Infinite interval problems for Hilfer fractional evolution equations with almost sectorial operators

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Abstract
The purpose of this paper is to investigate the Cauchy problem on an infinite interval for the fractional evolution equation with Hilfer fractional derivative, which is a generalization of both Riemann-Liouville and Caputo fractional derivatives. Our methods base on the generalized Ascoli-Arzela theorem, Schauder’s fixed point theorem, Wright function and Kuratowski’s measure of noncompactness. We obtain sufficient conditions of the existence for global mild solutions and attractive solutions when the semigroup associated with an almost sectorial operator is compact as well as noncompact. At last, two examples are provided to illustrate the results.

Keywords: Fractional evolution equations; Hilfer derivative; almost sectorial operator; infinite interval.

2010 MSC: 26A33; 34A08; 35R11.

1 Introduction
Fractional calculus is considered as a generalization of classical calculus. The order of fractional derivative can be an arbitrary (noninteger) positive real number or even complex number. In the past two decades, fractional calculus has been a research focus and attracted the attention of many researchers all over the world. On the one hand, it is due to the further development of fractional calculus theory itself. More than that, fractional calculus is more and more widely used in various disciplines, especially in fluid mechanics, physics, signal processing, materials science, electrochemistry, biology and so on.

In recent years, fractional differential equations are widely used in the mathematical modeling of real-world phenomena. These applications have motivated many researchers
in the field of differential equations to investigate fractional differential equations with different fractional order derivatives, see the monographs [1–6] and the recent references.

The main motivation of studying fractional evolution equation comes from two aspects. One is that many mathematical models in physics and fluid mechanics are characterized by fractional partial differential equations; Second, many types of fractional partial differential equations, such as fractional diffusion equations, wave equations, Navier-Stokes equations, Rayleigh-Stokes equations, Fokker-Planck equations, fractional Schrödinger equations, and so on, can be abstracted as fractional evolution equations [7–9]. Therefore, the study of fractional evolution equations is of great significance both in terms of theory and practical application.

The well-posedness of fractional evolution equations is an important research topic of evolution equations. However, it seems that there are few works concerned with the existence of fractional evolution equations on infinite intervals. Almost all of these results involve the existence of solutions for fractional evolution equations on a finite interval $[0, T]$, where $T \in (0, \infty)$ (see [12–17]).

Consider the initial value problem of fractional evolution equations on infinite interval

\[
\begin{align*}
H^\mu_{0+} y(t) &= Ay(t) + G(t, y(t)), \quad t \in (0, \infty), \\
I^{(1-\mu)(1-\beta)}_{0+} y(0) &= y_0,
\end{align*}
\]  

where $H^\mu_{0+}$ is the Hilfer fractional derivative of order $0 < \beta < 1$ and type $0 \leq \mu \leq 1$, $I^{(1-\mu)(1-\beta)}_{0+}$ is Riemann-Liouville integral of order $(1-\mu)(1-\beta)$, $A$ is an almost sectorial operator in Banach space $X$, $G : [0, \infty) \times X \to X$ is a function which is defined later.

The Hilfer fractional derivative is a natural generalization of Caputo derivative and Riemann-Liouville derivative [2]. It is obvious that fractional differential equations with Hilfer derivatives include fractional differential equations with Riemann-Liouville derivative or Caputo derivative as special cases.

In this paper, we prove existence theorems of mild solutions for the infinite interval problem (1.1) for two cases that the semigroup associated with the almost sectorial operator is compact as well as noncompact. And especially, we do not need to assume that the $G(t, \cdot)$ satisfies the Lipschitz condition. The considerations of this paper are based on the generalization of Ascoli-Arzela theorem on infinite interval, Schauder’s fixed point theorem and the concept of a measure of noncompactness.

2 Preliminaries

We first introduce some notations and definitions about almost sectorial operators, fractional calculus and solution operators. For more details, we refer to [2, 3, 17, 18].
Let \( \text{Hilfer fractional derivative, see [2]} \). 

Remark 2.1. 

Definition 2.2. (see [3]). The fractional integral of order \( \beta \) for a function \( x : [0, \infty) \rightarrow \mathbb{R} \) is defined as 

\[
I_{0+}^{\beta}x(t) = \frac{1}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta-1} x(s) ds, \quad \beta > 0, \ t > 0,
\]

provided the right side is point-wise defined on \([0, \infty)\), where \( \Gamma(\cdot) \) is the gamma function.

Definition 2.3. (Hilfer fractional derivative, see [2]). Let \( 0 < \beta < 1 \) and \( 0 \leq \mu \leq 1 \). The Hilfer fractional derivative of order \( \beta \) and type \( \mu \) for a function \( x : [0, \infty) \rightarrow \mathbb{R} \) is defined as 

\[
^{H}D_{0+}^{\beta, \mu}x(t) = I_{0+}^{\mu(1-\beta)} \frac{d}{dt} I_{0+}^{(1-\mu)(1-\beta)} x(t).
\]

Remark 2.1. (i) In particular, when \( \mu = 0, 0 < \beta < 1 \), then 

\[
^{H}D_{0+}^{0, \beta}x(t) = \frac{d}{dt} I_{0+}^{1-\beta} x(t) =: t^{\beta} D_{0+}^{\beta} x(t),
\]
where $L_{0+}^\beta$ is Riemann-Liouville derivative.

(ii) When $\mu = 1, \ 0 < \beta < 1$, we have

$$H_{0+}^{1,\beta}x(t) = \textit{I}_{0+}^{1-\beta} \frac{d}{dt} x(t) =: C_{0+}^\beta x(t),$$

where $C_{0+}^\beta$ is Caputo derivative.

Let $D$ be a nonempty subset of $X$. The Kuratowski’s measure of noncompactness $\chi$ is said to be:

$$\chi(D) = \inf \left\{ d > 0 : D \subset \bigcup_{j=1}^n M_j \text{ and } \text{diam}(M_j) \leq d \right\},$$

where the diameter of $M_j$ is given by $\text{diam}(M_j) = \sup\{\|x - y\| : x, y \in M_j\}, j = 1, \ldots, n$.

**Lemma 2.2.** [19] Let $\{u_n(t)\}_{n=1}^\infty : [0, \infty) \rightarrow X$ be a continuous function family. If there exists $\xi \in L^1[0, \infty)$ such that

$$|u_n(t)| \leq \xi(t), \quad t \in [0, \infty), \ n = 1, 2, \ldots.$$

Then $\chi(\{u_n(t)\}_{n=1}^\infty)$ is integrable on $[0, \infty)$, and

$$\chi\left(\left\{ \int_0^t u_n(t) dt : n = 1, 2, \ldots \right\}\right) \leq 2 \int_0^t \chi(\{u_n(t) : n = 1, 2, \ldots\}) dt.$$

**Definition 2.4.** [20] Define Wright function $W_\beta(\theta)$ by

$$W_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1 - n\beta)}, \quad 0 < \beta < 1, \ \theta \in \mathbb{C},$$

with the following property

$$\int_0^{\infty} \theta^\delta W_\beta(\theta) d\theta = \frac{\Gamma(1 + \delta)}{\Gamma(1 + \beta \delta)}, \quad \text{for } \delta \geq 0.$$

**Lemma 2.3.** The problem (1.1) is equivalent to the integral equation

$$y(t) = \frac{y_0}{\Gamma(\mu(1 - \beta) + \beta)} t^{(\mu - 1)(1 - \beta)} + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}[Ay(s) + G(s, y(s))] ds, \quad t \in (0, \infty).$$

(2.1)

**Proof.** This proof is similar to [21], so we omit it. \qed

**Lemma 2.4.** Assume that $y(t)$ satisfies integral equation (2.1). Then

$$y(t) = K_{\mu, \beta}(t)y_0 + \int_0^t Q_\beta(t - s)G(s, y(s)) ds, \quad t \in (0, \infty),$$

(2.2)

where

$$K_{\mu, \beta}(t) = \textit{I}_{0+}^{\mu(1 - \beta)} Q_\beta(t), \quad Q_\beta(t) = t^{\beta - 1} \mathcal{T}_\beta(t), \quad \text{and } \mathcal{T}_\beta(t) = \int_0^{\infty} \beta \theta W_\beta(\theta) Q(t^\beta \theta) d\theta.$$
This proof is similar to [12], so we omit it.

In view of Lemma 2.3, we have the following definition.

**Definition 2.5.** If \( y \in C((0, \infty), X) \) satisfies
\[
y(t) = K_{\mu, \beta}(t)y_0 + \int_0^t Q_\beta(t-s)G(s, y(s))ds, \quad t \in (0, \infty),
\]
Then \( u(t) \) is called a mild solution of the initial value problem (1.1).

**Definition 2.6.** The mild solution \( y(t) \) of the initial value problem (1.1) is called attractive if \( y(t) \to 0 \) as \( t \to \infty \).

**Lemma 2.5.** [5, 12] If \( Q(t) \) is a compact operator for any \( t > 0 \), then \( K_{\mu, \beta}(t) \) and \( T_\beta(t) \) are also compact operators for every \( t > 0 \).

**Lemma 2.6.** [13] For any fixed \( t > 0 \), \( \{T_\beta(t)\}_{t>0} \), \( \{Q_\beta(t)\}_{t>0} \) and \( \{K_{\mu, \beta}(t)\}_{t>0} \) are linear operators, and for any \( y \in X \)
\[
|T_\beta(t)y| \leq L_1 t^{\beta(k-1)}|y|, \quad |Q_\beta(t)y| \leq L_1 t^{\beta k-1}|y|, \quad \text{and} \quad |K_{\mu, \beta}(t)y| \leq L_2 t^{-1+\mu-\beta \mu + \beta k}|y|,
\]
where
\[
L_1 = \frac{\beta C_0 \Gamma(1+k)}{\Gamma(1+\beta k)} \quad \text{and} \quad L_2 = \frac{L_1 \Gamma(\beta k)}{\Gamma(1-\beta + \beta k)}.
\]

**Lemma 2.7.** [13] \( \{T_\beta(t)\}_{t>0}, \{Q_\beta(t)\}_{t>0} \) and \( \{K_{\mu, \beta}(t)\}_{t>0} \) are strongly continuous, that is, for \( \forall \ y \in X \) and \( t'' > t' > 0 \),
\[
|T_\beta(t'y) - T_\beta(t''y)| \to 0, \quad |Q_\beta(t'y) - Q_\beta(t''y)| \to 0 \quad \text{and} \quad |K_{\mu, \beta}(t'y) - K_{\mu, \beta}(t''y)| \to 0, \quad \text{as} \ t'' \to t'.
\]
Let
\[
E = \{u \in C([0, \infty), X) : \lim_{t \to \infty} \frac{|u(t)|}{1+t} = 0\}.
\]
Clearly, \((E, \| \cdot \|)\) is a Banach space with the norm \( \|u\| = \sup_{t \in [0, \infty)} |u(t)|/(1+t) < \infty \).

In the following, we state the generalization of Ascoli-Arzela theorem.

**Lemma 2.8.** [22] The set \( \Lambda \subset E \) is relatively compact if and only if the following conditions hold:

(a) for any \( h > 0 \), the set \( V = \{v : v(t) = x(t)/(1+t), x \in \Lambda\} \) is equicontinuous on \([0, h]\);

(b) \( \lim_{t \to \infty} |x(t)|/(1+t) = 0 \) uniformly for \( x \in \Lambda \);

(c) for any \( t \in [0, \infty) \), \( V(t) = \{v(t) : v(t) = x(t)/(1+t), x \in \Lambda\} \) is relatively compact in \( X \).
3 Some Lemmas

First, we introduce the following hypotheses:

(H1) for each \( t \in [0, \infty) \), the function \( G(t, \cdot) : X \to X \) is continuous and for each \( y \in X \), the function \( G(\cdot, y) : [0, \infty) \to X \) is strongly measurable.

(H2) there exists a function \( m : (0, \infty) \to (0, \infty) \) such that

\[
I_0^{\beta_k}m(t) \in C((0, \infty), (0, \infty)), \quad |G(t, y)| \leq m(t), \quad \text{for all } y \in X, \ t \in (0, \infty),
\]
and

\[
\lim_{t \to 0^+} t^{1-\mu+\beta_k}I_0^{\beta_k}m(t) = 0, \quad \lim_{t \to \infty} \frac{t^{1-\mu+\beta_k}}{1+t}I_0^{\beta_k}m(t) = 0.
\]

In this paper, we let

\[
C_\beta((0, \infty), X) = \left\{ y \in C((0, \infty), X) : \lim_{t \to 0^+} t^{1-\mu+\beta_k}y(t) \text{ exists and is finite, and } \lim_{t \to \infty} \frac{t^{1-\mu+\beta_k}}{1+t}|y(t)| = 0 \right\},
\]
with the norm

\[
\|y\|_\beta = \sup_{t \in [0, \infty)} \frac{t^{1-\mu+\beta_k}|y(t)|}{1+t}.
\]

Then \( C_\beta((0, \infty), X), \| \cdot \|_\beta \) is a Banach space (see Lemma 3.2 of [23]).

For any \( y \in C_\beta((0, \infty), X) \), define the mapping \( \Psi \) by

\[
(\Psi y)(t) = (\Psi_1 y)(t) + (\Psi_2 y)(t),
\]
where

\[
(\Psi_1 y)(t) = K_{\mu, \beta}(t)y_0, \quad (\Psi_2 y)(t) = \int_0^t Q_\beta(t-s)G(s, y(s))ds, \quad \text{for } t \in (0, \infty).
\]

Clearly, the problem (1.1) has a mild solution \( y^* \in C_\beta((0, \infty), X) \) if and only if \( \Psi \) has a fixed point \( y^* \in C_\beta((0, \infty), X) \).

Let

\[
y(t) = t^{-1-\mu+\beta_k}u(t), \quad \text{for any } u \in E, t \in (0, \infty).
\]

Clearly, \( y \in C_\beta((0, \infty), X) \). Define an operator \( \Phi \) by

\[
(\Phi u)(t) = (\Phi_1 u)(t) + (\Phi_2 u)(t),
\]
where

\[
(\Phi_1 u)(t) = \begin{cases} t^{1-\mu+\beta_k}(\Psi_1 y)(t), & \text{for } t \in (0, \infty), \\ 0, & \text{for } t = 0, \end{cases}
\]
\[
(\Phi_2 u)(t) = \begin{cases} 
  t^{1-\mu+\beta \mu-\beta k}(\Psi_2 y)(t), & \text{for } t \in (0, \infty), \\
  0, & \text{for } t = 0.
\end{cases}
\]

In view of (H2), there exists a constant \( r > 0 \) such that

\[
\sup_{t \in [0, \infty)} \left\{ L_2 \frac{|y_0|}{1 + t} + L_1 \frac{t^{1-\mu+\beta \mu-\beta k}}{1 + t} \int_0^t (t-s)^{\beta_{k-1} m(s)} ds \right\} \leq r. \tag{3.1}
\]

Let

\[
\Omega = \{ u \in E : \|u\| \leq r \}, \quad \widetilde{\Omega} = \{ y \in C_{\beta}((0, \infty), X) : \|y\|_{\beta} \leq r \}. \tag{3.2}
\]

Clearly, \( \Omega \) is a nonempty, convex and closed subset of \( E \), and \( \widetilde{\Omega} \) is a nonempty, convex and closed subset of \( C_{\beta}((0, \infty), X) \).

Let

\[
V := \{ v : v(t) = (\Phi u)(t)/(1 + t), \ u \in \Omega \}.
\]

To prove the results in this paper we need the following lemmas.

**Lemma 3.1.** Suppose that (H1) and (H2) hold. Then, the set \( V \) is equicontinuous.

**Proof.** **Step I.** We first prove that \( \{ v : v(t) = (\Phi_1 u)(t)/(1 + t), \ u \in \Omega \} \) is equicontinuous.

Since

\[
t^{1-\mu+\beta \mu-\beta k}K_{\mu,\beta}(t)y_0 = \frac{t^{1-\mu+\beta \mu-\beta k}}{\Gamma(\mu(1-\beta))} \int_0^t (t-s)^{\mu(1-\beta)-1}s^{\beta-1}T_{\beta}(s)y_0 ds
\]

\[
= \int_0^1 (1-z)^{\mu(1-\beta)-1}z^{\beta-1}T_{\beta}(t)z y_0 dz.
\]

Noting that \( \lim_{t \to 0^+} t^{\beta(1-k)}T_{\beta}(t)z y_0 = 0 \) and \( \int_0^1 (1-z)^{\mu(1-\beta)-1}z^{\beta-1} dz \) exists, we have

\[
\lim_{t \to 0^+} t^{1-\mu+\beta \mu-\beta k}K_{\mu,\beta}(t)y_0 = 0.
\]

Hence, for \( t_1 = 0, \ t_2 \in (0, \infty) \), we obtain

\[
\left| \frac{(\Phi_1 u)(t_2)}{1 + t_2} - (\Phi_1 u)(0) \right| \leq \left| \frac{1}{1 + t_2} t_2^{1-\mu+\beta \mu-\beta k}K_{\mu,\beta}(t_2)y_0 - 0 \right| \to 0, \quad \text{as } t_2 \to 0. \tag{3.3}
\]
For any $t_1, t_2 \in (0, \infty)$ and $t_1 < t_2$, we have

\[
\left| \frac{(\Phi_1 u)(t_2)}{1 + t_2} - \frac{(\Phi_1 u)(t_1)}{1 + t_1} \right| \\
\leq \left| \frac{t_2^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_2)y_0}{1 + t_2} - \frac{t_1^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_1)y_0}{1 + t_1} \right| \\
\leq \left| \frac{t_2^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_2)y_0}{1 + t_2} - \frac{t_2^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_2)y_0}{1 + t_1} \right| \\
+ \left| \frac{t_2^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_2)y_0}{1 + t_1} - \frac{t_1^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_1)y_0}{1 + t_1} \right| \\
\leq t_2^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_2)y_0 \left| \frac{t_2 - t_1}{(1 + t_2)(1 + t_1)} \right| \\
+ \left| t_2^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_2)y_0 - t_1^{1-\mu+\beta-\delta k}K_{\mu,\beta}(t_1)y_0 \right| \frac{1}{1 + t_1}
\]

(3.4)

\[
\rightarrow 0, \text{ as } t_2 \to t_1.
\]

Hence, \{ \{ v : v(t) = (\Phi_1 u)(t)/(1 + t), u \in \Omega \} \} is equicontinuous.

**Step II.** We prove that \{ \{ v : v(t) = (\Phi_2 u)(t)/(1 + t), u \in \Omega \} \} is equicontinuous.

Let \( y(t) = t^{1-\mu+\beta-\delta k}u(t) \), for any \( u \in \Omega, t \in (0, \infty) \). Then \( y \in \bar{\Omega} \).

By (H2), for \( \varepsilon > 0 \), there exists \( T > 0 \) such that

\[
L_1 \frac{t^{1-\mu+\beta-\delta k}}{1 + t} \int_0^t (t - s)^{\delta k - 1}m(s)ds < \frac{\varepsilon}{2}, \text{ for } t > T.
\]

(3.5)

For \( t_1, t_2 > T \), in virtue of (3.5), we get

\[
\left| \frac{(\Phi_2 u)(t_2)}{1 + t_2} - \frac{(\Phi_2 u)(t_1)}{1 + t_1} \right| \\
\leq \left| \frac{t_2^{1-\mu+\beta-\delta k}}{1 + t_2} \int_0^t Q_\beta(t_2 - s)G(s, y(s))ds \right| \\
+ \left| \frac{t_1^{1-\mu+\beta-\delta k}}{1 + t_1} \int_0^t Q_\beta(t_1 - s)G(s, y(s))ds \right| \\
\leq L_1 t_2^{1-\mu+\beta-\delta k} \int_0^t (t_2 - s)^{\delta k - 1}m(s)ds \\
+ \frac{L_1 t_1^{1-\mu+\beta-\delta k}}{1 + t_1} \int_0^t (t_1 - s)^{\delta k - 1}m(s)ds < \varepsilon.
\]

(3.6)
When $t_1 = 0$, $0 < t_2 < T$, we have

$$
\left| \frac{\Phi_2 u(t_2)}{1 + t_2} - \frac{\Phi_2 u(0)}{1 + t_2} \right| = \left| \frac{t_2^{1-\mu + \beta \mu - \beta k}}{1 + t_2} \int_0^{t_2} \mathcal{Q}_\beta(t_2 - s)G(s, y(s))ds \right|
$$

$$
\leq \frac{L_1 t_2^{1-\mu + \beta \mu - \beta k}}{1 + t_2} \int_0^{t_2} (t_2 - s)^{\beta k - 1}m(s)ds
$$

(3.7)

$$
\to 0 \quad \text{as } t_2 \to 0.
$$

When $0 < t_1 < t_2 \leq T$, we get

$$
\left| \frac{\Phi_2 u(t_2)}{1 + t_2} - \frac{\Phi_2 u(t_1)}{1 + t_1} \right|
$$

$$
\leq \left| \frac{t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \int_0^{t_1} (t_2 - s)^{\beta - 1}T_\beta(t_2 - s)G(s, y(s))ds \right|
$$

$$
+ \left| \frac{t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \int_0^{t_1} ((t_2 - s)^{\beta - 1} - (t_1 - s)^{\beta - 1})T_\beta(t_2 - s)G(s, y(s))ds \right|
$$

$$
+ \left| \frac{t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \int_0^{t_1} (t_1 - s)^{\beta - 1}(T_\beta(t_2 - s) - T_\beta(t_1 - s))G(s, y(s))ds \right|
$$

$$
+ \left| \frac{t_2^{1-\mu + \beta \mu - \beta k}}{1 + t_2} - \frac{t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \right| \int_0^{t_2} (t_2 - s)^{\beta - 1}T_\beta(t_2 - s)G(s, y(s))ds
$$

$$
\leq J_1 + J_2 + J_3 + J_4,
$$

where

\[
J_1 = \frac{L_1 t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \int_0^{t_1} (t_2 - s)^{\beta - 1}m(s)ds - \int_0^{t_1} (t_1 - s)^{\beta k - 1}m(s)ds,
\]

\[
J_2 = 2L_1 t_1^{1-\mu + \beta \mu - \beta k} \int_0^{t_1} ((t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1})(t_2 - s)^{\beta(k - 1)}m(s)ds,
\]

\[
J_3 = \frac{t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \int_0^{t_1} (t_1 - s)^{\beta - 1}(T_\beta(t_2 - s) - T_\beta(t_1 - s))G(s, y(s))ds,
\]

\[
J_4 = \frac{t_2^{1-\mu + \beta \mu - \beta k}}{1 + t_2} - \frac{t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \int_0^{t_1} (t_2 - s)^{\beta - 1}T_\beta(t_2 - s)G(s, y(s))ds.
\]

One can deduce that $\lim_{t_2 \to t_1} J_1 = 0$, since $\mathcal{T}_\beta m \in C((0, \infty), (0, \infty))$. Noting that $((t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1})(t_2 - s)^{\beta(k - 1)}m(s) \leq (t_1 - s)^{\beta k - 1}m(s)$, for $s \in [0, t_1)$, then Lebesgue dominated convergence theorem implies

$$
\int_0^{t_1} ((t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1})(t_2 - s)^{\beta(k - 1)}m(s)ds \to 0, \quad \text{as } t_2 \to t_1,
$$

so, $J_2 \to 0$ as $t_2 \to t_1$.

By (H2), for $\epsilon > 0$, we have

$$
J_3 \leq \frac{t_1^{1-\mu + \beta \mu - \beta k}}{1 + t_1} \int_0^{t_1 - \epsilon} (t_1 - s)^{\beta - 1} \| T_\beta(t_2 - s) - T_\beta(t_1 - s) \|_{\mathcal{L}(X)} G(s, y(s))ds.
$$
Assume that (H1) and (H2) hold. Then, implies that

Proof. Lemma 3.2. Therefore, where

Consequently, For 0 < t < T, if t → t1, then t2 → T and t1 → T. Thus, for u ∈ Ω

Consequently,

Therefore, \( \{v : v(t) = (\Phi_2u)(t)/(1 + t), u \in \Omega\} \) is equicontinuous. Furthermore, V is equicontinuous.

Lemma 3.2. Assume that (H1) and (H2) hold. Then, \( \lim_{t \to \infty} |(\Phi u)(t)|(1 + t) = 0 \) uniformly for any \( u \in \Omega \).

Proof. In fact, for any \( u \in \Omega \), by (H2) and Lemma 2.6, we get

Then, we derive

\[
\frac{|(\Phi u)(t)|}{1 + t} \leq L_2 \frac{|y_0|}{1 + t} + L_1 \frac{t^{1-\mu+\beta+\beta k}}{1 + t} \int_0^t (t-s)^{\beta k-1}m(s)ds. \quad (3.11)
\]
Indeed, let \( \sup \) Suppose that (H1) and (H2) hold. Then \( \Phi \) and \( H \) have

\[
\| \Phi(t) \| \leq L_2 \frac{|y(t)|}{1 + t} + L_1 \frac{t^{1-\mu+\beta \mu-\beta \kappa}}{1 + t} \int_0^t (t-s)^{\beta \kappa - 1} m(s) ds \leq r.
\]

(3.13)

For \( t > 0 \), \( |\Phi(0)| = 0 < r \). Therefore, \( \Phi \) is continuous. The proof is completed. \( \square \)

**Lemma 3.3.** Assume that (H1) and (H2) hold. Then \( \Phi \Omega \subset \Omega \).

**Proof.** From Lemmas 3.1 and 3.2, we know that \( \Phi \Omega \subset E \). For \( t > 0 \) and any \( u \in \Omega \), we have

\[
\| \Phi(t) \| \leq L_2 \frac{|y(t)|}{1 + t} + L_1 \frac{t^{1-\mu+\beta \mu-\beta \kappa}}{1 + t} \int_0^t (t-s)^{\beta \kappa - 1} m(s) ds \leq r.
\]

(3.13)

For \( t = 0 \), \( |\Phi(0)| = 0 < r \). Therefore, \( \Phi \Omega \subset \Omega \).

**Lemma 3.4.** Suppose that (H1) and (H2) hold. Then \( \Phi \) is continuous.

**Proof.** Indeed, let \( \{u_n\}_{n=1}^\infty \) be a sequence in \( \Omega \) which is convergent to \( u \in \Omega \). Consequently,

\[
\lim_{n \to \infty} u_n(t) = u(t), \quad \text{and} \quad \lim_{n \to \infty} t^{1-\mu+\beta \mu-\beta \kappa} u_n(t) = t^{1-\mu+\beta \mu-\beta \kappa} u(t), \quad \text{for} \ t \in (0, \infty).
\]

Let \( y(t) = t^{1-\mu+\beta \mu-\beta \kappa} u(t), \ y_n(t) = t^{1-\mu+\beta \mu-\beta \kappa} u_n(t) \ t \in (0, \infty) \). Then \( y, y_n \in \tilde{\Omega} \). In view of (H1), we have

\[
\lim_{n \to \infty} G(t, y_n(t)) = \lim_{n \to \infty} G(t, t^{1-\mu+\beta \mu-\beta \kappa} u_n(t)) = G(t, t^{1-\mu+\beta \mu-\beta \kappa} u(t)) = G(t, y(t)).
\]

For any \( \varepsilon > 0 \), there exists \( T > 0 \) such that (3.5) holds. Thus, for \( t > T \),

\[
\left| \frac{(\Phi u_n)(t)}{1 + t} - \frac{(\Phi u)(t)}{1 + t} \right| \leq 2L_1 \frac{t^{1-\mu+\beta \mu-\beta \kappa}}{1 + t} \int_0^t (t-s)^{\beta \kappa - 1} m(s) ds < \varepsilon,
\]

(3.14)

which implies that \( \| \Phi u_n - \Phi u \| \to 0 \) as \( n \to \infty \).

For each \( t \in (0, T], \ (t-s)^{\beta \kappa - 1} |G(s, y_n(s)) - G(s, y(s))| \leq 2(t-s)^{\beta \kappa - 1} m(s) \). By Lebesgue dominated convergence theorem, we obtain

\[
\int_0^t (t-s)^{\beta \kappa - 1} |G(s, y_n(s)) - G(s, y(s))| ds \to 0, \quad \text{as} \ n \to \infty.
\]

Thus, for \( t \in [0, T] \),

\[
\left| \frac{(\Phi u_n)(t)}{1 + t} - \frac{(\Phi u)(t)}{1 + t} \right| \leq \frac{t^{1-\mu+\beta \mu-\beta \kappa}}{1 + t} \int_0^t |Q_\beta(t-s)(G(s, y_n(s)) - G(s, y(s)))| ds \leq L_1 \frac{t^{1-\mu+\beta \mu-\beta \kappa}}{1 + t} \int_0^t (t-s)^{\beta \kappa - 1} |G(s, y_n(s)) - G(s, y(s))| ds \to 0, \quad \text{as} \ n \to \infty.
\]

So, \( \| \Phi u_n - \Phi u \| \to 0 \) as \( n \to \infty \). Hence, \( \Phi \) is continuous. The proof is completed. \( \square \)
4 Main Results

Theorem 4.1. Suppose that \( Q(t) \) is compact for \( t > 0 \). Further assume that (H1) and (H2) hold. Then there is at least one mild solution for the initial value problem (1.1).

Proof. Clearly, the problem (1.1) exists a mild solution \( y \in \Omega \) if and only if the operator \( \Phi \) has a fixed point \( u \in \Omega \), where \( u(t) = t^{1-\mu-\beta k}y(t) \). Hence, we only need to prove that the operator \( \Phi \) has a fixed point in \( \Omega \). From Lemmas 3.3 and 3.4, we know that \( \Phi \Omega \subset \Omega \) and \( \Phi \) is continuous. In order to prove that \( \Phi \) is a completely continuous operator, we need to prove that \( \Phi \Omega \) is a relatively compact set. In view of Lemmas 3.1 and 3.2, the set \( V = \{ v : v(t) = (\Phi u)(t)/(1 + t), u \in \Omega \} \) is equicontinuous and \( \lim_{t \to \infty} |(\Phi u)(t)/(1 + t)| = 0 \) uniformly for \( u \in \Omega \). According to Lemma 2.8, we only need to prove \( V(t) = \{ v(t) : v(t) = (\Phi u)(t)/(1 + t), u \in \Omega \} \) is relatively compact in \( X \) for \( t \in [0, \infty) \). Clearly, \( V(0) \) is relatively compact in \( X \). We only consider the case \( t > 0 \). For \( \forall \varepsilon \in (0, t) \) and \( \delta > 0 \), define \( \Phi_{\varepsilon, \delta} \) on \( \Omega \) as follow

\[
(\Phi_{\varepsilon, \delta} u)(t) := t^{1-\mu-\beta k}(\Phi_{\varepsilon, \delta} y)(t)
\]

Thus,

\[
\frac{(\Phi u)(t)}{1 + t} = \frac{t^{1-\mu-\beta k}}{1 + t} \left( K_{\mu, \beta}(t)y_0 + \int_0^{t-\varepsilon} \int_{\delta}^{\infty} \beta \theta(t - s)^{\beta - 1} W_{\beta}(\theta) Q((t - s)^{\beta \theta}) G(s, y(s)) d\theta ds \right).
\]

By Theorem 3 of [13], we know that \( K_{\mu, \beta}(t) \) is compact because \( Q(t) \) is compact for \( t > 0 \). Further, \( Q(\varepsilon \delta) \) is compact, then the set \( \{ \frac{(\Phi_{\varepsilon, \delta} u)(t)}{1 + t}, u \in \Omega \} \) is relatively compact in \( X \) for any \( \varepsilon \in (0, t) \) and for any \( \delta > 0 \). Moreover, for every \( u \in \Omega \), we find

\[
\left| \frac{(\Phi u)(t)}{1 + t} - \frac{(\Phi_{\varepsilon, \delta} u)(t)}{1 + t} \right| \leq \frac{t^{1-\mu-\beta k}}{1 + t} \left| \int_0^t \beta \theta(t - s)^{\beta - 1} W_{\beta}(\theta) Q((t - s)^{\beta \theta}) G(s, y(s)) d\theta ds \right|
\]

\[
+ \frac{t^{1-\mu-\beta k}}{1 + t} \left| \int_{t-\varepsilon}^t \int_{\delta}^{\infty} \beta \theta(t - s)^{\beta - 1} W_{\beta}(\theta) Q((t - s)^{\beta \theta}) G(s, y(s)) d\theta ds \right|
\]

\[
\leq \beta C_0 t^{1-\mu-\beta k} \int_0^t (s - \varepsilon)^{\beta k - 1} G(s, y(s)) ds \int_{\delta}^{\infty} \frac{\theta^k W_{\beta}(\theta) d\theta}{1 + t}
\]

\[
+ \beta C_0 t^{1-\mu-\beta k} \int_{t-\varepsilon}^t (s - \varepsilon)^{\beta k - 1} G(s, y(s)) ds \int_{\delta}^{\infty} \frac{\theta^k W_{\beta}(\theta) d\theta}{1 + t}
\]

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\[
\begin{align*}
\leq & \frac{\beta C_0 t^{1-\mu+\beta \mu-\beta k}}{1+t} \int_0^t (t-s)^{\beta k-1} m(s) ds \int_0^\delta \theta^k W_{\beta}(\theta) d\theta \\
+ & \frac{\beta C_0 t^{1-\mu+\beta \mu-\beta k}}{1+t} \int_t^t (t-s)^{\beta k-1} m(s) ds \int_0^\infty \theta^k W_{\beta}(\theta) d\theta \\
\to & 0, \quad \text{as } \varepsilon \to 0, \quad \delta \to 0.
\end{align*}
\]

Thus, \( V(t) \) is also a relatively compact set in \( X \) for \( t \in [0, \infty) \). Therefore, Schauder’s fixed point theorem implies that \( \Phi \) has at least a fixed point \( u^* \in \Omega \). Let \( y^*(t) = t^{-1-\mu+\beta \mu-\beta k} u^*(t) \). Thus

\[
y^*(t) = K_{\mu,\beta}(t) y_0 + \int_0^t Q_\beta(t-s) G(s, y^*(s)) ds, \quad t \in (0, \infty),
\]

which implies that \( y^* \) is a mild solution of (1.1). The proof is completed. \qed

**Theorem 4.2.** Suppose that \( Q(t) \) is compact for \( t > 0 \). Further assume that \( (H1), (H2) \) hold and \( \lim_{t \to \infty} I_{0+}^{\beta k} m(t) = 0 \). Then all mild solutions of the initial value problem (1.1) are attractive.

**Proof.** Let \( y(t) \) is a mild solution of (1.1). From (H2), we get

\[
|y(t)| = \left| K_{\mu,\beta}(t) y_0 + \int_0^t Q_\beta(t-s) G(s, y(s)) ds \right| \\
\leq & L_2 t^{1+\nu-\beta \mu+\beta k} |y_0| + L_1 \Gamma(\beta k) I_{0+}^{\beta k} m(s) \to 0, \quad \text{as } t \to \infty,
\]

which implies that \( y(t) \) is an attractive solution. This completes the proof. \qed

By Theorems 4.1 and 4.2, we have the following corollaries.

**Corollary 4.1.** Assume that \( Q(t) \) is compact for \( t > 0 \). Further suppose that \( (H1) \) and \( (H2)’ \) there exist a function \( m : (0, \infty) \to (0, \infty) \) and \( \eta \in (0, 1), M > 0 \) such that

\[
I_{0+}^{\beta k} m(t) \in C((0, \infty), (0, \infty)), \quad t^{1-\mu+\beta \mu-\beta k} I_{0+}^{\beta k} m(t) \leq M t^\eta,
\]

and

\[
|G(t, \cdot)| \leq m(t), \quad \text{for all } t \in (0, \infty),
\]

hold. Then there is at least one mild solution for the initial value problem (1.1).

**Corollary 4.2.** Suppose that \( Q(t) \) is compact for \( t > 0 \). Further assume that \( (H1), (H2)’ \) hold and \( \eta < 1 - \mu + \beta \mu - \beta k \). Then all mild solutions of the initial value problem (1.1) are attractive.

In the case that \( Q(t) \) is noncompact for \( t > 0 \), we impose the following hypothesis.
there exists a constant $K > 0$ such that for any bounded $D \subseteq X,$

$$\chi(G(t, D)) \leq K t^{1-\mu+\beta-\beta k} \chi(D), \quad \text{for a.e. } t \in [0, \infty),$$

where $\chi$ is the Kuratowski’s measure of noncompactness.

**Theorem 4.3.** Suppose that $Q(t)$ is noncompact for $t > 0$. Further assume that (H1), (H2) and (H3) hold. Then there is at least one mild solution for the initial value problem (1.1).

**Proof.** Let $u_0(t) = t^{1-\mu+\beta-\beta k}K_{\mu, \beta}(t)y_0$ for all $t \in [0, \infty)$ and $u_{n+1} = \Phi u_n$, $n = 0, 1, 2, \cdots$.

By Lemma 3.3, $(\Phi u_n)(t) \subset \Omega$, for $u_n \in \Omega$. Consider set $\mathcal{V} = \{v_n : v_n(t) = (\Phi u_n)(t)/(1 + t), u_n \in \Omega\}_{n=0}^{\infty}$, and we will prove set $\mathcal{V}$ is relatively compact. In view of Lemmas 3.1 and 3.2, the set $\mathcal{V}$ is equicontinuous and $\lim_{t \to \infty}|(\Phi u_n)(t)/(1 + t)| = 0$ uniformly for $u_n \in \Omega$. According to Lemma 2.8, we only need to prove $\mathcal{V}(t) = \{v_n(t) : v_n(t) = (\Phi u_n)(t)/(1 + t), u_n \in \Omega\}_{n=0}^{\infty}$ is relatively compact in $X$ for $t \in [0, \infty)$.

Let $y_n(t) = t^{-(1-\lambda)(1-\mu)}u_n(t)$, $t \in (0, \infty)$, $n = 0, 1, 2, \cdots$. By the condition (H3) and Lemma 2.2, we have

$$\chi(\mathcal{V}(t)) = \chi\left(\left\{\frac{\Phi u_n(t)}{1 + t} \right\}_{n=0}^{\infty}\right) = \chi\left(\left\{\frac{t^{1-\mu+\beta-\beta k}}{1 + t} K_{\mu, \beta}(t)y_0 + \frac{t^{1-\mu+\beta-\beta k}}{1 + t} \int_{0}^{t} Q_{\beta}(t-s)G(s, y_n(s))ds \right\}_{n=0}^{\infty}\right)$$

$$= \chi\left(\left\{\frac{t^{1-\mu+\beta-\beta k}}{1 + t} \int_{0}^{t} Q_{\beta}(t-s)G(s, y_n(s))ds \right\}_{n=0}^{\infty}\right)$$

$$\leq 2L_1 \frac{t^{1-\mu+\beta-\beta k}}{1 + t} \int_{0}^{t} (t-s)^{\beta k-1} \chi\left(G(s, \{s^{1-\mu+\beta-\beta k}u_n(s)\}_{n=0}^{\infty})\right) ds$$

$$\leq 2L_1 K M^* \int_{0}^{t} (t-s)^{\beta k-1} (1 + s) \chi\left(\{u_n(s)\}_{n=0}^{\infty}\right) ds$$

In the other hand, by the properties of measure of noncompactness, for any $t \in [0, \infty)$ we have

$$\chi\left(\left\{\frac{u_n(t)}{1 + t} \right\}_{n=0}^{\infty}\right) = \chi\left(\left\{\frac{u_0(t)}{1 + t} \cup \{u_n(t)\}_{n=1}^{\infty} \right\}\right) = \chi\left(\left\{\frac{u_n(t)}{1 + t} \right\}_{n=1}^{\infty}\right) = \chi(\mathcal{V}(t)).$$

Thus,

$$\chi(\mathcal{V}(t)) \leq 2L_1 K M^* \int_{0}^{t} (t-s)^{\beta k-1} (1 + s) \chi(\mathcal{V}(s)) ds,$$

where

$$M^* = \max_{t \in [0, \infty)} \left\{\frac{t^{1-\mu+\beta-\beta k}}{1 + t} \right\}.$$
From (4.1), we know that
\[ \chi(\mathcal{V}(t)) \leq 4L_1KM^* \int_0^t (t-s)^{\beta k-1} \chi(\mathcal{V}(s))ds, \]
or
\[ \chi(\mathcal{V}(t)) \leq 4L_1KM^* \int_0^t (t-s)^{\beta k-1}s \chi(\mathcal{V}(s))ds, \]
holds. Therefore, by the inequality in [24, p. 188], we obtain that \( \chi(\mathcal{V}(t)) = 0 \), then \( \mathcal{V}(t) \) is relatively compact. Consequently, it follows from Lemma 2.8 that set \( \mathcal{V} \) is relatively compact, i.e., there exists a convergent subsequence of \( \{u_n\}_{n=0}^\infty \). With no confusion, let \( \lim_{n \to \infty} u_n = u^* \), \( u^* \in \Omega \).

Thus, by continuity of the operator \( \Phi \), we have
\[ u^* = \lim_{n \to \infty} u_n = \lim_{n \to \infty} \Phi u_{n-1} = \Phi \left( \lim_{n \to \infty} u_{n-1} \right) = \Phi u^*. \]
Let \( y^*(t) = t^{-1+\mu-\beta \mu+\beta k}u^*(t) \). Thus, \( y^* \) is a mild solution of (1.1). The proof is completed. \( \square \)

**Theorem 4.4.** Suppose that \( Q(t) \) is noncompact for \( t > 0 \). Further assume that (H1), (H2) and (H3) hold and \( \lim_{t \to \infty} I_{0+}^{\beta k}m(t) = 0 \). Then all mild solutions of the initial value problem (1.1) are attractive.

By Theorems 4.3 and 4.4, we have the following corollaries.

**Corollary 4.3.** Assume that \( Q(t) \) is noncompact for \( t > 0 \). Further suppose that (H1), (H2)' and (H3) hold. Then there is at least one mild solution for the initial value problem (1.1).

**Corollary 4.4.** Suppose that \( Q(t) \) is noncompact for \( t > 0 \). Further assume that (H1), (H2)' and (H3) hold and \( \eta < 1 - \mu + \beta \mu - \beta k \). Then all mild solutions of the initial value problem (1.1) are attractive.

**Example 4.1.** Consider the following initial value problem on infinite interval
\[
\begin{cases}
\frac{H}{D}^{\alpha, \beta} y(t) = Ay(t) + g_1(t, y(t)), & t \in (0, \infty), \\
I_{0+}^{(1-\beta)(1-\mu)}y(0) = y_0,
\end{cases}
\tag{4.2}
\]
where \( g_1(t, \cdot) \) is Lebesgue measurable with respect to \( t \) on \((0, \infty)\), \( g_1(\cdot, y) \) is continuous with respect to \( y \) on \( X \), and \( |g_1(t, y(t))| \leq t^{-\alpha}, \alpha \in (\beta k, 1 - \mu + \mu \beta), t \in (0, \infty) \).

Let \( m(t) = t^{-\alpha}, t > 0 \). Then
\[ I_{0+}^{\beta k}m(t) = \frac{\Gamma(1-\alpha)}{\Gamma(1+\beta k - \alpha)} t^{\beta k-\alpha}, \quad t^{1-\mu+\beta \mu-\beta k}I_{0+}^{\beta k}m(t) \leq \frac{\Gamma(1-\alpha)}{\Gamma(1+\beta k - \alpha)} t^{\eta}, \]
where $\eta = 1 - \mu + \beta \mu - \alpha \in (0, 1)$. That means that the condition (H2)$'$ is satisfied. By Corollaries 4.1 and 4.2, the problem (4.2) has at least a mild solution, and all mild solutions of (4.2) are attractive.

**Example 4.2.** Consider the following initial value problem on infinite interval

\[
\begin{aligned}
  \frac{H D_{0+}^{\mu, \beta}}{\alpha} y(t) &= Ay(t) + g_2(t, y(t)), \quad t \in (0, \infty), \\
  I_{0+}^{(1-\beta)(1-\mu)} y(0) &= y_0,
\end{aligned}
\]

(4.3)

where $g_2(t, \cdot)$ is Lebesgue measurable with respect to $t$ on $(0, \infty)$. $g_2(\cdot, y)$ is continuous with respect to $y$ on $X$, and $|g_2(t, y(t))| \leq M$, $M \in (0, \infty)$, $t \in (0, \infty)$.

Let $m(t) = 1, t > 0$. Then

\[
(I_{0+}^\beta m(t)) = \frac{M \Gamma^{\beta k}}{\Gamma(1 + \beta k)} \in C((0, \infty), (0, \infty)), \quad t^{1-\mu+\beta \mu-\beta k} I_{0+}^{\beta k} m(t) \leq \frac{M \Gamma^{\eta}}{\Gamma(1 + \beta k)},
\]

where $\eta = 1 - \mu + \beta \mu \in (0, 1)$. That means that the condition (H2)$'$ is satisfied. By Corollary 4.1, the problem (4.3) has at least a mild solution.

5 Conclusions

In this paper, by using the generalized Ascoli-Arzela theorem, we investigated the existence of mild solutions and attractive solutions for Hilfer fractional evolution equations on infinite interval. We proved existence theorems of mild solutions and attractive solutions for two cases that the semigroup is compact and noncompact respectively. In particular, we do not need to assume that the $G(t, \cdot)$ satisfies the Lipschitz condition. It is worth mentioning that we also develop some new techniques, for example, structuring the space $C_\beta((0, \infty), X)$ which is the key in dealing with the existence of global solutions for fractional evolution equations on infinite intervals. The method in this paper can be applied to study infinite intervals problems for fractional evolution equations with instantaneous/non-instantaneous impulses, fractional stochastic evolution equations.

Acknowledgement

The work was supported by the Macau Science and Technology Development Fund (Grant No. 0074/2019/A2) from the Macau Special Administrative Region of the Peoples Republic of China.

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