Homogeneity degree of hyperspaces of arcs and simple closed curves

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Abstract

Given a continuum $X$ and $n \in \mathbb{N}$, let $C_n(X)$ (resp., $F_n(X)$) be the hyperspace of nonempty closed sets with at most $n$ components (resp., $n$ points). Given $1 \leq m \leq n$, we consider the quotient space $C_n(X)/F_m(X)$. The homogeneity degree of $X$, $\text{hd}(X)$, is the number of orbits of the group of homeomorphisms of $X$. In this paper we discuss lower bounds for the homogeneity degree of the hyperspaces $C_n(X), C_n(X)/F_m(X)$ when $X$ is a finite graph. In particular, we prove that for a finite graph $X$:

(a) $\text{hd}(C_n(X)/F_m(X)) = 1$ if and only if $X$ is a simple closed curve and $n = m = 1$,
(b) $\text{hd}(C_n(X)/F_m(X)) = 2$ if and only if $X$ is an arc and either $n = m = 1$ or $n = 2$ and $m \in \{1, 2\}$,
(c) $\text{hd}(C_n(X)/F_m(X)) = 3$ if and only if $X$ is a simple closed curve and $n = m = 2$, and
(d) $\text{hd}(C_n(X)/F_m(X)) = 4$ if and only if $X$ is a simple closed curve, $n = 2$ and $m = 1$.

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1 Introduction

A continuum is a compact connected metric space with more than one point. A subcontinuum of a continuum $X$ is a nonempty compact connected subspace of $X$, so one-point sets are also subcontinua.

If $m, n \in \mathbb{N}$ and $1 \leq m \leq n$, in this paper we consider the following hyperspaces of $X$:

$2^X = \{A \subset X : A$ is a nonempty closed subset of $X\}$,
$C_n(X) = \{A \in 2^X : A$ has at most $n$ components$,\}
$F_n(X) = \{A \in 2^X : A$ has at most $n$ points$\}$, and
the quotient space $C_n(X)/F_m(X)$.
The hyperspace $2^X$ is considered with the Hausdorff metric [13, Theorem 2.2]. A mapping is a continuous function. As usual $C_1(X)$ is denoted by $C(X)$.

Given a topological space $Z$, the homogeneity degree of $Z$, denoted by hd$(Z)$, is the number of orbits of the group of homeomorphisms of $X$. So, $Z$ is homogeneous when hd$(Z) = 1$. Spaces $Z$ for which hd$(Z) = n$ $(n ∈ ℕ)$ are also known as $\frac{1}{n}$-homogeneous.

The homogeneity degree of hyperspaces of continua has been studied by several authors [2], [6], [9], [16], [17], [21], [22] and [23]. In particular, in [17] and [18] continua for which hd$(C_n(X)/F_n(X)) = 1$ were studied. In [20] Sam B. Nadler, Jr. included several problems on the homogeneity degree of hyperspaces. Some interesting questions about continua $X$ such that hd$(C(X)/F_1(X)) = 2$ were included.

A finite graph is a continuum which is a finite union of arcs such that the intersection of each two of them is a finite set.

A simple $n$-od is a continuum homeomorphic to the cone over a discrete space with exactly $n$-points, the vertex of a simple $n$-od is the point corresponding to the vertex of the cone. Given a point $p$ in a finite graph $X$, the order of $p$ in $X$, denoted by $o(p, X)$, is equal to the positive integer $n$, if there exists a neighborhood $M$ such that $M$ is an $n$-od and $p$ is the vertex of $M$. The point $p$ is an end-point of $X$ if $o(p, X) = 1$, an ordinary point of $X$ if $o(p, X) = 2$, and $p$ is a ramification point if $o(p, X) ≥ 3$.

As usual we denote the unit circle in the plane by $S^1$. The following are the known results about models and homogeneity degree of the hyperspaces of $[0, 1]$ and $S^1$.

A. Since for every locally connected continuum $X$, $2^X$ is homeomorphic to the Hilbert cube [13, Theorem 11.3], we have that $2^{([0, 1])} = 1 = \text{hd}(2^{S^1})$.

B. $C([0, 1])$ and $C(S^1)$ are 2-cells [11, p. 41], so hd$(C(S^1)) = 2 = \text{hd}(C([0, 1]))$.

C. $C_2([0, 1])$ is a 4-cell [11, p. 50], so hd$(C_2([0, 1])) = 2$.

D. $C_2(S^1)$ is homeomorphic to the cone over a solid torus [8], so hd$(C_2(S^1)) = 3$.

E. $F_2([0, 1])$ is a 2-cell [11, p. 51], so hd$(F_2([0, 1])) = 2$.

F. $F_2(S^1)$ is homeomorphic to the Möbius strip [11, p. 53 and 54], so hd$(F_2(S^1)) = 2$.

G. $F_3([0, 1])$ is a 3-cell [11, p. 51], so hd$(F_3([0, 1])) = 2$.

H. $F_3(S^1)$ is homeomorphic to the 3-dimensional sphere [1], so hd$(F_3(S^1)) = 1$.

I. If $n ≥ 4$, then hd$(F_n([0, 1])) = 2n$ [6, Theorem 11].

J. If $n ≥ 4$, then hd$(F_n(S^1)) = n$ [6, Proposition 8].

K. If $X$ is locally connected, then hd$(C_n(X)) = 2$ if and only if either $n ∈ \{1, 2\}$ and $X$ is an arc or $n = 1$ and $X$ is a simple closed curve [9, Theorem 1.4].

L. If $X$ is locally connected, then hd$(C_n(X)) = 3$ if and only if $n = 2$ and $X$ is a simple closed curve [9, Theorem 1.4].
M. $C(S^1)/F_1(S^1)$ is a 2-sphere (follows from the model in [11, p. 41]), so \( \text{hd}(C(S^1)/F_1(S^1)) = 1 \).

N. \( \text{hd}(C_2(S^1)/F_1(S^1)) = 4 \) [12, Theorem 10].

O. \( \text{hd}(C_2(S^1)/F_2(S^1)) = 3 \) [12, Theorem 14].

P. $C([0,1])/F_1([0,1])$ is a 2-cell (follows from the model in [11, p. 41]), so \( \text{hd}(C([0,1])/F_1([0,1])) = 2 \).

Q. \( \text{hd}(C_2([0,1])/F_2([0,1])) = 2 \), since $C_2([0,1])/F_2([0,1])$ is a 4-cell [16, Theorem 4.6].

R. \( \text{hd}(C_2([0,1])/F_1([0,1])) = 2 \), since $C_2([0,1])/F_1([0,1])$ is a 4-cell [14, Theorem 3.3].

The main results of this paper are the following.

**Theorem 1** If $n \geq 3$ and $1 \leq m \leq n$, then:

(a) \( \text{hd}(C_n([0,1])) \geq 2n \),

(b) \( \text{hd}(C_n([0,1])/F_m([0,1])) \geq 2n - 1 \), and

(c) \( \text{hd}(C_n(S^1)) \geq 2n + 1 \).

**Theorem 2** Let $X$ be a finite graph and $1 \leq m \leq n$. Then

(a) \( \text{hd}(C_n(X)/F_m(X)) = 1 \) if and only if $X$ is homeomorphic to $S^1$ and $n = m = 1$,

(b) \( \text{hd}(C_n(X)/F_m(X)) = 2 \) if and only if $X$ is an arc and either $n = m = 1$ or $n = 2$ and $m \in \{1, 2\}$,

(c) \( \text{hd}(C_n(X)/F_m(X)) = 3 \) if and only if $X$ is homeomorphic to $S^1$ and $n = m = 2$, and

(d) \( \text{hd}(C_n(X)/F_m(X)) = 4 \) if and only if $X$ is homeomorphic to $S^1$, $n = 2$, and $m = 1$.

2 Lower bounds for the homogeneity degree

Throughout this section the letter $X$ will denote a finite graph with metric $d$. We denote by $R(X)$ the set of ramification points of $X$. Then $R(X) = \{ p \in X : \text{ord}_X(p) \geq 3 \}$. A useful tool for our results is the formula, developed in [19] by the third named author, for computing the dimension of an element in $C_n(X)$.

**Theorem 3** [19, section 2] For each $n \in \mathbb{N}$ and each $A \subset C_n(X)$, we have

\[
\text{dim}_A[C_n(X)] = 2n + \sum_{p \in R(X) \cap A} (\text{ord}_X(p) - 2).
\]

We will use the following sets inspired in definitions given in [7]. These sets have been very useful in the area of uniqueness of hyperspaces (e.g., [3] and [10]). Fix integers $k,n$, with $1 \leq k \leq n$. The hyperspace $C_n(X)/F_k(X)$ will be denoted by $\mathcal{Z}$. Let $\varphi : C_n(X) \rightarrow \mathcal{Z}$ be the quotient mapping. Let $Z_0 \subset \mathcal{Z}$ be such that $\{Z_0\} = \varphi(F_k(X))$. Observe that the mapping:

\[
\varphi|_{C_n(X) \setminus F_k(X)} : C_n(X) \setminus F_k(X) \rightarrow \mathcal{Z} \setminus \{Z_0\}
\]

is a homeomorphism.

Define:
\[C_0(X) = \emptyset,\]
\[\mathcal{M}_n(X) = \{ A \in C_n(X) \setminus C_{n-1}(X) : A \cap R(X) = \emptyset \},\]
\[\mathcal{L}_n(X) = \{ A \in C_n(X) : A \text{ has a } 2n\text{-cell neighborhood in } C_n(X) \},\]
\[\mathcal{E}_n(X) = \{ A \in C_n(X) \setminus \mathcal{M}_n(X) : A \text{ has a basis of open neighborhoods } \mathcal{B} \text{ in } C_n(X) \text{ such that, for each } \mathcal{U} \in \mathcal{B}, \mathcal{U} \cap \mathcal{L}_n(X) \text{ is arcwise connected and } \dim[\mathcal{U}] \leq 2n \}.
\]

Given \(3 \leq m \leq n\), we define a set \(\mathcal{R}_m(X)\) in the following way.

\[
\begin{align*}
\mathcal{R}_{n+1}(X) &= \emptyset, \\
\mathcal{R}_m(X) &= \{ A \in C_n(X) \setminus (\mathcal{R}_{m+1}(X) \cup \cdots \cup \mathcal{R}_{n+1}(X)) : A \text{ has a } m\text{-cell neighborhood in } C_n(X) \setminus (\mathcal{R}_{m+1}(X) \cup \cdots \cup \mathcal{R}_{n+1}(X)) \}.
\end{align*}
\]

The following result was proved in [7, Lemma 3.2, Lemma 3.3, Lemma 3.5 and Lemma 3.6].

**Theorem 4** For the finite graph \(X\), the following holds:

(a) \(\mathcal{M}_n(X) \subseteq \mathcal{L}_n(X)\),
(b) if \(n \geq 3\), then \(\mathcal{M}_n(X) = \mathcal{L}_n(X)\),
(c) if \(n \geq 3\), then \(\mathcal{E}_n(X) = \mathcal{M}_1(X)\).

**Lemma 5** Let \(3 \leq m \leq n\). Then \(\mathcal{R}_m(X) = \mathcal{M}_m(X)\).

**Proof.** In the case that \(n = m\), by definition, \(\mathcal{R}_n(X) = \mathcal{L}_n(X)\), and by Theorem 4 (b), \(\mathcal{L}_n(X) = \mathcal{M}_n(X)\). Thus \(\mathcal{R}_n(X) = \mathcal{M}_n(X)\).

Now, suppose that for each \(r \in \{m+1, \ldots, n\}\), \(\mathcal{R}_r(X) = \mathcal{M}_r(X)\). Define

\[
\mathcal{M} = C_n(X) \setminus (\mathcal{M}_{m+1}(X) \cup \cdots \cup \mathcal{M}_{n+1}(X)) = C_n(X) \setminus (\mathcal{R}_{m+1}(X) \cup \cdots \cup \mathcal{R}_{n+1}(X)).
\]

Given \(A \in \mathcal{R}_m(X)\) we have that \(A \in \mathcal{M}\), \(A\) has a \(2m\)-cell neighborhood \(\mathcal{D}\) in \(\mathcal{M}\) and since \(A \notin \mathcal{M}_{m+1}(X)\), either \(A \cap R(X) \neq \emptyset\) or \(A \in C_m(X)\).

We check that \(A \cap R(X) = \emptyset\). Suppose to the contrary that there exists a point \(p \in A \cap R(X)\). Let \(\mathcal{T} = \{ B \in C_n(X) : p \in B \}\). Observe that \(\mathcal{T} \subseteq \mathcal{M}\). Let \(\mathcal{N}\) be a \(2m\)-cell neighborhood of \(A\) in \(\mathcal{M}\). If \(A\) contains a simple closed curve \(S\), fix a point \(q \in S \setminus R(X)\). Removing, from \(S\) a small open subinterval centered at \(q\), we obtain an element \(A_1 \in \text{int}_\mathcal{M}(\mathcal{N})\) such that \(p \in A_1 \subset A\) and \(S\) is not contained in \(A_1\). If \(A_1\) contains simple closed curves, we repeat the procedure until we obtain an element \(A_2 \in \text{int}_\mathcal{M}(\mathcal{N})\) such that \(p \in A_2 \subset A\). If it is necessary, we add (a finite number of) small subarcs having \(p\) as endpoint to obtain an element \(A_3 \in \text{int}_\mathcal{M}(\mathcal{N})\) such that \(p \in \text{int}_X(A_3)\) and \(A_3\) does not contain simple closed curves. Thus there exists a neighborhood \(\mathcal{V}\) of \(A_3\) in \(C_n(X)\) such that every element in \(\mathcal{V}\) has the point \(p\). Hence \(\mathcal{V} \subseteq \mathcal{M}\).

Since \(A \in \text{int}_\mathcal{M}(\mathcal{N})\), we may assume that \(\mathcal{V} \subseteq \text{int}_\mathcal{M}(\mathcal{N})\). By Theorem 3,
$2n + 1 \leq \dim_{\mathcal{A}_k}(C_n(X)) \leq \dim[V] \leq \dim[N] = 2m$, a contradiction. Therefore $A \cap R(X) = \emptyset$.

Since $R(X)$ is closed in $X$, there exists an open neighborhood $\mathcal{W}$ of $A$ in $C_n(X)$ such that for each $B \in \mathcal{W}$, $B \cap R(X) = \emptyset$. This implies that $\mathcal{W} \cap (C_n(X) \setminus (\mathcal{M}_{m+1}(X) \cup \cdots \cup \mathcal{M}_{n+1}(X))) \subset C_m(X)$. Thus

$$\mathcal{W} \cap \mathcal{M} = \mathcal{W} \cap C_m(X).$$

Since $\mathcal{W} \cap \mathcal{M}$ is an open subset of $A$ in $\mathcal{M}$ and $D$ is a $2m$-cell neighborhood of $A$ in $\mathcal{M}$, there exists a $2m$-subcell $D_1$ of $D$ such that $D_1 \subset \mathcal{W} \cap \mathcal{M}$ and $A \notin \text{cl}_D(D \setminus D_1)$. Since $A \in \text{int}_\mathcal{M}(D)$, there is an open subset $\mathcal{Y}$ in $C_n(X)$ such that $A \in \mathcal{Y} \cap \mathcal{M} \subset \mathcal{D}$. Then $A \in (\mathcal{W} \cap \mathcal{Y}) \setminus \text{cl}_D(D \setminus D_1) \cap \mathcal{M} = (\mathcal{W} \cap \mathcal{Y}) \setminus \text{cl}_D(D \setminus D_1) \cap \mathcal{M} \subset D_1 \subset C_m(X)$. Thus $D_1$ is a $2m$-cell neighborhood of $A$ in $C_m(X)$. Therefore $A \in L_m(X) = \mathcal{M}_m(X)$.

Now take an element $A \in \mathcal{M}_m(X) \subset C_m(X)$. Then $A$ has exactly $m$ components and $A \cap R(X) = \emptyset$. Thus $A \notin \mathcal{M}_{m+1}(X) \cup \cdots \cup \mathcal{M}_{n+1}(X)$, so $A \in \mathcal{M}$. Since $\mathcal{M}_m(X) = L_m(X)$, there exists a $2m$-cell neighborhood $\mathcal{D}$ of $A$ in $C_m(X) \subset \mathcal{M}$. Let $\mathcal{W}$ be an open subset of $C_n(X)$ such that $A \in \mathcal{W} \cap C_m(X) \subset \mathcal{D}$ and for each $B \in \mathcal{W}$, $B \cap R(X) = \emptyset$.

Given $B \in \mathcal{W} \cap \mathcal{M}$, we have that $B \in C_n(X) \setminus (\mathcal{M}_{m+1}(X) \cup \cdots \cup \mathcal{M}_{n+1}(X))$ and $B \cap R(X) = \emptyset$. This implies that $B \in C_m(X)$, so $B \in \mathcal{D}$. Thus $\mathcal{W} \cap \mathcal{M} \subset \mathcal{D}$. Hence $D$ is a $2m$-cell neighborhood of $A$ in $\mathcal{M}$ and $A \in R_m(X)$.

We have shown that $\mathcal{M}_m(X) = R_m(X)$, so the lemma is proved.

**Lemma 6** Let $3 \leq m \leq n$ and $h : C_n(X) \to C_n(X)$ a homeomorphism. Then $h(\mathcal{M}_m(X)) = \mathcal{M}_m(X)$.

**Proof.** In the case that $n = m$, $R_m(X) = \{A \in C_n(X) : A$ has a $2n$-cell neighborhood in $C_n(X)\}$. Since $h$ is a homeomorphism, $h(R_m(X)) = R_m(X)$.

Now, we suppose that $3 \leq m < n$ and that $h(R_r(X)) = R_r(X)$ for each $r \in \{m+1, \ldots, n\}$. Then $h(C_n(X) \setminus (R_{m+1}(X) \cup \cdots \cup R_{n+1}(X))) = C_n(X) \setminus (R_{m+1}(X) \cup \cdots \cup R_{n+1}(X))$. Thus the equality $h(R_m(X)) = R_m(X)$ follows from the fact that $h$ is a homeomorphism.

**Lemma 7** Let $3 \leq m \leq n$ and $h : Z \to Z$ be a homeomorphism. Then $h(\varphi(\mathcal{M}_m(X)) \setminus \{Z_0\}) = \varphi(\mathcal{M}_m(X)) \setminus \{Z_0\}$.

**Proof.** We consider two cases.

**Case 1.** $m = n$.

First we check that $Z_0 \notin h(\varphi(\mathcal{M}_n(X)) \setminus \{Z_0\})$. Suppose to the contrary that there exists $B \in \varphi(\mathcal{M}_n(X)) \setminus \{Z_0\}$ such that $Z_0 = h(B)$ (and $B \neq Z_0$). Thus there exists $A \in \mathcal{M}_n(X)$ such that $B = \varphi(A)$. By Theorem 4 (b), $A \in L_n(X)$. Then there exists a $2n$-cell neighborhood $\mathcal{N}$ of $A$ in $C_n(X)$. Since $A \notin \varphi^{-1}(Z_0)$, we may assume that $\mathcal{N} \cap \varphi^{-1}(Z_0) = \emptyset$. Then $\varphi(\mathcal{N})$ is a $2n$-cell neighborhood of $B$ in $Z$, and $h(\varphi(\mathcal{N}))$ is a $2n$-cell neighborhood of $Z_0$ in $Z$. Fix a point $q \in X$. Then $\varphi(q) = Z_0 \in \text{int}_Z(h(\varphi(\mathcal{N})))$. Then it is possible to find an element $E \in C_n(X)$ (close to $\{q\}$) such that $E$ has exactly $n - 1$ components,
all of them are non-degenerate and $\varphi(E) \in \operatorname{int}_Z(h(\varphi(N)))$. Since $\varphi(E) \neq Z_0$, there exists a 2n-subcell $R$ of $h(\varphi(N))$ such that $\varphi(E) \in \operatorname{int}_Z(R)$ and $Z_0 \notin R$. Then $\varphi^{-1}(R)$ is a 2n-cell neighborhood of $E$ in $C_n(X)$. So $E \in L_n(X) = M_n(X)$, and $E$ has $n$ components, a contradiction. This ends the proof that $Z_0 \notin h(\varphi(M_n(X)) \setminus \{Z_0\})$.

We prove $h(\varphi(M_n(X)) \setminus \{Z_0\}) \subset \varphi(M_n(X)) \setminus \{Z_0\}$. Let $B \in h(\varphi(M_n(X)) \setminus \{Z_0\})$ and $A \in M_n(X) = L_n(X)$ be such that $B = h(\varphi(A))$ and $\varphi(A) \neq Z_0$. By the previous paragraph, $B \neq Z_0$. Let $N$ be a 2n-cell neighborhood of $A$ in $C_n(X)$. Since $A \notin \varphi^{-1}(Z_0)$, we may assume that $N \cap \varphi^{-1}(Z_0) = \emptyset$. Then $\varphi(N)$ is a 2n-cell neighborhood of $\varphi(A)$ in $Z$ and $h(\varphi(N))$ is a 2n-cell neighborhood of $B$ in $Z$. Since $B \neq Z_0$, there exists a 2n-cell neighborhood $R$ of $B$ in $Z$ such that $R \subset h(\varphi(N))$ and $Z_0 \notin R$. Then $\varphi^{-1}(R)$ is a 2n-cell neighborhood of $\varphi^{-1}(B)$ in $C_n(X)$. Thus $\varphi^{-1}(B) \in L_n(X) = M_n(X)$. Thus $B \in \varphi(M_n(X)) \setminus \{Z_0\}$. Therefore $h(\varphi(M_n(X)) \setminus \{Z_0\}) \subset \varphi(M_n(X)) \setminus \{Z_0\}$. Therefore $h(\varphi(M_n(X)) \setminus \{Z_0\}) = \varphi(M_n(X)) \setminus \{Z_0\}$.

This ends the proof for Case 1.

**Case 2.** Suppose that $3 \leq m < n$ and the conclusion holds for every $r \in \{m+1, \ldots, n\}$.

As in Case 1, first we check that $Z_0 \notin h(\varphi(M_m(X)) \setminus \{Z_0\})$. Suppose to the contrary that there exists $B \in \varphi(M_m(X)) \setminus \{Z_0\}$ such that $Z_0 = h(B)$. Thus there exists $A \in M_m(X)$ such that $B = \varphi(A)$. By Lemma 5, we have $A \in R_m(X)$. Let $W = C_n(X) \setminus (R_{m+1}(X) \cup \cdots \cup R_{n+1}(X)) = C_n(X) \setminus (M_{m+1}(X) \cup \cdots \cup M_{n+1}(X))$. Then $A$ has a 2m-cell neighborhood $D$ in $W$. Since $A \notin \varphi^{-1}(Z_0)$, we may assume that $D \cap \varphi^{-1}(Z_0) = \emptyset$.

Let $U$ be an open subset of $C_n(X)$ such that $A \in U \cap W \subset D$, we may assume that $U \cap \varphi^{-1}(Z_0) = \emptyset$. Then $\varphi(U)$ and $h(\varphi(U))$ are open in $Z$, $B \in \varphi(U)$ and $Z_0 \in h(\varphi(U))$. Fix a point $q \in X$. Then $\varphi(q) = Z_0 \in h(\varphi(U))$. Then it is possible to find an element $E \in C_n(X)$ (close to $\{q\}$) such that $E$ has exactly $m-1$ components, all of them are non-degenerate, $E \cap R(X) = \emptyset$, $h(Z_0) \neq h(E)$ and $\varphi(E) \in h(\varphi(U))$. Observe that $E \notin M_{m+1}(X) \cup \cdots \cup M_{n+1}(X)$. Let $D \in C_n(X)$ be such that $h(\varphi(D)) = \varphi(E)$.

If there exists $i \in \{m+1, \ldots, n+1\}$ such that $D \in M_i(X)$, since $\varphi(D) \neq Z_0$, by the hypothesis we have that $\varphi(E) = h(\varphi(D)) \in h(\varphi(M_i(X)) \setminus \{Z_0\}) = \varphi(M_i(X)) \setminus \{Z_0\}$. This implies that $E \in M_i(X)$, a contradiction. We have shown that $D \notin M_{m+1}(X) \cup \cdots \cup M_{n+1}(X)$. Therefore, $D \in W$. Since $h(\varphi(D)) = \varphi(E) \in h(\varphi(U))$ and $\varphi(D) \neq Z_0$, we obtain that $D \in U$. Hence $D \in D$.

Since $h(\varphi(D)) = \varphi(E) \neq Z_0$, there exists a 2m-subcell $D_0$ of $D$ such that $D \notin M_{m+1}(X)$ and $Z_0 \notin h(\varphi(D_0))$, so $D \in D_0$. Since $D \cap \varphi^{-1}(Z_0) = \emptyset$, we have that $\varphi(D_0)$, $h(\varphi(D_0))$ and $\varphi^{-1}(h(\varphi(D_0)))$ are homeomorphic to $D_0$. Thus the set $S = \varphi^{-1}(h(\varphi(D_0)))$ is a 2m-cell containing $E$.

We prove that $S \subset W$. Suppose to the contrary that there exists $F \in S$ such that $F \in M_i(X)$ for some $i \in \{m+1, \ldots, n+1\}$. Since $\varphi(F) \notin \varphi(E)$, we have that $\varphi(F) \notin \varphi(M_i(X)) \setminus \{Z_0\} = h(\varphi(M_i(X)) \setminus \{Z_0\})$. So there exists $G \in M_i(X)$ such that $h(\varphi(G)) = \varphi(F) \in \varphi(\varphi^{-1}(h(\varphi(D_0)))) = h(\varphi(D_0))$. Then
\( \varphi(G) \in \varphi(D_0) \). Since \( Z_0 \notin \varphi(D) \), we conclude that \( G \in D_0 \subset W \). Thus \( G \in W \cap M_i \), a contradiction. We have shown that \( E \subset W \).

Let \( Q = (\varphi^{-1}(h(\varphi(U))) \setminus (\varphi^{-1}((Z_0) \cup h(\varphi(\text{cl}_{C_m}(X) \setminus D \setminus D_0))) \cap W). \) Since \( h(\varphi(U)) \) is open in \( Z \), \( Q \) is open in \( W \).

We prove that \( Q \in E \). Given \( Q \in \mathbb{R} \), there exists \( P \in U \) such that \( \varphi(Q) = h(\varphi(P)) \), \( Q \notin \varphi^{-1}((Z_0) \cup h(\varphi(\text{cl}_{C_m}(X) \setminus D \setminus D_0))) \) and \( Q \in W \). We see that \( P \in W \). Suppose to the contrary that \( P \in M_i(X) \) for some \( i \in \{m + 1, \ldots, n + 1\} \). Since \( P \in U \), we have that \( \varphi(P) \notin Z_0 \). Then \( h(\varphi(P)) \in h(\varphi(M_i(X)) \setminus \{Z_0\}) = \varphi(M_i(X)) \setminus \{Z_0\} \). Thus \( \varphi(Q) \in \varphi(M_i(X)) \setminus \{Z_0\} \), and \( Q \in M_i(X) \). This contradicts the fact that \( Q \in W \) and proves that \( P \in W \). Since \( P \in U \cap W \subset D \), we have that \( Q \in \varphi^{-1}(h(\varphi(D))) \). Furthermore, \( Q \notin \varphi^{-1}(h(\varphi(D \setminus D_0)))\). This implies that \( Q \notin \varphi^{-1}(h(\varphi(D))) \). This completes the proof that \( Q \in E \).

Since \( \varphi(E) = h(\varphi(D)) \in h(\varphi(U)) \), \( D \notin \text{cl}_{C_m}(X) \setminus D \) and \( \varphi(D) \neq Z_0 \), we obtain that \( \varphi(E) \notin h(\varphi(\text{cl}_{C_m}(X) \setminus D \setminus D_0))) \). This implies that \( E \subset Q \).

We have shown that \( E \) is a 2m-cell neighborhood of \( E \) in \( W \). By the definition of \( R_m \), \( E \in R_m(X) = M_m(X) \) and \( E \) has exactly \( m \) elements. This contradicts the choice of \( E \) and completes the proof that \( Z_0 \notin h(\varphi(M_m(X)) \setminus \{Z_0\}) \).

Finally, we prove that \( h(\varphi(M_m(X)) \setminus \{Z_0\}) = \varphi(M_m(X)) \setminus \{Z_0\} \). Take \( B \in h(\varphi(M_m(X)) \setminus \{Z_0\}) \) and \( A \in M_m(X) \) such that \( h = h(\varphi(A)) \) and \( \varphi(A) \neq Z_0 \). Observe that \( A \cap R(X) = \emptyset \). By Theorem 3, \( \dim_A(C_m(X)) = 2n \).

Since \( A \cap \varphi^{-1}(Z_0) = \emptyset \), there exists an open subset \( U \) of \( C_m(X) \) such that \( A \in U \) and \( E \cap (R(X) \cup \varphi^{-1}(Z_0)) = \emptyset \) for each \( E \in U \). Since the number of components of \( A \) is \( m \), we can ask that for each \( E \in U \), \( E \) has at least \( m \) components.

Given \( E \in U \), let \( j \) be the number of components of \( E \). Then \( m \leq j \leq n \).

Since \( E \cap R(X) = \emptyset \), \( E \in M_j(X) \). By the induction hypothesis and what we have shown for \( j = m \), we have that \( Z_0 \notin h(\varphi(M_j(X)) \setminus \{Z_0\}) \).

Since \( \varphi(E) \neq Z_0 \), we obtain that \( h(\varphi(E)) \neq Z_0 \).

Let \( \psi : U \rightarrow C_m(X) \) be the mapping \( \psi = (\varphi^{-1} \circ h \circ \varphi)|_U \). Since \( Z_0 \notin \varphi(U) \) and \( Z_0 \notin h(\varphi(U)) \), we infer that \( \psi \) is a homeomorphism between \( U \) and the open subset \( V = \varphi^{-1}(h(\varphi(U))) \) of \( C_m(X) \).

Given \( E \in U \), by Theorem 3, \( \dim_E[C_m(X)] = 2n \). Then \( \dim_{\psi(E)}[C_m(X)] = 2n \). Theorem 3 also implies that \( \psi(E) \cap R(X) = \emptyset \). If there exists \( i \in \{m + 1, \ldots, n + 1\} \) such that \( \psi(E) \in M_i(X) \), then the hypothesis implies that \( h(\varphi(E)) \in \varphi(M_i(X)) \setminus \{Z_0\} = h(\varphi(M_i(X)) \setminus \{Z_0\}) \).

Thus \( \varphi(E) \in \varphi(M_i(X)) \setminus \{Z_0\} \).

Hence \( E \in M_i(X) \).

We have shown that, for each \( i \in \{m + 1, \ldots, n + 1\} \), \( E \notin M_i(X) \), then \( \psi(E) \notin M_i(X) \). Since \( \psi(E) \cap R(X) = \emptyset \), we have the following implication: if \( E \in U \cap C_m(X) \), then \( \psi(E) \in V \cap C_m(X) \). The converse implication can be proved in a similar way. Hence \( \psi(U \cap C_m(X)) = V \cap C_m(X) \).

Since the number of components of \( A \) is \( m \) and \( U \cap C_m(X) \) is an open subset of \( C_m(X) \) containing \( A \), by Theorem 4 (b), there exists a 2m-cell neighborhood \( A \) of \( A \) in \( C_m(X) \) and we may assume that \( A \subset U \cap C_m(X) \).

Since \( \psi \) is a homeomorphism, \( \psi(A) \) is a 2m-cell neighborhood of \( \psi(A) = \varphi^{-1}(B) \) in \( C_m(X) \).

Applying again Theorem 4 (b), we obtain that \( \varphi^{-1}(B) \in L_m(X) = M_m(X) \).

Hence \( B \in \varphi(M_m(X)) \setminus \{Z_0\} \). This ends the proof that \( h(\varphi(M_m(X)) \setminus \{Z_0\}) \subset \varphi(M_m(X)) \setminus \{Z_0\} \).
By the symmetry of the roles of $h$ and $h^{-1}$, $h^{-1}(\varphi(M_m(X)) \setminus \{Z_0\}) \subset \varphi(M_m(X)) \setminus \{Z_0\}$. Therefore $h(\varphi(M_m(X)) \setminus \{Z_0\}) = \varphi(M_m(X)) \setminus \{Z_0\}$. □

Given a topological space $Y$ and a point $y \in Y$, define

$$o(y) = \{v \in Y : \text{there is a homeomorphism } h : Y \to Y \text{ such that } h(y) = v\}.$$  

Fix an arc $J_0$ in $X$, with end points $a$ and $b$ such that $J \cap R(X) = \emptyset$, then the set $J = J_0 \setminus \{a, b\}$ is open in $X$. Given $2 \leq m \leq n$, fix elements $A_m$ and $B_m$ in $C_m(X) \setminus C_{m-1}(X)$ such that $A_m \cup B_m \subset J$, all the components of $A_m$ are non-degenerate and all but one of the components of $B_m$ are non-degenerate. Then $(A_m \cup B_m) \cap R(X) = \emptyset$. Also fix an arc $A_1$ contained in $J$ and a one-point set $B_1$ contained in $J$.

**Theorem 8** Let $3 \leq n$. Then

(a) the subsets $o(A_3), \ldots, o(A_n), o(B_3), \ldots, o(B_n)$ of $C_n(X)$ are pairwise distinct, and

(b) the subsets $o(\varphi(A_3)), \ldots, o(\varphi(A_n)), o(\varphi(B_3)), \ldots, o(\varphi(B_n))$ of $Z$ are pairwise distinct.

**Proof.** We prove (b), the proof for (a) is similar and easier.

Take $m \in \{3, \ldots, n\}$ and a homeomorphism $h : Z \to Z$. Since $A_m \in M_m(X)$ and $A_m \notin F_n(X)$, we have that $\varphi(A_m) \in \varphi(M_m(X)) \setminus \{Z_0\}$. By Lemma 7, $h(\varphi(A_m)) \in \varphi(M_m(X)) \setminus \{Z_0\}$.

If $h(\varphi(A_m)) \in \{\varphi(A_r), \varphi(B_r)\}$ for some $r \in \{3, \ldots, n\}$, then $\varphi(A_r) \in \varphi(M_m(X))$ and $A_r \in M_m(X)$, so $r = m$. Similarly, in the case that $h(\varphi(B_m)) \in \{\varphi(A_r), \varphi(B_r)\}$ for some $r \in \{3, \ldots, n\}$, we have that $r = m$. Hence the sets

$$\{o(\varphi(A_m)), o(\varphi(B_m)), \ldots, o(\varphi(A_n)), o(\varphi(B_n))\}$$

are pairwise disjoint.

**Claim 1.** If $m \in \{3, \ldots, n\}$, then $o(\varphi(A_m)) \neq o(\varphi(B_m))$.

In order to prove this claim, let

$$\mathfrak{A} = \{A \in \varphi(M_m(X)) \setminus \{Z_0\} : \text{there exists a } 2m\text{-cell neighborhood } A \text{ in } \varphi(M_m(X)) \setminus \{Z_0\} \text{ that contains } A \text{ in its interior as manifold}\},$$

and

$$\mathfrak{B} = \{B \in \varphi(M_m(X)) \setminus \{Z_0\} : \text{there exists a } 2m\text{-cell neighborhood } B \text{ in } \varphi(M_m(X)) \setminus \{Z_0\} \text{ that contains } B \text{ in its boundary as manifold}\}.$$

Clearly, $\mathfrak{A} \cap \mathfrak{B} = \emptyset$. Given a homeomorphism $h : Z \to Z$, Lemma 7 implies that $h(\mathfrak{A}) = \mathfrak{A}$ and $h(\mathfrak{B}) = \mathfrak{B}$.

In order to show that $o(A_m) \neq o(B_m)$, it is enough to show that $\varphi(A_m) \in \mathfrak{A}$ and $\varphi(B_m) \in \mathfrak{B}$. We prove $\varphi(B_m) \in \mathfrak{B}$. The proof that $\varphi(A_m) \in \mathfrak{A}$ is similar and easier. Fix pairwise disjoint subarcs $C_1, \ldots, C_m$ of $X$ such that for each $i \in \{1, \ldots, m\}$, $\text{int}_X(C_i)$ contains the $i^\text{th}$-component of $B_m$ and $(C_1 \cup \cdots \cup C_m) \cap R(X) = \emptyset$. Let

$$\mathcal{D} = \{D \in C_m(X) : D \subset C_1 \cup \cdots \cup C_m \text{ and } D \cap C_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\}\}.$$
Observe that \( \mathcal{D} \subset \mathcal{M}_m(X) \), \( B_m \in \text{int}_{C_m(X)}(\mathcal{D}) \) and \( B_m \in \text{int}_{\mathcal{M}_m(X)}(\mathcal{D}) \). It is easy to show that the function \( \psi : C(C_1) \times \cdots \times C(C_m) \to \mathcal{D} \) given by \( \psi(D_1, \ldots, D_m) = D_1 \cup \cdots \cup D_m \) is a homeomorphism. Since each hyperspace \( C(C_1) \) is a 2-cell, we obtain that \( \mathcal{D} \) is a 2m-cell. Since \( B_m \) has a degenerate component, we may assume that \( B_m \cap C_1 \) is a one-point set. Then [11, p. 41] \( B_m \cap C_1 \) belongs to the manifold boundary of the 2-cell \( C(C_1) \). Thus \( B_m \) belongs to the manifold boundary of the 2m-cell \( \mathcal{D} \). Since \( \varphi(B_m) \neq Z_0 \), there exists a 2m-subcell \( D_0 \) of \( \mathcal{D} \) such that \( B_m \in \text{int}_{\mathcal{M}_m(X)}(D_0) \) and \( D_0 \cap \varphi^{-1}(Z_0) = \emptyset \). Hence \( \varphi(D_0) \) is a 2m-cell neighborhood of \( \varphi(B_m) \) in \( \varphi(\mathcal{M}_m(X)) \setminus \{Z_0\} \) and \( \varphi(B_m) \) is in the manifold boundary of \( \varphi(D_0) \). Hence \( \varphi(B_m) \in \mathfrak{B} \). This finishes the proof of Claim 1.

The theorem is proved. ■

**Theorem 9** If \( 3 \leq n \), then:

(a) \( \text{hd}(C_n(X)) \geq 2n \),

(b) \( \text{hd}(Z) \geq 2n - 1 \),

(c) \( \text{hd}(C_n(S^1)) \geq 2n + 1 \).

**Proof.** Given \( m \geq 3 \), if \( o(\varphi(A_1)) = o(\varphi(A_m)) \), then there exists a homeomorphism \( h : Z \to Z \) such that \( h(\varphi(A_m)) = \varphi(A_1) \). By Lemma 7, we have that \( \varphi(A_1) \in h(\varphi(\mathcal{M}_m(X)) \setminus \{Z_0\}) = \varphi(\mathcal{M}_m(X)) \setminus \{Z_0\} \). Since \( A_1 \notin F_k(X) \), we conclude that \( A_1 \in \mathcal{M}_m(X) \), a contradiction. We have shown that \( o(\varphi(A_1)) \neq o(\varphi(A_m)) \) for all \( m \geq 3 \). Similarly, \( o(\varphi(A_1)) \neq o(\varphi(B_m)) \) for all \( m \geq 3 \). Therefore, for each \( m \geq 3 \), \( o(\varphi(A_1)) \notin \{o(\varphi(A_m)), o(\varphi(B_m))\} \).

Proceeding as in the previous paragraph, it is possible to show that for each \( m \geq 3 \), we have that \( \{o(\varphi(A_2)), o(\varphi(B_2))\} \cap \{o(\varphi(A_m)), o(\varphi(B_m))\} = \emptyset \) and \( \{o(A_1), o(B_1), o(A_2), o(B_2)\} \cap \{o(A_m), o(B_m)\} = \emptyset \).

Given a homeomorphism \( h : C_n(X) \to C_n(X) \), by Lemma 6, \( h(\mathcal{M}_n(X)) = \mathcal{M}_n(X) \). So all the properties included in the definition of \( \mathcal{E}_n(X) \) are preserved under homeomorphisms. Then \( h(\mathcal{E}_n(X)) = \mathcal{E}_n(X) \). By Theorem 4 (c), \( h(\mathcal{M}_n(X)) = \mathcal{M}_n(X) \). Then Lemma 6 implies that for each \( m \in \{1, 3, \ldots, n\} \), \( h(\mathcal{M}_m(X)) = \mathcal{M}_m(X) \). Theorem 3 implies that \( \{A \in C_n(X) : A \cap R(X) = \emptyset\} = \{A \in C_n(X) : \dim A C_n(X) \leq 2n\} \). Hence \( \{A \in C_n(X) : A \cap R(X) = \emptyset\} = \{A \in C_n(X) : A \cap R(X) = \emptyset\} \). Since \( \{A \in C_n(X) : A \cap R(X) = \emptyset\} = \bigcup \mathcal{M}_m(X) \), we conclude that \( \mathcal{M}_2(X) = \mathcal{M}_2(X) \). This implies that \( \{o(A_1), o(B_1), o(A_2), o(B_2)\} = \emptyset \).

**Claim 2.** \( \{o(\varphi(A_2)), o(\varphi(B_2))\} \cap \{o(\varphi(A_1))\} = \emptyset \).

We prove Claim 2. We only prove that \( o(\varphi(A_2)) \neq o(\varphi(A_1)) \). The proof that \( o(\varphi(B_2)) \neq o(\varphi(A_1)) \) is analogous.

Suppose to the contrary that there exists a homeomorphism \( h : Z \to Z \) such that \( h(\varphi(A_2)) = \varphi(A_1) \). Since \( A_2 \notin \mathcal{W}_1(X) \), by Theorem 4 (c), we have that \( A_2 \notin \mathcal{E}_n(X) \). Since \( A_2 \in C_n(X) \setminus \mathcal{M}_n(X) \), we have that \( A_2 \) does not have a basis of neighborhoods \( \mathfrak{B} \) in \( C_n(X) \) as the one described in the definition of \( \mathcal{E}_n(X) \). Then there exists an open neighborhood \( W \) of \( A_2 \in C_n(X) \) such that it is not possible to find an open neighborhood \( U \) of \( A_2 \) in \( C_n(X) \) such that
\( U \subset W, \dim[U] \leq 2n \) and \( U \cap L_n(X) \) is arcwise connected. Since \( \varphi(A_2) \neq Z_0 \) and \( \varphi(A_1) \neq Z_0 \), we may ask that \( W \cap (\varphi^{-1}(Z_0) \cup (h \circ \varphi)^{-1}(Z_0)) = \emptyset \). Since \( A_2 \cap R(X) = \emptyset \), we may also ask that for each \( B \in W, B \cap R(X) = \emptyset \). Then, by Theorem 3, \( \dim[W] = 2n \). Thus it is possible to consider the homeomorphism \( \psi = (\varphi^{-1} \circ h \circ \varphi)|_W : W \to \psi(W) \subset C_n(X) \). Since \( \psi(W) \) is an open subset of \( C_n(X) \) containing \( A_1 \) and \( A_1 \in M_1(X) = E_n(X) \), there exists an open neighborhood \( U_0 \) of \( A_1 \) in \( C_n(X) \) such that \( U_0 \subset \psi(W) \cap [U] \leq 2n \) and \( U_0 \cap L_n(X) \) is arcwise connected. Set \( U = \psi^{-1}(U_0) \subset W \). Hence \( \dim[U] \leq \dim[W] = 2n \).

Given \( A \in U_0 \cap L_n(X) \), we have that \( A \in U_0 \) and there exists a 2n-cell neighborhood \( N \) of \( A \) in \( C_n(X) \). Since \( U_0 \) is open in \( C_n(X) \), there exists a 2n-cell neighborhood \( N_0 \subset N \cap U_0 \) of \( A \) in \( C_n(X) \). Then \( \psi^{-1}(N_0) \) is a 2n-cell neighborhood of \( \psi^{-1}(A) \) in \( C_n(X) \). Thus \( \psi^{-1}(U_0 \cap L_n(X)) \subset U \cap L_n(X) \).

Similarly, \( \psi^{-1}(U_0 \cap L_n(X)) \supset U \cap L_n(X) \). Therefore \( \psi^{-1}(U_0 \cap L_n(X)) = U \cap L_n(X) \). Hence \( U \cap L_n(X) \) is arcwise connected. Since \( A_2 = \psi^{-1}(A_1) \in U \), we obtain a contradiction with the choice of \( W \). Therefore \( o(\varphi(A_2)) \neq o(\varphi(A_1)) \).

So, Claim 2 is proved.

Claim 3. \( o(\varphi(A_2)) \neq o(\varphi(B_2)) \) and \( o(A_2) \neq o(B_2) \).

We only prove that \( o(\varphi(A_2)) \neq o(\varphi(B_2)) \), the proof that \( o(A_2) \neq o(B_2) \) is similar. We suppose to the contrary that there exists a homeomorphism \( h : Z \to Z \) such that \( h(\varphi(A_2)) = \varphi(B_2) \).

Since \( \varphi(A_2) \neq Z_0 \) and \( \varphi(B_2) \neq Z_0 \), we can take an open neighborhood \( W \) of \( A_2 \) in \( C_n(X) \) such that \( W \cap (\varphi^{-1}(Z_0) \cup (h \circ \varphi)^{-1}(Z_0)) = \emptyset \). Then it is possible to consider the homeomorphism \( \psi : (\varphi^{-1} \circ h \circ \varphi)|_W : W \to \psi(W) \subset C_n(X) \).

Suppose that the components of \( B_2 \) are \( E_1 \) and \( E_2 \) and \( E_2 \) is a one-point set. Fix two disjoint arcs \( L_1 \) and \( L_2 \) contained in \( J \) such that for each \( i \in \{1, 2\}, E_i \subset \text{int}_X(L_i) \). Let \( V = \{ A \in C_n(X) : A \subset \text{int}_X(L_1) \cup \text{int}_X(L_2), A \cap \text{int}_X(L_1) \neq \emptyset \} \). Then \( V \) is an open subset of \( C_n(X) \) containing \( B_2 \).

Let \( U = \psi^{-1}(\psi(W) \cap V) \). Then \( U \) is an open subset of \( C_n(X) \) containing \( A_2 \).

Let \( D_1 \) and \( D_2 \) the components of \( A_2 \). By the choice of \( A_2 \), \( D_1 \) and \( D_2 \) are arcs contained in \( J \). Fix two disjoint arcs \( K_1, K_2 \) in \( J \) such that for each \( i \in \{1, 2\}, D_i \subset \text{int}_X(K_i) \). It is easy to show that the set \( D = \{ A \in C_2(X) : A \subset K_1 \cup K_2, A \cap K_1 \neq \emptyset \} \) is a 4-cell containing \( A_2 \) in its interior as manifold. Then there exists a 4-cell \( D_0 \) such that \( A_2 \) is in the interior as manifold of \( D_0 \), each element of \( D_0 \) is the union of two non-degenerate components and \( D_0 \subset D \cap U \). Since \( \psi \) is a homeomorphism and \( \psi(A_2) = B_2 \), \( \psi(D_0) \) is a 4-cell containing \( B_2 \) in its interior as manifold and \( \psi(D_0) \subset V \).

Given \( A \in D_0 \), \( \varphi(A) \) plays a similar role as \( \varphi(A_2) \) (it is the union of two non-degenerate components and it is contained in \( J \)). Proceeding as in the beginning of the proof of this theorem, we can conclude that \( h(\varphi(A)) \neq \varphi(M_m(X)) \) and \( \varphi(A) \neq M_m(X) \) for any \( m \in \{3, \ldots, n\} \). Thus \( \psi(A) \) is the union of two components, one contained in \( L_1 \) and other contained in \( L_2 \). We have shown that \( \psi(D_0) \) is a subset of the set \( \mathcal{E} = \{ B \in C_2(X) : B \subset L_1 \cup L_2 \} \). Observe that \( \mathcal{E} \) is a 4-cell. Since \( B_2 \cap L_2 \) is a one-point set, we obtain that \( B_2 \) is in the boundary as manifold of \( \mathcal{E} \). This contradicts...
the fact that \( \psi(D_0) \) is a 4-cell contained in \( E \) and containing \( B_2 \) in its interior as manifold. This finishes the proof of Claim 3.

By the first three paragraphs of the proof of Theorem 9, Claim 2 and Claim 3, we obtain that (b) holds.

Now we have the tools for proving the theorem. By the same reasons as in the (b), in order to finish the proof of (a), we only need to prove that \( o(A_1) \neq o(B_1) \).

Suppose to the contrary that there exists a homeomorphism \( h : C_n(X) \to C_n(X) \) such that \( h(A_1) = B_1 \). By Lemma 6, \( h(M_n(X)) = M_n(X) \). This implies that \( h(E_n(X)) = E_n(X) \), and by Theorem 4 (c), \( h(M_1(X)) = M_1(X) \).

Since \( J \) is an open subset of \( X \), the set \( W = \{ A \in C_n(X) : A \subset J \} \) is open in \( C_n(X) \). Since \( B_1 \in W \) and \( A_1 \subset J \), there exists an open subset \( U \) of \( C_n(X) \) such that \( A_1 \in U \subset W \) and \( h(U) \subset W \). Since \( C(J) \) is a 2-cell containing \( A_1 \) in its interior as manifold [11, p. 41], there exists a 2-cell \( D \subset C(J) \cap U \) such that \( A_1 \) is in the interior as manifold of \( D \). Then \( h(D) \) is a 2-cell containing \( B_1 \) in its interior as manifold.

Given \( A \in D \), we have that \( A \) is connected and \( A \cap R(X) = \emptyset \). So \( A \in M_1(X) \). Then \( h(A) \in M_1(X) \), so \( h(A) \in W \) and it is a connected subset of \( J \). Thus \( h(A) \in C(J) \). We have shown that \( h(D) \subset C(J) \). Since \( C(J) \) is a 2-cell and \( B_1 \) is a one-point set in \( C(J) \), we have that \( B_1 \) is in the boundary as manifold of \( C(J) \). This implies that \( B_1 \) is not in the interior as manifold of \( h(D) \), a contradiction. This completes the proof of (a).

Finally, we prove (c). It is enough to show that \( o(S^1) \neq \{ o(A_1), \ldots, o(A_n) \} \cup \{ o(B_1), \ldots, o(B_n) \} \). We have constructed the sets \( A_1, \ldots, A_n, B_1, \ldots, B_n \) in such a way that they are distinct from \( S^1 \). So it is enough to show that if \( h : C_n(S^1) \to C_n(S^1) \) is a homeomorphism, then \( h(S^1) = S^1 \).

Since \( R(S^1) = \emptyset, C_n(S^1) = \bigcup \{ M_m(S^1) : m \in \{1, \ldots, n\} \} \). Taking complements, Lemma 6, implies that \( h(C_2(S^1)) = h(M_1(S^1) \cup M_2(S^1)) = M_1(S^1) \cup M_2(S^1) = C_2(S^1) \). Thus \( h|_{C_2(S^1)} : C_2(S^1) \to C_2(S^1) \) is a homeomorphism.

By Lemma 4.2 of [18], \( S^1 \) is the only element in \( C_2(S^1) \) not having a 4-cell neighborhood in \( C_2(S^1) \). Thus \( h(S^1) = S^1 \). 

**Definition 10** Given \( A, B \in C(X) \) such that \( A \subsetneq B \), an order arc from \( A \) to \( B \) is a mapping \( \alpha : [0,1] \to C(X) \) with the following properties: \( \alpha(0) = A \), \( \alpha(1) = B \) and if \( 0 \leq s < t \leq 1 \), then \( \alpha(s) \subsetneq \alpha(t) \). Given \( m \in \mathbb{N} \), an \( m \)-od is a subcontinuum \( B \) of \( X \) for which there exists a subcontinuum \( A \) of \( B \) such that \( B \setminus A \) has at least \( m \) components.

Given an \( A, B \in C(X) \), such that \( A \subsetneq B \), the existence of order arcs from \( A \) to \( B \) is proved in [13, Theorem 14.6]. If \( B \) is an \( m \)-od and \( B \setminus A \) has at least \( m \) components, then there exist \( m \) pairwise separated nonempty sets \( C_1, \ldots, C_m \) such that \( B \setminus A = C_1 \cup \cdots \cup C_m \). Then for each \( i \in \{1, \ldots, m\} \), there exists an order arc \( \alpha_i : [0,1] \to A \cup C_i \) such that \( \alpha_i(0) = A \) and \( \alpha_i(1) = A \cup C_i \). Thus we can consider the mapping \( \sigma : [0,1]^m \to C(B) \) given by \( \sigma(t_1, \ldots, t_m) = \alpha_1(t_1) \cup \cdots \cup \alpha_m(t_m) \). Then [13, section 70] \( \sigma \) is an embedding. So, the set \( B = \text{Im} \sigma \) is an \( m \)-cell having \( A \) and \( B \) in its boundary as manifold. In the case that \( B \) is an \( m \)-od, \( D \) is an \( r \)-od and \( B \cap D = \emptyset \), we consider mappings \( \sigma_B : [0,1]^m \to C(B) \) and \( \sigma_D : [0,1]^r \to C(D) \) as before. Then the mapping
σ : [0, 1]m × [0, 1]r → C2(X) given by σ(s, t) = σB(s)∪σD(t) is an embedding and B∪D belongs to the boundary as manifold of the (m+r)-cell Im σ. Moreover, it is easy to identify the elements in the interior (and in the boundary) as manifold of Im σ.

**Theorem 11** If R(X) ≠ ∅, then for, n = 2, 5 ≤ hd(Z) and 5 ≤ hd(C2(X)).

**Proof.** Fix a point p ∈ R(X). Suppose that the order of p in X is m. Fix elements A1, A2, A3, A4 and A5 in C2(X) as follows.

Let A1 = B1 ∪ B2, where B1 ∩ B2 = ∅, p ∈ intX(B1), B1 is a simple m-od with p as its vertex, B2 is an arc and A1 ∩ R(X) = {p}; A2 = D1 ∪ D2, where D1 ∩ D2 = ∅, p ∈ intX(D1), D1 is a simple m-od with p as its vertex, D2 is a one-point set and A2 ∩ R(X) = {p}; A3 = E1 ∪ E2, where E1 and E2 are disjoint arcs, p is an end-point of E1 and A3 ∩ R(X) = {p}; A4 = G1 ∪ G2, where G1 and G2 are disjoint arcs and A4 ∩ R(X) = ∅; and A5 = H1 ∪ H2, where H1 ∩ H2 = ∅, H1 is an arc, H2 is a one-point set A5 ∩ R(X) = ∅. We suppose that A1 ∪ A2 ∪ A3 ∪ A4 ∪ A5 does not contain end-points of X. We are going to show that the sets o(ϕ(A1)), o(ϕ(A2)), o(ϕ(A3)), o(ϕ(A4)) and o(ϕ(A5)) are pairwise distinct, and the same happens with the sets o(A1), o(A2), o(A3), o(A4) and o(A5).

By Theorem 3, 5 ≤ dimA1[Cn(X)] = dimA2[Cn(X)] = dimA3[Cn(X)]. Then 5 ≤ dimϕ(A1)[Z] = dimϕ(A2)[Z] = dimϕ(A3)[Z]. Also, we have that 4 = dimA4[Cn(X)] = dimA5[Cn(X)] = dimϕ(A4)[Z] = dimϕ(A5)[Z].

Using ideas similar to those discussed in the paragraph before Theorem 11, it is possible to see that: A1 (resp., ϕ(A1)) has a (m + 2)-cell neighborhood M in C2(X) (resp., Z) such that A1, (resp., ϕ(A1)) is in the interior as manifold of M; A2 (resp., ϕ(A2)) has a (m + 2)-cell neighborhood M in C2(X) (resp., Z) such that A2 (resp., ϕ(A2)) is in the boundary as manifold of M; A3 (resp., ϕ(A3)) can be approximated by elements K in C2(X) (resp., Z) such that dim[K][C2(X)] = 4 (resp., dim[K][Z] = 4); A4 (resp., ϕ(A4)) has a 4-cell neighborhood M in C2(X) (resp., Z) such that A4 (resp., ϕ(A4)) is in the interior as manifold of M; and A5 (resp., ϕ(A5)) has a 4-cell neighborhood M in C2(X) (resp., Z) such that ϕ(A5) is in the boundary as manifold of M. Observing the topological properties of the sets A1, A2, A3, A4 and A5; and ϕ(A1), ϕ(A2), ϕ(A3), ϕ(A4) and ϕ(A5), we conclude that the sets o(ϕ(A1)), o(ϕ(A2)), o(ϕ(A3)), o(ϕ(A4)) and o(ϕ(A5)) are pairwise distinct, and the same happens with the sets o(A1), o(A2), o(A3), o(A4) and o(A5). Therefore, 5 ≤ hd(Z) and 5 ≤ hd(C2(X)).

**Theorem 12** If R(X) ≠ ∅ and n = 1, then, 5 ≤ hd(Z) and 5 ≤ hd(C(X)).

**Proof.** We consider two cases.

**Case 1.** X contains at least two ramification points.

In this case it is possible to find an arc J in X that joins two ramification points p and q such that J ∩ R(X) = {p, q}. Suppose that the order of p (resp., q) in X is m (resp., r). Take a simple m-od, T, in X and a simple r-od, S in X such that T ∩ S = ∅, (T ∪ S) ∩ R(X) = {p, q}, p ∈ S, q ∈ T, p is the vertex
of $T$, $q$ is the vertex of $S$, $p \in \text{int}_X(T)$, $q \in \text{int}_X(S)$ and $T \cup S$ does not contain end-points of $X$. Take subarcs $I$ and $L$ of $J$ such that $I \cap L = \emptyset$, $p \in I$ and $q \notin L$. Define $A_1 = T \cup J \cup S$, $A_2 = T \cup J$, $A_3 = T$, $A_4 = I$ and $A_5 = L$.

By Theorem 3, $m + r - 2 = \dim_{oA_1}[C(X)] = \dim_{oA_2}[C(X)]$. So $m + r - 2 = \dim_{o(\varphi(A_1))}[Z] = \dim_{o(\varphi(A_2))}[Z]$. Moreover, $m = \dim_{A_1}[C(X)] = \dim_{A_2}[C(X)] = \dim_{o(\varphi(A_3))}[Z] = \dim_{o(\varphi(A_4))}[Z]$. Finally, $2 = \dim_{A_5}[C(X)] = \dim_{o(\varphi(A_5))}[Z]$.

Using ideas similar to those discussed in the paragraph before Theorem 11, it is possible to show that: $A_1$ (resp., $\varphi(A_1)$) has a $(m + r - 2)$-cell neighborhood $M$ in $C(X)$ (resp., $\varphi(M)$) satisfying $\dim_{K}[C(X)] = m$ (resp., $\dim_{K}[\varphi(M)] = m$); $A_3$ has a $m$-cell neighborhood $N$ in $C(X)$ (resp., $\varphi(N)$); $A_4$ (resp. $\varphi(A_4)$) can be approximated by elements $K$ in $C(X)$ (resp., $\varphi(K)$) such that $\dim_{K}[C(X)] = 2$ (resp., $\dim_{K}[\varphi(K)] = 2$); and $\dim_{A_5}[C(X)] = 2$ (resp., $\dim_{\varphi(A_5)}[C(X)] = 2$).

Observing the topological properties of the sets $A_1, A_2, A_3, A_4$ and $A_5$; and $\varphi(A_1), \varphi(A_2), \varphi(A_3), \varphi(A_4)$ and $\varphi(A_5)$, we conclude that the sets $o(\varphi(A_1)), o(\varphi(A_2)), o(\varphi(A_3))$ and $o(\varphi(A_4))$ are pairwise distinct, and the same happens with the sets $o(A_1), o(A_2), o(A_3), o(A_4)$ and $o(A_5)$. Therefore, in this case, $5 \leq \text{hd}(Z)$ and $5 \leq \text{hd}(C(X))$.

**Case 2.** $X$ contains exactly one ramification point $p$.

Let $m$ be the order of $p$ in $X$ and $G = \{A \in C(X) : p \in A\}$. By [4, 5.2, p. 271], $G$ is an $m$-cell neighborhood of $X$ in $C(X)$ such that $X$ is in the boundary as manifold of $G$. Let $A_1 = X$. Let $A_2$ be a simple $m$-od such that $p \in \text{int}_X(A_2)$, $p$ is the vertex of $A_2$. Then $A_2 \in \text{int}_{C(X)}(G)$ and $A_2$ belongs to the interior as manifold of $G$. Let $A_3$ be an arc in $X$ having $p$ as its end-point. Let $A_4$ be an arc in $X$ such that $p \notin A_4$. Finally, let $q$ be a point in $X$ such that $q(p, X) = 2$, and set $A_5 = \{q\}$. We also ask that $A_2 \cup A_3 \cup A_4 \cup A_5$ does not contain end-points of $X$.

Theorem 3 implies: $m = \dim_{A_1}[C(X)] = \dim_{A_2}[C(X)] = \dim_{A_3}[C(X)] = \dim_{o(\varphi(A_1))}[Z] = \dim_{o(\varphi(A_2))}[Z] = \dim_{o(\varphi(A_3))}[Z]$ and $\dim_{A_5}[C(X)] = \dim_{o(\varphi(A_5))}[Z] = 2 = \dim_{A_5}[C(X)]$.

Observe that $A_3$ can be approximated by elements $K$ in $C(X)$ such that $2 = \dim_{K}([C(X)]) = \dim_{\varphi(K)}[Z]$.

Observing the model for $C([0, 1])$ [11, p. 41], we conclude that $A_4$ (resp., $\varphi(A_4)$) has a 2-cell neighborhood $M$ in $C(X)$ (resp., $\varphi(M)$) such that $A_4$ (resp., $\varphi(A_4)$) is in the interior as manifold of $M$. Moreover, $A_5$ has a 2-cell neighborhood $N$ in $C(X)$ such that $A_5$ is in the interior as manifold of $M$.

Observing the topological properties of the sets $A_1, A_2, A_3, A_4$ and $A_5$; and $\varphi(A_1), \varphi(A_2), \varphi(A_3)$ and $\varphi(A_4)$, we conclude that the sets $o(\varphi(A_1)), o(\varphi(A_2)), o(\varphi(A_3))$ and $o(\varphi(A_4))$ are pairwise distinct, and the same happens with the sets $o(A_1), o(A_2), o(A_3), o(A_4)$ and $o(A_5)$.

Since $\varphi(p) = \varphi(q)$, $\varphi(A_5)$ can be approximated by elements $K$ and $L$ in $Z$ such that $\dim_{K}[Z] = m$ and $\dim_{L}[Z] = 2$. This implies that $o(\varphi(A_5)) \notin \{o(\varphi(A_1)), o(\varphi(A_2)), o(\varphi(A_4))\}$. Thus it only remains to check that $o(\varphi(A_5)) \neq o(\varphi(A_3))$.

Let $J_1, \ldots, J_{m^*}$ be the components of $X \setminus \{p\}$. Since $p$ is a ramification point of $X$, $m^* \geq 2$. For each $i \in \{1, \ldots, m^*\}$, let $J_i = \{A \in C(X) : A \subset J_i\}$. 

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By Theorem 3, \( \{ A \in C(X) : \dim_A[ C(X) ] \leq 2n \} = \{ A \in C(X) : p \notin A \} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_{m^*} \). Since the sets \( \mathcal{J}_1, \ldots, \mathcal{J}_{m^*} \) are connected, open in \( C(X) \) and pairwise disjoint, we conclude that they are the components of the set

\[
\mathcal{G} = \{ A \in C(X) : \dim_A[ C(X) ] \leq 2n \}.
\]

We may assume that \( A_k \subset J_1 \).

We prove that \( o(\varphi(A_2)) \neq o(\varphi(A_3)) \). Suppose to the contrary that there exists a homeomorphism \( h : Z \to Z \) such that \( h(Z_0) = h(\varphi(A_3)) = \varphi(A_3) \). Let \( Z_1 \in Z \) be such that \( h(Z_1) = Z_0 \). Fix an arc \( L_0 \) contained in \( J_2 \). Then \( L_0 \in \mathcal{J}_2 \). Let \( L_1 = h^{-1}(\varphi(L_0)) \in Z \). Since \( L_0 \neq A_3 \) and they are arcs, we have that \( h(L_1) = \varphi(L_0) \neq \varphi(A_3) = h(Z_0) \), so \( L_1 \neq Z_0 \) and there exists a unique \( L \in C(X) \setminus F_1(X) \) such that \( \varphi(L) = L_1 \). Then \( \varphi(L) \neq Z_1 \).

By Theorem 3, \( 2 = \dim_{\varphi(0)}[ C(X) ] = \dim_{\varphi(t_0)}[ Z ] = \dim_{h(\varphi(t_0))}[ Z ] = \dim_{\varphi(t_0)}[ C(X) ] \), so \( p \notin L \) and we conclude that \( L \) is an arc and there exists \( j \in \{ 1, \ldots, m^* \} \) such that \( L \subset J_j \). Fix a point \( q_0 \in L \). Take an order arc \( \alpha : [0, 1] \to C(L) \), from \( \{ q_0 \} \) to \( L \). Given \( t \in (0, 1] \), \( \alpha(t) \) is a subarc of \( L \), so \( \varphi^{-1}(\varphi(\alpha(t))) = \{ \alpha(t) \} \). This implies that there is at most one number \( t \in (0, 1] \) such that \( \varphi(\alpha(t)) = Z_1 \). Thus it is possible to construct the order arc \( \alpha \) in such a way that \( \varphi(\alpha(t)) \neq Z_1 \) for all \( t \in (0, 1] \), and then \( h(\varphi(\alpha(t))) \neq Z_0 \).

This implies that for each \( t \in (0, 1] \), \( 2 = \dim_{\varphi(0)}[ C(X) ] = \dim_{\varphi(0)}[ C(X) ] = \dim_{h(\varphi(t))}[ Z ] = \dim_{h(\varphi(t))}[ Z ] = \dim_{h(\varphi(t))}[ C(X) ] = \dim_{h(\varphi(t))}[ C(X) ] \). Hence \( p \notin \varphi^{-1}(h(\varphi(\alpha(t)))) \). Therefore the set \( \mathcal{K} = \varphi^{-1}(h(\varphi(\alpha((0, 1])))) \) is a connected subset of \( \mathcal{G} \). Therefore, there exists \( i_0 \in \{ 1, \ldots, m^* \} \) such that \( \mathcal{K} \subset \mathcal{J}_{i_0} \). In particular, \( L_0 = \varphi^{-1}(h(L_1)) = \varphi^{-1}(h(\varphi(L))) = \varphi^{-1}(h(\varphi(1))) \subset \mathcal{J}_{i_0} \). Since \( L_0 \subset J_2 \), we conclude that \( i_0 = 2 \). Thus for each \( t \in (0, 1] \), we have that \( \varphi^{-1}(h(\varphi(\alpha(t)))) \subset J_2 \). Since \( \lim_{n \to \infty} h(\varphi(\alpha(\frac{1}{n}))) = h(\varphi(\{ q_0 \})) = h(Z_0) = \varphi(A_3) \) and \( A_3 \) is non-degenerate, we have that it is possible to consider \( A_3 = \varphi^{-1}(h(\varphi(A_3))) = \lim_{n \to \infty} \varphi^{-1}(h(\varphi(\alpha(\frac{1}{n})))) \). Hence \( A_3 \subset J_2 \cup \{ p \} \). This contradicts the fact that \( A_3 \subset J_1 \) and \( A_3 \) is non-degenerate.

We have shown that \( o(\varphi(A_2)) \neq o(\varphi(A_3)) \). Therefore the sets \( o(\varphi(A_1)), o(\varphi(A_2)), o(\varphi(A_3)), o(\varphi(A_4)) \) and \( o(\varphi(A_5)) \) are pairwise distinct. This finishes the proof that \( 5 \leq \text{hd}(Z) \) and \( 5 \leq \text{hd}(C(X)) \).

### 3 Applications

R. Schori asked whether \( C_3([0, 1]) \) is homeomorphic to \( [0, 1]^6 \). This question was included as Question 7.4.2 in the book [15]. The second named author of this paper, in his review of the book for the journal Zentralblatt, mentioned in 2005 that Lemma 3.5 of [7] (published in 2003) implies that the answer for this question is negative. In fact, as a consequence of Theorem 1.4 of [9] (published in 2007), we have that \( \text{hd}(C_n([0, 1])) = 2 \) if and only if \( n \in \{ 1, 2 \} \). However, in the second edition of the book [15], published in 2018, S. Macías insisted that J. Camargo, S. Macías and M. Ruiz answered the question in 2019.

As a consequence of Theorem 9, we obtain that \( \text{hd}(C_3([0, 1])) \geq 6 \). This implies that \( C_3([0, 1]) \) is far from being a 6-cell. In fact the inequality \( \text{hd}(C_n([0, 1])) \geq \)
2n implies that $C_n([0, 1])$ is a very complicated space when $n$ is large. It would be interesting to know if $\text{hd}(C_3([0, 1])) = 6$. Moreover, it would be interesting to find reasonable upper bounds for the numbers $\text{hd}(C_n([0, 1]))$.

The hyperspace $C_3([0, 1])$ is rigid if for each homeomorphism $h : C_3([0, 1]) \to C_3([0, 1])$ we have that $h(F_1([0, 1])) = F_1([0, 1])$. An important open problem is to determine if $C_3([0, 1])$ is rigid [5, Question 4.5].

Related to the topic of this paper, another interesting problem is to determine if $C_n([0, 1])$ can be embedded in $\mathbb{R}^{2n}$ for each (for some) $n \geq 4$ [11, Problem 10.1].

Proofs of the main theorems

Proof. Theorem 1 was proved in Theorem 9. Now we prove Theorem 2. The sufficiency of each of the points (a)-(d) is known, we included the corresponding references in points M-R in the introduction.

In order to prove the necessities of the points (a)-(d), suppose that $X$ is a finite graph and $1 \leq m \leq n$ are such that $\omega(C_n(X)/F_m(X)) \leq 4$. By Theorem 9 (b), $n \in \{1, 2\}$. By Theorems 11 and 12, $R(X) = \emptyset$. Then $X$ is either an arc or a simple closed curve. Therefore, the necessities in Theorem 12 follow from the equalities: $\text{hd}(C(S_1)/F_1(S_1)) = 1$, $\text{hd}(C_2(S_1)/F_1(S_1)) = 4$, $\text{hd}(C_2(S_1)/F_2(S_1)) = 3$, $\text{hd}(C([0, 1])/F_1([0, 1])) = 2$, $\text{hd}(C_2([0, 1])/F_2([0, 1])) = 2$ and $\text{hd}(C_2([0, 1])/F_1([0, 1])) = 2$. □

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