ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS TO NONLOCAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, we consider the asymptotic behavior of solutions on the half line to nonlocal fractional differential equations of the form \( k \ast u' = Au(t) + f(t, u(t)) \), where \( k \ast u' \) stands for the nonlocal derivative of \( u \) in Caputo’s sense corresponding to a singular kernel \( k \), \( A \) is a positively definite, selfadjoint operator; the nonlinearity \( f \) is either locally Lipschitz continuous with respect to the second variable or of sublinear growth for small time and Lipschitz continuous for large time. Our main results show that if \( f \) is an asymptotically almost periodic function on \( t \) then the problem possesses asymptotically almost periodic solutions.

1. Introduction

In the present paper, we investigate the following problem

\[
\begin{align*}
(1) & \quad \frac{d}{dt}[k \ast (u - u_0)](t) + Au(t) = f(t, u(t)), \quad t > 0, \\
(2) & \quad u(0) = u_0,
\end{align*}
\]

where the unknown function \( u \) takes values in a separable Hilbert space \( H \); the kernel \( k \in L^1_{\text{loc}}(\mathbb{R}^+) \), \( A : D(A) \subset H \to H \) is an unbounded linear operator; the function \( f : \mathbb{R}^+ \times H \to H \); and \( \ast \) denotes the Laplace convolution, namely \( (k \ast v)(t) = \int_0^t k(t - s)v(s)ds \).

The motivation of our research comes from two directions. From one hand, nonlocal equations appear in the mathematical modelling of various problems concerning dynamic processes in materials with memory (see, e.g., [8, 12, 21]). They are also employed to describe anomalous diffusion phenomena, where \( A \) is specifically the Laplace operator associated with a Dirichlet/Neumann type boundary condition on Euclidean spaces and particular choice of the kernel \( k \) models slow/ultraslow diffusion, see [23].

From the other hand, asymptotic behavior of solutions on the half line is also an active topic in the theory of differential equations. We refer the readers to the monograph [11] and the works [2, 4, 5, 7, 10] for a complete presentation of the methods as well as results in the last decades concerning almost automorphic, almost periodic and asymptotically almost periodic solutions.

Such types of behavior have been shown for solutions of various fractional differential equations containing the Caputo derivative \( D^\alpha_C \) of fractional order \( \alpha \), see [1, 3, 9, 15, 22, 26]. The fractional derivative \( D^\alpha_C \) is a special nonlocal differential operator for a specific kernel \( k(t) = g_{1-\alpha}(t) := t^{-\alpha}/\Gamma(1-\alpha), \alpha \in (0, 1) \). At the best of our knowledge, the periodicity of solutions of more general nonlocal differential equations as (1) is an open question.

Motivated by these facts, we aim at studying the asymptotic behavior of bounded solutions of problem (1)-(2). We assume the following hypotheses.

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Key words and phrases. Nonlocal Differential Equations, asymptotic almost periodic solutions, Almost periodic solutions to PDEs, Fixed-point theorems.
(A) The operator $A : D(A) \to H$ is densely defined, self-adjoint, and positively definite with compact resolvent.

(K) The kernel function $k \in L^1_{\text{loc}}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $k \ast l = 1$ on $(0, \infty)$.

(L) $l \notin L^1(\mathbb{R}^+)$. 

Our work is organized as follows: in Section 2, we recall preliminary results on almost periodic functions and solution operators generated by the operator $A$; in Section 3, we present the existence of mild solutions to the linear Cauchy problem and basic estimates on the resolvent family; then we prove our main theorems in Section 4; finally, in Section 5 we mention important examples which serve as applications of our results.

2. Notations and Preliminaries

2.1. Notation. In our work, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of real and positive real numbers, respectively; by $BC(\mathbb{R}, H), BUC(\mathbb{R}, H)$ the spaces of all $H$-valued bounded continuous and bounded uniformly continuous functions on $\mathbb{R}$. We also use the following functional spaces:

$C_0(H) := \{ f \in BUC(\mathbb{R}^+ ; H) : \lim_{t \to +\infty} \| f(t) \| = 0 \}$,

$AP(H) := \{ f \in BUC(\mathbb{R}; H) : f$ is Bohr almost periodic $\}$,

$AAP(\mathbb{R}^+ ; H) := \{ g : g = f \mid_{\mathbb{R}^+} \text{ for some } f \in AP(H) \}$,

We know from [25], the space $AAP(\mathbb{R}^+ ; H)$ is an closed subspace of the Banach space $BUC(\mathbb{R}^+ ; H)$.

2.2. Bohr almost periodic functions. We summarize in this part facts about Bohr almost periodic functions.

Definition 2.1. [10, Definition 1.10] A subset $E \subset \mathbb{R}$ is said to be relatively dense if there exists a number $\varepsilon > 0$ such that every interval $[a, a + \varepsilon]$ contains at least one point of $E$.

Definition 2.2. [10, Definition 1.12] A continuous function $f : \mathbb{R} \to X, X$ is a Banach space, is called Bohr almost periodic if for every $\varepsilon > 0$, there exists a relatively dense set $T(\varepsilon, f)$ such that

$$\sup_{t \in \mathbb{R}} \| f(t + \tau) - f(t) \| \leq \varepsilon, \forall \tau \in T(\varepsilon, f).$$

It is proved that an almost periodic function $f$ can be approximated uniformly by trigonometric polynomials

$$P_n(t) := \sum_{k=1}^{N(n)} a_{n,k} e^{i\lambda_{n,k} t}, \quad n = 1, 2, \ldots; \lambda_{n,k} \in \mathbb{R}, a_{n,k} \in H, t \in \mathbb{R},$$

see [17, Chap. 2, Approximation Theorem]). Moreover, the exponents $\lambda_{n,k}$ of the trigonometric polynomials $P_n(t)$ can be chosen from the Bohr spectrum $\sigma_b(f)$ of $f$, such that the following integrals (Fourier coefficients)

$$a(\lambda, f) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt$$

are different from 0. The Bohr spectrum is at most a countable set.

By definition of almost periodicity, it is easy to prove that for all $f \in AP(H)$, we have

$$\sup_{t \in \mathbb{R}} \| f(t) \| = \sup_{t \in \mathbb{R}^+} \| f(t) \|.$$
Therefore, the restriction operator of an almost periodic function on the whole line to the half line $\mathbb{R}^+$ is actually an invertible isometry from $AP(H)$ onto $AP(\mathbb{R}^+; H)$. Later on we will sometimes identify the function $f \in AP(H)$ with its restriction on $AP(\mathbb{R}^+; H)$.

### 2.3. The solution operators.
Consider the following scalar Volterra equations

\begin{align}
(3) & \quad s(t) + \mu (l * s)(t) = 1, \quad t \geq 0, \\
(4) & \quad r(t) + \mu (l * r)(t) = l(t), \quad t > 0,
\end{align}

where $l$ is a given kernel, $\mu$ is a parameter, and the functions $s$ and $r$ are to be found. We denote by $s(\cdot, \mu)$ and $r(\cdot, \mu)$ the solutions of $(3)$ and $(4)$, respectively, to observe also the dependence of $s$ and $r$ on $\mu$. Their existence and uniqueness were examined in [18]. The kernel $l$ is said to be completely positive iff $s(\cdot, \mu)$ and $r(\cdot, \mu)$ take nonnegative values for every $\mu > 0$. It is shown in [8] that, $l$ is completely positive iff there exist $\alpha \geq 0$ and nonnegative and nonincreasing function $k \in L^1_{loc}(\mathbb{R}^+)$ which satisfy $\alpha l + l * k = 1$. Our hypothesis $(K)$ is a special case when $\alpha = 0$, hence, it implies that $l$ is completely positive. As mentioned in [24], the nonnegativity of $l$ implies that $s(\cdot, \mu) > 0$ and $r(\cdot, \mu) > 0$ even for $\mu \leq 0$. Certain class of kernels satisfying condition $(K)$ with the slowly varying behaviour and the operations on these class have been recently shown [20]. We now remind some further properties of these functions [8, 13, 14].

**Proposition 2.3.** [13, Proposition 2.1] Let the hypothesis $(K)$ hold. Then for every $\mu > 0$, $s(\cdot, \mu), r(\cdot, \mu) \in L^1_{loc}(\mathbb{R}^+)$. In addition, we have:

1. The function $s(\cdot, \mu)$ is nonnegative and nonincreasing. Moreover,

\[
s(t, \mu) \left[ 1 + \mu \int_0^t l(\tau)d\tau \right] \leq 1, \quad \forall t \geq 0,
\]

hence if $l \notin L^1(\mathbb{R}^+)$ then $\lim_{t \to \infty} s(t, \mu) = 0$ for every $\mu > 0$.

2. The function $r(\cdot, \mu)$ is nonnegative and one has

\[
s(t, \mu) = 1 - \mu \int_0^t r(\tau, \mu)d\tau = k*r(\cdot, \mu)(t), \quad t \geq 0,
\]

so $\int_0^t r(\tau, \mu)d\tau \leq \mu^{-1}, \quad \forall t > 0$. If $l \notin L^1(\mathbb{R}^+)$ then $\int_0^\infty r(\tau, \mu)d\tau = \mu^{-1}$ for every $\mu > 0$.

3. For each $t > 0$, the functions $\mu \mapsto s(t, \mu)$ and $\mu \mapsto r(t, \mu)$ are nonincreasing.

4. Equation $(3)$ is equivalent to the problem

\[
\frac{d}{dt}[k*(s-1)](t) + \mu s(t) = 0, \quad t > 0, \quad s(0) = 1.
\]

5. Let $v(t) = s(t, \mu)v_0 + (r(\cdot, \mu) * g)(t)$, here $g \in L^\infty(\mathbb{R}^+)$. Then $v$ solves the problem

\[
\frac{d}{dt}[k*(v-v_0)](t) + \mu v(t) = g(t), \quad v(0) = v_0.
\]

Under the hypothesis $(A)$, the operator $A$ has only positive eigenvalues, which can be arranged as $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \to \infty$ as $n \to \infty$. The corresponding eigenfunctions $\{e_n\}_{n=1}^\infty$ constitutes an orthonormal basis of $H$. The domain of $A$ is

\[
D(A) = \left\{ v = \sum_{n=1}^\infty v_ne_n : \sum_{n=1}^\infty \lambda_n^2v_n^2 < \infty \right\}, \quad Av = \sum_{n=1}^\infty \lambda_n v_ne_n.
\]

For $s \in \mathbb{R}$, one can define the fractional power of $A$ as follows

\[
V_s := D(A^s) = \left\{ v = \sum_{n=1}^\infty v_ne_n : \sum_{n=1}^\infty \lambda_n^{2s}v_n^2 < \infty \right\} \text{ and } A^s v = \sum_{n=1}^\infty \lambda_n^{s}v_ne_n.
\]
The domain $V_s$ is a Hilbert space furnished with the inner product
\[(u, v)_{V_s} = \sum_{n=1}^{\infty} \lambda_n^{2s} u_n v_n.\]

Furthermore, for $s > 0$, $V_s$ is compactly embedded in $L^2(\Omega)$ and its dual space $V_s^*$ of $V_s$ can be identified with $V_{-s}$.

Now we define the solution operators
\[(5) \quad S(t)v = \sum_{n=1}^{\infty} s(t, \lambda_n) v_n e_n, \quad t \geq 0, \quad v \in H, \]
\[(6) \quad R(t)v = \sum_{n=1}^{\infty} r(t, \lambda_n) v_n e_n, \quad t > 0, \quad v \in H.\]

It is obvious that $S(t)$ and $R(t)$ are linear, self-adjoint operators in $H$. We utilize later some basic properties of these operators.

**Lemma 2.4.** [13, Lemma 2.3] Let $\{S(t)\}_{t \geq 0}$ and $\{R(t)\}_{t > 0}$, be the families of linear operators defined by (5) and (6), respectively. Then

1. For each $v \in H$ and $T > 0$, $S(\cdot)v \in C([0, T]; H)$ and $AS(\cdot)v \in C((0, T]; H)$. Moreover,
   \[\|S(t)v\| \leq s(t, \lambda_1)\|v\|, \quad t \in [0, T],\]

2. Let $v \in H, T > 0$ and $g \in C([0, T]; H)$. Then $R(\cdot)v \in C((0, T]; H)$, $R \ast g \in C([0, T]; H)$ and $A(R \ast g) \in C([0, T]; V_{-2})$. Furthermore,
   \[\|R(t)v\| \leq r(t, \lambda_1)\|v\|, \quad t \in (0, T],\]
   \[\|(R \ast g)(t)\| \leq \int_0^t r(t - \tau, \lambda_1)\|g(\tau)\|d\tau, \quad t \in [0, T],\]

3. Properties of resolvent families for linear Cauchy problem

We consider the linear Cauchy problem of differential form
\[(7) \quad k \ast u'(t) + Au(t) = g(t), u(0) = u_0,\]
where $g \in BC(\mathbb{R}^+; H), A : D(A) \subset H \to H$ satisfies the assumption (A). The authors in [13] showed that there exists a mild solution (and actually a weak solution) to (7), that is a function $u : \mathbb{R}^+ \to H$ given by the following formula
\[(8) \quad u(t) = S(t)x + \int_0^t R(t-s)g(s)ds, \quad t \geq 0.\]

The wellposedness of the Cauchy problem is a consequence of the properties of the resolvent operators $S(t)$ and $R(t)$.

We now observe further properties of $S(t)$ and $R(t)$ which concern almost periodic functions.

**Proposition 3.1.** The family of operators $\{R(t)\}_{t > 0}$ is uniformly integrable, namely
\[\int_0^\infty \|R(t)\|dt < \infty.\]

It holds $\lim_{t \to \infty} \|S(t)\| = 0.$
**Proof.** By Lemma 2.4, the monotonicity of \( s(t, \cdot), r(t, \cdot) \) in Proposition 2.3 and the choice \( v = e_1 \) in (5), (6), we obtain
\[
\|S(t)\| = s(t, \lambda_1), \quad t \geq 0,
\]
\[
\|R(t)\| = r(t, \lambda_1), \quad t > 0.
\]
Therefore,
\[
\int_0^\infty \|R(t)\| dt = \int_0^\infty r(t, \lambda_1) dt = \frac{1}{\lambda_1} < \infty,
\]
the family of operators \( \{R(t)\}_{t > 0} \) is uniformly integrable.

Using Proposition 2.3, since \( l \notin L^1(\mathbb{R}^+) \), it holds \( \lim_{t \to \infty} s(t, \lambda_1) = 0 \). Thus
\[
\lim_{t \to \infty} \|S(t)\| = 0.
\]
Moreover, again by Proposition 2.3 and Lemma 2.4, as \( l \notin L^1(\mathbb{R}^+) \), we obtain
\[
\int_{-\infty}^t \|R(t-s)\| ds = \int_{-\infty}^t r(t-s, \lambda_1) ds = \int_0^\infty r(s, \lambda_1) ds = \frac{1}{\lambda_1}.
\]

**Lemma 3.2.** Let \( g \in \text{AP}(H) \), and set
\[
w(t) = \int_{-\infty}^t R(t-s)g(s) ds.
\]
Then \( w \) belongs to \( \text{AP}(H) \).

**Proof.** If \( g \in \text{AP}(H) \), by definition, for each \( \varepsilon > 0 \) there exists a relatively dense set \( T(\varepsilon, g) \) such that
\[
\sup_{\tau \in \mathbb{R}} \|g(t+\tau) - g(t)\| \leq \varepsilon, \forall \tau \in T(\varepsilon, g).
\]
For each \( \tau \in T(\varepsilon, g) \), we have
\[
\sup_{\tau \in \mathbb{R}} \|w(t+\tau) - w(t)\| = \sup_{\tau \in \mathbb{R}} \left\| \int_{-\infty}^t R(t-s)[g(s+\tau) - g(s)] ds \right\|
\]
\[
\leq \varepsilon \int_{-\infty}^t \|R(t-s)\| ds \leq \frac{\varepsilon}{\lambda_1},
\]
and therefore, \( w \) is almost periodic. \( \square \)

**Lemma 3.3.** Let \( g \in \text{AAP}(\mathbb{R}^+; H) \), and \( Qg \) is defined by
\[
(Qg)(t) = \int_{-\infty}^t R(t-s)g(s) ds.
\]
Then \( Qg \) belongs to \( \text{AAP}(\mathbb{R}^+; H) \).

**Proof.** If \( g \in \text{AAP}(\mathbb{R}^+; H) \), we can write \( g = g_1 + g_2 \), where \( g_1 \in \text{AP}(H) \) and \( g_2 \in C_0(H) \), then we have that \( (Qg)(t) = Gg(t) + Hg(t) \), where
\[
(Gg)(t) = \int_{-\infty}^t R(t-s)g_1(s) ds,
\]
\[
(Hg)(t) = \int_0^t R(t-s)g_2(s) ds - \int_{-\infty}^0 R(t-s)g_1(s) ds.
\]
By Lemma 3.2, \( Gg \in AP(H) \). Next, we will prove \( Hg \in C_0(H) \). Because \( g_2 \in C_0(H) \), for each \( \varepsilon > 0 \) there exists a number \( T > 0 \) such that \( \|g_2(s)\| \leq \varepsilon, \forall s \geq T \). Then for all \( t \geq 2T \), we have

\[
\|(Hg)(t)\| \leq \|g_2\| \int_0^{t/2} \|R(t-s)\| ds + \int_{t/2}^t \|R(t-s)\| ds +
\]

\[
+ \|g_1\| \int_{-\infty}^{0} \|R(t-s)\| ds
\]

\[
\leq (\|g_1\| + \|g_2\|) \int_{t/2}^{\infty} \|R(s)\| ds + \frac{\varepsilon}{\lambda_1}.
\]

By Proposition 3.1,

\[
\int_0^{\infty} \|R(s)\| ds < \frac{1}{\lambda_1} < \infty,
\]

it yields that \((Hg)(t) \to 0 \) as \( t \to \infty \). Hence, \( Qg \in AAP(\mathbb{R}^+; H) \).

As an immediate consequence of Lemma 3.3, we obtain the following:

**Theorem 3.4.** Let \((A)\) and \((K)\) hold. Then, for each \( g \in AAP(\mathbb{R}^+; H) \) all mild solutions \( u \) of \((7)\) are in \( AAP(\mathbb{R}^+; H) \).

4. Asymptotic behavior of mild solution for semilinear problem

In this section, we study the existence of asymptotically almost periodic mild solutions to problem \((1)-(2)\). The nonlinearity \( f \) comes from the following functional space.

**Definition 4.1.** [25, p. 33] We denote by \( AAPU(\mathbb{R}^+ \times H; H) \) the set of all functions \( f : \mathbb{R}^+ \times H \to H \) such that \( f(\cdot, x) \in AAP(\mathbb{R}^+; H) \) uniformly on compact subsets of \( H \), that is, \( f \) is continuous and for all compact subset \( K \) of \( H \), for all \( \varepsilon > 0 \), there exist numbers \( T = T(K, \varepsilon) \geq 0 \) and \( \ell = \ell(K, \varepsilon) > 0 \) such that, for all \( r \in \mathbb{R}^+ \), there exists \( \tau \in [r, r + \ell] \) satisfying \( \|f(t + \tau, x) - f(t, x)\| \leq \varepsilon \) for all \( x \in K \) and for all \( t \geq T \).

Let \( f \in AAPU(\mathbb{R}^+ \times H; H) \), the Nemytskii superposition operator is defined as follows

\[
\mathcal{N}(\varphi)(\cdot) := f(\cdot, \varphi(\cdot))
\]

for \( \varphi \in AAP(\mathbb{R}^+; H) \). The following result is due to [25], which is also explicitly stated in [6, Lemma 8.3]; more general asymptotic properties of Nemytskii superposition operator can be found in [16, Theorem 4.1].

**Theorem 4.2.** [25] Given \( f \in AAPU(\mathbb{R}^+ \times H; H) \), suppose that there exists a constant \( L > 0 \) such that

\[
\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \text{ for all } t \in \mathbb{R}^+, x, y \in H.
\]

Then for \( \varphi \in AAP(\mathbb{R}^+; H) \), \( \mathcal{N}(\varphi) \in AAP(\mathbb{R}^+; H) \).

Motivated by the formula \((8)\), we have the following definition of mild solutions for semilinear problem.

**Definition 4.3.** A function \( u : \mathbb{R}^+ \to H \) is called a mild solution to the problem \((1)-(2)\) if \( u \in C([0, \infty); H) \) and satisfies the following integral identity

\[
u(t) = S(t)u_0 + \int_0^t R(t-s)f(s, u(s))ds, \text{ } t \geq 0.
\]
Theorem 4.4. Suppose that (A), (K), (L) are fulfilled, and $f: \mathbb{R}^+ \times H \to H$ is a function in $\text{AAP}(\mathbb{R}^+ \times H; H)$. Assume further that there exists a bounded integrable function $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying
\[ \|f(t,x) - f(t,y)\| \leq \rho(t)\|x - y\|, \]
for all $t \in \mathbb{R}^+, x, y \in H$. Then problem (1)-(2) possesses a unique asymptotically almost periodic mild solution provided that $\rho := \sup_{t \geq 0} \|\rho(t)\| < \lambda_1$.

Proof. We define the operator $\Phi$ on the space $\text{AAP}(\mathbb{R}^+, X)$ by
\[ \Phi(u)(t) = S(t)u_0 + \int_0^t R(t-s)f(s,u(s))ds := S(t)u_0 + Q(t). \]
Note that, by the assumptions (A), (K), we have $\lim_{t \to \infty} \|S(t)u_0\| = 0$. It implies that $S(t)u_0 \in C_0(X)$.

It follows from Theorem 4.2 that the functions $s \mapsto f(s,u(s))$ is in $\text{AAP}(\mathbb{R}^+; H)$; then according to Lemma 3.3, $Q(t) \in \text{AAP}(\mathbb{R}^+; H)$. Hence, $\Phi$ is well defined on $\text{AAP}(\mathbb{R}^+; H)$. Then $u$ is an asymptotically almost periodic mild solution of problem (1)-(2) iff it is a fixed point of the operator $\Phi$.

Let $u, v$ be in $\text{AAP}(\mathbb{R}^+; H)$. Because $f(t, \cdot)$ is Lipschitz continuous, we have following estimate
\[ \|\Phi(u)(t) - \Phi(v)(t)\| = \int_0^t R(t-s)[f(s,u) - f(s,v)]ds \]
\[ \leq \|u - v\| \int_0^t \|R(t-s)\|\rho(s)ds \]
\[ \leq \frac{\rho}{\lambda_1}\|u - v\|. \]
It yields that $\Phi$ is a contraction mapping provided that $\rho < \lambda_1$. Hence, by Banach fixed point theorem, $\Phi$ has a unique fixed point $u \in \text{AAP}(\mathbb{R}^+; H)$. This completes the proof. \qed

We also obtain the existence of at least one asymptotically almost periodic mild solution in a more general nonlinear term by utilizing fixed point arguments as follows

Theorem 4.5. Let $f \in \text{AAPU}(\mathbb{R}^+ \times H, H)$ obey the condition
\[ \|f(t,u)\| \leq \alpha(t) + \beta(t)\|u\|, \quad \forall t \geq 0, u \in H, \]
where $\alpha \in L^1_{loc}(\mathbb{R}^+, \mathbb{R}^+)$ and $\beta \in BC_{loc}([0, +\infty))$. Furthermore, assume that $f$ is Lipschitz continuous for large time, that is, there is a positive number $T_1$ such that
\[ \|f(t,u) - f(t,v)\| \leq \rho(t)\|u - v\|, \quad \forall t \geq T_1, u, v \in H. \]
Then the problem (1)-(2) has a mild solution $u$ which is also asymptotically almost periodic provided that $\limsup_{t \to +\infty} \rho(t) < \lambda_1$.

Remark 4.6. The result is applicable for nonlipschitz continuous nonlinearity. The assumption requires only sublinear growth and Lipschitz continuity for large time. Without Lipschitz continuity, the uniqueness is violated.

The condition of Lipschitz continuity for large time is vital. Without this assumption, the problem may admit a mild solution, which might not belong to $\text{AAP}(\mathbb{R}^+; H)$. Indeed, consider the simple scalar model
\[ ku' + \lambda_1 u = Lu, \quad t > 0, u(0) = u_0. \]
This problem has a unique solution $u(t) = s(t, \lambda_1 - L)$. If $L > \lambda_1$ then $s(t, \lambda_1 - L) \notin \text{AAP}(\mathbb{R}^+)$ since it is not bounded on $[0, \infty)$. 

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\textbf{Proof.} We first show that the sublinear nonlinearity leads to the existence (without uniqueness) of mild solutions on an arbitrary finite interval \([0, T]\). Then by constructing a suitable invariant subset of \(BC([0, \infty); H)\) and utilizing Lipschitz continuity for large time, we gain the existence of a mild solution on the half interval \([0, \infty)\) via fixed point argument. As in the proof of Theorem 4.4, the operator \(\Phi\) is well defined on \(AAP(\mathbb{R}^+; H)\), \(u\) is an asymptotically almost periodic mild solution of problem (1)-(2) iff it is a fixed point of the operator \(\Phi\).

\textbf{Step 1.} Fix an arbitrary \(T > 0\). We first show the existence of a mild solution \(U \in C([0, T]; H)\).

By [19, Proposition 2.3] (for convenience of the reader, see Proposition 6.1 in the appendix of this paper), \(Q(t)\) and consequently \(\Phi(t)\) are compact operators. We need to find a closed convex set in \(AAP(\mathbb{R}^+; H)\) which is invariant under \(\Phi\).

Indeed, by the growth condition of \(f(t, x)\) and Lemma 2.4, we estimate

\[
|\Phi(u)(t)| = \left| S(t)u_0 + \int_0^t R(t-s)f(s, u(s))ds \right|
\leq s(t, \lambda_1)|u_0| + \int_0^t r(t-s, \lambda_1) (\alpha(s) + \beta(s)|u(s)|)ds
\leq \|u_0\| + \sup_{[0, T]} (r(\cdot, \lambda_1) \ast \alpha)(t) + \max_{\lambda_1} \beta \int_0^t r(t-s, \lambda_1)|u(s)|ds
\leq M + \beta_0 \int_0^t r(t-s, \lambda_1)|u(s)|ds,
\]

where \(M = \|u_0\| + \sup_{[0, T]} (r(\cdot, \lambda_1) \ast \alpha)(t)\), \(\beta_0 = \max_{\lambda_1} \beta\). Here, we have used the monotonicity of \(s(\cdot, \lambda_1)\) and \(s(0, \lambda_1) = 1\).

Let \(v \in C([0, T]; \mathbb{R}^+)\) be the unique solution of the integral equation

\[
v(t) = M + \beta_0 \int_0^t r(t-s, \lambda_1)v(s)ds, t \in [0, T].\]

We define the set \(D = \{ \omega \in C([0, T]; H) : \|\omega(t)\| \leq v(t), \forall t \in [0, T] \}\). Obviously, \(D\) is a bounded closed convex nonempty subset of \(C([0, T]; H)\). Moreover, \(D\) is an invariant subset of \(C([0, T]; H)\) under \(\Phi\) as it holds

\[
\|\Phi(u)(t)\| \leq v(t), \forall t \in [0, T].
\]

Therefore, \(\Phi\) has a fixed point \(U \in D \subset C([0, T]; H)\) by Schauder fixed point theorem.

\textbf{Step 2.} By Step 1, we choose a mild solution \(U(t)\) in \(BC([0, T])\) where \(T_2 > T_1\) is a number such that \(\rho_i = \sup_{t \geq T_1} \rho(t) < \lambda_1\). Consider the subset

\[
\mathcal{M} = \{ u \in AAP(\mathbb{R}^+; H) : u(t) = U(t) \forall t \in [0, T_2] \}.
\]

Obviously, \(\mathcal{M}\) is a nonempty complete metric space with the induced metric in \(BC([0, \infty); H)\). Because \(U\) is a mild solution in \([0, T_2]\), \(\mathcal{M}\) is invariant under the mapping \(\Phi\). For all \(u, v \in \mathcal{M}\), we have for \(t \geq T_2\)

\[
|\Phi(u)(t) - \Phi(v)(t)| \leq \int_0^t \|R(t-s)\| \|f(s, u(s)) - f(s, v(s))\| ds
\leq \int_0^{T_2} \|R(t-s)\| \|f(s, u(s)) - f(s, v(s))\| ds +
\leq \int_0^{T_2} r(t-s, \lambda_1) \|f(s, u(s)) - f(s, v(s))\| ds
\leq \int_0^{T_2} r(t-s, \lambda_1) \rho \|u(s) - v(s)\| ds
\]
we use here \( u(s) = v(s) = U(s) \) for all \( s \in [0, T_i] \). Note that \( \Phi(u)(t) = U(t) \) for all \( t \in [0, T_i] \).

Therefore, we obtain

\[
\sup_{t \geq 0} \| \Phi(u)(t) - \Phi(v)(t) \| \leq \frac{\rho_{\infty}}{\lambda_1} \| u - v \|_{C((0, \infty); H)}.
\]

This means that \( \Phi \) is a contraction mapping on the metric space \( \mathcal{M} \). Consequently, there is a fixed point \( u \in \mathcal{M} \). This completes the proof. \( \square \)

**Remark 4.7.** In this proof, we use fixed point arguments twice. First, we apply the Schauder fixed point theorem to construct a solution in \( C([0, T]; H) \). Second, we use fixed point argument for a contraction mapping in a suitable complete metric space. Note that we cannot apply directly the Schauder fixed point theorem on the space \( AP(\mathbb{R}_+; H) \) because we only have the compactness of the solution operator in \( C([0, T]; H) \) and as mentioned in [10], it is difficult to testify the compactness in \( AP(\mathbb{R}_+; H) \) or \( AAP(\mathbb{R}_+; H) \). Instead, we utilize the Lipschitz continuity for large time to establish the contraction of the solution mapping on \( AAP(\mathbb{R}_+; H) \). Hence, Lipschitz continuity for large time is an essential assumption, which we suspend only on finite intervals.

### 5. Examples

**5.1. Slow diffusion case.** Consider the case \( k(t) = g_1 - a(t) \) and \( l(t) = g_a(t), t > 0 \), then the fractional differential operator becomes the fractional derivative in the sense of Caputo. The equation (1) reads as

\[
D^\alpha_C u(t, x) + Au(t, x) = f(t, u(t, x)),
\]

(14)

\[
u(0, x) = \varphi(x), \; x \in \Omega,
\]

(15)

where \( \Omega \) is a bounded domain with sufficiently smooth boundary in \( \mathbb{R}^n \).

Let \( H = L^2(\Omega), A = -\Delta \) with \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \). Then the problem (14)-(15) is of the abstract model (1)-(2).

As the matter of fact, \( A \) generates a selfadjoint positive operator with compact resolvent. By the variational characterization, the first eigenvalue of the operator \( A \) is given by

\[
\lambda_1 = \inf_{u \in C_0^\infty(\Omega), \| u \| = 1} \| \nabla u \|^2 > 0
\]

as a consequence of the Poincaré inequality. We obtain the following result.

**Theorem 5.1.** Let \( f : \mathbb{R}_+ \times H \rightarrow H \) be a function in \( AAPU(\mathbb{R}_+ \times H; H) \) and assume that there exists a bounded integrable function \( \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfying

\[
\| f(t, x) - f(t, y) \| \leq \rho(t) \| x - y \|,
\]

(14) for all \( t \in \mathbb{R}_+, x, y \in H \). Then the unique mild solution \( u \) of (14)-(15) is asymptotically almost periodic whenever \( \rho := \sup_{t \geq 0} \| \rho(t) \| < \lambda_1 \).

**5.2. Ultraslow diffusion case.** Consider \( k(t) = \int_0^1 g_B(t) d\beta \) and \( l(t) = \int_0^\infty \frac{e^{-pt}}{1 + p} d\beta, t > 0 \). The equation (1) then has distributed order.

Similar to the previous example, (A) holds for \( H = L^2(\Omega), A = (-\Delta)^s, 0 < s < 1, \) with \( D(A) = V_s(\Omega) \). As shown in [23, Example 4.3], there is a positive constant \( T_1 \) such that

\[
l \ast 1(t) \geq \frac{1}{2} \ln t \text{ for all } t > T_1.
\]
Hence, (K) and (L) are testified. Then the problem
\[
\frac{d}{dt} (k * (u - u_0))(t, x) + Au(t, x) = f(t, u(t, x)),
\]
\[
u(0, x) = \varphi(x), \ x \in \Omega,
\]
fits in the abstract model (1)-(2). We obtain the following result.

**Theorem 5.2.** Let \( f : \mathbb{R}^+ \times H \rightarrow H \) be a function on \( AAPU(\mathbb{R}^+ \times H; H) \) and assume that there exists a bounded integrable function \( \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfying
\[
\|f(t, x) - f(t, y)\| \leq \rho(t)\|x - y\|,
\]
for all \( t \in \mathbb{R}^+, x, y \in H \). Then the unique mild solution \( u \) of problem (14)-(15) is asymptotically almost periodic whenever \( \rho := \sup_{t \geq 0} \|\rho(t)\| < \lambda_1 \).

### 5.3. Multi-term fractional differential equation

Consider the following multi-term fractional-in-time partial differential equation
\[
\sum_{i=1}^{m} \mu_i \partial^{\alpha_i}_t u(t, x) + (-\Lambda)^s u(t, x) = F \left( \int_{\Omega} u^2(t, x) dx \right) G(x, u(t, x)), \ t \geq 0, x \in \Omega,
\]
\[
u(t, x) = 0, \ for \ t \geq 0, x \in \partial \Omega,
\]
\[
u(0, x) = u_0(x), \ for \ x \in \Omega,
\]
where \( 0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < 1, s > 0, \mu_i > 0, \) and \( \partial^{\alpha_i}_t \) stands for the Caputo derivatives of order \( \alpha_i \) in \( t, 1 \leq i \leq m \). The operator \( \Lambda \) with
\[
D(\Lambda) = \{ u \in H_0^1(\Omega), Au = \sum_{i,j=1}^{N} \partial_{ij}(a_{ij}(x) \partial^2_{ij} u) \in L^2(\Omega) \},
\]
whose coefficients \( a_{ij} \in L^{\infty}(\Omega), a_{ij} = a_{ji}, 1 \leq i, j \leq N \), is assumed to be strongly elliptic i.e. \( \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \) for all \( \xi \in \mathbb{R}^N \) for some \( \theta > 0 \).

Moreover, we assume that
\[
\bullet \ F \in C^1(\mathbb{R}) \ satisfies \ the \ growth \ condition \ |F(r)| \leq a + b|r|^v, \ where \ a, b, v \geq 0.
\]
\[
\bullet \ G : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \ is \ a \ Carathéodory \ function \ which \ is \ Lipschitz \ continuous \ in \ the \ second \ variable, \ namely
\]
\[
|G(x, y_1) - G(x, y_2)| \leq h(x)|y_1 - y_2|, \forall x \in \Omega, y_1, y_2 \in \mathbb{R},
\]
where \( 0 \leq h \in L^{\infty}(\Omega) \). Moreover, we suppose \( G(x, 0) = 0 \) for almost all \( x \in \Omega \).

The nonlinearity in our model can be considered as a perturbation which depends on both the state and the energy of the system. This model fits the Theorem 4.4 by the following functional setting:
\[
H = L^2(\Omega), A = (-\Lambda)^s, k(t) = \sum_{i=1}^{m} \mu_i g_1 - \alpha_i(t), \text{ and } f(v)(x) = \int_{\Omega} v^2(x) dx \ G(x, v(x)).
\]
As mentioned in [23], the kernel \( k \) is completely monotonic, i.e. \( (-1)^n k^{(n)}(t) \geq 0 \) for \( t \in (0, \infty) \).

Hence, it admits a resolvent function \( I \), i.e. \( k * I = 1 \) on \( (0, \infty) \). In this case, \( (k * I)(t) \sim \mu_1^{-1} g_1 + \alpha_1(t) \) as \( t \rightarrow +\infty \). Therefore,
\[
s(t, \lambda_1) \leq \frac{1}{1 + \lambda_1 (1 * I)(t)} \rightarrow 0 \ as \ t \rightarrow +\infty.
\]

It is verified in [13] that \( f : L^2(\Omega) \rightarrow L^2(\Omega) \) is locally Lipschitzian, namely,
\[
\|f(v_1) - f(v_2)\| \leq \kappa(\rho)\|v_1 - v_2\|, \ for \ all \ v_1, v_2 \in L^2(\Omega), \|v_1\|, \|v_2\| \leq \rho,
\]
where $\kappa(\rho) = 2\rho^2\|h\|_\infty\sup_{r\in[0,\rho^2]} |F'(r)| + (a + b\rho^{2r})\|h\|_\infty$. By applying Theorem 4.4, the given problem has a unique asymptotically almost periodic solution.

**6. Appendix**

We summarize now the proof of compactness of the operator $Q(t)$ on $X_s := C([0,T];D(A^s))$, which is taken from the preprint [19].

**Proposition 6.1.** Let assumptions (A) and (K) hold. Then the operator

$$Q : C([0,T];D(A^s)) \rightarrow C([0,T];D(A^{s+1/2})), f \mapsto Qf(t) := R*f(t)$$

is compact for any $s \in \mathbb{R}$.

**Proof.** Based on the approximation argument, the proof is divided into several steps.

**Step 1.** For a function $g \in C[0,T]$, the convolution map $\mathcal{C}_g : C[0,T] \rightarrow C[0,T], v \mapsto g*v$ is compact. Take a bounded subset $D \subset C[0,T]$, for all $v \in D, \|v\| \leq R$ for a certain $R > 0$. We have

$$\|g * v(t)\| \leq \|g\|_{L^1(0,T)} \max_{t \in [0,T]} |v(t)|, v \in D,$$

which implies the point-wise bounded of $\mathcal{C}_g(D)$.

On the other hand, by the uniform continuity of $g$ on $[0,T]$, for any $\varepsilon > 0$, one chooses $0 < \delta < \varepsilon/(2R\|g\| + 1)$ such that

$$|g(s_1) - g(s_2)| \leq \frac{\varepsilon}{2RT} \text{ for any } s_1, s_2 \in [0,T], |s_1 - s_2| \leq \delta.$$

For any $t_1, t_2 \in [0,T], 0 < t_2 - t_1 < \delta$ and $v \in D$, one has

$$|g * v(t_2) - g * v(t_1)| \leq \int_0^{t_1} |g(t_1 - \tau) - g(t_2 - \tau)| |v(\tau)|d\tau + \int_{t_1}^{t_2} |g(t_2 - \tau)v(\tau)|d\tau$$

$$\leq \frac{\varepsilon}{2RT} \int_0^{t_1} |v(\tau)|d\tau + \max_{\tau \in [0,T]} |v(\tau)| |t_1 - t_2| \max_{\tau \in [0,T]} |g(\tau)|$$

$$\leq \left( \frac{\varepsilon}{2RT} T + \|g\|\delta \right) \sup_{v \in D} \|v\| \leq \varepsilon.$$

Therefore, $\mathcal{C}_g(D)$ is equicontinuous, hence by Azelà-Ascoli Theorem, $\mathcal{C}_g(D)$ is relatively compact in $C[0,T]$.

**Step 2.** Take $g \in L^1(0,T)$, by density of smooth functions in $L^1(0,T)$, one can choose a sequence of continuous function $g_n$ such that $g_n \rightarrow g$ in $L^1(0,T)$. We have

$$\|\mathcal{C}_{g_n} - \mathcal{C}_g\|_{C([0,T])} = \sup_{t \in [0,T]} \left| \int_0^t (g_n(t - \tau) - g(t - \tau))v(\tau)d\tau \right|$$

$$\leq \sup_{t \in [0,T]} \int_0^t |g_n(t - \tau) - g(t - \tau)||v(\tau)|d\tau$$

$$\leq \|g_n - g\|_{L^1(0,T)}\|v\|.$$ 

Thus $\lim_{n \rightarrow \infty} \|\mathcal{C}_{g_n} - \mathcal{C}_g\|_{C([0,T])} = 0$, which implies the compactness of $\mathcal{C}_g$.

**Step 3.** For any $f = \sum_{k=1}^n f_k(t)e_k \in C([0,T];D(A^s))$, denote

$$Q_nf(t) = \sum_{k=1}^n \left( \int_0^r r(t - \tau, \lambda_k) f_k(\tau)d\tau \right) e_k$$

By Step 2, $Q_n$ is a compact operator from $C([0,T];D(A^s))$ to $C([0,T];D(A^{s+1/2}))$.
Step 4. By Step 3, it reduces to show that $Q_n$ converges to $Q$ with respect to the operator norm in $\mathcal{L}(X_s, X_{s+1/2})$. Indeed, we have

$$
\| (Q - Q_n) v(t) \|_{D(A^{s+1/2})}^2 = \sum_{k > n} |\lambda_k^{1 + 2s} \int_0^t r(t - \tau, \lambda_k) v_k(\tau) d\tau |^2
$$

$$
= \sum_{k > n} \int_0^t \lambda_k r(t - \tau, \lambda_k) d\tau \cdot \int_0^t r(t - \tau, \lambda_k) |\lambda_k^{1 + 2s} v_k(\tau) |^2 d\tau
$$

$$
\leq \int_0^t r(t - \tau, \lambda_n) \sum_{k > n} |\lambda_k^2 v_k(\tau) |^2 d\tau
$$

$$
\leq \int_0^t r(t - \tau, \lambda_n) \sup_{s \in [0, T]} \| v(\tau) \|_{D(A^s)}^2 d\tau
$$

$$
\leq \left( \int_0^t r(t - \tau, \lambda_n) d\tau \right) \frac{1 - s(t, \lambda_n)}{\lambda_n} \| v \|_{X_s}^2.
$$

Hence, we obtain

$$
\| Q_n - Q \|_{\mathcal{L}(X_s, X_{s+1/2})} \leq \lambda_n^{-1/2} \rightarrow 0 \text{ as } n \rightarrow \infty.
$$

Therefore, $\| Q_n - Q \|_{\mathcal{L}(X_s, X_{s+1/2})} \leq \lambda_n^{-1/2} \rightarrow 0$ as $n \rightarrow \infty$. □

Remark 6.2. The standard argument for checking the compactness of a subset in $C([0, T], D(A^{s+1/2}))$ is applying Arzelà-Ascoli Theorem directly. However, due to the singularity of the kernel $l$ (so $r(\cdot, \lambda)$), it is difficult to testify the equicontinuity of $Q(D)$ directly without further regularity assumption on the function $l$.

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References

ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS


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