THE 2-RANK OF THE REAL PURE QUARTIC NUMBER FIELD

\[ K = \mathbb{Q}(\sqrt{pd^2}) \]

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ABSTRACT. In this paper, we consider the real pure quartic number field \( K = \mathbb{Q}(\sqrt{pd^2}) \), where \( p \) is a prime number and \( d \) is a square-free positive integer such that \( d \) is prime to \( p \). We compute \( r_2(K) \) the 2-rank of the class group of \( K \) and as an application we exhibit all possible forms of \( d \) for which the 2-class group of \( K \) is trivial (equivalently: the class number of \( K \) is odd), cyclic or isomorphic to \( \mathbb{Z}/2^{n_1}\mathbb{Z} \times \mathbb{Z}/2^{n_2}\mathbb{Z} \), where \( n_i \in \mathbb{N}^* \).

1. Introduction

The study of the structure of the class group of a number field and the computation of its class number has been a deep and difficult problem in algebraic number theory since the time of Gauss (1777-1855).

Let \( K \) be a number field, \( Cl(K) \) its class group and \( Cl_2(K) \) the 2-Sylow subgroup of \( Cl(K) \). We recall that the the 2-rank of \( Cl(K) \) is the 2-rank of \( Cl_2(K) \); it will be denoted by \( r_2(K) \). Immediately: \( r_2(K) = \dim_{\mathbb{F}_2} (Cl_2(K)/Cl_2(K)^2) \), where \( \mathbb{F}_2 \) is the finite field with two elements.

Let us give an example (useful in the present paper) for the computation of \( r_2(K) \). Let \( K \) be a number field and \( K/k \) be a quadratic extension. Let \( u \) (resp. \( u_\infty \)) be the number of the primes (resp. infinite primes) of \( k \) ramified in \( K/k \). Let \( E_k \) be the unit group of \( k \) and \( e \) the integer satisfying \( 2^e = [E_k : E_k \cap N_{K/k}(K^\times)] \), where \( N_{K/k} \) is the norm map in \( K/k \) (the quotient group \( E_k/E_k \cap N_{K/k}(K^\times) \) is an elementary abelian 2-group because \( \forall x \in E_k, N_{K/k}(x) = x^2 \)). Assume that the class number \( h(k) \) of \( k \) is odd. The following formula is well known (see for instance [13, Proposition 2.4])

\[
 r_2(K) = u + u_\infty - e - 1. \tag{1.1}
\]

Let us give a quick proof of (1.1). Let \( Am(K/k) \) denote the group of \( c \) in \( Cl(K) \) fixed by \( Gal(K/k) = \langle \sigma \rangle \) (\( c \) is called an ambiguous class in \( K/k \)), and \( Am_2(K/k) \) its 2-Sylow subgroup. Since \( K/k \) is a cyclic extension of prime degree 2, thanks to a well known formula due to C. Chevalley (see [7]) the order \( | Am(K/k) | \) of \( Am(K/k) \) is

\[
 | Am(K/k) | = \frac{h(k)}{[E_k : E_k \cap N_{K/k}(K^\times)]}, \prod e(P), [K : k]^{2u_\infty},
\]

where \( \prod e(P) \) is the product of all ramification indexes of primes of \( k \). Therefore

\[
 | Am(K/k) | = h(k)2^{u+u_\infty-e-1}.
\]

Since \( h(k) \) is odd, we have

\[
 | Am_2(K/k) | = 2^{u+u_\infty-e-1}.
\]

The surjective morphime: \( Cl_2(K) \rightarrow Cl_2(K)^2 \), \( c \mapsto c^2 \), has kernel \( Am_2(K/k) \). Indeed: Let \( c \in Cl_2(K) \); the order of \( c \) is a power of 2, so is \( N_{K/k}(c) \), which gives \( N_{K/k}(c) = 1 \) because
$h(k)$ is odd; consequently $\sigma(c) = c^{-1}$, so: $c^2 = 1 \Leftrightarrow \sigma(c) = c$. Therefore the quotient group $\text{Cl}_2(K)/\text{Am}_2(K/k)$ is isomorphic to $\text{Cl}_2(K)^2$. We deduce that

$$| \text{Cl}_2(K) / \text{Cl}_2(K)^2 | / | \text{Am}_2(K/k) | = | \text{Cl}_2(K)^2 | .$$

Hence

$$| \text{Cl}_2(K)/\text{Cl}_2(K)^2 | / | \text{Am}_2(K/k) | = 2^{u+u_\infty-e-1}.$$  

Since $\text{Cl}_2(K)/\text{Cl}_2(K)^2$ is an elementary abelian 2-group, we have $\dim_{\mathbb{F}_2} (\text{Cl}_2(K)/\text{Cl}_2(K)^2) = u + u_\infty - e - 1$. This ends the proof of (1.1).

Using genus theory a significant number of papers contributed to the determination of the 2-rank of the class group of a quartic number field $K$; when $K$ is biquadratic or cyclic one can see [1, 2, 3, 4, 5, 6]. But for a pure quartic number field a few researchers addressed this problem. We point out that C. Parry in [15] considered the problem of determining the exact power of 2 dividing the class number of a pure quartic number field, and in [16] determined a necessary and sufficient conditions for some real pure quartic number fields to have an odd class number (equivalently $r_2(K) = 0$).

In the present paper we consider the real pure quartic number field $K = \mathbb{Q}(\sqrt{pd^2})$, where $p$ is a prime number and $d$ is a square-free positive integer such that $d$ is prime to $p$. We denote $k = \mathbb{Q}(\sqrt{p})$ the quadratic subextension of $K/\mathbb{Q}$. Since $K/k$ is quadratic and the class number of $k$ is odd (see [8]), it follows from (1.1) that

$$r_2(K) = u + u_\infty - e - 1.$$  

The main result is the computation of $r_2(K)$ for all $p$ and $d$ by distinguishing all the possible cases of $p$: $p = 2$ (see Theorem A), $p \equiv 3$ or $5$ (mod 8) (see Theorem B), $p \equiv 7$ (mod 8) (see Theorem C), $p \equiv 1$ (mod 8) (see Theorem D). As an application we exhibit in §4 all possible forms of $d$ for which the 2-class group of $K$ is trivial (see Corollary 4.1), cyclic (see Corollary 4.2) or is of the type $(2n_1, 2n_2)$-i.e. isomorphic to $\mathbb{Z}/2n_1\mathbb{Z} \times \mathbb{Z}/2n_2\mathbb{Z}$ (see Corollary 4.3), where $n_i \in \mathbb{N}^*$. The proofs use the above formula and calculations of Hilbert symbols. In addition, we will give in §4 some numerical examples (see Examples 4.4 - 4.15) computed with the Pari-GP calculator (see [14], version 2.9.3).

We point out that Theorem B in the case $p \equiv 5$ (mod 8) is Theorem 1.1 in [10].

Throughout this paper, we will use the following notations:

- $K = \mathbb{Q}(\sqrt{pd^2})$.
- $k = \mathbb{Q}(\sqrt{p})$ the quadratic subextension of $K/\mathbb{Q}$ and $O_k$ its ring of integers.
- $i_{p,\infty}$ : the $\mathbb{Q}$-embedding of $k$ in $\mathbb{C}$ defined by $\sqrt{p} \mapsto -\sqrt{p}$.
- $p$ : the infinite prime corresponding to $i_{p,\infty}$.
- Let $\mathcal{P}$ be a finite or infinite prime of $k$.
  - $[\frac{a}{\mathcal{P}}]$ : the quadratic residue symbol over $k$ (the generalized Legendre symbol). If $\mathcal{P} = (a)$ is principal, the symbol will be denoted $[\frac{a}{k}]$.
  - $[\frac{a}{\mathcal{P}}]_\infty$ : the Hilbert symbol over $k$.
- $(\frac{m}{n})$ : the Legendre symbol.
- $(\frac{m}{n})_\delta$ : the rational biquadratic symbol. If $q \equiv 1$ (mod 8), then $(\frac{q}{2})_4 := (-1)^{\frac{q-1}{2}}$.
- $\delta = d\sqrt{p}$ and $\varepsilon_p$ is the fundamental unit of $k$.
- $N_{k/\mathbb{Q}}$ : the norm map in $k/\mathbb{Q}$.
- For $z \in k$, $z'$ denotes the conjugate of $z$ over $\mathbb{Q}$.
- Let $d = 2^r \prod_{i=1}^s q_i q_i'$ be the unique factorization of $d$ into distinct primes $2$, $q_i$ and $q_i'$, with the convention $t = s = 0$ for $d = 2^r$, and where:
  - $r \in \{0, 1\}$;  
  - for all $i$, $1 \leq i \leq s$, $\left(\frac{p}{q_i}\right) = -1$ (\Leftrightarrow $q_i$ remains a prime in $k/\mathbb{Q}$) ; we denote by $\tilde{q}_i$ the prime ideal of $O_k$ above $q_i$ so that $\tilde{q}_i = q_i O_k$.
Theorem D. \[ \Delta \]

where

\[ q_i \]

under the assumptions on applications and examples are in §4. We point out that the following theorems are complete under the assumptions on \( p \) and \( d \).

Before stating Theorem A, let us recall: \( \left( \frac{2}{q_i} \right) = -1 \Leftrightarrow q_i \equiv 3, 5 \pmod{8} \), \( \left( \frac{2}{q_i} \right) = 1 \Leftrightarrow q_i' \equiv 1, 7 \pmod{8} \).

**Theorem A.** Let \( K = \mathbb{Q}(\sqrt[p]{d^2}) \) (here \( p = 2 \)), with \( d = q_1 \cdots q_s q_1' \cdots q_s' \).

1. If there exists \( i, 1 \leq i \leq s \), such that \( q_i \equiv 3 \pmod{8} \), then
   \[ r_2(K) = 2t + s - 1. \]

2. If there exists \( i, 1 \leq i \leq t \), such that \( q_i' \equiv 7 \pmod{8} \), then
   \[ r_2(K) = 2t + s - 1. \]

3. Suppose that for all \( i, 1 \leq i \leq s \), \( q_i \equiv 5 \pmod{8} \) and for all \( i, 1 \leq i \leq t \), \( q_i' \equiv 1 \pmod{8} \), then
   \[ r_2(K) = \begin{cases} 
   2t + s - 1, & \text{if } \Delta_0 \neq \emptyset; \\
   2t + s, & \text{if } \Delta_0 = \emptyset;
   \end{cases} \]

where \( \Delta_0 \) is the set defined by \( \Delta_0 = \{ i \mid \left( \frac{2}{q_i} \right)_4 (-1)^{q_i' - 1} = -1 \} \).

**Theorem B.** Let \( K = \mathbb{Q}(\sqrt[p]{d^2}) \), with \( p \equiv 3 \) or \( 5 \pmod{8} \). Then
\[ r_2(K) = 2t + s. \]

**Theorem C.** Let \( K = \mathbb{Q}(\sqrt[p]{d^2}) \), with \( p \equiv 7 \pmod{8} \). Then
\[ r_2(K) = \begin{cases} 
   2t + s + 1, & \text{if } \Delta_1 = \emptyset \text{ and } d \text{ is odd}; \\
   2t + s, & \text{if } (\Delta_1 = \emptyset \text{ and } d \text{ is even}) \text{ or } (\Delta_1 \neq \emptyset \text{ and } d \text{ is odd}); \\
   2t + s - 1, & \text{if } \Delta_1 \neq \emptyset \text{ and } d \text{ is even},
   \end{cases} \]

where \( \Delta_1 \) is the set defined by \( \Delta_1 = \{ i \mid 1 \leq i \leq t, \left( \frac{2}{q_i} \right) = -1 \} \).

**Theorem D.** Let \( K = \mathbb{Q}(\sqrt[p]{d^2}) \), with \( p \equiv 1 \pmod{8} \).

1. If there exists \( i, 1 \leq i \leq s \), such that \( q_i \equiv 3 \pmod{4} \) or \( i, 1 \leq i \leq t \), such that \( q_i' \equiv 3 \pmod{4} \), then
   \[ r_2(K) = \begin{cases} 
   2t + s, & \text{if } d \text{ is odd}; \\
   2t + s + 1, & \text{if } d \text{ is even}.
   \end{cases} \]

2. Suppose that for all \( i, 1 \leq i \leq s \), \( q_i \equiv 1 \pmod{4} \) and for all \( i, 1 \leq i \leq t \), \( q_i' \equiv 1 \pmod{4} \),
   \begin{enumerate}
   \item If \( d \) is odd, then
     \[ r_2(K) = \begin{cases} 
     2t + s, & \text{if } \Delta_2 \neq \emptyset; \\
     2t + s + 1, & \text{if } \Delta_2 = \emptyset.
     \end{cases} \]
   \item Assume that \( d \) is even.
     \begin{enumerate}
     \item If \( \Delta_2 \neq \emptyset \) then
       \[ r_2(K) = 2t + s + 1. \]
     \end{enumerate}
   \end{enumerate}
(b) Suppose that $\Delta_2 = \emptyset$.

- If $\left(\frac{2}{p}\right)_4 (-1)^{\frac{p-1}{4}} = -1$, then
\[
r_2(K) = 2t + s + 1.
\]

- If $\left(\frac{2}{p}\right)_4 (-1)^{\frac{p-1}{4}} = 1$, then
\[
r_2(K) = 2t + s + 2.
\]

where $\Delta_2$ is set define by $\Delta_2 = \left\{ i \mid 1 \leq i \leq t, \left( \frac{p}{q_i} \right)_4 \left( \frac{q_i}{p} \right)_4 = -1 \right\}$.

2. Preliminaries

The purpose of this section is to state some lemmas and results which will be useful for the proofs of the main results of the present article.

Recall that (see (1.1))
\[
r_2(K) = u + u_\infty - e - 1,
\]
where $u$ (resp. $u_\infty$) is the number of the finite (resp. infinite) primes ramified in $K/k$, and $e$ is defined by
\[
2^e = [E_k : E_k \cap N_{K/k}(K^*)].
\]

Let us start by computing $u$ and $u_\infty$.

The real $\mathbb{Q}$-embeddings of $k$ in $\mathbb{C}$ are $p_\infty$ and $\text{Id} : \sqrt{p} \mapsto \sqrt{p}$. We have $K = k(\sqrt{δ})$ and the minimal polynomial of $\sqrt{δ}$ over $k$ is $T = X^2 - δ$. Recall that we can define the extensions of $\text{Id}$ (resp. $p_\infty$) to $K$ by the roots of $\text{Id}(T) = X^2 - \text{Id}(δ) = X^2 - δ$ (resp. $p_\infty(T) = X^2 - p_\infty(δ) = X^2 + δ$). Since the roots of $T$ are real, $\text{Id}$ is unramified in $K/k$. The polynomial $p_\infty(T)$ has at least one non real root, therefore $p_\infty$ is ramified in $K/k$. We conclude that $u_\infty = 1$.

Concerning the computation of $u$, we will calculate the discriminant $Δ_{K/k}$ of the extension $K/k$ thanks to [11, Theorem 1] (use in that Theorem the formula of $Δ_{K_1/k_1}$; in the notation of [11]: $f = p, g = d, h = 1, K_1 = K, k_1 = k$).

- If $p = 2$ and $d = q_1 \cdots q_s q'_1 \cdots q'_t$, then $Δ_{K/k} = (4d\sqrt{2})(= 4d\sqrt{2}O_k)$. Therefore the finite primes of $O_K$ ramified in $K/k$ are $(\sqrt{2}) = \sqrt{2}O_k, \bar{q}_1, \cdots, \bar{q}_s, \bar{π}_1, \cdots, \bar{π}_t, \bar{π}_1', \cdots, \bar{π}_t'$. We have $u = 2t + s + 1$.

- If $p \equiv 5 \pmod{8}$, then $Δ_{K/k} = (4d\sqrt{p})(= 4d\sqrt{p}O_k)$. The finite primes ramified in $K/k$ are $(\sqrt{p}), \bar{q}_1, \cdots, \bar{q}_s, \bar{π}_1, \cdots, \bar{π}_t, \bar{π}_1', \cdots, \bar{π}_t'$ and $\bar{2}$, where $pO_k = (\sqrt{p})^2, 2O_k = \bar{2}$ (2 remains prime in $k$ since $p \equiv 5 \pmod{8}$). We have $u = 2t + s + 2$.

- If $p \equiv 3 \pmod{8}$, then
\[
Δ_{K/k} = \begin{cases} (4d\sqrt{p}), & \text{if } d \text{ is odd;} \\ (d\sqrt{p}), & \text{if } d \text{ is even.} \end{cases}
\]

Since $p \equiv 3 \pmod{4}, 2$ is ramified in $k/\mathbb{Q}$; put $2O_k = \bar{2}^2$. The finite primes ramified in $K/k$ are $(\sqrt{p}), \bar{q}_1, \cdots, \bar{q}_s, \bar{π}_1, \cdots, \bar{π}_t, \bar{π}_1', \cdots, \bar{π}_t'$ and $\bar{2}$. We have $u = 2t + s + 2$.

- If $p \equiv 7 \pmod{8}$, then
\[
Δ_{K/k} = \begin{cases} (4d\sqrt{p}), & \text{if } d \text{ is odd;} \\ (d\sqrt{\frac{d}{2}\sqrt{p}}), & \text{if } d \text{ is even.} \end{cases}
\]
The set of finite primes ramified in \( K/k \) is:
\[
\mathcal{R} = \begin{cases} 
\{((\sqrt{d}), \tilde{2}, \tilde{q}_1, \cdots, \tilde{q}_k, \pi_1, \cdots, \pi_t, \bar{\pi}_1, \cdots, \bar{\pi}_t) \}, & \text{if } d \text{ is odd;} \\
\{((\sqrt{d}), \tilde{2}, \tilde{q}_1, \cdots, \tilde{q}_k, \pi_1, \cdots, \pi_t, \bar{\pi}_1, \cdots, \bar{\pi}_t) \}, & \text{if } d \text{ is even;}
\end{cases}
\]
where \( 2O_k = \tilde{2}^2 \) (Since \( p \equiv 3 \pmod{4} \), 2 is ramified in \( k/Q \)). If \( d \) is odd, then \( u = 2t + s + 2 \). If \( d \) is even, then \( u = 2t + s + 1 \).

- If \( p \equiv 1 \pmod{8} \), then
  \[
  \Delta_{K/k} = \begin{cases} 
  \tilde{2}^2(d\sqrt{d}), & \text{if } d \text{ is odd;} \\
  (4d\sqrt{d}), & \text{if } d \text{ is even.}
  \end{cases}
  \]

The set of finite primes ramified in \( K/k \) is:
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\{((\sqrt{d}), \tilde{2}, \tilde{q}_1, \cdots, \tilde{q}_k, \pi_1, \cdots, \pi_t, \bar{\pi}_1, \cdots, \bar{\pi}_t) \}, & \text{if } d \text{ is odd;} \\
\{((\sqrt{d}), \tilde{2}, \tilde{q}_1, \cdots, \tilde{q}_k, \pi_1, \cdots, \pi_t, \bar{\pi}_1, \cdots, \bar{\pi}_t) \}, & \text{if } d \text{ is even;}
\end{cases}
\]
where \( 2O_k = \tilde{2}_1\tilde{2}_2 \) (2 splits in \( k/Q \) because \( p \equiv 1 \pmod{8} \)). If \( d \) is odd, then \( u = 2t + s + 2 \). If \( d \) is even, then \( u = 2t + s + 3 \).

In the following, until the end of this section, we will give lemmas in order to compute \( e \).

According to Dirichlet’s Unit Theorem: \( E_k = \{ \pm \epsilon^n_k, n \in \mathbb{Z} \} \). Since for all \( x \in E_k \), \( N_{K/k}(x) = x^2 \), we have
\[
E_k/E_k \cap N_{K/k}(K^\times) = \{ \bar{1}, \bar{-1}, \bar{\epsilon}_p, \bar{-\epsilon}_p \},
\]
where \( \pi \) is the coset of \( x \). Therefore \( e = 0, 1, \) or \( 2 \). Immediately we have:

- \( e = 0 \Leftrightarrow -1 \) and \( \epsilon_p \) are norm in \( K/k \).
- \( e = 1 \Leftrightarrow (-1 \text{ is a norm in } K/k \text{ and } \epsilon_p \text{ is not a norm in } K/k) \) or \( (-1 \text{ is not a norm in } K/k \text{ and } \epsilon_p \text{ or } -\epsilon_p \text{ is a norm in } K/k) \).
- \( e = 2 \Leftrightarrow -1, \epsilon_p \) and \( -\epsilon_p \) are not norm in \( K/k \).

Since \( K/k \) is a cyclic extension, the Hasse Norm Theorem (see for instance [9, Theorem 6.2, p. 179]) tells us that a necessary and sufficient condition for a nonzero \( a \in k \) to be the norm of an element of \( K \) is that \( a \) be a local norm everywhere for \( K/k \). Because \( K = k(\sqrt{\delta}) \) (recall that \( \delta = d\sqrt{d} \)), this is equivalent to:

For all finite or infinite prime \( \mathcal{P} \) of \( k \), \( \left(\frac{a, \delta}{\mathcal{P}}\right) = 1 \),

where we recall that if \( a \) and \( b \) are nonzero elements of \( k \), \( \left(\frac{a, b}{\mathcal{P}}\right) \) is the (quadratic local) Hilbert symbol (see [18, Chap. XIV, case \( n = 2 \)])

Now it is clear that in order to compute \( e \) we need the calculations of \( \left(\frac{a, b}{\mathcal{P}}\right) \), where \( a \in \{-1, \epsilon_p, -\epsilon_p\} = I \).

Let us recall some explicit computations of \( \left(\frac{a, b}{\mathcal{P}}\right) \). For all \( \mathcal{P} \), \( \left(\frac{a, b}{\mathcal{P}}\right) = \pm 1 \). The product formula is: \( \prod_{\mathcal{P}} \left(\frac{a, b}{\mathcal{P}}\right) = 1 \). If \( \mathcal{P} \) is an infinite complex prime, \( \left(\frac{a, b}{\mathcal{P}}\right) = 1 \). If \( \mathcal{P} \) is an infinite real prime, \( \left(\frac{a, b}{\mathcal{P}}\right) = 1 \) if and only if \( i_{\mathcal{P}}(a) > 0 \) or \( i_{\mathcal{P}}(b) > 0 \), where \( i_{\mathcal{P}} \) is the \( \mathbb{Q} \)-embedding of \( k \) in \( \mathbb{R} \) (see [9, Note, p. 198; pp. 11–12 for the notations used in Note]). Let \( \mathcal{P} \) a finite prime of \( O_k, v_{\mathcal{P}} \) the \( \mathcal{P} \)-adic valuation, \( k_{\mathcal{P}} \) the completion of \( k \) at \( \mathcal{P} \) and \( \bar{k}_{\mathcal{P}} \) its residual field. Let \( c = (-1)^{v_{\mathcal{P}}(a)v_{\mathcal{P}}(b)}a^{v_{\mathcal{P}}(b)}b^{-v_{\mathcal{P}}(a)} \), and \( \bar{c} \) the coset of \( c \) in \( \bar{k}_{\mathcal{P}} \). If \( \mathcal{P} \) is not above 2, then (see [18, Proposition 8 and its corollary, pp. 210-211])
\[
\left(\frac{a, b}{\mathcal{P}}\right) = \bar{c}^{(N_{k/q}(\mathcal{P}^{-1}))/2}.
\]

(2.1)
Let $a \in I = \{-1, \epsilon_p, -\epsilon_p\}$. If $\mathcal{P}$ corresponds to the trivial real $\mathbb{Q}$-embedding of $k$ in $\mathbb{R}$, $x \mapsto x$, we have $(\frac{a,\delta}{P}) = 1$, because $\delta > 0$. Now $\mathcal{P}$ is finite and not above 2. Suppose that $\nu_{\mathcal{P}}(\delta) = 0$, then $c = 1$ because $\nu_{\mathcal{P}}(a) = 0$ and by (2.1) we have

$$(\frac{a,\delta}{\mathcal{P}}) = 1.$$ 

If $\nu_{\mathcal{P}}(\delta) \neq 0$, it is immediate that $\nu_{\mathcal{P}}(\delta) = 1$; we then have

$$(\frac{a,\delta}{\mathcal{P}}) = \left[ \frac{a}{\mathcal{P}} \right],$$

indeed: by (2.1): $(\frac{a,\delta}{\mathcal{P}}) = \tilde{a}(N_{k/\mathbb{Q}}(P)-1)/2$; we conclude thanks to the definition of the generalized Legendre symbol (see for instance [12, Chap. 4, §4.1, (4.2), p. 111]).

Let $q$ be an odd prime number and $\tilde{q}$ a prime ideal of $O_k$ above $q$. Let $\alpha \in O_k$. Since $k/\mathbb{Q}$ is abelian, according to [12, Chap. 4, Proposition 4.2 (iii), p. 112] we have

$$\left[ \frac{\alpha}{qO_k} \right] = \left( \frac{N_{k/\mathbb{Q}}(\alpha)}{q} \right).$$

If $\alpha \in \mathbb{Z}$ and the residual (inertia) degree of $\tilde{q}$ is 1, then by [12, Chap. 4, Proposition 4.2 (ii), p. 112] we have

$$\left[ \frac{\alpha}{\tilde{q}} \right] = \left( \frac{\alpha}{q} \right).$$

We point out that in the statements of the following lemmas we will use our previous notations. Let us recall known results: $\varepsilon_2 = 1 + \sqrt{2}$, so $\varepsilon_2' < 0$; if $p \equiv 1$ (mod 4), then $N_{k/\mathbb{Q}}(\varepsilon_p) = -1$, so $\varepsilon_p' < 0$; if $p \equiv 3$ (mod 4), then $N_{k/\mathbb{Q}}(\varepsilon_p) = 1$, so $\varepsilon_p' > 0$.

**Lemma 2.1.** Recall that $p_{\infty}$ is the infinite prime which corresponds to the $\mathbb{Q}$-embeddings $i_{p_{\infty}} : \sqrt{d} \mapsto -\sqrt{d}$. We have:

(i) $(\frac{-1,d}{p_{\infty}}) = -1.$

(ii) If $p \equiv 3$ (mod 4), then $(\frac{\varepsilon_p,d}{p_{\infty}}) = 1$ and $(\frac{-\varepsilon_p,d}{p_{\infty}}) = -1.$

(iii) If $p = 2$ or $p \equiv 1$ (mod 4), then $(\frac{\varepsilon_p,d}{p_{\infty}}) = -1$ and $(\frac{-\varepsilon_p,d}{p_{\infty}}) = 1.$

**Proof.** We have $p_{\infty}(\delta) = -d\sqrt{d} < 0$.

(i) Because $p_{\infty}(-1) = -1 < 0$ and $p_{\infty}(\delta) < 0$.

(ii) Because $p_{\infty}(\varepsilon_p) = \varepsilon_p' > 0$ for the first equality, and $p_{\infty}(-\varepsilon_p) < 0$ and $p_{\infty}(\delta) < 0$ for the second.

(iii) Because $p_{\infty}(\varepsilon_p) < 0$ and $p_{\infty}(-\varepsilon_p) < 0$.

Recall the following notations:

- $d = 2^r \prod_{i=1}^{s} q_i \prod_{i=1}^{t} \tilde{q}_i$;
- for all $i$, $1 \leq i \leq s$, $(\frac{p}{q_i}) = -1$ ($\iff$ $q_i$ remains a prime in $k/\mathbb{Q}$); $\tilde{q}_i$ is the prime ideal of $O_k$ above $q_i$ so that $\tilde{q}_i = q_iO_k$;
- for all $i$, $1 \leq i \leq t$, $(\frac{p}{\tilde{q}_i}) = 1$ ($\iff$ $\tilde{q}_i$ splits in $k/\mathbb{Q}$); $\pi_i$ and $\tilde{\pi}_i$ are the prime ideals of $O_k$ above $q_i$ so that $\pi_i\tilde{\pi}_i = q_iO_k$.

**Lemma 2.2.** Under the previous notations, for all $i$, $1 \leq i \leq s$, we have:

(i) $(\frac{-1,d}{q_i}) = 1.$
(ii)\[ (\varepsilon_p, \delta) = \begin{cases} \left( \frac{-1}{q_i} \right) & \text{if } p = 2 \text{ or } p \equiv 1 \pmod{4}, \\ \left( \frac{1}{q_i} \right) & \text{if } p \equiv 3 \pmod{4}. \end{cases} \]

Proof. Let \( a = -1 \) or \( a = \varepsilon_p \). Thanks to (2.2) we have \( \left( \frac{a, \delta}{q_i} \right) = \left( \frac{a}{q_i} \right) \).

Since \( q_i = q_i O_k \), by (2.3) we have \( \left( \frac{a}{q_i} \right) = \left( \frac{N_{k/Q}(a)}{q_i} \right) \). Therefore \( \left( \frac{a, \delta}{q_i} \right) = \left( \frac{N_{k/Q}(a)}{q_i} \right) \).

(i) We have (i) because \( N_{k/Q}(-1) = 1 \).

(ii) We have (ii) according to \( N_{k/Q}(\varepsilon_p) = \pm 1 \).

\[ \Box \]

Lemma 2.3. Let \( p \) be an odd prime and \( \varepsilon_p \) the fundamental unit of \( k = \mathbb{Q}(\sqrt{p}) \), then
\[ \left[ \frac{\varepsilon_p}{\sqrt{p}} \right] = \left( \frac{2}{p} \right). \]

Proof.

- If \( p \equiv 1 \pmod{4} \), then \( N_{k/Q}(\varepsilon_p) = -1 \). According to [15, Lemma V, p. 105],
  \[ \left[ \frac{\varepsilon_p}{\sqrt{p}} \right] = 1 \text{ (resp. } -1 \text{) if } p \equiv 1 \pmod{8} \text{ (resp. } p \equiv 5 \pmod{8} \).\]
  Knowing that \( \left( \frac{2}{p} \right) = (-1)^{(p^2-1)/8} \), we obtain \( \left[ \frac{\varepsilon_p}{\sqrt{p}} \right] = \left( \frac{2}{p} \right) \).

- If \( p \equiv 3 \pmod{4} \), by [17, Lemma 3.2(1), p. 126], there exists \( u_p \in k^\times \) such that \( \varepsilon_p = 2u_p^2 \), so
  \[ \left[ \frac{\varepsilon_p}{\sqrt{p}} \right] = \left[ \frac{2u_p^2}{\sqrt{p}} \right] = \left[ \frac{2}{\sqrt{p}} \right]. \]

It follows from (2.4) that \( \left[ \frac{2}{\sqrt{p}} \right] = \left( \frac{2}{p} \right) \) because the residue degree of the prime ideal \( \sqrt{p}O_k \) equals 1.

\[ \Box \]

We point out that in order to simplify the writing, in the following we will use several times the assertions (2.2), (2.3) and (2.4) without mentioning them.

Lemma 2.4. We have:

(i) \( \left( -\frac{1, \delta}{\sqrt{p}} \right) = \begin{cases} -1, & \text{if } p = 2 \text{ or } p \equiv 3 \pmod{4}, \\ 1, & \text{if } p \equiv 1 \pmod{4}. \end{cases} \)

(ii) \( \left( \frac{\varepsilon_p, \delta}{\sqrt{2}} \right) = \begin{cases} -1, & \text{if } p \equiv 3, 5 \pmod{8}, \\ 1, & \text{if } p \equiv 1, 7 \pmod{8}. \end{cases} \)

(iii) \( \left( \frac{\varepsilon_p, \delta}{\sqrt{2}} \right) = -\prod_{i=1}^{s} \left( \frac{1}{q_i} \right) \prod_{i=1}^{t} \left( \frac{1}{q_i'} \right). \)

Proof.

(i) If \( p \neq 2 \), then \( \left( -\frac{1, \delta}{\sqrt{p}} \right) = \left[ -\frac{1}{\sqrt{p}} \right] = \left( -1 \right) = (-1)^{(p-1)/2}. \) The conclusion is then immediate.

(ii) Suppose that \( p = 2 \). Using the product formula for Hilbert symbols we obtain
\[ \left( -\frac{1, \delta}{\sqrt{2}} \right) = \left( -\frac{1, \delta}{2^{\infty}} \right) \prod_{i=1}^{s} \left( -\frac{1}{q_i} \right) \prod_{i=1}^{t} \left( -\frac{1}{q_i} \right) = \left( -\frac{1, \delta}{2^{\infty}} \right) \prod_{i=1}^{s} \left( -\frac{1}{q_i} \right) \prod_{i=1}^{t} \left( -\frac{1}{q_i} \right). \]

We conclude that \( \left( -\frac{1, \delta}{\sqrt{2}} \right) = -1 \) thanks to Lemma 2.1(i) and Lemma 2.2(i).
(ii) We have \( \left( \frac{\varepsilon_p \delta}{\sqrt{p}} \right) = \left[ \frac{\varepsilon_p}{\sqrt{p}} \right] \), so according to Lemma 2.3 \( \left[ \frac{\varepsilon_p}{\sqrt{p}} \right] = \left( \frac{2}{p} \right) \). Hence \( \left( \frac{\varepsilon_p \delta}{\sqrt{p}} \right) = (-1)^{(p^2 - 1)/8} \).

(iii) The product formula gives

\[
\left( \frac{\varepsilon_p \delta}{\sqrt{2}} \right) = \left( \frac{\varepsilon_p \delta}{\sqrt{2}} \right) \prod_{i=1}^{s} \left( \frac{\varepsilon_p \delta}{q_i} \right) \Pi_{i=1}^{t} \left( \frac{\varepsilon_p \delta}{q_i} \right) = \left( \frac{\varepsilon_p \delta}{q_i} \right) \Pi_{i=1}^{t} \left( \frac{\varepsilon_p \delta}{q_i} \right).
\]

We conclude thanks to: \( \left( \frac{\varepsilon_p \delta}{q_i} \right) = \left( \frac{-1}{q_i} \right) \) by Lemma 2.2(ii), \( \left( \frac{\varepsilon_p}{q_i Q_{O_k}} \right) = \left( \frac{N_{k/O_k}(\varepsilon_p)}{q_i} \right) = \left( \frac{-1}{q_i} \right) \), and \( \left( \frac{\varepsilon_p \delta}{\sqrt{2}} \right) = -1 \) by Lemma 2.1(iii).

\[ \square \]

**Lemma 2.5.** Let \( q \neq p \) be a prime such that \( q \neq 2 \) and \( \left( \frac{p}{q} \right) = 1 \) (then \( q \) can be one of the \( q_i \) and splits in \( k/O_k \)). Let \( \pi, \tilde{\pi} \) the prime ideals of \( O_k \) satisfying \( q O_k = \pi \tilde{\pi} \). We have:

(i) \( \left( \frac{-1, \delta}{\pi} \right) = \left( \frac{-1, \delta}{\tilde{\pi}} \right) = \left( \frac{-1}{q} \right) \).

(ii)

\[
\left( \frac{\varepsilon_p \delta}{\pi} \right) = \left( \frac{\varepsilon_p \delta}{\tilde{\pi}} \right) = \left\{ \begin{array}{ll}
\left( \frac{2}{q} \right)_4 \left( -1 \right)^{q+1} & \text{if } p = 2 \text{ and } q \equiv 1 \pmod{8}, \\
\left( \frac{2}{q} \right)_4 & \text{if } p = 7 \pmod{8}, \\
\left( \frac{2}{q} \right)_4 \left( \frac{q}{p} \right)_4 & \text{if } p = 1 \pmod{8} \text{ and } q \equiv 1 \pmod{4}.
\end{array} \right.
\]

Moreover, if \( (p = 2 \text{ and } q \equiv 7 \pmod{8}) \) or \( (p = 1 \pmod{8} \text{ and } q \equiv 3 \pmod{4}) \), then \( \left( \frac{\varepsilon_p \delta}{\pi} \right) \left( \frac{\varepsilon_p \delta}{\tilde{\pi}} \right) = -1 \).

**Proof.** Note that \( \left( \frac{\varepsilon_p \delta}{\pi} \right) = \left[ \frac{\varepsilon_p}{\pi} \right] \left[ \frac{\varepsilon_p}{\tilde{\pi}} \right] = \left[ \frac{\varepsilon_p}{\pi} \right] \left[ \frac{\varepsilon_p}{\pi} \right] = \left( \frac{N_{k/O_k}(\varepsilon_p)}{q} \right) \).

(i) We have \( \left( \frac{-1, \delta}{\pi} \right) = \left[ \frac{-1}{\pi} \right] = \left( \frac{-1}{q} \right) \) and \( \left( \frac{-1, \delta}{\tilde{\pi}} \right) = \left[ \frac{-1}{\tilde{\pi}} \right] = \left( \frac{-1}{q} \right) \).

(ii) We divide the proof into three parts.

(1) Assume \( p = 2 \).

- We have \( \left[ \frac{\varepsilon_p}{\pi} \right] \left[ \frac{\varepsilon_p}{\tilde{\pi}} \right] = \left( \frac{N_{k/O_k}(\varepsilon_p)}{q} \right) = \left( \frac{-1}{q} \right) \). If \( q \equiv 7 \pmod{8} \), then \( \left( \frac{-1}{q} \right) = -1 \).

- If \( q \equiv 1 \pmod{8} \), by Scholz’s reciprocity law (see [12, Proposition 5.8, p. 160]), we have

\[
\left[ \frac{\varepsilon_p}{\pi} \right] = \left( \frac{2}{q} \right)_4 \left( \frac{q}{2} \right)_4 = \left( \frac{2}{q} \right)_4 \left( -1 \right)^{q+1}.
\]

(2) Suppose \( p = 7 \pmod{8} \). From \( N_{k/O_k}(\varepsilon_p) = 1 \) we obtain \( \left[ \frac{\varepsilon_p}{\pi} \right] \left[ \frac{\varepsilon_p}{\tilde{\pi}} \right] = 1 \), hence \( \left[ \frac{\varepsilon_p}{\pi} \right] = \left[ \frac{\varepsilon_p}{\tilde{\pi}} \right] \). Since \( p = 3 \pmod{4} \), according to [17, Lemma 3.2(1), p. 126] there exists \( u_p \in k^\times \) such that \( 2 \varepsilon_p = u_p^2 \). So \( \left[ \frac{\varepsilon_p}{\pi} \right] = \left[ \frac{2u_p^2}{\pi} \right] = \left( \frac{2}{q} \right) \).

(3) Suppose \( p = 1 \pmod{8} \).

- If \( q \equiv 1 \pmod{4} \) and \( \left( \frac{p}{q} \right) = 1 \), by Scholz’s reciprocity law we have

\[
\left[ \frac{\varepsilon_p}{\pi} \right] = \left[ \frac{\varepsilon_p}{\tilde{\pi}} \right] = \left( \frac{p}{q} \right)_4 \left( \frac{q}{p} \right)_4.
\]

- If \( q \equiv 3 \pmod{4} \), we have \( \left[ \frac{\varepsilon_p}{\pi} \right] \left[ \frac{\varepsilon_p}{\tilde{\pi}} \right] = \left( \frac{N_{k/O_k}(\varepsilon_p)}{q} \right) = \left( \frac{-1}{q} \right) = -1 \).

\[ \square \]
Lemma 2.6. Under the previous assumptions and notations, we have

(i) If $p \equiv 7 \pmod{8}$, then $\left(\frac{\varepsilon_p \delta}{2}\right) = \left(\frac{-1}{2}\right) = 1$.

(ii) If $p \equiv 1 \pmod{8}$, then

- If $d$ is odd, we denote $\hat{2}_2$ the prime ideal above 2 in $k$ unramified in $K/k$. Then

  \[
  \left(\frac{-1}{\hat{2}_2}\right) = \left(\frac{\varepsilon_p \delta}{2}\right) = 1, \quad \left(\frac{-1}{\hat{2}_2}\right) = -1 \quad \text{and} \quad \left(\frac{\varepsilon_p \delta}{2}\right) = -\prod_{i=1}^{s} \left(\frac{-1}{q_i}\right).
  \]

- If $d$ is even, we put $\delta = 2\delta'$ and denote $\hat{2}_2$ the prime ideal above 2 in $k$ unramified in $k(\sqrt{\delta'})/k$. Then

  \[
  \left(\frac{-1}{\hat{2}_2}\right) = -\left(\frac{-1}{\hat{2}_2}\right) = -1,
  \]

  \[
  \left(\frac{\varepsilon_p \delta}{2}\right) = \prod_{i=1}^{s} \left(\frac{-1}{q_i}\right) \left(\frac{2}{p}\right) 4 \left(\frac{-1}{8}\right),
  \]

  \[
  \left(\frac{\varepsilon_p \delta}{2}\right) = \left(\frac{2}{p}\right) 4 \left(\frac{-1}{8}\right).
  \]

Proof.

(i) If $p \equiv 7 \pmod{8}$, using the product formula for Hilbert symbols we obtain:

\[
\left(\frac{-1}{\hat{2}_2}\right) = \left(\frac{-1}{\sqrt{\delta'}}\right) \prod_{i=1}^{s} \left(\frac{-1}{\varepsilon_p \delta \pi_i}\right) \left(\frac{-1}{\varepsilon_p \delta \pi_i}\right) = \left(\frac{-1}{\sqrt{\delta'}}\right) \prod_{i=1}^{s} \left(\frac{-1}{\varepsilon_p \delta \pi_i}\right) \prod_{i=1}^{t} \left(\frac{-1}{\varepsilon_p \delta \pi_i}\right) 2^2.
\]

We then have $\left(\frac{-1}{\hat{2}_2}\right) = 1$ because: $\left(\frac{-1}{\sqrt{\delta'}}\right) = -1$ by Lemma 2.4(i), $\left(\frac{-1}{\sqrt{\delta'}}\right) = -1$ by Lemma 2.1(i), and $\left(\frac{-1}{\hat{2}_2}\right) = 1$ by Lemma 2.2(i).

We have

\[
\left(\frac{\varepsilon_p \delta}{2}\right) = \left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) \prod_{i=1}^{s} \left(\frac{\varepsilon_p \delta \pi_i}{\varepsilon_p \delta \pi_i}\right) \prod_{i=1}^{t} \left(\frac{\varepsilon_p \delta \pi_i}{\varepsilon_p \delta \pi_i}\right) = \left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) \prod_{i=1}^{s} \left(\frac{\varepsilon_p \delta \pi_i}{\varepsilon_p \delta \pi_i}\right) \prod_{i=1}^{t} \left(\frac{2}{q_i}\right) 2^2.
\]

It follows that $\left(\frac{\varepsilon_p \delta}{2}\right) = 1$ thanks to: $\left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) = 1$ by Lemma 2.4(ii), $\left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) = 1$ by Lemma 2.1(ii) and $\left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) = 1$ by Lemma 2.1(ii).

(ii) Using the product formula as in (i) we obtain:

\[
\left(\frac{-1}{\hat{2}_2}\right) \left(\frac{-1}{\hat{2}_2}\right) = \left(\frac{-1}{\sqrt{\delta'}}\right) \left(\frac{-1}{\sqrt{\delta'}}\right).
\]

Since $\left(\frac{-1}{\sqrt{\delta'}}\right) = 1$ by Lemma 2.4(i) and $\left(\frac{-1}{\sqrt{\delta'}}\right) = -1$ by Lemma 2.1(i), we deduce that

\[
\left(\frac{-1}{\hat{2}_2}\right) \left(\frac{-1}{\hat{2}_2}\right) = -1.
\]

We have

\[
\left(\frac{\varepsilon_p \delta}{2}\right) \left(\frac{\varepsilon_p \delta}{2}\right) = \left(\frac{\varepsilon_p \delta}{\sqrt{\delta'}}\right) \prod_{i=1}^{s} \left(\frac{\varepsilon_p \delta}{q_i}\right) = -\prod_{i=1}^{s} \left(\frac{-1}{q_i}\right).
\]

Thanks to: $\left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) = 1$ by Lemma 2.4(ii), $\left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) = -1$ by Lemma 2.1(iii) and $\left(\frac{\varepsilon_p \delta \pi_i}{\sqrt{\delta'}}\right) = \left(\frac{-1}{q_i}\right)$ by Lemma 2.2(ii).
• Assume that $d$ is odd. Then $2_2$ is not ramified in $k(\sqrt{d})/k$. It follows from [9, Proposition 7.4.3 (viii), p. 205] that for $a = -1$ or $a = \xi_p$ we have

$$\left( \frac{a, \delta}{2_2} \right) = \left( \frac{\delta}{2_2} \right)^{v_{2_2}(a)}.$$  

Since $v_{2_2}(a) = 0$, we obtain

$$\left( \frac{-1, \delta}{2_2} \right) = \left( \frac{\xi_p, \delta}{2_2} \right) = 1. $$

Therefore

$$\left( \frac{-1, \delta}{2_1} \right) = -1,$$

and

$$\left( \frac{\xi_p, \delta}{2_1} \right) = -\prod_{i=1}^{s} \left( \frac{-1}{q_i} \right).$$

• If $d$ is even, put $\delta = 2\delta'$ so that $\delta' = \delta \sqrt{p}$, where $\delta'$ is odd. According to [1, The end of the proof of Theorem 2, p. 2748]), we have

$$\left( \frac{\xi_p, 2}{2_1} \right) = \left( \frac{2}{p} \right)^4 (-1)^{\frac{p-1}{2}}.$$  

Thanks to the previous case ($d$ odd) we have $\left( \frac{-1, \delta'}{2_1} \right) = -1$ and $\left( \frac{\xi_p, \delta'}{2_1} \right) = -\prod_{i=1}^{s} \left( \frac{-1}{q_i} \right)$. Therefore

$$\left( \frac{-1, \delta'}{2_1} \right) = \left( \frac{-1, 2}{2_1} \right) \left( \frac{-1, \delta'}{2_1} \right) = 1 \times -1 = -1,$$

because $\left( \frac{-1, 2}{2_1} \right) = 1$ ($-1$ is a norm in $k(\sqrt{2})/k$), and

$$\left( \frac{\xi_p, \delta'}{2_1} \right) = \left( \frac{\xi_p, 2}{2_1} \right) \left( \frac{\xi_p, \delta'}{2_1} \right) = -\prod_{i=1}^{s} \left( \frac{-1}{q_i} \right) \left( \frac{2}{p} \right)^4 (-1)^{\frac{p-1}{2}}.$$

As

$$\left( \frac{\xi_p, \delta}{2_2} \right) = -\prod_{i=1}^{s} \left( \frac{-1}{q_i} \right),$$

we obtain

$$\left( \frac{\xi_p, \delta}{2_2} \right) = \left( \frac{2}{p} \right)^4 (-1)^{\frac{p-1}{2}}. $$

\[\square\]

**Remark 2.7.** To compute $\left( \frac{-\xi_p, \delta}{P} \right)$ for a prime ideal $P$ of $O_k$, we use the multiplicative property of Hilbert symbols: $\left( \frac{-\xi_p, \delta}{P} \right) = \left( \frac{-1, \delta}{P} \right) \left( \frac{\xi_p, \delta}{P} \right)$.

3. **Proof of the main results**

3.1. **Proof of Theorem A : The case $p = 2$.** In this case, the number of the (finite or infinite) primes ramified in $K/k$ is $2t+s+2$ (see the computation of $u$ and $u_\infty$ in the beginning of §2, the case $p = 2$). Using the formula (1.1) in §1 we obtain $r_2(K) = (2t+s+2) - e - 1 = 2t + s + 1 - e$.

According to the assertions $(i)$ and $(iii)$ of Lemma 2.1 we have $\left( \frac{-1, \delta}{2_\infty} \right) = \left( \frac{\xi_2, \delta}{2_\infty} \right) = -1$. Therefore $-1$ and $\xi_2$ are not norm in $K/k$.

(1) Let $1 \leq i \leq s$. We have $\left( \frac{-\xi_2, \delta}{q_i} \right) = \left( \frac{-1, \delta}{q_i} \right) \left( \frac{\xi_2, \delta}{q_i} \right)$. By Lemma 2.2(i), $\left( \frac{-1, \delta}{q_i} \right) = 1$.

Suppose that there exists $i$, $1 \leq i \leq s$, such that $q_i \equiv 3 \pmod{8}$. Lemma 2.2(ii) gives us $\left( \frac{\xi_2, \delta}{q_i} \right) = \left( \frac{-1}{q_i} \right) = -1$; thus $\left( \frac{-\xi_2, \delta}{q_i} \right) = -1$ and then $-\xi_2$ is not a norm in $K/k$.

We conclude that the units $-1, \xi_2, -\xi_2$ are not norm in $K/k$, hence $e = 2$. Therefore

$$r_2(K) = 2t + s - 1.$$
(2) Let $i, 1 \leq i \leq t$. By Lemma 2.5(i) we have 
\[
\left( -\frac{\varepsilon_2, \delta}{\pi_i} \right) \left( -\frac{\varepsilon_2, \delta}{\pi_i} \right) = \left( \frac{\varepsilon_2, \delta}{\pi_i} \right) \left( \frac{\varepsilon_2, \delta}{\pi_i} \right).
\]
If there exists $1 \leq i \leq t$ such that $q_i \equiv 7 \pmod{8}$, then by Lemma 2.5(ii) (see the part in Moreover) 
\[
\left( \frac{\varepsilon_2, \delta}{\pi_i} \right) \left( \frac{\varepsilon_2, \delta}{\pi_i} \right) = -1, \text{ so } -\varepsilon_2 \text{ is not norm in } K/k. \text{ Consequently } e = 2 \text{ and } r_2(K) = 2t + s - 1.
\]
(3) Recall that
\[
\Delta_0 = \left\{ \left[ \frac{2}{q_i} \right] \right\}_4 (1 - 1) = -1.\]
- If $\Delta_0 \neq 0$, then there exists $1 \leq i \leq t$ such that $\left( \frac{2}{q_i} \right)_4 (1 - 1) = -1$. We have 
\[
\left( \frac{-\varepsilon_2, \delta}{\pi_i} \right) = \left( \frac{-1, \delta}{\pi_i} \right) \left( \frac{\varepsilon_2, \delta}{\pi_i} \right) = \left( \frac{-1}{q_i} \right)_4 (1 - 1) = -1 \text{ thanks to Lemma 2.5(i)(ii) since } q_i \equiv 1 \pmod{8}. \text{ So } -\varepsilon_p \text{ is not a norm in } K/k. \text{ we conclude that } e = 2 \text{ and } r_2(K) = 2t + s - 1.
\]
- If $\Delta_0 = 0$, then for all $1 \leq i \leq t$, $\left( \frac{2}{q_i} \right)_4 (1 - 1) = 1$, so $\left( \frac{-\varepsilon_2, \delta}{\pi_i} \right) = \left( \frac{-\varepsilon_2, \delta}{\pi_i} \right) = 1$ by Lemma 2.5.
Thanks to Lemma 2.1 we have: for all $1 \leq i \leq s$, $\left( \frac{-\varepsilon_2, \delta}{q_i} \right) = 1$ because $q_i \equiv 1 \pmod{4}$. Lemma 2.2 gives $\left( \frac{-\varepsilon_2, \delta}{2\infty} \right) = 1$. Finally the product formula implies that $\left( \frac{-\varepsilon_2, \delta}{\sqrt{2}} \right) = 1$. We conclude that $-\varepsilon_2$ is a norm in $K/k$, hence $e = 1$ and $r_2(K) = 2t + s$.

**Remark 3.1.** If $(t, s) = (0, 0)$, then $d = 1$, so $K = \mathbb{Q}(\sqrt{2})$. The number of the primes ramified in $K/k$ is $2$ and $e = 1$ (we are in the case $\Delta_0 = \emptyset$; the condition on the $q_i$ is empty), hence $r_2(K) = 0$.

3.2. **Proof of Theorem B** : The cases $p \equiv 3, 5 \pmod{8}$
First, the number of the primes ramified in $K/k$ is $2t + s + 3$ (see §1). Therefore $r_2(K) = 2t + s + 2 - e$ (thanks to (1.1)).

According to Lemma 2.1(i) and Lemma 2.4(ii), 
\[
\left( \frac{-1, \delta}{p_\infty} \right) = \left( \frac{-\varepsilon_p, \delta}{p_\infty} \right) = -1; \text{ therefore } -1, \varepsilon_p \text{ are not norm in } K/k. \text{ By Lemma 2.1(i), we have } \left( \frac{-\varepsilon_p, \delta}{p_\infty} \right) = -1, \text{ if } p \equiv 3 \pmod{8}. \text{ By Lemma 2.4(i)(ii), we have } \left( \frac{-\varepsilon_p, \delta}{\sqrt{p}} \right) = -1, \text{ if } p \equiv 5 \pmod{8}. \text{ It follows that the units } -1, \varepsilon_p, -\varepsilon_p \text{ are not norm in } K/k, \text{ hence } e = 2 \text{ and } r_2(K) = 2t + s.
\]

**Remark 3.2.** If $(t, s) = (0, 0)$ and $p \equiv 3, 5 \pmod{8}$, then $d = 1$ or $2$, so $K = \mathbb{Q}(\sqrt{p})$ or $K = \mathbb{Q}(\sqrt{4p})$. We have $r_2(K) = 0$.

3.3. **Proof of Theorem C** : The case $p \equiv 7 \pmod{8}$. The number of the primes ramified in $K/k$ is (see §1)
\[
\begin{align*}
&\left\{ 
\begin{array}{ll}
2t + s + 3, & \text{if } d \text{ is odd;} \\
2t + s + 2, & \text{if } d \text{ is even.}
\end{array}
\right.
\end{align*}
\]
According to Lemma 2.1(ii)(ii) we have $\left( \frac{-1, \delta}{p_\infty} \right) = \left( \frac{-\varepsilon_p, \delta}{p_\infty} \right) = -1$. Consequently $-1, -\varepsilon_p$ are not norm in $K/k$.
Recall that $\Delta_1 = \left\{ i \left[ \frac{2}{q_i} \right] \right\}_4 (-1) = -1$.
- If $\Delta_1 \neq 0$, then there exists $1 \leq i \leq t$ such that $\left( \frac{2}{q_i} \right) = -1$. We have $\left( \frac{\varepsilon_p, \delta}{\pi_i} \right) = -1$ thanks to Lemma 2.5(ii); so $\varepsilon_p$ is not a norm in $K/k$. We conclude that $e = 2$. 

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220106-Sodaigui Version 2 - Submitted to Rocky Mountain J. Math.
• If $\Delta_1 = \emptyset$, then for all $1 \leq i \leq t$, $\left(\frac{p}{q_i^t}\right) = 1$ and $\left(\frac{2}{q_i^t}\right) = 1$; hence $\left(\frac{e_{p,\delta}}{\pi_i}\right) = 1$ by Lemma 2.5(ii). For all $1 \leq i \leq s$, $\left(\frac{e_{p,\delta}}{q_i}\right) = 1$ comes from Lemma 2.2(ii). Lemma 2.4(ii) implies that $\left(\frac{e_{p,\delta}}{\sqrt{p}}\right) = 1$. We have $\left(\frac{e_{p,\delta}}{p_\infty}\right) = 1$ by Lemma 2.1(ii). By the product formula we have $\left(\frac{e_{p,\delta}}{2}\right) = 1$. From the above discussion it follows that $e_p$ is a norm in $K/k$. We conclude that $e = 1$.

Using the formula (1.1) and distinguishing the four cases depending on $d$ and $\Delta_1$ allow us to finish the proof. For instance if $d$ is odd and $\Delta_1 = \emptyset$, then $r_2(K) = (2t + s + 3) - 1 - 1 = 2t + s + 1$.

**Remark 3.3.** If $(t, s) = (0, 0)$ and $p \equiv 7 \pmod{8}$, then $d = 1$ or $d = 2$.

- If $d = 1$ then $K = \mathbb{Q}(\sqrt{p})$. Here, we have $r_2(K) = 1$,
- If $d = 2$ then $K = \mathbb{Q}(\sqrt{4p})$ and $r_2(K) = 0$.

3.4. **Proof of Theorem D : The case $p \equiv 1 \pmod{8}$.** The number of the primes ramified in $K/k$ is (see §1)

$$
\begin{cases}
2t + s + 3 & \text{if } d \text{ is odd}, \\
2t + s + 4 & \text{if } d \text{ is even}.
\end{cases}
$$

By (1.1) we have

$$
\begin{align*}
\begin{cases}
r_2(K) = 2t + s + 2 - e & \text{if } d \text{ is odd}, \\
r_2(K) = 2t + s + 3 - e & \text{if } d \text{ is even}.
\end{cases}
\end{align*}
$$

According to Lemma 2.1(i)(iii), $\left(\frac{-1,\delta}{p_\infty}\right) = \left(\frac{e_{p,\delta}}{p_\infty}\right) = -1$. Therefore $-1, e_p$ are not norm in $K/k$.

Recall that $\Delta_2$ is defined by

$$\Delta_2 = \{i \mid \left(\frac{p}{q_i^t}\right) \left(\frac{q_i^t}{p}\right) = -1\}.$$

(1) Suppose that there exists $1 \leq i \leq s$, such that $q_i \equiv 3 \pmod{4}$. Then by Lemma 2.2(i)(ii) we have $\left(\frac{-e_{p,\delta}}{q_i}\right) = \left(\frac{-1}{q_i}\right) = -1$; hence $e = 2$.

If there exists $1 \leq i \leq t$, such that $q_i^t \equiv 3 \pmod{4}$, then by Lemma 2.5(i)(ii) we have $\left(\frac{-e_{p,\delta}}{q_i}\right) \left(\frac{-e_{p,\delta}}{\pi_i}\right) = -1$; hence $e = 2$.

(2) Suppose that for all $1 \leq i \leq s$, $q_i \equiv 1 \pmod{4}$ and for all $1 \leq i \leq t$, $q_i^t \equiv 1 \pmod{4}$.

- If $\Delta_2 = \emptyset$, then for all $1 \leq i \leq t$ such that $\left(\frac{2}{q_i^t}\right) \left(\frac{q_i^t}{2}\right) = -1$. By Lemma 2.6(i)(ii), $\left(\frac{-e_{p,\delta}}{q_i}\right) \left(\frac{e_{p,\delta}}{q_i}\right) = \left(\frac{-1}{q_i}\right) \left(\frac{2}{q_i^t}\right) \left(\frac{q_i^t}{2}\right) = -1$, so $e = 2$.
- If $\Delta_2 = \emptyset$, then for all $1 \leq i \leq t$, $\left(\frac{2}{q_i^t}\right) \left(\frac{q_i^t}{2}\right) = 1$; hence $\left(\frac{-e_{p,\delta}}{q_i}\right) = \left(\frac{-e_{p,\delta}}{\pi_i}\right) = 1$ by Lemma 2.5(i)(ii).

For all $1 \leq i \leq s$, $\left(\frac{-e_{p,\delta}}{q_i^t}\right) = 1$ comes from Lemma 2.2(i)(ii). Lemma 2.4(i)(ii) implies that $\left(\frac{-e_{p,\delta}}{\sqrt{p}}\right) = 1$. We have $\left(\frac{-e_{p,\delta}}{p_\infty}\right) = 1$ by Lemma 2.1(iii).

If $d$ is odd, Lemma 2.6(ii) gives $\left(\frac{-e_{p,\delta}}{2}\right) = 1$. By the product formula (or Lemma 2.6(ii)) we have $\left(\frac{-e_{p,\delta}}{2}\right) = 1$. It follows that $-e_p$ is a norm in $K/k$. We conclude that $e = 1$ in this case.
If \( d \) even, by Lemma 2.6(ii) we have
\[
\left( \frac{-\varepsilon_p \cdot \delta}{2^2} \right) = \left( \frac{-1, \delta}{2^2} \right) \left( \frac{-\varepsilon_p, \delta}{2^2} \right) = 1 \times \left( \frac{2}{p} \right)_4 (-1)^{\frac{d-1}{8}}.
\]
Suppose that \( \left( \frac{2}{p} \right)_4 (-1)^{\frac{d-1}{8}} = -1 \). Then \( \left( \frac{-\varepsilon_p, \delta}{2^2} \right) = -1 \). So \( -\varepsilon_p \) is not a norm in \( K/k \), hence \( e = 2 \).

Now suppose that \( \left( \frac{2}{p} \right)_4 (-1)^{\frac{d-1}{8}} = 1 \). Then \( \left( \frac{-\varepsilon_p, \delta}{2^2} \right) = 1 \) and by the product formula \( \left( \frac{-\varepsilon_p \cdot \delta}{2^2} \right) = 1 \); we conclude that \( -\varepsilon_p \) is a norm in \( K/k \), hence \( e = 1 \).

**Remark 3.4.** If \((t, s) = (0, 0)\) and \( p \equiv 1 \) (mod \( 8 \)), then \( d = 1 \) or \( d = 2 \). In this case the hypothesis of Theorem D(2) is empty, thus satisfied, and \( \Delta_2 = \emptyset \).

- If \( d = 1 \), then \( K = \mathbb{Q}(\sqrt[p]{p}) \). Theorem D(2)(i) gives \( r_2(K) = 1 \).
- If \( d = 2 \), then \( K = \mathbb{Q}(\sqrt[4]{p}) \). Theorem D(2)(ii) gives

\[
r_2(K) = \begin{cases} 
1 \text{ if } \left( \frac{2}{p} \right)_4 (-1)^{\frac{d-1}{8}} = -1, \\
2 \text{ if } \left( \frac{2}{p} \right)_4 (-1)^{\frac{d-1}{8}} = 1.
\end{cases}
\]

4. **Corollaries and Examples**

In this section we give some corollaries of our theorems and we illustrate our results with a variety of numerical examples calculated with the Pari-GP calculator ([14, version 2.13.1]).

**Corollary 4.1.** Let \( K = \mathbb{Q}(\sqrt[pd^2]{}) \). Then the class number of \( K \) is odd only in the following cases:

1. \( p = 2 \), and \( d = q \equiv 3 \) (mod \( 8 \)) or \( d = 1 \).
2. \( p \equiv 3, 5 \) (mod \( 8 \)), and \( d = 1 \) or \( d = 2 \).
3. \( p \equiv 7 \) (mod \( 8 \)) and \( d = 2 \).

**Proof.** Since \( r_2(K) \) is the number of the components of of the primary decomposition of the finite abelian group \( Cl_2(K) \) we have: the class number of \( K \) is odd if and only if \( r_2(K) = 0 \).

For the proof, we will distinguish all the possible cases for \( p \).

1. Assume \( p = 2 \).
   - If there exists \( i, 1 \leq i \leq s \), such that \( q_i \equiv 3 \) (mod \( 8 \)), then \( r_2(K) = 2t + s - 1 \) by Theorem A(1). Since \( s \geq 1 \) We have: \( r_2(K) = 0 \iff (t, s) = (0, 1) \). Therefore \( d = q_i \equiv 3 \) (mod \( 8 \)).
   - If there exists \( i, 1 \leq i \leq t \), such that \( q_i' \equiv 7 \) (mod \( 8 \)), then by Theorem A(2) \( r_2(K) = 2t + s - 1 \). Since \( t \geq 1 \), \( r_2(K) \neq 0 \).
   - Suppose that for all \( i, 1 \leq i \leq s \), \( q_i \equiv 5 \) (mod \( 8 \)) and for all \( i, 1 \leq i \leq t \), \( q_i' \equiv 1 \) (mod \( 8 \)). Below we will use Theorem A(2) and its notation. Note that \( \Delta_0 \neq \emptyset \) \( \iff t \geq 1 \). If \( \Delta_0 \neq \emptyset \), then \( r_2(K) = 2t + s - 1 \neq 0 \).
   - If \( \Delta_0 = \emptyset \), then \( r_2(K) = 2t + s \). We have: \( r_2(K) = 0 \iff (t, s) = (0, 0) \). Therefore \( d = 1 \).

2. Suppose \( p \equiv 3, 5 \) (mod \( 8 \)). By Theorem B, \( r_2(K) = 2t + s \). So \( r_2(K) = 0 \iff (t, s) = (0, 0) \). Immediately we have (2).

3. Suppose that \( p \equiv 7 \) (mod \( 8 \)). We will use the notation and the results of Theorem C.
   - If \( \Delta_1 \neq \emptyset \), then \( t \geq 1 \) and \( r_2(K) \neq 0 \) thanks to Theorem C.
   - If \( \Delta_1 = \emptyset \), then \( r_2(K) \) can be 0 only in the case \( d \) is even. In this case \( r_2(K) = 2t + s \).
   - Hence \( r_2(K) = 0 \iff (t, s) = (0, 0) \). We conclude that \( d = 2 \).
Corollary 4.2. Let \( K = \mathbb{Q}(\sqrt[4]{pq}) \). Let \( q_1, q_2 \) and \( q \) be odd primes. Then the 2-class group of \( K \) is cyclic only in the following cases:

(1) \( p = 2 \) and \( d \) takes one of the following forms:
   (a) \( d = q \equiv 5, 7 \pmod{8} \).
   (b) \( d = q \equiv 1 \pmod{8} \), such that \( \left( \frac{2}{q} \right) = (-1)^{\frac{q-1}{2}} = -1 \).
   (c) \( d = q_1q_2 \), with \( (q_1, q_2) \equiv (3, 5), (3, 3) \pmod{8} \).

(2) \( p \equiv 3, 5 \pmod{8} \), and \( d \) takes one of the following forms:
   (a) \( d = q, 2q \), such that \( \left( \frac{q}{q} \right) = -1 \).

(3) \( p \equiv 7 \pmod{8} \), and \( d \) takes one of the following forms:
   (a) \( d = 2q \), with \( \left( \frac{2}{q} \right) = -1 \).
   (b) \( d = 2q \), with \( q \equiv 3, 5 \pmod{8} \) such that \( \left( \frac{2}{q} \right) = 1 \).
   (c) \( d = 1 \).

(4) \( p \equiv 1 \pmod{8} \), and \( d \) takes one of the following forms:
   (a) \( d = q \equiv 3 \pmod{4} \), such that \( \left( \frac{2}{q} \right) = -1 \).
   (b) \( d = 2 \), and \( \left( \frac{2}{p} \right) = (-1)^{\frac{p-1}{2}} = -1 \).
   (c) \( d = 1 \).

Proof. We have: \( Cl_2(K) \) is cyclic if and only if \( r_2(K) = 1 \). We will use the same method as in the proof of Corollary 4.1 replacing \( r_2(K) = 0 \) by \( r_2(K) = 1 \). Also, we will try to be fairly concise.

(1) Suppose \( p = 2 \).
   - Assume there exists \( i \) such that \( q_i \equiv 3 \pmod{8} \). According to Theorem A-1, \( r_2(K) = 1 \Leftrightarrow (t, s) = (0, 2) \). Then we obtain (c).
   - Assume that there exists \( i \), \( 1 \leq i \leq t \), such that \( q_i' \equiv 7 \pmod{8} \). According to Theorem A-2, \( r_2(K) = 1 \Leftrightarrow (t, s) = (1, 0) \). Then we obtain the second part of (a) \( (q \equiv 7 \pmod{8}) \).
   - Suppose that for all \( i \), \( 1 \leq i \leq s \), \( q_i \equiv 5 \pmod{8} \) and for all \( i \), \( 1 \leq i \leq t \), \( q_i' \equiv 1 \pmod{8} \). According to Theorem A-3, \( r_2(K) = 1 \Leftrightarrow ((t, s) = (1, 0) \text{ or } (t, s) = (0, 1)) \); \( (t, s) = (1, 0) \) gives (b) and \( (t, s) = (0, 1) \) gives the first part of (a) \( (q \equiv 5 \pmod{8}) \).

(2) Now suppose \( p \equiv 3, 5 \pmod{8} \). Thanks to Theorem B, \( r_2(K) = 1 \Leftrightarrow (t, s) = (0, 1) \).
   Then \( d = q_1 \) or \( d = 2q_1 \) (recall that \( q_i \) satisfies \( \left( \frac{2}{q_i} \right) = -1 \)).

(3) Suppose \( p \equiv 7 \pmod{8} \). We use the notation of Theorem C.
   - Assume \( t \geq 1 \). Then \( \Delta_1 \neq \emptyset \).
     By Theorem C we have: \( r_2(K) = 2t + s - 1 = 1 \Leftrightarrow (t, s) = (1, 0) \) and \( d \) is even.
     This gives (b) \( \Delta_1 \neq \emptyset \Rightarrow \left( \frac{2}{q} \right) = -1 \Leftrightarrow q \equiv 3, 5 \pmod{8} \).
   - Assume \( t = 0 \). Then \( \Delta_1 = \emptyset \).
By Theorem C we have: \( r_2(K) = s + 1 = 1 \Leftrightarrow (t, s) = (0, 0) \) and \( d \) is odd, then \( d = 1 \) (we then have (c)), or \( r_2(K) = s = 1 \Leftrightarrow (t, s) = (0, 1) \) and \( d \) is even, then \( d = 2 \) (we then have (a)).

(4) Suppose \( p \equiv 1 \pmod{8} \). We use the notation of Theorem D.

- Assume \( t \geq 1 \). Then \( r_2(K) \geq 2 \) by Theorem D, hence \( r_2(K) \neq 1 \).
- Assume \( t = 0 \). Then \( \Delta_2 = 0 \).

By Theorem D(1) we have: \( r_2(K) = 2t + s = 1 \Leftrightarrow (t, s) = (0, 1) \) and \( d \) is odd. This gives (a).

By Theorem D(2)(i) we have: \( r_2(K) = 2t + s + 1 = 1 \Leftrightarrow (t, s) = (0, 0) \) and \( d \) is even. This gives (c).

By Theorem D (2)(ii)(b) we have: \( r_2(K) = 2t + s + 1 = 1 \Leftrightarrow (t, s) = (0, 0) \) and \( d \) is even, we then have (b).

\[ \square \]

**Corollary 4.3.** Let \( K = \mathbb{Q}(\sqrt[p]{d^2}) \). Let \( q_1, q_2 \) and \( q \) be odd primes. Then the 2-class group of \( K \) is the type \((2^{n_1}, 2^{n_2})\) (i.e., \( \text{Cl}_2(K) \cong \mathbb{Z}/2^{n_1}\mathbb{Z} \times \mathbb{Z}/2^{n_2}\mathbb{Z} \)), where \( n_i \in \mathbb{N}^* \), only in the following cases:

(1) \( p = 2 \) and \( d \) take one of the following forms:

- (a) \( d = q_1q_2 \) with \( (q_1, q_2) \equiv (3, 1), (3, 7), (5, 1), (5, 7), (5, 5) \pmod{8} \).
- (b) \( d = q_1q_2 \) with \( (q_1, q_2) \equiv (1, 5) \pmod{8} \) and \( \left( \frac{2}{q_1} \right)_4 (-1)^{\frac{q_1-1}{8}} = -1 \).
- (c) \( d = q \equiv 1 \pmod{8} \). Such that \( \left( \frac{2}{q} \right)_4 (-1)^{\frac{q-1}{8}} = 1 \).
- (d) \( d = q_1q_2q_3 \) with \( (q_1, q_2, q_3) \equiv (3, 3, 1), (3, 3, 5), (3, 5, 5) \pmod{8} \).

(2) \( p \equiv 3, 5 \pmod{8} \) and \( d \) take one of the following forms:

- (a) \( d = q, 2q \) such that \( \left( \frac{q}{q} \right) = 1 \).
- (b) \( d = q_1q_2, 2q_1q_2 \) such that \( \left( \frac{q_1}{q} \right)_4 \left( \frac{q}{q_2} \right)_4 = -1 \).

(3) \( p \equiv 7 \pmod{8} \) and \( d \) take one of the following forms:

- (a) \( d = q \) with \( q \) such that \( \left( \frac{q}{q} \right) = -1 \).
- (b) \( d = 2q \) with \( q \equiv 1, 7 \pmod{8} \) such that \( \left( \frac{q}{q} \right) = 1 \).
- (c) \( d = q \) with \( q \equiv 3, 5 \pmod{8} \) such that \( \left( \frac{q}{q} \right) = 1 \).
- (d) \( d = 2q_1q_2 \) with \( q_1, q_2 \) are odd prime such that \( \left( \frac{q_1}{q_2} \right)_4 \left( \frac{q_2}{q_2} \right)_4 = -1 \).
- (e) \( d = 2q_1q_2 \) with \( q_1 \equiv 3, 5 \pmod{8} \) such that \( \left( \frac{q_1}{q_2} \right)_4 = \left( \frac{q_2}{q_2} \right)_4 = 1 \).

(4) \( p \equiv 1 \pmod{8} \) and \( d \) take one of the following forms:

- (a) \( d = q \equiv 3 \pmod{4} \) such that \( \left( \frac{q}{q} \right) = 1 \).
- (b) \( d = q_1q_2 \) with \( q_1, q_2 \) such that \( q_1 \equiv 3 \pmod{4} \) and \( \left( \frac{q_1}{q_2} \right)_4 = \left( \frac{q_2}{q_2} \right)_4 = -1 \).
- (c) \( d = 2q \) such that \( q \equiv 3 \pmod{4} \) and \( \left( \frac{q}{q} \right)_4 = -1 \).
- (d) \( d = q, \) where \( d = q \equiv 1 \pmod{4} \) such that \( \left( \frac{q}{q} \right)_4 = 1 \) and \( \left( \frac{q}{q_2} \right)_4 \left( \frac{q_2}{q_2} \right)_4 = -1 \).
- (e) \( d = q \equiv 1 \pmod{4} \) such that \( \left( \frac{q}{q} \right)_4 = -1 \).
- (f) \( d = 2q \) with \( q \equiv 1 \pmod{4} \) such that \( \left( \frac{q}{q} \right)_4 = -1 \) and \( \left( \frac{q_2}{q_2} \right)_4 (-1)^{\frac{q_2-1}{8}} = -1 \).
- (g) \( d = 2 \) such that \( \left( \frac{2}{q} \right)_4 (-1)^{\frac{q-1}{8}} = 1 \).

\[ \text{Proof.} \] We have: \( \text{Cl}_2(K) \cong \mathbb{Z}/2^{n_1}\mathbb{Z} \times \mathbb{Z}/2^{n_2}\mathbb{Z} \) if and only if \( r_2(K) = 2 \). In the following, we we will mainly sketch the proof.
Assume \( t \geq 2 \). Then \( r_2(K) \geq 3 \) by Theorems A, B, C, D.

(1) suppose \( p = 2 \).
- Assume \( t = 1 \).
  Theorem A(1)(2) give: \((t, s) = (1, 1)\). We then have (a) without the case \((5, 5)\).
  Theorem A(3) gives:
  - if \( \Delta_0 \neq \emptyset \), \((t, s) = (1, 1)\). We then have (b).
  - if \( \Delta_0 = \emptyset \), \((t, s) = (1, 0)\). We then have (c).
- Assume \( t = 0 \). Then \( \Delta_0 = \emptyset \).
  Theorem A(1) gives: \((t, s) = (0, 3)\). We then have (d).
  Theorem A(3) gives: \((t, s) = (0, 2)\). We then have the case \((5, 5)\) of (a).

(2) Now suppose \( p \equiv 3, 5 \pmod{8} \)
  Theorem B(3) gives: \((t, s) = (1, 0)\), we then have (a), or \((t, s) = (0, 2)\) which implies (b).

(3) suppose \( p \equiv 7 \pmod{8} \)
- Assume \( t = 1 \).
  If \( \Delta_1 = \emptyset \) and \( d \) is even, then \((t, s) = (1, 0)\). We then have (b).
  If \( \Delta_1 \neq \emptyset \) and \( d \) is odd, then \((t, s) = (1, 0)\). We then have (c).
  If \( \Delta_1 \neq \emptyset \) and \( d \) is even, then \((t, s) = (1, 1)\). We then have (e).
- Assume \( t = 0 \). Then \( \Delta_1 = \emptyset \).
  If \( d \) is odd, then \((t, s) = (0, 1)\). We then have (a).
  If \( d \) is even, then \((t, s) = (0, 2)\). We then have (d).

(4) suppose \( p \equiv 1 \pmod{8} \).
- Assume \( t = 1 \). Then \( s = 0 \) by Theorem D.
  Theorem D(1) gives: \((t, s) = (1, 0)\), when \( d \) is odd. We then have (a).
  Theorem D(2)(i) gives: \((t, s) = (1, 0)\), when \( d \) is odd and \( \Delta_2 \neq \emptyset \). We then have the first part of (d) \((d = q)\).
- Assume \( t = 0 \). Then \( \Delta_2 = \emptyset \).
  Theorem D(1) gives:
  - \((t, s) = (0, 2)\), when \( d \) is odd. We then have (b).
  - \((t, s) = (0, 1)\), when \( d \) is even. We then have (c).
  Theorem D(2)(i) gives: \((t, s) = (0, 1)\), when \( d \) is odd. We then have (e).
  Theorem D(2)(ii)(b) gives:
  - \((t, s) = (0, 1)\), when \( d \) is even. We then have (f).
  - \((t, s) = (0, 0)\), when \( d \) is even. We then have (g).

\( \square \)

**Example 4.4.** Let \( k = \mathbb{Q}(\sqrt{2}) \) and \( K = \mathbb{Q}(\sqrt[4]{2q^2}) \).

<table>
<thead>
<tr>
<th>( q )</th>
<th>( q \pmod{8} )</th>
<th>( \frac{q}{4} )</th>
<th>(-1)^{\frac{q-1}{4}})</th>
<th>Conditions</th>
<th>( r_2(K) )</th>
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<td>( \frac{3}{4} )</td>
<td>(-1)^{\frac{3-1}{4}})</td>
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<td>( \frac{5}{4} )</td>
<td>(-1)^{\frac{5-1}{4}})</td>
<td>Corollary 4.2(1)-a</td>
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**Example 4.5.** Let \( k = \mathbb{Q}(\sqrt{2}) \) and \( K = \mathbb{Q}(\sqrt[4]{2q^2}) \).
Table 2. Numerical examples.

<table>
<thead>
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<th>$q_2$</th>
<th>$q_1 \pmod{8}$</th>
<th>$q_2 \pmod{8}$</th>
<th>$(\frac{q}{q_1})_4 (-1)^{n_2-1}$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
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<td>Corollary 4.3(1)-b</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>113</td>
<td>3, 11, 19, 43, 59, 67</td>
<td>1</td>
<td>3</td>
<td>Corollary 4.3(1)-c</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>3, 11, 19, 59, 67, 83</td>
<td>3</td>
<td>5</td>
<td>Corollary 4.2(1)-c</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Example 4.6. Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt{pq^2})$ and $p \equiv 3, 5 \pmod{8}$.

- The case: $p \equiv 3 \pmod{8}$

Table 3. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \pmod{8}$</th>
<th>$(\frac{q}{p})_4$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>43</td>
<td>3, 7, 13, 17, 19, 41, 53, 71, 97, 101, 109, 131, 151</td>
<td>3</td>
<td>1</td>
<td>Corollary 4.3(2)-a</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>7, 11, 13, 23, 29, 37, 41, 43, 47, 53, 83, 89, 97</td>
<td>3</td>
<td>-1</td>
<td>Corollary 4.2(2)-a</td>
<td>1</td>
</tr>
</tbody>
</table>

- The case: $p \equiv 5 \pmod{8}$

Table 4. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \pmod{8}$</th>
<th>$(\frac{q}{p})_4$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>3, 7, 11, 41, 47, 53, 67, 71, 73, 83, 101, 107, 127</td>
<td>5</td>
<td>1</td>
<td>Corollary 4.3(2)-a</td>
<td>2</td>
</tr>
<tr>
<td>29</td>
<td>3, 11, 17, 19, 31, 37, 41, 43, 47, 61, 73, 79, 89, 97</td>
<td>5</td>
<td>-1</td>
<td>Corollary 4.2(2)-a</td>
<td>1</td>
</tr>
</tbody>
</table>

Example 4.7. Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt{pq^2})$ and $p \equiv 3, 5 \pmod{8}$.

- The case: $p \equiv 3 \pmod{8}$

Table 5. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$(\frac{q_1}{p})_4$</th>
<th>$(\frac{q_2}{p})_4$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>7, 17, 19, 29, 31, 41, 43, 53, 67, 79, 89, 101, 103</td>
<td>-1</td>
<td>-1</td>
<td>Corollary 4.3(2)-b</td>
<td>2</td>
</tr>
</tbody>
</table>

- The case: $p \equiv 5 \pmod{8}$

Table 6. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$(\frac{q_1}{p})_4$</th>
<th>$(\frac{q_2}{p})_4$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97, 103</td>
<td>-1</td>
<td>-1</td>
<td>Corollary 4.3(2)-b</td>
<td>2</td>
</tr>
</tbody>
</table>

Example 4.8. Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt{4pq^2})$ and $p \equiv 3, 5 \pmod{8}$.
- The case: $p \equiv 3 \pmod{8}$

Table 7. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \pmod{8}$</th>
<th>$(p \pmod{8})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>3, 5, 17, 31, 59, 61, 67, 71, 73, 79, 101, 103, 107</td>
<td>3</td>
<td>1</td>
<td>Corollary 4.3(2)-b</td>
<td>2</td>
</tr>
<tr>
<td>43</td>
<td>5, 11, 23, 29, 31, 37, 47, 59, 61, 67, 73, 79, 83, 89</td>
<td>3</td>
<td>-1</td>
<td>Corollary 4.2(2)-a</td>
<td>1</td>
</tr>
</tbody>
</table>

- The case: $p \equiv 5 \pmod{8}$

Table 8. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \pmod{8}$</th>
<th>$(p \pmod{8})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>53</td>
<td>7, 11, 13, 17, 29, 37, 43, 47, 59, 89, 97, 107, 113</td>
<td>5</td>
<td>1</td>
<td>corollary 4.3(2)-a</td>
<td>2</td>
</tr>
<tr>
<td>37</td>
<td>5, 13, 17, 19, 23, 29, 31, 43, 59, 61, 79, 89, 97, 103</td>
<td>5</td>
<td>-1</td>
<td>Corollary 4.2(2)-a</td>
<td>1</td>
</tr>
</tbody>
</table>

Example 4.9. Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt[4]{pq_1q_2})$ and $p \equiv 3, 5 \pmod{8}$.

- The case: $p \equiv 3 \pmod{8}$

Table 9. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$(p \pmod{8})$</th>
<th>$(p \pmod{8})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7, 17, 19, 29, 31, 41, 43, 53, 67, 79, 89, 101, 103, 113</td>
<td>7</td>
<td>-1</td>
<td>-1</td>
<td>Corollary 4.3(2)-b</td>
<td>2</td>
</tr>
</tbody>
</table>

- The case: $p \equiv 5 \pmod{8}$

Table 10. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$(p \pmod{8})$</th>
<th>$(p \pmod{8})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97, 103, 107</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>Corollary 4.3(2)-b</td>
<td>2</td>
</tr>
</tbody>
</table>

Example 4.10. Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt[4]{pq_1q_2})$ and $p \equiv 7 \pmod{8}$

Table 11. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$q \pmod{8}$</th>
<th>$(p \pmod{8})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>71</td>
<td>23, 31, 47, 73, 89, 127, 233, 239, 313, 383, 409, 439</td>
<td>1, 7</td>
<td>1</td>
<td>Corollary 4.3(3)-b</td>
<td>2</td>
</tr>
<tr>
<td>23</td>
<td>3, 5, 17, 31, 37, 47, 53, 59, 61, 71, 89, 97</td>
<td>-1</td>
<td>-1</td>
<td>Corollary 4.2(3)-a</td>
<td>1</td>
</tr>
<tr>
<td>47</td>
<td>11, 19, 37, 43, 53, 61, 67, 101, 107</td>
<td>3, 5</td>
<td>1</td>
<td>Corollary 4.2(3)-b</td>
<td>1</td>
</tr>
</tbody>
</table>

Example 4.11. Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt[4]{pq_1q_2})$ and $p \equiv 7 \pmod{8}$.
### Table 12. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$q$ (mod 8)</th>
<th>$(\frac{p}{q})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>3, 5, 17, 31, 37, 47, 53, 59, 61, 71, 89, 97</td>
<td>-1</td>
<td>Corollary 4.3(3)-a</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>3, 5, 11, 43, 83, 101, 109, 139, 149, 157, 173, 179</td>
<td>3, 5</td>
<td>Corollary 4.3(3)-c</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

**Example 4.12.** Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt{\sqrt{4pq^2}})$ and $p \equiv 7$ (mod 8).

### Table 13. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_1$ (mod 8)</th>
<th>$(\frac{p}{q_1})$</th>
<th>$(\frac{p}{q_2})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>67</td>
<td>5, 11, 13, 17, 23, 41, 43, 61, 71, 73</td>
<td>-1</td>
<td>-1</td>
<td>Corollary 4.3(3)-d</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>19</td>
<td>3, 5, 17, 31, 37, 47, 53, 59, 61, 71</td>
<td>3, 5</td>
<td>1</td>
<td>Corollary 4.3(3)-e</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

**Example 4.13.** Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt{\sqrt{4pq^2}})$ and $p \equiv 1$ (mod 8).

### Table 14. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$q$ (mod 4)</th>
<th>$(\frac{p}{q})$</th>
<th>$(\frac{2}{p})_4$</th>
<th>$(-1)^{\frac{p-1}{2}}$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>73</td>
<td>7, 11, 31, 43, 47, 59, 83, 103, 107, 131</td>
<td>3</td>
<td>-1</td>
<td>Corollary 4.3(4)-c</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>73</td>
<td>5, 13, 17, 29, 53, 101, 113, 157, 193</td>
<td>1</td>
<td>-1</td>
<td>Corollary 4.3(4)-f</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 4.14.** Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt{pq^2})$ and $p \equiv 1$ (mod 8).

### Table 15. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$q$ (mod 4)</th>
<th>$(\frac{p}{q})$</th>
<th>$(\frac{p}{q})_4$</th>
<th>$(\frac{q}{p})_4$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>19, 43, 47, 59, 67, 83, 103, 127, 151</td>
<td>3</td>
<td>1</td>
<td>Corollary 4.3(4)-a</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>3, 7, 11, 23, 31, 71, 79, 107, 131</td>
<td>3</td>
<td>-1</td>
<td>Corollary 4.2(4)-a</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>13, 89, 101, 137, 229, 257, 373, 389</td>
<td>1</td>
<td>1</td>
<td>Corollary 4.3(4)-d</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>5, 29, 37, 41, 61, 73, 97, 109, 113, 173</td>
<td>1</td>
<td>-1</td>
<td>Corollary 4.3(4)-e</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 4.15.** Let $k = \mathbb{Q}(\sqrt{p}), K = \mathbb{Q}(\sqrt{pq^2})$ and $p \equiv 1$ (mod 8).

### Table 16. Numerical examples.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_1$ (mod 4)</th>
<th>$(\frac{p}{q_1})$</th>
<th>$(\frac{p}{q_2})$</th>
<th>Conditions</th>
<th>$r_2(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>79</td>
<td>3, 7, 11, 13, 17, 19, 29, 47, 53, 67, 71</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>Corollary 4.3(4)-b</td>
<td>2</td>
</tr>
</tbody>
</table>
References

[1] A. Azizi, A. Mouhib, *Sur le rang du 2-groupe de classes de \( \mathbb{Q}(\sqrt{m}, \sqrt{d}) \), où \( m = 2 \) ou un premier \( p \equiv 1 \) (mod 4)*, Trans. Amer. Math. Soc. 353 (7) (2001) 2741-2752.


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