma 2.5. For constants \( t > \lambda_1 \) and \( \frac{1}{2} \leq \gamma \leq 1 \), the following inequality holds:

\[
A(t) = \frac{t}{C_{2\gamma} + \mu t^{2\gamma}} \leq C_3 \mu^{-\frac{1}{p}},
\]

where, \( C_3 = C_3(\gamma, C) > 0 \) are independent on \( \mu, t \).

Proof. For \( \frac{1}{2} \leq \gamma \leq 1 \), there exists a unique \( t_0 = C(2\gamma - 1)^{-\frac{1}{p}} \mu^{-\frac{1}{p}} \geq 0 \) such that \( A'(t_0) = 0 \). We have

\[
A(t) \leq A(t_0) \leq \frac{1}{2\gamma} C^{1-2\gamma}(2\gamma - 1)^{\frac{2\gamma - 1}{p}} \mu^{-\frac{1}{p}} := C_3(\gamma, C) \mu^{-\frac{1}{p}}.
\]

\[ \square \]

Lemma 2.6. For the constants \( t \geq \lambda_1 > 0 \) and \( \frac{1}{2} \leq \gamma \leq 1 \), we have

\[
B(t) = \frac{\mu t^{4\gamma - p}}{C_4^{2\gamma} + \mu t^{2\gamma}} \leq \begin{cases} \ C_5 \mu^{\frac{1}{p}}, & 0 < p < 4\gamma, \\ C_6 \mu, & p \geq 4\gamma, \end{cases}
\]

where, \( C_5 = C_5(\gamma, p, C_4) > 0 \), \( C_6 = C_6(\gamma, p, \lambda_1) > 0 \) are independent on \( t \).

Proof. If \( 0 < p < 4\gamma \), then \( \lim_{t \to 0} B(t) = \lim_{t \to \infty} B(t) = 0 \). Thus there exists \( t_0 = C_4^{\frac{1}{p}}(\frac{4\gamma - p}{pp})^{\frac{1}{p}} \geq 0 \) which is a global maximizer such that \( B'(t_0) = 0 \). We have

\[
B(t) \leq B(t_0) \leq C_4^{\frac{1}{p}}(\frac{4\gamma - p}{p} - 1)^{-\frac{1}{p}} \mu^{\frac{1}{p}} := C_5(\gamma, p, C_4) \mu^{\frac{1}{p}}.
\]

If \( p \geq 4\gamma \), then we have

\[
B(t) \leq \frac{\mu t^{4\gamma - p}}{C_4^{2\gamma}} = \frac{1}{C_4^{2\gamma} t^{4\gamma - p}} \mu \leq 1 \frac{1}{C_4^{2\gamma} \lambda_1^{4\gamma}} \mu := C_6(\gamma, p, \lambda_1, C_4) \mu.
\]

\[ \square \]

Lemma 2.7. For the constants \( t \geq \lambda_1 > 0 \) and \( \frac{1}{2} \leq \gamma \leq 1 \), we have

\[
C(t) = \frac{\mu t^{2\gamma - p - 1}}{C_7^{2\gamma} + \mu t^{2\gamma}} \leq \begin{cases} \ C_7 \mu^{\frac{p+1}{p+1}}(\frac{p+1}{p+1})^{\frac{1}{p+1}} \mu^{\frac{p+1}{p+1}}, & 0 < p < 2\gamma - 1, \\ C_7 \lambda_1^{2\gamma - 2\gamma} \mu, & p \geq 2\gamma - 1, \end{cases}
\]
where, \( C_\gamma = 1 - \frac{\alpha}{\lambda} \).

**Proof.** If \( 0 < p < 2\gamma - 1 \), then \( \lim_{t \to 0} C(t) = \lim_{t \to \infty} C(t) = 0 \). Thus there exists \( t_0 = C_\gamma \left( \frac{2\gamma - p - 1}{\mu(p+1)} \right) \geq 0 \) which is a global maximizer such that \( C'(t_0) = 0 \). We have \( C(t) \leq \sup_{t \in (0, \infty)} C(t) \leq C(t_0) \), thus, we have

\[
C(t) \leq C(t_0) = \frac{C_\gamma^{2\gamma-p-1} \mu \left( \frac{2\gamma-p-1}{\mu(p+1)} \right)^{\frac{2\gamma-p-1}{p+1}}}{C_\gamma^{2\gamma} + C_\gamma^{2\gamma} \mu \frac{2\gamma-p-1}{\mu(p+1)}} \leq \frac{\mu^{\frac{p+1}{p\gamma}} \left( \frac{2\gamma-p-1}{\mu(p+1)} \right)^{1-\frac{p+1}{p\gamma}}}{C_\gamma^{p+1} \frac{2\gamma-p-1}{p+1}} \leq \frac{1}{C_\gamma^{p+1} \left( \frac{p+1}{2\gamma-p} \right)^{\frac{p+1}{p\gamma}}} \mu^{\frac{p+1}{p\gamma}}.
\]

If \( p \geq 2\gamma - 1 \), then we have

\[
C(t) \leq \frac{\mu t^{2\gamma-p-1}}{C_\gamma^{2\gamma}} = \frac{1}{C_\gamma^{2\gamma} t^{p+1-2\gamma} \mu} \leq \frac{1}{C_\gamma^{2\gamma} \lambda^{p+1-2\gamma} \mu}.
\]

\( \square \)

**Lemma 2.8.** The two operators can be commutative, that is to say, \( K_{2,2} K_{2,1} = K_{2,1} K_{2,2} \), we can prove them by calculation.

**Proof.** Using equation (2.7)

\[
(K_{2,1}(\phi)) (x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_1^{\alpha}) \phi_k \varphi_k(x),
\]

then

\[
K_{2,2} (K_{2,1}(\phi)) (x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_1^{\alpha}) \phi_k E_{\alpha,1}(-\lambda_k T_2^{\alpha}) \phi_k(x).
\]

Similarly,

\[
K_{2,1} (K_{2,2}(\phi)) (x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_2^{\alpha}) \phi_k E_{\alpha,1}(-\lambda_k T_1^{\alpha}) \phi_k(x).
\]

Hence, \( K_{2,2} K_{2,1} = K_{2,1} K_{2,2} \). \( \square \)
Through linear superposition, the solution \( u(x,t) \) satisfying the problem (1.1) can be divided into the components \( u_1(x,t) \) and \( u_2(x,t) \), which are the solutions of the two subproblems respectively,

\[
\begin{align*}
\frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} + Lu_1(x,t) &= f(x), \quad (x,t) \in \Omega \times [0,T], \\
u_1(x,t) &= 0, \quad (x,t) \in \partial \Omega \times [0,T], \\
u_1(x,0) |_{t=0} &= 0, \quad x \in \overline{\Omega}.
\end{align*}
\]

(2.2)

\[
\begin{align*}
\frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} + Lu_2(x,t) &= 0, \quad (x,t) \in \Omega \times [0,T], \\
u_2(x,t) &= 0, \quad (x,t) \in \partial \Omega \times [0,T], \\
u_2(x,0) |_{t=0} &= \phi(x), \quad x \in \overline{\Omega}.
\end{align*}
\]

(2.3)

Therefore,

\[ u(x,t) = u_1(x,t) + u_2(x,t). \]

Suppose \( \lambda_n \in \mathbb{R} \) be the eigenvalues of the operator \( L \), and corresponding orthogonal eigenfunctions \( \varphi_k(x) \in H^2(\Omega) \cap H_0^1(\Omega) \). We have \( L\varphi_k = \lambda_k \varphi_k \). Thanks to \( L \) is a symmetric strongly elliptic operator, we can set

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty. \]

Using separation of variables and the results of Lemma 2.2, the formal solutions of the direct problem (2.2) and (2.3) can be constructed as:

\[
u_1(x,t) = \sum_{k=1}^{\infty} 1 - E_{\alpha,1}(-\lambda_k t^\alpha) \beta f_k \varphi_k(x),
\]

(2.4)

\[
u_2(x,t) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k t^\alpha) \beta \phi_k \varphi_k(x),
\]

(2.5)

where \( f_k = \langle f(x), \varphi_k(x) \rangle \), \( \phi_k = \langle \phi(x), \varphi_k(x) \rangle \).
For given source term $f(x)$ and initial function $\phi(x)$, we can define a pair of linear operators $K_1$ and $K_2$ to solve problem (1.1):

$$K_1 : (f, \phi) \mapsto u(x, T_1),$$
$$K_2 : (f, \phi) \mapsto u(x, T_2).$$

Similarly, for the problems (2.2) and (2.3), we can define four linear operators respectively.

$$K_{1,i} : f \mapsto u_1(x, T_i), \quad i = 1, 2,$$
$$K_{2,i} : \phi \mapsto u_2(x, T_i), \quad i = 1, 2.$$  

By the solution expressions (2.4) and (2.5), we can obtain operator equations:

$$(K_{1,i}(f)) (x) = \sum_{k=1}^{\infty} \frac{1 - E_{\alpha,1}(-\lambda_k T_i^\alpha)}{\lambda_k} f_k \varphi_k(x), \quad i = 1, 2,$$

$$(K_{2,i}(\phi)) (x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_i^\alpha) \phi_k \varphi_k(x), \quad i = 1, 2.$$  

Using the property of linear superposition, then we have the following expressions for the operator equations

$$(K_1(f)) (x) = (K_{1,1}(f)) (x) + (K_{2,1}(\phi)) (x) = g_1(x),$$
$$(K_2(f)) (x) = (K_{1,2}(f)) (x) + (K_{2,2}(\phi)) (x) = g_2(x).$$

Now, we wish to solve the inverse problem and find the functions $(f, \phi)$ in problem (2.2) and (2.3). By equations (2.8) and (2.9), the solutions $(f, \phi)$ of the system can be obtained:

$$(K_{2,1}\phi + K_{1,1}f = g_1, \quad K_{2,2}\phi + K_{1,2}f = g_2.$$  

Applying operator $K_{2,2}$ to the first equation in the system (2.10) and operator $K_{2,1}$ to the second one yields:

$$K_{2,2}K_{2,1}\phi + K_{2,2}K_{1,1}f = K_{2,2}g_1,$$
$$K_{2,1}K_{2,2}\phi + K_{2,1}K_{1,2}f = K_{2,1}g_2.$$
By subtracting (2.11) from (2.12) and exploiting the semi-groups properties, we have

\[(K_{2,1}K_{1,2} - K_{2,2}K_{1,1})f = K_{2,1}g_2 - K_{2,2}g_1.\]

Similarly, we apply operator \(K_{1,2}\) to the first equation in the system (2.10) and \(K_{1,1}\) to the second one,

\[(2.13) \quad K_{1,2}K_{2,1}\phi + K_{1,2}K_{1,1}f = K_{1,2}g_1,\]

\[(2.14) \quad K_{1,1}K_{2,2}\phi + K_{1,1}K_{1,2}f = K_{1,1}g_2.\]

We subtract the equation (2.13) from (2.14) as follows

\[(K_{1,2}K_{2,1} - K_{1,1}K_{2,2})\phi = K_{1,2}g_1 - K_{1,1}g_2.\]

Thus, (2.10) is equivalent to the system

\[(2.15) \quad \begin{cases} Kf = \eta_1, \\ K\phi = \eta_2. \end{cases}\]

Where

\[K = K_{1,2}K_{2,1} - K_{1,1}K_{2,2}, \quad \eta_1 = K_{2,1}g_2 - K_{2,2}g_1, \quad \eta_2 = K_{1,2}g_1 - K_{1,1}g_2.\]

Using the properties of singular values, the singular values of the operators \(K_{1,1}, K_{1,2}, K_{2,1}, K_{2,2}\) are obtained as

\[\sigma_{1k} = \frac{1 - E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k}, \]

\[\sigma_{2k} = \frac{1 - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}, \]

\[\sigma_{3k} = E_{\alpha,1}(-\lambda_k T_1^\alpha), \]

\[\sigma_{4k} = E_{\alpha,1}(-\lambda_k T_2^\alpha).\]

Therefore, it is easy to obtain the singular values of operator \(K\):

\[(2.16) \quad \sigma_k = \frac{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}, \quad k = 1, 2, \ldots .\]

For arbitrary $\psi \in L^2(\Omega)$, we define
\begin{equation}
D((L)^p) = \left\{ \psi \in L^2(\Omega) : \left( \sum_{k=1}^{\infty} \lambda_k^{2p} |\langle \psi, \varphi_k \rangle|^2 \right)^{\frac{1}{2}} < \infty \right\},
\end{equation}
where $\langle \cdot, \cdot \rangle$ is the inner product in $L_2(\Omega)$, then $D((L)^p)$ is a Hilbert space with the norm
\begin{equation}
\|\psi\|_{D((L)^p)} = \left( \sum_{k=1}^{\infty} \lambda_k^{2p} |\langle \psi, \varphi_k \rangle|^2 \right)^{\frac{1}{2}}.
\end{equation}

In the paper, the study of the (2.8)-(2.9) is reduced to the study of the system (2.15), that is, the study of the first kind operator equation in $L^2(\Omega)$:
\[ Kb = \eta. \]

From the properties of $K$, we obtain
\[ b = K^{-1} \eta = \sum_{k=1}^{\infty} \frac{1}{\sigma_k} \eta_k \varphi_k. \]

We call $1/\sigma_k$ the magnifying factor of the problem. Since $1/\sigma_k \to \infty$ as $k \to \infty$, the problem is ill-posed, that is, the solution does not persistently dependent on the given data.

Furthermore, since the measured data $g_1(\cdot)$ and $g_2(\cdot)$ are not accurate in practice, our goal is to construct stable approximate solutions for $\phi$ and $f$ in the system:
\begin{equation}
\left\{ \begin{array}{l}
Kf = \eta_1^\delta, \\
K\phi = \eta_2^\delta,
\end{array} \right.
\end{equation}
where $\eta_1^\delta = K_{2,1}g_2 - K_{2,2}g_1$, $\eta_2^\delta = K_{1,2}g_2 - K_{1,1}g_1$, $g_1$ and $g_2$ are perturbed data functions that satisfying
\begin{equation}
\|g_1(\cdot) - g_1^\delta(\cdot)\| \leq \delta, \quad \|g_2(\cdot) - g_2^\delta(\cdot)\| \leq \delta.
\end{equation}

**Theorem 3.1.** If $f$ and $\phi \in D((L)^p) \subset H^p$ ($p > 0$) satisfy the a-priori bound condition
\begin{equation}
\max \left\{ \|f(\cdot)\|_{D((L)^p)}, \|\phi(\cdot)\|_{D((L)^p)} \right\} \leq E,
\end{equation}
then we have
\[ \|f\| \leq CE^{\frac{1}{p+1}} \left( \|g_1\| + \left\| \left( \frac{T_2}{T_1} \right)^{a} \|g_2\| \right\| \right)^{\frac{1}{p+1}}, \]
\[ \|\phi\| \leq CE^{\frac{1}{p+1}} \left( \frac{1}{1 - E_{\alpha,1}(-\lambda_1 T_1)} \|g_1\| + \|g_2\| \right)^{\frac{1}{p+1}}, \]
where \( C \) is a constant depending on \( \alpha, T_i, p, \lambda_1 \).

**Proof.** The proof can be found in [19].

### 4. Fractional Tikhonov Regularization and Convergence Analysis.

In this section, we will use the fractional Tikhonov regularization method to solve the problem, and present the convergence analysis under two regularization parameter choice rules. The standard theory of the fractional Tikhonov regularisation method can be found in [30].

Now, the solutions of fractional Tikhonov regularization method with noisy data and exact data are present by

\[
\begin{align*}
&f_\mu = \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{2\gamma+\mu}} (\eta_1, \varphi_k) \varphi_k, \\
&\phi_\mu = \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{2\gamma+\mu}} (\eta_2, \varphi_k) \varphi_k,
\end{align*}
\]
\[(4.1)\]

and

\[
\begin{align*}
&f_\delta = \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{2\gamma+\mu}} (\eta_1^\delta, \varphi_k) \varphi_k, \\
&\phi_\delta = \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{2\gamma+\mu}} (\eta_2^\delta, \varphi_k) \varphi_k,
\end{align*}
\]
\[(4.2)\]

where \( \mu > 0 \) acts as the regularization parameter, and \( \gamma \) is called the fractional parameter. Especially, it is the classical Tikhonov regularization method while \( \gamma = 1 \).

We define
\[
\begin{align*}
&K_1(f, \phi) \circ (x) = (K_{1,1}(f)) \circ (x) + (K_{2,1}(\phi)) \circ (x) = h_1(x) + h_3(x), \\
&K_2(f, \phi) \circ (x) = (K_{1,2}(f)) \circ (x) + (K_{2,2}(\phi)) \circ (x) = h_2(x) + h_4(x).
\end{align*}
\]
FRACTIONAL TIKHONOV REGULARIZATION METHOD FOR SIMULTANEOUS INVERSION OF THE SOURCE TERM AND INITIAL DATA IN A TIME-FRACTIONAL DIFFUSION EQUATION

Since $K_{i,j} (i,j = 1, 2, 3, 4)$ are linear compact operators, we define the operators $R_{1,\mu}$ and $R_{2,\mu} : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$,

(4.5) \[ R_{1,\mu} g_1 = \sum_{k=1}^{\infty} \frac{\sigma_{1k}^{2\gamma-1}}{\sigma_{1k}^{2\gamma} + \mu} h_{1,k} \varphi_k + \sum_{k=1}^{\infty} \frac{\sigma_{3k}^{2\gamma-1}}{\sigma_{3k}^{2\gamma} + \mu} h_{3,k} \varphi_k, \]

(4.6) \[ R_{2,\mu} g_2 = \sum_{k=1}^{\infty} \frac{\sigma_{2k}^{2\gamma-1}}{\sigma_{2k}^{2\gamma} + \mu} h_{2,k} \varphi_k + \sum_{k=1}^{\infty} \frac{\sigma_{4k}^{2\gamma-1}}{\sigma_{4k}^{2\gamma} + \mu} h_{4,k} \varphi_k, \]

where $h_{i,k} = (h_i, \varphi_k)$. ($i = 1, 2, 3, 4$)

The convergence results are given in the following theorems.

4.1. A-priori regularization parameter choice rule.

The convergence rate of the fractional Tikhonov regularized solutions to exact solution can be obtained under the a-priori regularization parameter choice rule.

**Theorem 4.1.** Suppose the a-priori condition (3.5) and the noise assumption (3.4) hold, then:

(1) If we choose $0 < p < 4\gamma$ and regularization parameter $\mu = (\delta E)^{\frac{2\gamma}{2+2\gamma}}$, then we have the following estimates:

\[ \| f_\mu^\delta (\cdot) - f(\cdot) \| + \| \phi_\mu^\delta (\cdot) - \phi(\cdot) \| \leq C_8 \delta E^{\frac{2\gamma}{2+2\gamma}} \delta^{\frac{2\gamma}{2+2\gamma}}, \]

(2) If we choose $p \geq 4\gamma$ and regularization parameter $\mu = (\delta E)^{\frac{2\gamma}{2+2\gamma}}$, then we have the following estimates:

\[ \| f_\mu^\delta (\cdot) - f(\cdot) \| + \| \phi_\mu^\delta (\cdot) - \phi(\cdot) \| \leq C_9 \delta E^{\frac{2\gamma}{2+2\gamma}} \delta^{\frac{2\gamma}{2+2\gamma}}, \]

where $C_8 = \frac{C_3(2+C_{1\mu})}{\lambda_1} + 2C_5$, $C_9 = \frac{C_3(2+C_{1\mu})}{\lambda_2} + 2C_6$, and they are positive constants.

**Proof.** By the triangle inequality, we have

(4.7) \[ \| f_\mu^\delta (\cdot) - f(\cdot) \| \leq \| f_\mu^\delta (\cdot) - f_\mu (\cdot) \| + \| f_\mu (\cdot) - f(\cdot) \| = I_1 + I_2. \]
From (3.4) and Remark 2.4, we have

\[ I_1 = \|f_\delta^\mu(\cdot) - f_\mu^\delta(\cdot)\| = \left\| \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{\gamma} + \mu} \delta_{1,k}^\mu \varphi_k - \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{\gamma} + \mu} \eta_{1,k} \varphi_k \right\| 
\]

\[ = \left\| \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{\gamma} + \mu} (\delta_{1,k} - \eta_{1,k}) \varphi_k \right\| 
\]

\[ = \left\| \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{\gamma} + \mu} (K_{2,1}g_{2,k}^\delta - K_{2,1}g_{1,k}^\delta - K_{2,1}g_{2,k}^\delta + K_{2,1}g_{1,k}^\delta) \varphi_k \right\| 
\]

\[ \leq \left( \|K_{2,1}(g_{2,k}^\delta - g_{2,k})\| + \|K_{2,1}(g_{2,k}^\delta - g_{2,k})\| \right) \sup_{\sigma_k > 0} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{\gamma} + \mu} \leq \sup_{\lambda_k > 0} \left( E_{\alpha,1}(-\lambda_k T_1^\alpha) + E_{\alpha,1}(-\lambda_k T_2^\alpha) \right) \left( \sup_{\sigma_k > 0} G(n) \right) \delta, \]

where,

\[ E_{\alpha,1}(-\lambda_k T_1^\alpha) + E_{\alpha,1}(-\lambda_k T_2^\alpha) \leq C_1 \frac{C_3}{\lambda_k} + C_2 \frac{C_3}{\lambda_k} \leq C_{10} \frac{C_3}{\lambda_1} \leq C_{10} \frac{C_3}{\lambda_1}, \]

\[ C_1 \text{ depending on } \alpha, \ T_1; \ C_2 \text{ depending on } \alpha, \ T_2; \ C_{10} = C_1 + C_2. \]

In the following equation,

\[ G(n) = \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{\gamma} + \mu}, \quad \sigma_k = \frac{E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}. \]

By Lemma 2.5, we have

\[ (4.8) \quad \sup_{\sigma_k > 0} G(n) \leq C_3 \mu^{-\frac{1}{\lambda_k}}. \]

Then

\[ (4.9) \quad \|f_\delta^\mu(\cdot) - f_\mu^\delta(\cdot)\| \leq \frac{C_{10}}{\lambda_1} C_3 \mu^{-\frac{1}{\lambda_1}} \delta. \]

For the second term on the right side of (4.13), using a-priori bound
FRACTIONAL Tikhonov Regularization Method for Simultaneous Inversion of the Source Term and Initial Data in a Time-Fractional Diffusion Equation

According to Condition (3.5), we obtain:

\[
I_2 = \| f(\cdot) - f_\mu(\cdot) \| = \left\| \sum_{k=1}^{\infty} \frac{1}{\sigma_k^p} \eta_{1,k} \varphi_k - \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^2 + \mu} \eta_{1,k} \varphi_k \right\| \\
= \left\| \sum_{k=1}^{\infty} \frac{\mu \lambda_k^{-p} \lambda_k^p \eta_{1,k}}{\sigma_k^2 + \mu} \varphi_k \right\| \\
\leq E \sup_{\lambda_k > 0, \sigma_k > 0} \frac{\mu \lambda_k^{-p}}{\sigma_k^2 + \mu}.
\]

According to Remark 2.4, we have

\[
\frac{C_1}{\lambda_k} - \frac{C_2}{\lambda_k} \leq E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha) \leq \frac{C_1}{\lambda_k} - \frac{C_2}{\lambda_k}.
\]

We define \( \frac{C_1}{\lambda_k} - \frac{C_2}{\lambda_k} = \frac{C_4}{\lambda_k} \), hence \( \frac{C_4}{\lambda_k} \leq E_{\alpha,1}(-\lambda_k T_1^\alpha) - E_{\alpha,1}(-\lambda_k T_2^\alpha) = \sigma_n. \)

So, we have

\[
I_2 \leq E \sup_{\lambda_k > 0, \sigma_k > 0} \frac{\mu \lambda_k^{-p}}{\sigma_k^2 + \mu} \leq E \sup_{\lambda_k > 0, \sigma_k > 0} \frac{\mu \lambda_k^{\gamma-p}}{C_4^2 + \mu \lambda_k^{4\gamma}}.
\]

Hence, using Lemma 2.6, we get:

\[
I_2 = \| f_\mu(\cdot) - f(\cdot) \| \leq \left\{ \begin{array}{ll}
EC_4^{-\frac{\gamma}{2}} \left( \frac{\mu}{\gamma} - 1 \right)^{-\frac{\gamma}{2}} \mu \frac{\lambda_k}{\sigma_k}, & 0 < p < 4\gamma, \\
EC_4^{-\frac{\gamma}{2}} \left( \frac{\mu}{\gamma} - 1 \right) \mu \frac{\lambda_k}{\sigma_k}, & p \geq 4\gamma.
\end{array} \right.
\]

Choose the regularization parameter \( \mu \) by

\[
\mu = \left\{ \begin{array}{ll}
\left( \frac{\delta}{p} \right)^{\frac{\gamma}{2}}, & 0 < p < 4\gamma, \\
\left( \frac{\delta}{p} \right)^{\frac{\gamma}{2}}, & p \geq 4\gamma.
\end{array} \right.
\]

Combining (4.9) and (4.10), we obtain:

\[
\| f_\mu^p(\cdot) - f(\cdot) \| \leq \left\{ \begin{array}{ll}
\left( \frac{C_4}{\lambda_k} + C_4^{-\frac{\gamma}{2}} \left( \frac{\mu}{\gamma} - 1 \right)^{-\frac{\gamma}{2}} \right) \delta^\frac{\gamma}{2} E^\frac{\gamma}{2}, & 0 < p < 4\gamma, \\
\left( \frac{C_4}{\lambda_k} + \frac{1}{C_4} \left( \frac{\mu}{\gamma} - 1 \right) \delta^\frac{\gamma}{2} E^\frac{\gamma}{2}, & p \geq 4\gamma.
\end{array} \right.
\]
By the triangle inequality, we have

\[(4.13) \quad \| \phi^h(\cdot) - \phi(\cdot) \| \leq \| \phi^h(\cdot) - \phi(\cdot) \| + \| \phi(\cdot) - \phi(\cdot) \| = I_3 + I_4.\]

By the same calculation used to obtain (4.10) as follows

\[I_4 = \| \phi(\cdot) - \phi(\cdot) \| \leq \begin{cases} C_{4^{-\frac{1}{2}}(4^2 p - 1)^{-\frac{p}{q}}} E_{\mu}^{\frac{p}{q}}, & 0 < p < 4\gamma, \\ \frac{1}{C_{4^{-\frac{1}{2}}(4^2 p - 1)^{-\frac{p}{q}}}} E_{\mu}, & p \geq 4\gamma. \end{cases}\]

On the other hand,

\[I_3 = \| \phi^h(\cdot) - \phi(\cdot) \| = \left\{ \begin{array}{ll} \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma - 1}}{\sigma_k^{\gamma} + \mu} \eta_k^2 \varphi_k - \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma - 1}}{\sigma_k^{\gamma} + \mu} \eta_2^2 \varphi_k \\ \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma - 1}}{\sigma_k^{\gamma} + \mu} (\eta_k^2 - \eta_2^2) \varphi_k \\ \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma - 1}}{\sigma_k^{\gamma} + \mu} (K_{1,2} g_{1,k}^2 - K_{1,1} g_{2,k}^2 - \mu K_{1,2} g_{1,k} + \mu K_{1,1} g_{2,k}) \varphi_k \\ \leq \left( \| K_{1,2} (g_{1,k}^2 - g_{1,k}) \| + \| K_{1,1} (g_{2,k} - g_{2,k}) \| \right) \sup_{\sigma_k > 0} \frac{\sigma_k^{2\gamma - 1}}{\sigma_k^{\gamma} + \mu} \\ \leq \left( \frac{1 - E_{\alpha_1}(-\lambda_k T_{1}^2)}{\lambda_k} + \frac{1 - E_{\alpha_1}(-\lambda_k T_{1}^2)}{\lambda_k} \right) C_{\gamma} \mu^{-\frac{\gamma}{4}} \delta. \end{array} \right.\]

using Lemma 2.3, we have:

\[(4.14) \quad \| \phi^h(\cdot) - \phi(\cdot) \| \leq \frac{2}{\lambda_k} C_{\gamma} \mu^{-\frac{\gamma}{4}} \delta < \frac{2}{\lambda_k} C_{\gamma} \mu^{-\frac{\gamma}{4}} \delta.\]

If we select \( \mu \) in equation (4.11), then we have:

\[(4.15) \quad \| \phi^h(\cdot) - \phi(\cdot) \| \leq \begin{cases} \left( \frac{2C_{\gamma}}{\lambda_k} + C_{4^{-\frac{1}{2}}(4^2 p - 1)^{-\frac{p}{q}}} \right) \delta^{\frac{1}{4}} E_{\mu}^{\frac{1}{2}} , & 0 < p < 4\gamma, \\ \left( \frac{2C_{\gamma}}{\lambda_k} + \frac{1}{C_{4^{-\frac{1}{2}}(4^2 p - 1)^{-\frac{p}{q}}}} \right) \delta^{\frac{1}{4}} E_{\mu}^{\frac{1}{2}} , & p \geq 4\gamma. \end{cases}\]

Finally, we add formula (4.12) and formula (4.15) to get:

\[
\| f^h(\cdot) - f(\cdot) \| + \| \phi^h(\cdot) - \phi(\cdot) \| \leq \begin{cases} C_{\gamma} \delta^{\frac{1}{4}} E_{\mu}^{\frac{1}{2}} , & 0 < p < 4\gamma, \\ C_{\gamma} \delta^{\frac{1}{4}} E_{\mu}^{\frac{1}{2}} , & p \geq 4\gamma. \end{cases}
\]
4.2. A-posteriori regularization choice rule.

In this section, we consider the a-posteriori regularization parameter choice rule. Then we use the Morozov’s discrepancy principle [39] to determine the regularization parameter $\mu$ by using the a-posteriori choice rule, and we give the rate of convergence for the regularized solution.

The general a-posteriori rule can be summarized as follows:

\begin{equation}
\|u^\delta_\mu(\cdot, T_1) - g^\delta_1(\cdot)\| + \|u^\delta_\mu(\cdot, T_2) - g^\delta_2(\cdot)\| = \tau \delta.
\end{equation}

Where $\tau > 2$ is constant independent of $\delta$, $f^\delta_\mu$, $\phi^\delta_\mu$ are the fractional Tikhonov regularization solutions defined in (4.2).

**Lemma 4.2.** Let $\rho(\mu) = \|u^\delta_\mu(\cdot, T_1) - g^\delta_1(\cdot)\| + \|u^\delta_\mu(\cdot, T_2) - g^\delta_2(\cdot)\|$. Then we have the following results:

1. $\rho(\mu)$ is a continuous function;
2. $\lim_{\mu \to 0} \rho(\mu) = 0$;
3. $\lim_{\mu \to \infty} \rho(\mu) = \|g^\delta_1(\cdot)\| + \|g^\delta_2(\cdot)\|$;
4. $\rho(\mu)$ is a strictly decreasing function over $(0, \infty)$.

**Proof.** From (4.2) and (4.3) (4.4), we obtain:

\[
\rho(\mu) = \left( \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{1,k}^2 + \mu} h^\delta_{1,k} + \frac{\mu}{\sigma_{3,k}^2 + \mu} h^\delta_{3,k} \right)^2 \right)^{\frac{1}{2}}
+ \left( \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{2,k}^2 + \mu} h^\delta_{2,k} + \frac{\mu}{\sigma_{4,k}^2 + \mu} h^\delta_{4,k} \right)^2 \right)^{\frac{1}{2}}.
\]

Hence, $\lim_{\mu \to 0} \rho(\mu) = \left( \sum_{k=1}^{\infty} (h^\delta_{1,k} + h^\delta_{3,k})^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (h^\delta_{2,k} + h^\delta_{4,k})^2 \right)^{\frac{1}{2}} = \|g^\delta_1(\cdot)\| + \|g^\delta_2(\cdot)\|$.

The above results (1)-(4) are easily obtained. $\square$
Lemma 4.3. If \( \mu \) is the solution of equation (4.16), the following inequality should be established:

\[
\mu^{-\frac{1}{2\gamma}} \leq \begin{cases} 
\left( \frac{C_1}{\gamma} \right) \frac{1}{\gamma} + \left( \frac{C_2}{\gamma} \right) \frac{1}{\gamma} & , \quad 0 < p < 2\gamma - 1, \\
\left( \frac{C_1}{\gamma} \right) \frac{1}{\gamma} + \left( \frac{C_2}{\gamma} \right) \frac{1}{\gamma} & , \quad p \geq 2\gamma - 1.
\end{cases}
\]

Proof. From the definition of \( \mu \), and using the triangle inequality we have:

\[
\tau \delta 
\leq \left\| u_{\mu-1}(\cdot, T_1) - g_1(\cdot) \right\| + \left\| u_{\mu-1}(\cdot, T_2) - g_2(\cdot) \right\|
\]

\[
= \left\| \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{1k}^2 + \mu} h_{1,k}^\delta \varphi_k + \frac{\mu}{\sigma_{3k}^2 + \mu} h_{3,k}^\delta \varphi_k \right) \right\| + \left\| \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{2k}^2 + \mu} h_{2,k}^\delta \varphi_k + \frac{\mu}{\sigma_{4k}^2 + \mu} h_{4,k}^\delta \varphi_k \right) \right\|
\]

\[
\leq \left\| \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{1k}^2 + \mu} h_{1,k}^\delta \varphi_k + \frac{\mu}{\sigma_{3k}^2 + \mu} h_{3,k}^\delta \varphi_k \right) \right\| + \left\| \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{2k}^2 + \mu} h_{2,k}^\delta \varphi_k + \frac{\mu}{\sigma_{4k}^2 + \mu} h_{4,k}^\delta \varphi_k \right) \right\|
\]

\[
\leq \sum_{k=1}^{\infty} \left( h_{1,k}^\delta - h_{3,k}^\delta - h_{3,k}^\delta \right) \varphi_k + \sum_{k=1}^{\infty} \left( h_{2,k}^\delta - h_{2,k}^\delta - h_{4,k}^\delta \right) \varphi_k
\]

\[
+ \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{1k}^2 + \mu} \sigma_{1,k} \lambda_k^p \gamma_k^p \varphi_k + \frac{\mu}{\sigma_{3k}^2 + \mu} \sigma_{3,k} \lambda_k^p \gamma_k^p \varphi_k \right) + \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_{2k}^2 + \mu} \sigma_{2,k} \lambda_k^p \gamma_k^p \varphi_k \right)
\]

\[
\leq 2\delta + E \left( \sup_{\sigma_{1k}>0} C_{1\gamma} \frac{\mu}{\sigma_{1k}^2 + \mu} s_{2\gamma} \gamma - p - 1 \right) + \sup_{\sigma_{3k}>0} C_{1\gamma} \frac{\mu}{\sigma_{3k}^2 + \mu} s_{2\gamma} \gamma - p - 1
\]

\[
+ E \left( \sup_{\sigma_{2k}>0} C_{1\gamma} \frac{\mu}{\sigma_{2k}^2 + \mu} s_{2\gamma} \gamma - p - 1 \right) + \sup_{\sigma_{3k}>0} C_{1\gamma} \frac{\mu}{\sigma_{3k}^2 + \mu} s_{2\gamma} \gamma - p - 1
\].
where \( C_7 = 1 - \frac{C_1}{\lambda_1}, C_7 = 1 - \frac{C_2}{\lambda_1} \).

According to Lemma 2.7, we get:

\[
\tau \delta \leq 2\delta + E \left\{ \begin{array}{ll}
C_{11} \mu^{\frac{p+1}{\gamma}} & , \quad 0 < p < 2\gamma - 1, \\
C_{12} \mu & , \quad p \geq 2\gamma - 1,
\end{array} \right.
\]

where, \( C_{11} = \left( \frac{1}{C_7^{\frac{1}{\gamma}}} + \frac{C_1}{C_7^{\frac{1}{\gamma}}} + \frac{1}{\lambda_1} + \frac{C_2}{\lambda_1} \right) \left( \frac{\mu^{p+1}}{2\gamma - p - 1} \right)^{\frac{p+1}{\gamma}}, \)

\( C_{12} = \left( \frac{1}{C_7^{\frac{1}{\gamma}}} + \frac{C_1}{C_7^{\frac{1}{\gamma}}} + \frac{1}{\lambda_1^{\frac{1}{\gamma}}} + \frac{C_2}{\lambda_1^{\frac{1}{\gamma}}} \right) \mu, \)

by simple calculation, we have:

\[
\mu^{\frac{1}{\gamma}} \left\{ \begin{array}{ll}
\left( \frac{C_{11}}{\tau} \right)^{\frac{1}{\gamma}} \frac{E}{\mu} & , \quad 0 < p < 2\gamma - 1, \\
\left( \frac{C_{12}}{\tau} \right)^{\frac{1}{\gamma}} \frac{E}{\mu} & , \quad p \geq 2\gamma - 1.
\end{array} \right.
\]

\[\square\]

**Theorem 4.4.** Assume the a-priori condition (3.5) and the noise assumption (3.4) hold, and regularization parameter \( \mu > 0 \) is given by Lemma 4.3, we can obtain the error estimates as follows:

(i) If \( 0 < p < 2\gamma - 1 \), we have the convergence estimate:

\[
\| f_{\mu}^\delta (\cdot) - f(\cdot) \| + \| \phi_{\mu}^\delta (\cdot) - \phi(\cdot) \| \leq C_{13} E^{\frac{1}{\gamma - 1}} \delta^{\frac{p}{\gamma - 1}},
\]

(ii) If \( p \geq 2\gamma - 1 \), we have the convergence estimate:

\[
\| f_{\mu}^\delta (\cdot) - f(\cdot) \| + \| \phi_{\mu}^\delta (\cdot) - \phi(\cdot) \| \leq C_{14} E^{\frac{1}{\gamma - 1}} \delta^{1 - \frac{p}{\gamma - 1}},
\]
where

\[ C_{13} = C_3 \left( \frac{C_{10} + 2}{\lambda_1} \right)^{\frac{1}{\tau_2 - 2}} + (C_{15})^{\frac{1}{\tau_2}} + (C_{16})^{\frac{1}{\tau_2}}, \]

\[ C_{14} = C_3 \left( \frac{C_{11} + 2}{\lambda_1} \right)^{\frac{1}{\tau_2 - 2}} + (C_{15})^{\frac{1}{\tau_2 - 2}} + (C_{16})^{\frac{1}{\tau_2 - 2}}, \]

\[ C_{15} = \left( \frac{1}{E_{\alpha,1}(-\lambda_1 T_1^\alpha)} - 1 \right) \left[ 1 + \left( \frac{T_2}{T_1} \right)^{\alpha} + \tau \right], \]

\[ C_{16} = \left( \frac{1}{E_{\alpha,1}(-\lambda_1 T_1^\alpha)} - 1 \right) \left[ 1 + \left( \frac{1 - E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{1 - E_{\alpha,1}(-\lambda_1 T_1^\alpha)} + \tau \right]. \]

**Proof.** By the triangle inequality, we know

\[ \| f_\mu^\delta(\cdot) - f(\cdot) \| \leq \| f_\mu^\delta(\cdot) - f_\mu(\cdot) \| + \| f_\mu(\cdot) - f(\cdot) \| = I_5 + I_6, \]

\[ \| \phi_\mu^\delta(\cdot) - \phi(\cdot) \| \leq \| \phi_\mu^\delta(\cdot) - \phi_\mu(\cdot) \| + \| \phi_\mu(\cdot) - \phi(\cdot) \| = I_7 + I_8. \]

(i) For \( 0 < p < 2\gamma - 1 \), we first estimate \( I_3 \), using (4.9) and Lemma 4.3, we obtain:

\[ I_5 = \| f_\mu^\delta(\cdot) - f_\mu(\cdot) \| \leq \frac{C_3}{\lambda_1} C_{10} \delta \mu^{\frac{1}{\tau_2 - 2}} \leq \frac{C_{10} C_3}{\lambda_1 \tau - 2} E^{\frac{1}{\tau_2 - 2}} \delta^{\frac{1}{\tau_2 - 2}}. \]
Now, we estimate the second item $I_4$, we can deduce that

\[
I_6 = \| f_\mu(\cdot) - f(\cdot) \| \leq \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_k^2 + \mu} \right) \left( \frac{(g_{1,k} - g_{2,k} \sigma_{3k}/\sigma_{4k})}{(\sigma_{1k} - \sigma_{2k} \sigma_{3k}/\sigma_{4k})^2} \right)^{\frac{1}{2(p+1)}} \left( \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_k^2 + \mu} \right) \left( g_{1,k} - g_{2,k} \sigma_{3k}/\sigma_{4k} \right)^2 \right)^{\frac{1}{2}}
\]

where we have used the Hölder inequality, and we also have

\[| \sigma_{1k} - \sigma_{2k} \sigma_{3k}/\sigma_{4k} | = \frac{1}{\lambda_k} \left( \frac{E_{\alpha,1}(\lambda_k T_1^{\alpha})}{E_{\alpha,1}(\lambda_k T_2^{\alpha})} - 1 \right).\]

It is easy to verify that $\frac{E_{\alpha,1}(-T_1^{\alpha} t)}{E_{\alpha,1}(-T_2^{\alpha} t)}$ is a nondecreasing function greater than 1 for any $t > 0$, hence

\[| \sigma_{1k} - \sigma_{2k} \sigma_{3k}/\sigma_{4k} | = \frac{1}{\lambda_k} \left( \frac{E_{\alpha,1}(\lambda_k T_1^{\alpha})}{E_{\alpha,1}(\lambda_k T_2^{\alpha})} - 1 \right) \geq \frac{1}{\lambda_k} \left( \frac{E_{\alpha,1}(\lambda_1 T_1^{\alpha})}{E_{\alpha,1}(\lambda_1 T_2^{\alpha})} - 1 \right),\]
and $\sigma_{3k}/\sigma_{4k} = \frac{E_{0,1}(-T^*_{2} T_{1})}{E_{0,1}(-T^*_{2} T_{1})} \leq \lim_{t \to \infty} \frac{E_{0,1}(-T^*_{2} t)}{E_{0,1}(-T^*_{2} t)} = \left( \frac{T_{2}}{T_{1}} \right)^{\alpha}$, so we obtain:

$$\|f_{\mu}(\cdot) - f(\cdot)\|^2 \leq \left( \sum_{k=1}^{\infty} \frac{1}{(\sigma_{1k} - \sigma_{2k} \sigma_{3k}/\sigma_{4k})^{2p} f_{k}^2} \right)^{\frac{1}{2p}} \left\{ \left( \sum_{k=1}^{\infty} \left( g_{1,k} - g_{2,k} \sigma_{3k}/\sigma_{4k} \right)^2 \right) \right\}^{\frac{1}{2}}$$

$$+ \left( \sum_{k=1}^{\infty} (g_{1,k}^2) + \sum_{k=1}^{\infty} (g_{2,k}^2) (\sigma_{3k}/\sigma_{4k})^2 \right)^{\frac{1}{2p}}$$

$$\leq \left( \frac{\lambda_{k}^{2p} f_{k}^2}{\frac{E_{0,1}(-\lambda_{1} T_{1})}{E_{0,1}(-\lambda_{1} T_{2})} - 1} \right)^{\frac{1}{2p}} \left\{ \left( \sum_{k=1}^{\infty} (g_{1,k} - g_{2,k}^2)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\}^{\frac{1}{2p}}$$

$$+ \left( \sum_{k=1}^{\infty} (\sigma_{3k}/\sigma_{4k})^2 (g_{2,k}^2 - g_{2,k}^2)^2 \right)^{\frac{1}{2p}} + \frac{\tau}{p}$$

$$\leq \left( \frac{1}{\frac{E_{0,1}(-\lambda_{1} T_{1})}{E_{0,1}(-\lambda_{1} T_{2})} - 1} \right)^{\frac{2p}{\alpha}} E^{\frac{2p}{\alpha}} \left[ 1 + \left( \frac{T_{2}}{T_{1}} \right)^{\alpha} + \tau \right]^{\frac{2p}{\alpha}} \delta^{\frac{2p}{\alpha}} \leq (C_{15})^{\frac{2p}{\alpha}} E^{\frac{2p}{\alpha}} \delta^{\frac{2p}{\alpha}}.$$

So,

$$\|f_{\mu}(\cdot) - f(\cdot)\| \leq (C_{15})^{\frac{2p}{\alpha}} E^{\frac{2p}{\alpha}} \delta^{\frac{2p}{\alpha}},$$

where $C_{15} = \frac{1}{\frac{E_{0,1}(-\lambda_{1} T_{1})}{E_{0,1}(-\lambda_{1} T_{2})} - 1} \left[ 1 + \left( \frac{T_{2}}{T_{1}} \right)^{\alpha} + \tau \right].$

Therefore, according (4.19) and the above result we have:

$$\|f_{\mu}^{\delta}(\cdot) - f(\cdot)\| \leq \left( \frac{C_{10} C_{4}}{\lambda_{1}} \left( \frac{C_{11}}{\tau - 2} \right)^{\frac{2p}{\alpha}} + (C_{15})^{\frac{2p}{\alpha}} \right) E^{\frac{2p}{\alpha}} \delta^{\frac{2p}{\alpha}}.$$
By the same calculation, we obtain
\[
\|\phi(\cdot) - \phi_\mu(\cdot)\|^2
\leq \left\| \sum_{k=1}^{\infty} \frac{1}{\sigma_k} \eta_{2,k} \varphi_k - \sum_{k=1}^{\infty} \frac{\sigma_k^{2\gamma-1}}{\sigma_k^{\gamma} + \mu} \eta_{2,k} \varphi_k \right\|^2
\leq \sum_{k=1}^{\infty} \left( \frac{\mu}{\sigma_k^{\gamma} + \mu} \right) \left( \eta_{2,k} \right)^2
\]
\[
\leq \left( \sum_{k=1}^{\infty} \left( \frac{g_{2,k} - g_{1,k} \sigma_{2k} / \sigma_{1k}}{\sigma_{4k} - \sigma_{2k} \sigma_{3k} / \sigma_{1k}} \right)^2 \right)^{\frac{1}{p+1}} \left( \sum_{k=1}^{\infty} \left( g_{2,k} - g_{1,k} \sigma_{2k} / \sigma_{1k} \right)^2 \right)^{\frac{p}{p+1}}
\]
and we have,
\[
|\sigma_{4k} - \sigma_{2k} \sigma_{3k} / \sigma_{1k}| = \frac{E_{a,1}(-\lambda_2 T_1^\gamma) - E_{a,1}(-\lambda_2 T_2^\gamma)}{1 - E_{a,1}(-\lambda_2 T_1^\gamma)}
\]
\[
= \left( \frac{E_{a,1}(-\lambda_2 T_1^\gamma)}{E_{a,1}(-\lambda_2 T_2^\gamma)} - 1 \right) \frac{E_{a,1}(-\lambda_2 T_2^\gamma)}{1 - E_{a,1}(-\lambda_2 T_1^\gamma)}
\]
\[
\geq \frac{C_3}{\lambda_k} \left( \frac{E_{a,1}(-\lambda_1 T_1^\gamma)}{E_{a,1}(-\lambda_1 T_2^\gamma)} - 1 \right),
\]
so, we can obtain
\[
(4.20) \quad \|\phi_\mu(\cdot) - \phi(\cdot)\| \leq (C_{16})^{\frac{p}{p+1}} E^{\frac{p}{p+1}} \delta^{\frac{1}{p+1}},
\]
where \( C_{16} = \left( \frac{1}{\frac{C_3}{\lambda_k} \left( \frac{E_{a,1}(-\lambda_1 T_1^\gamma)}{E_{a,1}(-\lambda_1 T_2^\gamma)} - 1 \right)} \right) \left[ 1 + \left( \frac{1}{1 - E_{a,1}(-\lambda_1 T_1^\gamma)} \right) \right]. \)
By (4.14) and Lemma 4.3,
\[
(4.21) \quad \|\phi_\mu(\cdot) - \phi_\mu^\delta(\cdot)\| \leq \frac{2}{\lambda_1} C_3 \mu^{\frac{1}{\tau}} \delta \leq \frac{2}{\lambda_1} C_3 \left( \frac{C_{11}}{\tau - 2} \right)^{\frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{\frac{1}{\tau}}.
\]
Thus, from (4.20) and (4.21), we have

$$\| \phi_\delta^\mu (\cdot) - \phi(\cdot) \| \leq \left( C_{16} \frac{C_3}{\lambda_1} \left( \frac{C_{11}}{\tau - 2} \right)^{1/\tau} \right) E^{-\frac{1}{\tau}} \delta^{\frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{\frac{1}{\tau}}.$$

So,

(4.22)

$$\| f_\delta^\mu (\cdot) - f(\cdot) \| + \| \phi_\delta^\mu (\cdot) - \phi(\cdot) \| \leq \left( C_3 \frac{C_{10} + 2}{\lambda_1} \left( \frac{C_{11}}{\tau - 2} \right)^{1/\tau} + (C_{15})^{1/\tau} + (C_{16})^{1/\tau} \right) E^{\frac{1}{\tau}} \delta^{\frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{\frac{1}{\tau}}.$$

(ii) For $p \geq 2\gamma - 1$, we first estimate $I_3$, using (4.9) and Lemma 4.3, we obtain:

$$I_5 = \| f_\mu^\delta (\cdot) - f_\mu (\cdot) \| \leq \frac{C_3}{\lambda_1} C_{10} \delta \mu^{-\frac{1}{\tau}} \leq \frac{C_{10} C_3}{\lambda_1} \left( \frac{C_{12}}{\tau - 2} \right)^{\frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{1 - \frac{1}{\tau}}.$$

Then, we estimate the second item $I_4$, the process is similar to that in (i), so we have:

$$I_6 = \| f_\mu (\cdot) - f(\cdot) \| \leq \left( \frac{1}{T_{\alpha,1}(-\lambda_1 T_1^\alpha)} - 1 \right) \left[ 1 + \left( \frac{T_2}{T_1} \right)^\alpha + \tau \right]^{1 - \frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{1 - \frac{1}{\tau}} \leq (C_{15})^{1 - \frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{1 - \frac{1}{\tau}}.$$

Similarly,

$$I_7 = \| \phi_\delta^\mu (\cdot) - \phi_\mu (\cdot) \| \leq \frac{2}{\lambda_1} C_3 \mu^{-\frac{1}{\tau}} \delta \leq \frac{2}{\lambda_1} C_3 \left( \frac{C_{12}}{\tau - 2} \right)^{1/\tau} E^{\frac{1}{\tau}} \delta^{1 - \frac{1}{\tau}},$$

and

$$I_8 = \| \phi_\mu (\cdot) - \phi(\cdot) \| \leq \left( \frac{1}{C_2 T_{\alpha,1}(-\lambda_1 T_1^\alpha)} - 1 \right) \left[ 1 + \frac{1}{1 - E_{\alpha,1}(-\lambda_1 T_1^\alpha) + \tau} \right]^{1 - \frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{1 - \frac{1}{\tau}} \leq (C_{16})^{1 - \frac{1}{\tau}} E^{\frac{1}{\tau}} \delta^{1 - \frac{1}{\tau}}.$$
FRACTIONAL TIKHONOV REGULARIZATION METHOD FOR SIMULTANEOUS INVERSION OF THE SOURCE TERM AND INITIAL DATA IN A TIME-FRACTIONAL DIFFUSION EQUATION

So,

$$\|f_{\mu}^\delta(\cdot) - f(\cdot)\| + \|\phi_{\mu}^\delta(\cdot) - \phi(\cdot)\| \leq \left( C_3 \frac{C_{10} + 2}{\lambda_1} \left( \frac{C_{12}}{\tau - 2} \right)^{\frac{1}{\tau}} + (C_{15})^{1 - \frac{1}{\tau}} + (C_{16})^{1 - \frac{1}{\tau}} \right) E^{\frac{1}{2}} \delta^{1 - \frac{1}{\tau}}.$$ 

5. Numerical examples.

In order to illustrate the stability and effectiveness of the proposed method, we give four numerical examples. The main purpose of this paper is to analyze the numerical error of solving the simultaneous inversion problem by using the fractional Tikhonov method.

The noisy data are generated by adding a random perturbation, i.e.

$$g_i^\delta = g_i + \varepsilon \cdot g_i \cdot (2 \text{randn(size}(g_i)) - 1). \quad (i = 1, 2.)$$

In our computation, the observation times are chosen to be $T_1 = 1/2$ and $T_2 = 1$ for the equation. To measure the accuracy of numerical solution, we use the discrete $L^2$ error as

$$E(f, \varepsilon) = \|f_{\mu}^\delta(\cdot) - f(\cdot)\|,$$
$$E(\phi, \varepsilon) = \|\phi_{\mu}^\delta(\cdot) - \phi(\cdot)\|,$$

and the relative error in $L^2(\Omega)$ norm denoted by

$$\varepsilon_f = \|f_{\mu}^\delta(\cdot) - f(\cdot)\|/\|f(\cdot)\|,$$
$$\varepsilon_\phi = \|\phi_{\mu}^\delta(\cdot) - \phi(\cdot)\|/\|\phi(\cdot)\|,$$

where $\delta$ is the noisy level.

In this case, the regularized solution can be computed with $\lambda_n = n^2$ and $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, for $n = 1, 2, \ldots$. In general, the a-priori bound $E$ is difficult to obtain, so we give the results on the a-posteriori regularization parameter choice rule. For the regularized solution $f_{\mu}^\delta(\cdot)$ and $\phi_{\mu}^\delta$, the regularization parameter is chosen by (4.16) with $\tau = 2.1$.

In the following two examples, when calculating the solution of the regularized solution the number of truncations we take as 10 and 15,
Table 1. Absolute error of regularization methods of Example 1 for different $\varepsilon$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\varepsilon = 0.01%$</th>
<th>$\varepsilon = 0.1%$</th>
<th>$\varepsilon = 0.2%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_f$</td>
<td>0.0185</td>
<td>0.0251</td>
<td>0.0367</td>
</tr>
<tr>
<td>$\varepsilon_\phi$</td>
<td>0.0045</td>
<td>0.0178</td>
<td>0.0347</td>
</tr>
</tbody>
</table>

and truncate 1000 and 500 items for $x$. Then we perform computational analysis using the fractional Tikhonov regularization.

**Example 1.** Consider the smooth heat source and initial value:

$$f(x) = -\sin(\pi x),$$

and

$$\phi(x) = \frac{\pi^2 - 1}{\pi^2} \sin(\pi x) + \sin(2\pi x).$$

In this example, the initial value and the source term are sine functions, we give the exact solution,

$$u(x, t) = E_{1/2,1}(\pi^2 t^{1/2}) \sin(\pi x) + E_{1/2,1}(-4\pi^2 t^{1/2}) \sin(2\pi x) - \frac{\sin(\pi x)}{\pi^2}. $$

The reconstructed solutions $\phi(x)$ and $f(x)$ with exact input data are shown in Fig. 1. We consider input data with various noise levels to test the stability of our algorithm, when $\varepsilon = 0.01\%$, $\varepsilon = 0.1\%$ and $\varepsilon = 0.2\%$. The reconstructed solutions under the a-posteriori conditions are also shown in Fig. 1, in which satisfactory estimated solutions with the noise data are obtained. As can be seen from Table 1, when we fix the time-fractional order $\alpha$, as the noise level increases, the numerical effect becomes worse and worse.

**Example 2.** Consider the respective smooth heat source and the non-smooth initial value:

$$f(x) = 2x(1 - x),$$

and

$$\phi(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2}, \\ (1 - x), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

In this example, it is easy to see that the source term is a quadratic function and the initial value is a continuous but not smooth function
Table 2. Absolute error of regularization methods of Example 2 for different $\varepsilon$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\varepsilon = 0.01%$</th>
<th>$\varepsilon = 0.1%$</th>
<th>$\delta = 0.2%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_f$</td>
<td>0.0460</td>
<td>0.0925</td>
<td>0.1189</td>
</tr>
<tr>
<td>$\varepsilon_\phi$</td>
<td>0.0201</td>
<td>0.0757</td>
<td>0.1210</td>
</tr>
</tbody>
</table>

Table 3. Absolute error of regularization methods of Example 3 for different $\varepsilon$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$\varepsilon = 0$</th>
<th>$\varepsilon = 0.01%$</th>
<th>$\varepsilon = 0.02%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_f$</td>
<td>0.1880</td>
<td>0.1922</td>
<td>0.1974</td>
</tr>
<tr>
<td>$\varepsilon_\phi$</td>
<td>0.2235</td>
<td>0.2429</td>
<td>0.2442</td>
</tr>
</tbody>
</table>

at $x = 1/2$. Because there is a sharp point at $x = 1/2$, it is usually difficult to be reconstructed.

The exact solution and the reconstructed solution of $f(x)$ and $\phi(x)$ from the noisy data $g_1^\delta(x)$ and $g_2^\delta(x)$ with a-posteriori conditions are given in Fig. 2, where $\alpha = 0.5$, $\beta = 1$, and the noise level $\varepsilon$ is taken as $0.01\%$, $0.2\%$ and $0.3\%$, respectively. We can see that the heat source is recovered very well under the a-posterior conditions, but the shape of the unknown initial temperature value is recovered not well, taking into consideration the non-smooth and the ill-posedness of the problem, the result presented is reasonable. By Table 2, we find as the noise level increases, the error of the reconstructed also increases.

**Example 3.** Choose the exact source term and initial value

$$f(x) = \begin{cases} 
1, & 0 \leq x < 0.2, \\
5, & 0.2 \leq x < 0.4, \\
1 + \sin 8\pi x, & 0.4 \leq x < 1,
\end{cases}$$

and

$$\phi(x) = \begin{cases} 
0, & 0 \leq x < 0.2, \\
2, & 0.2 \leq x < 0.4, \\
\sin 8\pi x, & 0.4 \leq x < 1.
\end{cases}$$
Table 4. Absolute error of regularization methods of Example 3 for different $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
<th>0.5</th>
<th>0.65</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_f$</td>
<td>0.1964</td>
<td>0.1942</td>
<td>0.1927</td>
<td>0.1922</td>
<td>0.1913</td>
<td>0.1911</td>
<td>0.1907</td>
</tr>
<tr>
<td>$\epsilon_\phi$</td>
<td>0.2503</td>
<td>0.2443</td>
<td>0.2423</td>
<td>0.2429</td>
<td>0.2560</td>
<td>0.3226</td>
<td>0.4763</td>
</tr>
</tbody>
</table>

In this example, we truncate 20 terms for $x$, 1000 terms for the sum term $k$. Fig. 3 present the exact solution and the regularization solution for various noise levels $\varepsilon = 0, 0.01\%, 0.02\%$ in case of $\alpha = 0.5$, $T_1 = 1/2$, $T_2 = 1$. And the regularization parameters are $\mu_1 = 1 \times 10^{-3}$, $\mu_2 = 5 \times 10^{-3}$ and $\mu_3 = 9.5 \times 10^{-3}$, respectively. By Table 3, when the time-fractional order $\alpha$ is a fixed constant, as the noise level increases, the error of the reconstructed also increases. In Table 4, when we fix the noise level to 0.01%, as the $\alpha$ increases, the error of the source term getting smaller and smaller, but there are no rules for the error of the initial value.

**Example 4.** Consider a discontinuous exact solution which does not satisfy the compatible conditions $f(0) = f(\pi) = 0$ and $\phi(0) = \phi(\pi) = 0$:

$$f(x) = \begin{cases} 
-1, & 0 \leq x < \frac{\pi}{4}, \\
1, & \frac{\pi}{4} \leq x < \frac{3\pi}{4}, \\
-1, & \frac{3\pi}{4} \leq x < \pi,
\end{cases}$$

and

$$\phi(x) = \begin{cases} 
-2, & 0 \leq x < \frac{\pi}{4}, \\
2, & \frac{\pi}{4} \leq x < \frac{3\pi}{4}, \\
-2, & \frac{3\pi}{4} \leq x < \pi,
\end{cases}$$

In this example, we truncate 15 terms for $x$, 100 terms for the sum term $k$. Fig. 4 and Fig. 5 present the exact solution and the regularization solution for various noise levels in case of $\alpha = 0.65$, $T_1 = 1/2$, $T_2 = 1$. We can see that the fractional Tikhonov method is better than the classical Tikhonov method with the condition of
the same parameters, but the classical Tikhonov regularized solution oversmooths.

From the numerical experiments of Examples 1 – 4, we can find that the smaller $\varepsilon$ is, the better the fitting effect between the exact solution and the regularization solution is. Moreover, numerical examples verify the validity and accuracy of the proposed method.

6. Conclusion.

In this paper, we have proposed the fractional Tikhonov regularization method to identify an unknown source term $f$ and unknown initial condition $\phi$ in a class of inverse boundary value problems of time-fractional diffusion equation (1.1). Our measurements are derived from the additional temperature at two fixed times. This inverse problem is reformulated as the first kind operator equation based on the Fourier method. Using the fractional Tikhonov regularization method, the solution of the problem of the corresponding conjugate operator equation is constructed. Error estimates between the exact solution and the regularized solution are shown and proved. Compared with the standard Tikhonov regularization method, four numerical examples are conducted for showing the effectiveness of our proposed method.

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FRACTIONAL TIKHONOV REGULARIZATION METHOD FOR SIMULTANEOUS INVERSION OF THE SOURCE TERM AND INITIAL DATA IN A TIME-FRACTIONAL DIFFUSION EQUATION

Figure 1. Numerical results for source term and initial value at different noise levels in Example 1.

Figure 2. Numerical results for source term and initial value at different noise levels in Example 2.
Figure 3. Numerical results for source term and initial value at different noise levels in Example 3.

Figure 4. Numerical results for source term at different noise levels in Example 4.
(a) initial value of the fractional Tikhonov method
(b) initial value of the classical Tikhonov method

Figure 5. Numerical results for initial value at different noise levels in Example 4.