ENTIRE AND MEROMORPHIC SOLUTIONS FOR SEVERAL FERMAT TYPE PARTIAL DIFFERENTIAL DIFFERENCE EQUATIONS IN $\mathbb{C}^2$

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Abstract. Our main aim of this paper is devoted to exploring the existence and the forms of entire and meromorphic solutions for the partial differential-difference equations with more general forms of

$$\left( \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} \right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1, z_2)},$$

and

$$\left( \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} \right)^2 + \left[ f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) \right]^2 = e^{g(z_1, z_2)},$$

where $g(z_1, z_2)$ is a polynomial in $\mathbb{C}^2$, and $\alpha, \beta$ are constants in $\mathbb{C}$. Some results about the forms of solutions for these equations are obtained are great improvements of the previous theorems given by Xu, Cao, Liu, Yang [16, 17, 18, 26, 27]. It is important that some examples show that there exist some significant differences in the forms of transcendental entire solutions of finite order of the equations with between several complex variables and a single complex variable.

1. Introduction and Main Results

For the classical Fermat type functional equation

$$(1.1) \quad f^m + g^n = 1,$$

Gross [6] had discussed the existence of solutions of equation (1.1) and showed that the entire solutions are $f = \cos a(z), g = \sin a(z)$ for $m = n = 2$, where $a(z)$ is an entire function. Montel [19] proved that there are no nonconstant entire solutions for equation (1.1) as $m = n > 2$.

Of late, with the establishment of difference Nevanlinna theory for meromorphic functions (see [3, 7, 8, 9]), many scholars have paid an increasing interest in studying the properties of solutions for the difference analogues of the Fermat type functional equation (including [16, 17]). Here, we just recall some results related to the Fermat type difference equations. In 2012, Liu, et al. [18] considered the Fermat type difference equation

$$(1.2) \quad f(z)^2 + f(z + c)^2 = 1,$$

and obtained that the transcendental entire solutions of finite order of Eq. (1.2) must satisfy $f(z) = \sin(Az + B)$, where $B$ is a constant and $A = \frac{(4k+1)\pi}{2c}$, $k$ is
an integer (see [18, Theorem 1.4]). In 2019, Han and Lü [10] investigated the complex difference equation with more general form \( f(z)^2 + f(z + c)^2 = e^{\alpha z + \beta} \), and obtained that all the nontrivial meromorphic solutions of this equation are the functions \( f(z) = de^{\alpha z + \beta} \) with \( d^2(1 + e^{\alpha c}) = 1 \) (see [10, pp. 99]).

More than four decades ago, Naftalevich [20, 21] initiated the study of complex differential-difference equations. He studied the meromorphic solutions of complex differential-difference equations by making use of operator theory and iteration method. But recently, many researchers have begun to study this kind of equations by utilizing Nevanlinna theory. In particular, Liu, et al. [18] studied the existence of solutions for some complex differential-difference equations and obtained Theorem A (see [18, Theorem 1.3]).

The transcendental entire solutions of finite order of

\[
f'(z)^2 + f(z + c)^2 = 1,
\]

must satisfy \( f(z) = \sin(z \pm Bi) \), where \( B \) is a constant and \( c = 2k\pi \) or \( c = (2k + 1)\pi \), \( k \) is an integer.

Theorem B (see [18, Theorem 1.5]). The transcendental entire solutions of finite order of

\[
f'(z)^2 + [f(z + c) - f(z)]^2 = 1,
\]

must satisfy \( f(z) = 12 \sin(2z + Bi) \), where \( c = (2k + 1)\pi \), \( k \) is an integer, and \( B \) is a constant.

Now, let us recall some results about the Fermat type function equations with several variable. The solutions of Fermat-type partial differential equations were originally investigated by [15, 23]. In 1995, Khavinson [12] pointed out that any entire solution of the partial differential equation

\[
\left( \frac{\partial f}{\partial z_1} \right)^2 + \left( \frac{\partial f}{\partial z_2} \right)^2 = 1 \quad \text{in} \quad \mathbb{C}^2
\]

is necessarily linear. This partial differential equations in real variable case occur in the study of characteristic surfaces and wave propagation theory, and it is the two dimensional eiconal equation, one of the main equations of geometric optics (see [4, 5]). Later, Li [14] in 2005 discussed the partial differential equation of Fermat type

\[
(1.3) \quad \left( \frac{\partial u}{\partial z_1} \right)^2 + \left( \frac{\partial u}{\partial z_2} \right)^2 = e^g,
\]

where \( g \) is a polynomial or an entire function in \( \mathbb{C}^2 \), and obtained some results on the forms of entire solution of equation (1.3) as follows

Theorem C ([14, Theorem 2.1]). Let \( g \) be a polynomial in \( \mathbb{C}^2 \). Then \( u \) is an entire solution of the partial differential equation (1.3), if and only if

(i) \( u = f(c_1 z_1 + c_2 z_2) \); or

(ii) \( u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2) \),

where \( f \) is an entire function in \( \mathbb{C} \) satisfying that \( f'(c_1 z_1 + c_2 z_2) = \pm e^{\frac{1}{2}g(z)} \), \( c_1 \) and \( c_2 \) are two constants satisfying that \( c_1^2 + c_2^2 = 1 \), and \( \phi_1 \) and \( \phi_2 \) are entire functions in \( \mathbb{C} \) satisfying that \( \phi_1'(z_1 + iz_2)\phi_2'(z_1 - iz_2) = \frac{1}{4}e^{g(z)} \).
Very recently, Xu and Cao [2, 26, 27] investigated the existence of the entire and meromorphic solutions for some Fermat-type partial differential-difference equations with several variables by making use of Nevanlinna theory and the recent result on difference logarithmic derivative lemma of several complex variables [1, 2, 13], and obtained the following theorems.

**Theorem D** (see [26, Theorem 1.2]). Let \( c = (c_1, c_2) \in \mathbb{C}^2 \). Then any transcendental entire solutions of finite order of the partial differential-difference equation

\[
\left( \frac{\partial f(z_1, z_2)}{\partial z_1} \right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1
\]

has the form of \( f(z_1, z_2) = \sin(Az_1 + B) \), where \( A \) is a constant on \( \mathbb{C} \) satisfying \( Ae^{i\pi c_1} = 1 \), and \( B \) is a constant on \( \mathbb{C} \); in the special case whenever \( c_1 = 0 \), we have \( f(z_1, z_2) = \sin(\pi + B) \).

**Theorem E** (see [26, Theorem 1.3]). Let \( c = (c_1, c_2) \in \mathbb{C}^2 \). Then any nonconstant meromorphic solutions of the partial difference-differential equation

\[
\left( \frac{\partial f(z_1, z_2)}{\partial z_1} \right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1
\]

has the form of \( f(z_1, z_2) = \frac{h(z - c) - \frac{1}{z} \cdot \frac{1}{n + 1}}{2^n} \), where \( h \) is a nonzero meromorphic function on \( \mathbb{C} \) satisfying \( i(h(z + c) + \frac{1}{n(z + c)}) = \frac{\partial h}{\partial z} \left( 1 + \frac{1}{n(z + c)} \right) \). In the special case whenever \( c_1 = c_2 = 0 \), we have \( f(z_1, z_2) = \sin(z_1 - i\alpha(z_2)) \), where \( \alpha(z_2) \) is a meromorphic function in one complex variable \( z_2 \).

**Remark 1.1.** In fact, one can get that all the meromorphic solutions of this equation are the entire solutions in the case \( c_1 = c_2 = 0 \).

Theorems C, D and E suggest the following questions as open problems.

**Question 1.1.** What can be said about the solutions of equations when the constant 1 is replaced by a function \( e^g \) in Theorems D and E, where \( g \) be a polynomial in \( \mathbb{C}^2 \).

**Question 1.2.** What can be said about the solutions of equations when \( \frac{\partial f(z_1, z_2)}{\partial z_1} \) is replaced by a function \( \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} \) in Theorems D and E, where \( \alpha, \beta \in \mathbb{C} \).

2. **Results**

Inspired by the above questions, this paper is concerned with description of entire and meromorphic solutions for several partial differential-difference of Fermat-type by utilizing the Nevanlinna theory and difference Nevanlinna theory of several complex variables. We extend Theorems D and E for the Fermat type partial differential-difference equations with the more general forms, and obtain some results which improve the previous theorems given by Xu and Cao, Liu, Cao and Cao [18, 26]. Throughout this paper, let \( z + w = (z_1 + w_1, z_2 + w_2) \) for any \( z = (z_1, z_2), w = (w_1, w_2) \). Here, we list our main results as follows.

**Theorem 2.1.** Let \( c = (c_1, c_2) \in \mathbb{C}^2 \), and \( \alpha, \beta \) be constants in \( \mathbb{C} \) that are not zero at the same time. If the partial differential-difference equation

\[
(\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2})^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1, z_2)}
\]

(2.1)
admits a transcendental entire solution of finite order, then \( g(z) \) must be a polynomial function of the form \( g(z) = L(z) + H(s_1) + B \), where \( L(z) \) is a linear function of the form \( L(z) = A_1 z_1 + A_2 z_2 + B \), \( H(s_1) \) is a polynomial in \( s_1 = c_2 z_1 - c_1 z_2 \), \( A_1, A_2, B \in \mathbb{C} \). Further, \( f(z) \) must satisfy one of the following cases

(i)

\[
f(z_1, z_2) = \pm e^{\frac{1}{2}g(z-c)},
\]

where \( g(z) = \phi(\beta z_1 - \alpha z_2) \), \( \phi \) is a polynomial in \( \mathbb{C} \);

(ii)

\[
f(z_1, z_2) = \frac{\xi^2 + 1}{\xi(\alpha A_1 + \beta A_2)} e^{\frac{1}{2}(L(z) + H(s_1) + B)},
\]

where \( \xi \neq 0 \), \( A_1, A_2, B \in \mathbb{C} \) satisfying

\[
(\alpha c_2 - \beta c_1)H' \equiv 0, \quad \frac{1}{2} \frac{\xi^2 - 1}{(\xi^2 + 1)i} (\alpha A_1 + \beta A_2) = e^{\frac{1}{2}(A_1 c_1 + A_2 c_2)};
\]

(iii)

\[
f(z_1, z_2) = \frac{e^{L_1(z)+H_1(s_1)+B_1}}{2(\alpha A_{11} + \beta A_{12})} + \frac{e^{L_2(z)+H_2(s_1)+B_2}}{2(\alpha A_{21} + \beta A_{22})},
\]

where \( L_1(z) = A_{11} z_1 + A_{12} z_2 + B_1, L_2(z) = A_{21} z_1 + A_{22} z_2 + B_2, A_{j1}, A_{j2}, B_j \in \mathbb{C}, (j = 1, 2) \) satisfy

\[
L_1(z) + H_1(s_1) \neq L_2(z) + H_2(s_1), \quad g(z) = L_1(z) + L_2(z) + H_1(s_1) + H_2(s_1) + B_1 + B_2,
\]

and

\[
(\alpha c_2 - \beta c_1)H'_j \equiv 0, \quad -i(\alpha A_{11} + \beta A_{12})e^{-L_1(c)} = i(\alpha A_{21} + \beta A_{22})e^{-L_2(c)} = 1,
\]

where \( H_j(s_1), j = 1, 2 \) are polynomials in \( s_1 = c_2 z_1 - c_1 z_2 \).

The following examples show that the forms of solutions are precise to some extent.

**Example 2.1.** Let \( \alpha = 1, \beta = -1 \), and

\[
f(z_1, z_2) = \pm e^{\frac{1}{2}(z_1 + z_2)},
\]

then \( \rho(f) = 1 \) and \( f(z_1, z_2) \) is a transcendental entire solution of equation (2.1) with \( g(z) = z_1 + z_2, c_1 = \pi i \) and \( c_2 = \pi i \).

**Example 2.2.** Let \( \alpha = 2, \beta = 1 \), \( L(z) = z_1 + 2 z_2, H(s_1) = \pi^4(2 z_2 - z_1)^4 \), and

\[
f(z_1, z_2) = \sqrt{\frac{5}{3}} e^{\frac{1}{2}z_1+2z_2+\pi^4(z_2-z_1)^4},
\]

then \( \rho(f) = 4 \) and \( f(z_1, z_2) \) is a transcendental entire solution of equation (2.1) with \( g(z) = z_1 + 2 z_2 + \pi^4(2 z_2 - z_1)^4, c_1 = 2 \pi i \) and \( c_2 = \pi i \).

**Example 2.3.** Let \( \alpha = \frac{1+2i}{3}, \beta = \frac{2+i}{3} \), \( L_1(z) = z_1 + 2 z_2, L_2(z) = 2 z_1 + z_2, H_1(s_1) = H_2(s_1) = 0 \), and

\[
f(z_1, z_2) = \frac{e^{z_1+2z_2} - e^{2z_1+z_2}}{2i},
\]

then \( \rho(f) = 1 \) and \( f(z_1, z_2) \) is a transcendental entire solution of equation (2.1) with \( g(z) = 3 z_1 + 3 z_2, c_1 = \frac{3}{2} \pi i \) and \( c_2 = -\pi i \).
Example 2.4. Let $\alpha = 1, \beta = 1$, $L_1(z) = i(z_2 - 2z_1)$, $L_2(z) = -i(z_2 - 2z_1)$, $H_1(s_1) = 4\pi^2(z_1 - z_2)^3$, $H_2(s_1) = 8\pi^3(z_1 - z_2)^3$, and 
\[
f(z_1, z_2) = \frac{e^{i(z_2 - 2z_1) + 4\pi^2(z_1 - z_2)^2} - e^{-i(z_2 - 2z_1) + 8\pi^3(z_1 - z_2)^3}}{2i},
\]
then $\rho(f) = 3$ and $f(z_1, z_2)$ is a transcendental entire solution of equation (2.1) with $g(z) = 4\pi^2(z_1 - z_2)^2 + 8\pi^3(z_1 - z_2)^3$, $c_1 = 2\pi$ and $c_2 = 2\pi$.

From Theorem 2.1, it is easy to get the following corollaries.

Corollary 2.1. Let $f$ be a finite order transcendental entire solution of the partial differential equation
\[
(2.2) \quad \left( \frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2} \right)^2 + f(z_1, z_2)^2 = 1.
\]
Then $f(z_1, z_2)$ must be the form of
\[
f(z_1, z_2) = \frac{e^{L(z) + B} - e^{-L(z) - B}}{2i} = \sin(-i(L(z) + B)),
\]
where $L(z) = A_1z_1 + A_2z_2$, $A_1, A_2, B \in \mathbb{C}$ satisfy $A_1 + A_2 = i$.

In addition, in view of Theorem 2.1, one can obtain the conclusions of Theorem 1.2 in [26] if $\alpha = 1, \beta = 0$, $g(z) = 2k\pi i$, $k \in \mathbb{Z}$ in equation (2.1).

For the following partial differential-difference equations
\[
(2.3) \quad \left( \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} \right)^2 + |f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2)|^2 = e^{g(z_1, z_2)},
\]
we have

Theorem 2.2. Let $c = (c_1, c_2) \in \mathbb{C}^2$, $\alpha \neq 0$, $\beta$ be constants in $\mathbb{C}$ that are not zero at the same time and $\alpha c_2 - \beta c_1 \neq 0$. Let $f(z)$ be a finite order transcendental entire solution of the partial differential-difference equation (2.3), then $f(z)$ must satisfy one of the following cases

(i) $f(z_1, z_2) = \varphi_1(\beta z_1 - \alpha z_2)$, $e^{\frac{1}{2}g(z)} = \varphi_1(\beta z_1 - \alpha z_2 + \beta c_1 - \alpha c_2) - \varphi_1(\beta z_1 - \alpha z_2)$, where $\varphi_1$ is a finite order transcendental entire function;

(ii) $g(z) = A_1z_1 + A_2z_2 + H(s_1) + B$ and
\[
f(z_1, z_2) = \pm \frac{1}{\alpha} \int_0^{\frac{2\pi}{\alpha}} e^{\frac{1}{2}(A_1z_1 + A_2z_2 + H(s_1) + B)dz_1} + G \frac{\alpha z_2 - \beta z_1}{\alpha},
\]
where $e^{\frac{1}{2}(A_1c_1 + A_2c_2)} = 1$, $H(s_1)$ is a polynomial in $s_1 = c_2z_1 - c_1z_2$, $G$ is a finite order period entire function with the period $\frac{\alpha c_2 - \beta c_1}{\alpha}$, $A_1, A_2 \in \mathbb{C}$;

(iii) $g(z) = A_1z_1 + A_2z_2 + B$ and
\[
f(z_1, z_2) = (\xi + 1) \left( \frac{1}{\xi} \frac{e^{\frac{1}{2}(A_1z_1 + A_2z_2 + B)}}{\alpha A_1 + \beta A_2} + G \frac{\alpha z_2 - \beta z_1}{\alpha} \right),
\]
where $A_1, A_2, B \in \mathbb{C}$, $G$ is a finite order entire period function with period $\frac{\alpha c_2 - \beta c_1}{\alpha}$, $\xi \neq 0$, $A_1, A_2, B \in \mathbb{C}$ satisfying
\[
\frac{\xi^2 - 1}{2i(\xi^2 + 1)} (\alpha A_1 + \beta A_2) + 1 = e^{\frac{1}{2}(A_1c_1 + A_2c_2)};
\]
(iv) \(g(z) = A_1z_1 + A_2z_2 + B\) and
\[
f(z_1, z_2) = \frac{e^{L_1(z)+B_1}}{2(\alpha A_{11} + \beta A_{12})} + \frac{e^{L_2(z)+B_2}}{2(\alpha A_{21} + \beta A_{22})} + G\left(\frac{\alpha z_2 - \beta z_1}{\alpha}\right),
\]
where \(A_1, A_2, B \in \mathbb{C}, G\) is a finite order entire period function with period \(\frac{\alpha c_2 - \beta c_1}{\alpha}\),
\(L_1(z) = A_{11}z_1 + A_{12}z_2 + B_1, L_2(z) = A_{21}z_1 + A_{22}z_2 + B_2, A_{ij}, B_i \in \mathbb{C}\) satisfy
\(L_1(z) \neq L_2(z), g(z) = L_1(z) + L_2(z) + B_1 + B_2,\)
and
\[|1 - i(\alpha A_{11} + \beta A_{12})| e^{-L_1(c)} = \left|1 + i(\alpha A_{21} + \beta A_{22})\right| e^{-L_2(c)} = 1.\]

The following examples explain the existence of transcendental entire solutions with finite order of (2.3).

**Example 2.5.** Let \(\alpha = 1, \beta = 0, H(s_1) = 0, A_1 = A_2 = 1,\) and
\[
\rho(f) = 1 \text{ and } f(z_1, z_2) = 2e^{\frac{c}{2}(z_1+z_2)} + e^{c_1},
\]
then \(\rho(f) = 1\) and \(f(z_1, z_2)\) is a transcendental entire solution of the equation (2.3) with \(g(z) = z_1 + z_2, c_1 = 2\pi i\) and \(c_2 = -2\pi i.\)

**Example 2.6.** Let \(\alpha = -3, \beta = 1, L(z) = z_1 + z_2, G(z) = G(\frac{3z_2 + 4z_1}{3}) = e^{\frac{c}{2}(z_1 + 3z_2)},\)
and
\[
f(z_1, z_2) = \frac{-\sqrt{5}}{3} e^{\frac{c}{2}(z_1+z_2)} + e^{\frac{c}{2}(z_1+3z_2)},
\]
then \(\rho(f) = 1\) and \(f(z_1, z_2)\) is a transcendental entire solution of the equation (2.3) with \(g(z) = z_1 + z_2, c_1 = 2\pi i\) and \(c_2 = 2\pi i.\)

**Example 2.7.** Let \(\alpha = -(1+4i), \beta = 1+3i, L_1(z) = 2z_1 + 2z_2, L_2(z) = 2z_1 + 3z_2,\)
\(G_1(z) = G_1(s) = e^{\frac{s}{4(1+4i)s}}\) and
\[
f(z_1, z_2) = \frac{e^{2z_1+2z_2} - e^{2z_1+z_2}}{-4i} + \frac{e^{2z_1+3z_2}}{2(1+i)} + e^{\frac{c}{2}\pi i[(1+3i)z_1+(1+4i)z_2]},
\]
then \(f(z_1, z_2)\) is a transcendental entire solution with finite order of the equation (2.3) with \(g(z) = 4z_1 + 5z_2, c_1 = \pi i\) and \(c_2 = -\frac{\pi}{2} i.\)

**Corollary 2.2.** Let \(c = (c_1, c_2) \in \mathbb{C}^2\) and \(c_1 \neq c_2.\) If \(g(z_1, z_2)\) is not a linear function of the form \(L(z) = A_1z_1 + A_2z_2 + B_0.\) Then the partial differential-difference equation
\[
(2.4) \quad \left(\frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + [f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2)]^2 = e^{g(z_1, z_2)}
\]
does not possess any transcendental entire solution of finite order.

The following example shows that the condition ”\(c_1 \neq c_2\)” in Corollary 2.2 cannot be removed.

**Example 2.8.** Let \(c_1 = c_2 \neq 0\) and
\[
f(z_1, z_2) = e^{\frac{c}{2}[(z_1+z_2) + (c_2z_1 - c_1z_2)^3]},
\]
Then \(f(z_1, z_2)\) is a transcendental entire solution of finite order of equation (2.4) with \(g(z_1, z_2) = (z_1 + z_2) + (c_2z_1 - c_1z_2)^3.\)

Similar to the argument as the above, we can obtain for the difference equation with two complex variables as follows.
Theorem 2.3. Let $c = (c_1, c_2) \in \mathbb{C}^2$. If the difference equation

\begin{equation}
\label{eq:transcendental}
f(z_1, z_2)^2 + f(z_1 + c_1, z_2 + c_2)^2 = e^{g(z_1, z_2)}
\end{equation}

admits a transcendental entire solution of finite order, then $g(z)$ must be a polynomial function of the form $g(z) = L(z) + H(s_1) + B$, where $L(z)$ is a linear function of the form $L(z) = A_1 z_1 + A_2 z_2$, $H(s_1)$ is stated as in Theorem 2.1. $A_1, A_2 \in \mathbb{C}$. Further, $f(z)$ must satisfy one of the following cases

(i)

\begin{equation}
f(z_1, z_2) = \begin{cases} 
+ \frac{1}{\sqrt{e^{L(c)} + 1}} e^{\frac{1}{2}(L(z) + H(s_1) + B)}, & \xi = 0, \\
\frac{\xi^2 + 1}{\xi} e^{\frac{1}{2}(L(z) + H(s_1) + B)}, & \xi \neq 0,
\end{cases}
\end{equation}

where $g(z) = L(z) + H(s_1) + B$, $L(c) = A_1 c_1 + A_2 c_2$, $A_1, A_2, c_1, c_2, B \in \mathbb{C}$ satisfying $e^{L(c)} + 1 \neq 0$ for $\xi = 0$, and

$$\frac{\xi^2 - 1}{(\xi^2 + 1)i} = e^{\frac{1}{2}(A_1 c_1 + A_2 c_2)}; \text{ for } \xi \neq 0;$$

(ii)

\begin{equation}
f(z_1, z_2) = \frac{e^{L_1(z) + H_1(s_1) + B_1} + e^{L_2(z) + H_2(s_1) + B_2}}{2},
\end{equation}

where $L_1(z) = A_{11} z_1 + A_{12} z_2 + B_1$, $L_2(z) = A_{21} z_1 + A_{22} z_2 + B_2$, $A_{1j}, A_{2j}, B_j \in \mathbb{C}, (j = 1, 2)$ satisfy $L_1(z) + H_1(s_1) \neq L_2(z) + H_2(s_1)$, $g(z) = L_1(z) + L_2(z) + H_1(s_1) + H_2(s_1) + B_1 + B_2$, and

$$-ie^{-L_1(c)} = ie^{-L_2(c)} = 1,$$

where $H_j(s_1), j = 1, 2$ are stated as in Theorem 2.1.

If $g(z_1, z_2) = 0$, one can obtain the following corollary from the conclusions of Theorem 2.3.

Corollary 2.3. Let $f$ be a finite order transcendental entire solution of the difference equation

\begin{equation}
f(z_1, z_2)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1.
\end{equation}

Then $f(z_1, z_2)$ must be the form of

$$f(z_1, z_2) = \frac{e^{L(z) + H(s_1) + B} + e^{L(z) - H(s_1) - B}}{2} = \cos(i(L(z) + H(s_1) + B));$$

where $L(z) = A_1 z_1 + A_2 z_2$, $H(s_1)$ is stated as in Theorem 2.1, $A_1, A_2, B \in \mathbb{C}$ satisfy $A_1 c_1 + A_2 c_2 = \frac{\pi}{2} i + 2k\pi i, k \in \mathbb{Z}$.

The following example shows the form of transcendental entire solution of equation (2.6) is precise in the conclusion of Corollary 2.3.

Example 2.9. Let $L(z) = z_1 + z_2$, $H(s_1) = (z_1 + z_2)^4$, $B = 0$ and

$$f(z_1, z_2) = \frac{1}{\sqrt{2}} e^{\frac{1}{2}(z_1 + z_2 + (z_1 + z_2)^4)};$$

then $\rho(f) = 4$ and $f(z_1, z_2)$ is a transcendental entire solution of equation (2.6) with $c_1 = \pi i$ and $c_2 = -\pi i$. 

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Example 2.10. Let $L(z) = z_1 + z_2$, $H(s_1) = s_1^2 = -\pi^2\left(\frac{1}{2}z_2 + z_1\right)^2$, $B = 0$ and

$$f(z_1, z_2) = \frac{e^{z_1 + z_2 - \pi^2\left(\frac{1}{2}z_2 + z_1\right)^2} + e^{-(z_1 + z_2 - \pi^2\left(\frac{1}{2}z_2 + z_1\right)^2)}}{2},$$

then $\rho(f) = 2$ and $f(z_1, z_2)$ is a transcendental entire solution of equation (2.6) with $c_1 = \pi i$ and $c_2 = -\frac{1}{2} \pi i$.

Corresponding to Theorem E, we obtain the following theorem on the existence of meromorphic solution of the partial differential-difference equation including both partial differential and difference.

**Theorem 2.4.** Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $\alpha, \beta$ be constants in $\mathbb{C}$ that are not zero at the same time. Then any nonconstant meromorphic solution of the partial differential-difference equation

$$(2.7) \quad \left( \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} \right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1$$

has the form of

$$f(z_1, z_2) = \frac{h(z - c) - \frac{1}{h(z - c)}}{2i},$$

where $h$ is a nonzero meromorphic function on $\mathbb{C}$ satisfying

$$i \left( h(z + c) + \frac{1}{h(z + c)} \right) = \left( \alpha \frac{\partial h}{\partial z_1} + \beta \frac{\partial h}{\partial z_2} \right) \left( 1 + \frac{1}{h(z)^2} \right).$$

In the special case where $c_1 = c_2 = 0$, we have $f(z_1, z_2) = \sin\left(\frac{z_1}{\alpha} - \frac{i g(\alpha z_2 - \beta z_1)}{\alpha}\right)$, where $g(s_2)$ is a meromorphic function in one complex variable $s_2 = \frac{\alpha z_2 - \beta z_1}{\alpha}$.

**Corollary 2.4.** Any nonconstant meromorphic solution of the partial differential equation

$$(2.8) \quad \left( \frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2} \right)^2 + f(z_1, z_2)^2 = 1$$

has the form of $f(z_1, z_2) = \sin(z_1 - i g(z_2 - z_1))$, where $g(s_2)$ is a meromorphic function in one complex variable $s_2$.

**Remark 2.1.** Similar to Remark 1.1, one can get that all the meromorphic solutions of equation (2.8) are the entire solutions.

3. Some Lemmas

The following lemmas play the key roles in proving our results.

**Lemma 3.1.** ([24, 25]). For an entire function $F$ on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function $f_F$ and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, $f_F$ is the canonical product of Weierstrass.

**Remark 3.1.** Here, denote $\rho(n_F)$ to be the order of the counting function of zeros of $F$. 


Lemma 3.2. ([22]). If g and h are entire functions on the complex plane \( \mathbb{C} \) and \( g(h) \) is an entire function of finite order, then there are only two possible cases: either

(a) the internal function \( h \) is a polynomial and the external function \( g \) is of finite order; or else

(b) the internal function \( h \) is not a polynomial but a function of finite order, and the external function \( g \) is of zero order.

Lemma 3.3. ([11, Theorem 1.106]). Suppose that \( a_0(z), a_1(z), \ldots, a_n(z) (n \geq 1) \) are meromorphic functions on \( \mathbb{C}^m \) and \( g_0(z), g_1(z), \ldots, g_n(z) \) are entire functions on \( \mathbb{C}^m \) such that \( g_j(z) - g_k(z) \) are not constants for \( 0 \leq j < k \leq n \). If the following conditions

\[
\sum_{j=0}^{n} a_j(z)e^{\beta_j(z)} = 0
\]

and

\[
\| T(r, a_j) = o(T(r)), \quad j = 0, 1, \ldots, n
\]

hold, where \( T(r) = \min_{0 \leq j \leq n} T(r, e^{\gamma_j}) \), then \( a_j(z) \equiv 0 (j = 0, 1, 2, \ldots, n) \).

Lemma 3.4. ([11, Lemma 3.1]). Let \( f_j \neq 0, j = 1, 2, 3 \), be meromorphic functions on \( \mathbb{C}^m \) such that \( f_1 \) is not constant, and \( f_1 + f_2 + f_3 = 1 \), and such that

\[
\sum_{j=1}^{3} \left\{ N_2(r, \frac{1}{f_j}) + 2 \mathcal{N}(r, f_j) \right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),
\]

for all \( r \) outside possibly a set with finite logarithmic measure, where \( \lambda < 1 \) is a positive number. Then either \( f_2 = 1 \) or \( f_3 = 1 \).

Remark 3.2. Here, \( N_2(r, \frac{1}{f}) \) is the counting function of the zeros of \( f \) in \( |z| \leq r \), where the simple zero is counted once, and the multiple zero is counted twice.

4. The Proof of Theorem 2.1

Proof. Suppose that \( f(z_1, z_2) \) is a transcendental entire solution of finite order of equation (2.1). We firstly rewrite (2.1) as the following form

\[
\left( \frac{\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}}{e^{\frac{\alpha f(z_1, z_2)}{2}}} \right)^2 + \left( \frac{f(z_1 + c_1, z_2 + c_2)}{e^{\frac{\alpha f(z_1, z_2)}{2}}} \right)^2 = 1,
\]

or

\[
\left( \frac{\alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2}}{e^{\frac{\alpha f(z)}{2}}} + i \frac{f(z + c)}{e^{\frac{\alpha f(z)}{2}}} \right) \left( \frac{\alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2}}{e^{\frac{\alpha f(z)}{2}}} - i \frac{f(z + c)}{e^{\frac{\alpha f(z)}{2}}} \right) = 1.
\]

Since \( f \) is a finite order transcendental entire function and \( g \) is a polynomial, by Lemmas 3.1 and 3.2, there thus exists a polynomial \( p(z) \) in \( \mathbb{C}^2 \) such that

\[
\begin{align*}
\left\{ \frac{\alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2}}{e^{\frac{\alpha f(z)}{2}}} + i \frac{f(z + c)}{e^{\frac{\alpha f(z)}{2}}} \right\} &= e^{p(z)}, \\
\left\{ \frac{\alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2}}{e^{\frac{\alpha f(z)}{2}}} - i \frac{f(z + c)}{e^{\frac{\alpha f(z)}{2}}} \right\} &= e^{-p(z)},
\end{align*}
\]
Denote
\begin{equation}
\gamma_1(z) = \frac{g(z)}{2} + p(z), \quad \gamma_2(z) = \frac{g(z)}{2} - p(z).
\end{equation}
By combining with (4.2), it follows that
\begin{equation}
\alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} = \frac{e^{\gamma_1(z)} + e^{\gamma_2(z)}}{2},
\end{equation}
\begin{equation}
f(z + c) = \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2i}.
\end{equation}
This leads to
\begin{equation}
-iQ_1(z)e^{\gamma_1(z)} - \gamma_1(z+c) + iQ_2(z)e^{\gamma_2(z)} - \gamma_1(z+c) - e^{\gamma_2(z)} - \gamma_1(z+c) = 1,
\end{equation}
where
\begin{equation}
Q_1(z) = \alpha \frac{\partial \gamma_1(z)}{\partial z_1} + \beta \frac{\partial \gamma_1(z)}{\partial z_2}, \quad Q_2(z) = \alpha \frac{\partial \gamma_2(z)}{\partial z_1} + \beta \frac{\partial \gamma_2(z)}{\partial z_2}.
\end{equation}

Next, two cases will be considered.

**Case 1.** If \(e^{\gamma_2(z+c)} - \gamma_1(z+c)\) is a constant. Then \(\gamma_2(z+c) - \gamma_1(z+c)\) is a constant, set \(\gamma_2(z+c) - \gamma_1(z+c) = \kappa, \kappa \in \mathbb{C}\). In view of (4.3), we obtain that \(p(z)\) is a constant. Let \(\xi = e^{\gamma_2(z)}\), then Eqs. (4.4)-(4.5) become
\begin{equation}
\alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} = K_1 e^{\frac{\xi(z)}{2}}, \quad f(z + c) = K_2 e^{\frac{\xi(z)}{2}},
\end{equation}
where \(K_1 = \frac{\xi - \xi^{-1}}{2}, \quad K_2 = \frac{\xi - \xi^{-1}}{2}\) and \(K_1^2 + K_2^2 = 1\). Obviously, \(K_2 \neq 0\).

**Subcase 1.1.** If \(K_1 = 0\), that is, \(\xi = e^p = i\). Eq. (4.7) become
\begin{equation}
\alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} = 0, \quad f(z + c) = \pm e^{\frac{i\xi(z)}{2}}.
\end{equation}
The first equation of (4.8) implies
\begin{equation}
f(z) = \phi_0(\beta z_1 - \alpha z_2),
\end{equation}
where \(\phi_0(\beta z_1 - \alpha z_2)\) is a finite order transcendental entire function. Moreover, it follows from the second equation of (4.8) that
\begin{equation}
f(z) = \pm e^{\frac{1}{2}g(z-c)}.
\end{equation}
Combining with (4.9) and (4.10), we have \(g(z) = \phi(\beta z_1 - \alpha z_2), \) where \(\phi(\beta z_1 - \alpha z_2) = 2\ln[\pm \phi_0(\beta z_1 - \alpha z_2 + \beta c_1 - \alpha c_2)]\). This is the conclusion (i) of Theorem 2.1.

**Subcase 1.2.** If \(K_1 \neq 0\). Thus, it follows from (4.7) that
\begin{equation}
\frac{K_2}{2K_1} \left( \alpha \frac{\partial g(z)}{\partial z_1} + \beta \frac{\partial g(z)}{\partial z_2} \right) = e^{\frac{g(z+c)-g(z)}{2}}.
\end{equation}
Since \(g(z)\) is a polynomial, then (4.11) implies \(g(z + c) - g(z) = \zeta, \) where \(\zeta\) is a constant in \(\mathbb{C}\). Thus, it follows that \(g(z) = L(z) + H(s_1) + B, \) where \(L(z) = A_1 z_1 + A_2 z_2, \) \(H(s_1)\) is a polynomial in \(s_1 = c_2 z_1 - c_1 z_2\). Equation (4.11) implies
\begin{equation}
\alpha \frac{\partial L(z)}{\partial z_1} + \beta \frac{\partial L(z)}{\partial z_2} + \alpha \frac{\partial H(s_1)}{\partial z_1} + \beta \frac{\partial H(s_1)}{\partial z_2} \equiv 0,
\end{equation}
that is,
\begin{equation}
\alpha \frac{\partial H(s_1)}{\partial z_1} + \beta \frac{\partial H(s_1)}{\partial z_2} \equiv (\alpha c_2 - \beta c_1) H' \equiv \vartheta,
\end{equation}
where \(\vartheta\) is a constant.
where \( \vartheta = \vartheta_0 - \alpha A_1 - \beta A_2 \), \( \zeta_0 \in \mathbb{C} \). If \( \alpha c_2 - \beta c_1 \neq 0 \), then \( H' \) is a constant, that is, \( H(s_1) = a_0 s_1 + b_0 = a_0 (c_2 z_1 - c_1 z_2) + b_0 \), where \( a_0, b_0 \in \mathbb{C} \), then \( L(z) + H(s_1) \) is also a linear function of the form \( A_1 z_1 + A_2 z_2 \). For convenience, we still denote \( g(z) = L(z) + B \), which implies \( H(s_1) = 0 \). Thus, it follows \( \zeta = 0 \). If \( \alpha c_2 - \beta c_1 = 0 \), then \( \zeta = 0 \). Hence, it yields that \( \zeta = 0 \), that is,

\[
(\alpha c_2 - \beta c_1) H' \equiv 0.
\]

By combining with (4.6)-(4.12), we conclude that

\[
f(z_1, z_2) = K_2 e^{iQ_2(z)} = K_2 e^{i[A_1 z_1 + A_2 z_2 + H(s_1) + B] - (A_1 c_1 + A_2 c_2)} = K_2 e^{i[L(z) + H(s_1) + B]},
\]

and

\[
B = B_0 - (A_1 c_1 + A_2 c_2), \quad \frac{\xi^2 - 1}{2i(\xi^2 + 1)} (\alpha A_1 + \beta A_2) = e^{\frac{i}{2}(A_1 c_1 + A_2 c_2)}.
\]

**Case 2.** If \( e^{\gamma_2(z+c) - \gamma_1(z+c)} \) is not a constant. Obviously, \( Q_1(z) \equiv 0 \) and \( Q_2(z) \equiv 0 \) can not hold at the same time. Otherwise, it follows from (4.6) that \( e^{\gamma_2(z+c) - \gamma_1(z+c)} = -1 \), a contradiction. If \( Q_1(z) \equiv 0 \) and \( Q_2(z) \equiv 0 \), then from (4.6), we have

\[
iQ_2(z) e^{\gamma_2(z) - \gamma_1(z+c)} - e^{\gamma_2(z+c) - \gamma_1(z+c)} \equiv 1.
\]

Thus, we conclude that \( e^{\gamma_2(z) - \gamma_1(z+c)} \) is not a constant because \( e^{\gamma_2(z+c) - \gamma_1(z+c)} \) is not a constant. Moreover, \( e^{\gamma_2(z+c) - \gamma_1(z+c)} \) is not a constant. Otherwise, \( \gamma_2(z+c) = \gamma_2(z) + \zeta \), where \( \zeta \in \mathbb{C} \). Then, from (4.13), we have \( iQ_2(z) e^{-\zeta} - 1 e^{\gamma_2(z+c) - \gamma_1(z+c)} \equiv 1 \), which is a contradiction with the nonconstant \( e^{\gamma_2(z+c) - \gamma_1(z+c)} \). Thus, (4.13) can be written as the following form

\[
iQ_2(z) e^{\gamma_2(z)} - e^{\gamma_2(z+c)} - e^{\gamma_1(z+c)} \equiv 0.
\]

By applying Lemma 3.3 for (4.14), it is easy to get a contradiction. If \( Q_2(z) \equiv 0 \) and \( Q_1(z) \equiv 0 \), by using the same argument as in the above, we can get a contradiction. Hence, we have that \( Q_1(z) \neq 0 \) and \( Q_2(z) \neq 0 \).

Since \( \gamma_1(z) \), \( \gamma_2(z) \) are polynomials and \( e^{\gamma_2(z+c) - \gamma_1(z+c)} \) is not a constant, and by applying Lemma 3.4 for (4.6), it follows that

\[
-iQ_1(z) e^{\gamma_1(z) - \gamma_1(z+c)} \equiv 1, \text{ or } iQ_2(z) e^{\gamma_2(z) - \gamma_1(z+c)} \equiv 1.
\]

**Subcase 2.1** Suppose that \( -iQ_1(z) e^{\gamma_1(z) - \gamma_1(z+c)} \equiv 1 \). Then it follows from (4.6) that \( iQ_2(z) e^{\gamma_2(z) - \gamma_2(z+c)} \equiv 1 \). Thus, it means that \( \gamma_1(z) - \gamma_1(z+c) = \zeta_1, \gamma_2(z) - \gamma_2(z+c) = \zeta_2 \), where \( \zeta_1, \zeta_2 \in \mathbb{C} \). Hence, we have that \( \gamma_1(z) = L_1(z) + H_1(s_1) + B_1 \) and \( \gamma_2(z) = L_2(z) + H_2(s_1) + B_2 \), where \( L_j(z) = A_{j1} z_1 + A_{j2} z_2, H_j(s_1), j = 1, 2 \) are polynomials in \( s_1 = c_2 z_1 - c_1 z_2, A_{j1}, A_{j2}, B_j \in \mathbb{C}, j = 1, 2 \). Obviously, \( L_1(z) + H_1(s_1) \neq L_2(z) + H_2(s_1) \). Otherwise, \( \gamma_2(z+c) - \gamma_1(z+c) \) is a constant, which implies that \( e^{\gamma_2(z+c) - \gamma_1(z+c)} \) is a constant, a contradiction. Substituting these into (4.6), we have

\[
i[\alpha A_{11} + \beta A_{12}] e^{\gamma_1(z+c)} = -1, \quad i[\alpha A_{21} + \beta A_{22}] e^{\gamma_2(z+c)} = 1.
\]

By using the same argument as in Case 1, we have \( (\alpha c_2 - \beta c_1) H'_1 \equiv 0 \) and \( (\alpha c_2 - \beta c_1) H'_2 \equiv 0 \), which means that

\[
i[\alpha A_{11} + \beta A_{12}] e^{\gamma_1(z+c)} = -1, \quad i[\alpha A_{21} + \beta A_{22}] e^{\gamma_2(z+c)} = 1.
\]
In view of (4.5), we have
\[ f(z) = \frac{e^{L_1(z) + H_1(s_1) + B_1 - L_1(c)} - e^{L_2(z) + H_2(s_1) + B_2 - L_2(c)}}{2i}. \]
From the definitions of \( \gamma_1(z) \) and \( \gamma_2(z) \), we can see that
\[ g(z) = \gamma_1(z) + \gamma_2(z) = L(z) + H(s_1) + B, \]
where \( L(z) = L_1(z) + L_2(z), H(s_1) = H_1(s_1) + H_2(s_1), B = B_1 + B_2. \)

**Subcase 2.2** Suppose that \( iQ_1(z)e^{\gamma_1(z)} - e^{\gamma_2(z)} \equiv 1 \). Then it follows from (4.6) that
\[ -iQ_1(z)e^{\gamma_1(z)} - e^{\gamma_2(z)} \equiv 1. \]
These mean that \( \gamma_2(z) - \gamma_1(z + c) = \zeta_1 \) and \( \gamma_1(z) - \gamma_2(z + c) = \zeta_2 \). We can obtain that \( \gamma_1(z) = L(z) + H(s_1) + B_1 \) and \( \gamma_2(z) = L(z) + H(s_1) + B_2 \), where \( L(z) = a_1 z_1 + a_2 z_2, H(s_1) \) is a polynomial in \( s_1 = c_2 z_1 - c_1 z_2, a_1, a_2, B_1, B_2 \in \mathbb{C} \). Hence, it yields that \( \gamma_2(z + c) = \gamma_1(z + c) = B_2 - B_1 \), which implies that \( e^{\gamma_2(z + c) - \gamma_1(z + c)} \) is a constant. This is a contradiction.

Therefore, this completes the proof of Theorem 2.1. □

5. The Proof of Theorem 2.2

**Proof.** Suppose that \( f(z_1, z_2) \) is a transcendental entire solution of finite order of (2.3). We firstly rewrite (2.3) as the following form
\[
\left( \frac{\partial f(z_1, z_2)}{\partial z_1} \right)^2 + \left( \frac{\partial f(z_1, z_2)}{\partial z_2} \right)^2 + \left( f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) \right) = 1,
\]
or
\[
(5.1) \left( \frac{\partial f(z_1, z_2)}{\partial z_1} + i f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) \right) \left( \frac{\partial f(z_1, z_2)}{\partial z_2} - i f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) \right) = 1.
\]

Since \( f \) is a finite order transcendental entire function and \( g \) is a polynomial, by Lemmas 3.1 and 3.2, there thus exists a polynomial \( p(z) \) in \( \mathbb{C}^2 \) such that
\[
\begin{cases}
\frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2} + i f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) = e^{p(z)}, \\
\frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2} - i f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) = e^{-p(z)},
\end{cases}
\]
Denote
\[
(5.3) \gamma_1(z) = \frac{g(z)}{2} + p(z), \quad \gamma_2(z) = \frac{g(z)}{2} - p(z).
\]
By combining with (5.2), it follows that
\[
\begin{align*}
\frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2} &= \frac{e^{\gamma_1(z) + e^{\gamma_2(z)}}}{2}, \\
f(z_1 + c_1, z_2 + c_2) - f(z_1, z_2) &= \frac{e^{\gamma_1(z)} - e^{\gamma_2(z)}}{2i}.
\end{align*}
\]
This leads to
\[
(5.6) Q_3(z)e^{\gamma_1(z)} - e^{\gamma_1(z + c)} + Q_4(z)e^{\gamma_2(z)} - e^{\gamma_2(z + c)} - e^{\gamma_1(z + c) - e^{\gamma_2(z + c)}} \equiv 1,
\]

where

\[ Q_3(z) = 1 - i \left( \alpha \frac{\partial \gamma_1(z)}{\partial z_1} + \beta \frac{\partial \gamma_1(z)}{\partial z_2} \right), \quad Q_4(z) = 1 + i \left( \alpha \frac{\partial \gamma_2(z)}{\partial z_1} + \beta \frac{\partial \gamma_2(z)}{\partial z_2} \right). \]

Next, two cases will be considered.

**Case 1.** If \( e^{\gamma_2(z+c) - \gamma_1(z+c)} \) is a constant. Then \( \gamma_2(z+c) - \gamma_1(z+c) \) is a constant, set \( \gamma_2(z+c) - \gamma_1(z+c) = \kappa, \kappa \in \mathbb{C} \). In view of (5.3), we have that \( p(z) \) is a constant.

Let \( \xi = e^{\rho(z)} \), then Eqs. (5.4)-(5.5) become

\[ \alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} = K_1 e^{\frac{\alpha(z)}{2}}, \quad f(z) = f(z) = K_2 e^{\frac{\alpha(z)}{2}}, \]

where \( K_1 = \frac{\xi + \xi^{-1}}{2} \), \( K_2 = \frac{\xi - \xi^{-1}}{2} \) and \( K_2^2 = 1 \).

**Subcase 1.1.** If \( K_1 = 0 \), then \( K_2 = \pm 1 \). Thus, it follows from (5.7) that

\[ \alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} = 0, \quad f(z) = f(z) = \pm e^{\frac{\alpha(z)}{2}}, \]

which implies that

\[ \gamma(z) = f(\beta z_1 - \alpha z_2), \quad \gamma(z) = f(z) \in \mathbb{C} \] and \( f(z) = f(z) = \pm e^{\frac{\alpha(z)}{2}}, \]

where \( \varphi \) is a finite order transcendental entire function. Thus, this is the conclusion (i) of Theorem 2.2.

**Subcase 1.2.** If \( K_2 = 0 \), then \( K_1 = \pm 1 \). Thus, it follows from (5.7) that

\[ \alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} = \pm e^{\frac{\alpha(z)}{2}}, \quad f(z) = f(z) = 0. \]

So, we have

\[ \alpha \frac{\partial f(z + c)}{\partial z_1} + \beta \frac{\partial f(z + c)}{\partial z_2} = \alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2}, \]

which implies that \( e^{\frac{\partial g(z + c) - g(z)}{z}} = 1 \). It yields that \( g(z) = A_1 z_1 + A_2 z_2 + H(s_1) + B \), where \( H \) is a polynomial in \( s_1 = c_2 z_1 - c_1 z_2, \ B \in \mathbb{C} \) and \( A_1 c_1 + A_2 c_2 = 4k \pi, k \in \mathbb{Z} \).

Solving the first equation of (5.9), we have

\[ f(z_1, z_2) = \frac{1}{\alpha} \int_{z_2}^{1} e^{\frac{1}{2}(A_1 z_1 + A_2 z_2 + H(c_2 z_1 - c_1 z_2) + B) d z_1} + G(z_2 - \frac{\alpha z_1}{\alpha})], \]

where \( G \) is a finite order entire function. Substituting it to the second equation of (5.9), we have \( G(z_2 - \frac{\alpha z_1}{\alpha}) = G(2z_2 - \frac{\alpha z_1}{\alpha}) \), this shows that \( G \) is a period function with the period \( \frac{2 \alpha z_2 - \beta z_1}{\alpha} \). Thus, this is the conclusion (ii) of Theorem 2.2.

**Subcase 1.3.** If \( K_1 \neq 0 \) and \( K_2 \neq 0 \). Thus, it follows from (5.7) that

\[ \frac{K_2}{2K_1} \left( \alpha \frac{\partial g(z)}{\partial z_1} + \beta \frac{\partial g(z)}{\partial z_2} \right) + 1 = e^{\frac{g(z + c) - g(z)}{2}}. \]

Since \( g(z) \) is a polynomial, then (5.11) implies \( g(z + c) - g(z) = \zeta, \) where \( \zeta \) is a constant in \( \mathbb{C} \). Thus, it follows that \( g(z) = L_0(z) + H(s_1) + B_0 \), where \( L_0(z) = A_1 z_1 + A_2 z_2, \ H(s_1) \) is a polynomial in \( s_1 = c_2 z_1 - c_1 z_2 \). Equation (5.11) implies that

\[ \alpha \frac{\partial L_0(z)}{\partial z_1} + \beta \frac{\partial L_0(z)}{\partial z_2} + \alpha \frac{\partial H(s_1)}{\partial z_1} + \beta \frac{\partial H(s_1)}{\partial z_2} = \zeta, \]
that is,
\[
\frac{\partial H(s_1)}{\partial z_1} + \frac{\beta}{\partial z_2} \equiv (\alpha c_2 - \beta c_1)H' \equiv \vartheta,
\]
where \( \vartheta = \vartheta_0 - \alpha A_10 - \beta A_20, \vartheta_0 \in \mathbb{C} \). In view of \( \alpha c_2 - \beta c_1 \neq 0 \), then \( H' \) is a constant, that is, \( H(s_1) = a_0 s_1 + b_0 = a_0(c_2 z_1 - c_1 z_2) + b_0 \), where \( a_0, b_0 \in \mathbb{C} \). Hence, we have
\[
(5.12) \quad g(z) = L_0(z) + H(s_1) + B_0 = L(z) + B = A_1 z_1 + A_2 z_2 + B,
\]
where \( A_1 = A_{10} + a_0 c_2, A_2 = A_{20} - a_0 c_1, B = B_0 + b_0 \). In view of (5.11) and (5.12), it follows that
\[
(5.13) \quad \frac{K_2}{2K_1} (\alpha A_1 + \beta A_2) + 1 = e^{\frac{1}{2}(A_1 c_1 + A_2 c_2)}.
\]

The characteristic equations for the first equation in (5.7) are
\[
\frac{dz_1}{dt} = \alpha, \quad \frac{dz_2}{dt} = \beta, \quad \frac{df}{dt} = K_1 e^{\frac{1}{2}g(z)},
\]
Using the initial conditions: \( z_1 = 0, z_2 = s \), and \( f = f(0, s) := G_0(s) \) with a parameter \( s \). Thus, we obtain the following parametric representation for the solutions of the characteristic equations: \( z_1 = \alpha t, z_2 = \beta t + s \),
\[
f(t, s) = \int_0^t K_1 e^{\frac{1}{2}[A_1 + \beta A_2](t + \alpha s + B)] \frac{dt}{\alpha A_1 + \beta A_2} G_0(s),
\]
where \( G(s) = G_0(s) - 2K_1 e^{\frac{1}{2}[A_1 + \beta A_2]s} \) is an entire function of finite order in \( s \). Then, by combining with \( t = \frac{z_1}{\alpha} \) and \( s = z_2 - \frac{\beta}{\alpha} z_1 \), the solution of equation (5.7) is of the form
\[
(5.14) \quad f(z_1, z_2) = 2K_1 e^{\frac{1}{2}[(L(z) + B)]} \frac{A_2}{\alpha A_1 + \beta A_2} + G\left(\frac{A_2 s_2 - B z_1}{\alpha}\right).
\]

Substituting (5.14) into the second equation in (5.7), and combining with (5.13), it yields that \( G(s + s_0) = G(s) \), this means that \( G(s) \) is a finite order entire period functions with period \( s_0 = \frac{\alpha c_2 - \beta c_1}{\alpha} \).

**Case 2.** If \( e^{\gamma_2(z+c)-\gamma_1(z+c)} \) is not a constant. Obviously, \( Q_3(z) \equiv 0 \) and \( Q_4(z) \equiv 0 \) can not hold at the same time. Otherwise, it follows from (5.6) that \( e^{\gamma_2(z+c)-\gamma_1(z+c)} = -1 \), a contradiction. If \( Q_3(z) \equiv 0 \) and \( Q_4(z) \neq 0 \), then from (5.6), we obtain that
\[
(5.15) \quad Q_4(z)e^{\gamma_2(z)-\gamma_1(z+c)} - e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1.
\]

Thus, we conclude that \( e^{\gamma_2(z)-\gamma_1(z+c)} \) is not a constant because \( e^{\gamma_2(z+c)-\gamma_1(z+c)} \) is not a constant. Moreover, \( e^{\gamma_2(z+c)-\gamma_2(z)} \) is not a constant. Otherwise, \( \gamma_2(z+c) = \gamma_2(z) + \zeta \), where \( \zeta \in \mathbb{C} \). Then, from (5.15), we have \( [Q_4(z)e^{-\zeta} - 1]e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1 \), which is a contradiction with the nonconstant \( e^{\gamma_2(z+c)-\gamma_1(z+c)} \). Thus, (5.15) can be written as the following form
\[
(5.16) \quad Q_4(z)e^{\gamma_2(z)} - e^{\gamma_2(z+c)} - e^{\gamma_1(z+c)} \equiv 0.
\]
By applying Lemma 3.3 for (5.16), it is easy to get a contradiction. If \(Q_4(z) \equiv 0\) and \(Q_3(z) \neq 0\), by using the same argument as in the above, we can get a contradiction. Hence, we have that \(Q_3(z) \neq 0\) and \(Q_4(z) \neq 0\).

Since \(\gamma_1(z), \gamma_2(z)\) are polynomials and \(e^{\gamma_2(z+c)}-\gamma_1(z+c)\) is a nonconstant function, and by applying Lemma 3.4 for (5.6), it follows that

\[(5.17) \quad Q_3(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1, \quad \text{or} \quad Q_4(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1.

Subcase 2.1 Suppose that \(Q_3(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1\). Then it follows from (5.6) that \(Q_4(z)e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1\). Thus, it means that \(\gamma_1(z) - \gamma_1(z + c) = \zeta_1, \gamma_2(z) - \gamma_2(z + c) = \zeta_2\), where \(\zeta_1, \zeta_2 \in \mathbb{C}\). Hence, we have that \(\gamma_1(z) = L_1(z) + H_1(s_1) + B_1\) and \(\gamma_2(z) = L_2(z) + H_2(s_1) + B_2\), where \(L_j(z) = A_1j_1z_1 + A_2j_2z_2, H_j(s_1), j = 1, 2\) are polynomials in \(s_1 = c_2z_1 - c_1z_2, A_1, A_2, B_j \in \mathbb{C}, j = 1, 2\). Since \(\alpha c_2 - \beta c_1 \neq 0\), similar to the argument as in Case 1 in Theorem 2.2, we have that \(H_j(s_1)\) is a polynomial in \(s_1\). Then we can see that \(L_j(z) + H_j(s_1)\) is a finite order entire period functions with period \(s_1 = c_2z_1 - c_1z_2\).

Substituting these into (5.6), we have

\[(5.18) \quad [1 - i(\alpha A_{11} + \beta A_{12})]e^{-L_1(c)} = 1, \quad [1 + i(\alpha A_{21} + \beta A_{22})]e^{-L_2(c)} = 1.

In view of (5.4), we have

\[(5.19) \quad \alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} = \frac{e^{L_1(z)+B_1} + e^{L_2(z)+B_2}}{2}.

The characteristic equations for the first equation in (5.7) are

\[
\frac{dz_1}{dt} = \alpha, \quad \frac{dz_2}{dt} = \beta, \quad \frac{df(z)}{dt} = \frac{e^{L_1(z)+B_1} + e^{L_2(z)+B_2}}{2}.
\]

Using the initial conditions: \(z_1 = 0, z_2 = s, f = f(0, s) := G_2(s)\) with a parameter \(s\). Thus, we obtain the following parametric representation for the solutions of the characteristic equations: \(z_1 = \alpha t, z_2 = \beta t + s, f(t, s) = \int_0^t \frac{e^{L_1(z)+B_1} + e^{L_2(z)+B_2}}{2} dt + G_2(s)\)

\[
= \frac{e^{(\alpha A_{11} + \beta A_{12})t + A_{12}s + B_1}}{2(\alpha A_{11} + \beta A_{12})} + \frac{e^{(\alpha A_{21} + \beta A_{22})t + A_{22}s + B_2}}{2(\alpha A_{21} + \beta A_{22})} + G_1(s),
\]

where \(G_1(s)\) is an entire function of finite order in \(s\), and

\[
G_1(s) = G_2(s) - \frac{e^{A_{12} + B_1}}{\alpha A_{11} + \beta A_{12}} - \frac{e^{A_{22}s + B_2}}{\alpha A_{21} + \beta A_{22}}.
\]

Then, by combining with \(t = \frac{z_1}{\alpha}\) and \(s = z_2 - \frac{\beta}{\alpha} z_1\), the solution of equation (5.7) is of the form

\[(5.20) \quad f(z_1, z_2) = \frac{e^{L_1(z)+B_1}}{\alpha A_{11} + \beta A_{12}} + \frac{e^{L_2(z)+B_2}}{\alpha A_{21} + \beta A_{22}} + G_1\left(\frac{\alpha z_2 - \beta z_1}{\alpha}\right),
\]

Substituting (5.20) into (5.5), and combining with (5.18), we have \(G_2(s + s_0) = G_2(s)\), which means that \(G_2(s)\) is a finite order entire period functions with period \(s_0 = \frac{\alpha c_2 - \beta c_1}{\alpha}\).
From the definitions of $\gamma_1(z)$ and $\gamma_2(z)$, we can see that

$$g(z) = \gamma_1(z) + \gamma_2(z) = L(z) + B,$$

where $L(z) = L_0(z) + L_2(z)$, $B = B_1 + B_2$.

**Subcase 2.2** Suppose that $Q_4(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$. Then it follows from (5.6) that $Q_3(z)e^{\gamma_1(z)-\gamma_2(z+c)} \equiv 1$. These mean that $\gamma_2(z) - \gamma_1(z + c) = \zeta_1$ and $\gamma_1(z) - \gamma_2(z + c) = \zeta_2$, where $\zeta_1, \zeta_2 \in \mathbb{C}$. Thus, it follows that $\gamma_1(z + 2c) - \gamma_1(z) = -\zeta_1 - \zeta_2$ and $\gamma_2(z + c) - \gamma_2(z) = -\zeta_1 - \zeta_2$. We can obtain that $\gamma_1(z) = L(z) + H(s_1) + B_1$ and $\gamma_2(z) = L(z) + H(s_1) + B_2$, where $L(z) = a_1z + a_2z^2$, $H(s_1)$ is a polynomial in $s_1 = c_2z_1 - c_1z_2$, $a_1, a_2, B_1, B_2 \in \mathbb{C}$. Hence, we have that $\gamma_2(z + c) - \gamma_1(z + c) = B_2 - B_1$, which implies that $e^{\gamma(z+c) - \gamma(z+c)}$ is a constant. This is a contradiction.

Therefore, this completes the proof of Theorem 2.2. \qed

### 6. The Proof of Theorem 2.4

**Proof.** Assume that $f$ is a nonconstant meromorphic solution of equation (2.7), then

$$(6.1) \quad \left( \alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} \right) + if(z + c) \left[ \left( \alpha \frac{\partial f(z)}{\partial z_1} + \beta \frac{\partial f(z)}{\partial z_2} \right) - if(z + c) \right] = 1.$$

Let

$$(6.2) \quad \left( \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} \right) + if(z_1 + c_1, z_2 + c_2) = h(z_1, z_2),$$

and

$$(6.3) \quad \left( \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} \right) - if(z_1 + c_1, z_2 + c_2) = \frac{1}{h(z_1, z_2)},$$

where $h$ is a nonzero meromorphic function on $\mathbb{C}^2$. Hence, it follows from (6.2) and (6.3) that

$$(6.4) \quad f(z_1, z_2) = \frac{h(z_1 - c_1, z_2 - c_2) - \frac{1}{h(z_1 - c_1, z_2 - c_2)}}{2i},$$

and

$$(6.5) \quad \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2} = \frac{h(z_1, z_2) + \frac{1}{h(z_1, z_2)}}{2}.$$

In view of (6.2)-(6.5), we have

$$\alpha \frac{\partial f(z + c)}{\partial z_1} + \beta \frac{\partial f(z + c)}{\partial z_2} = \frac{h(z + c) + \frac{1}{h(z + c)}}{2}$$

$$= \frac{1}{2i} \left( \alpha \frac{\partial h(z + c)}{\partial z_1} + \beta \frac{\partial h(z + c)}{\partial z_2} \right) \left( 1 + \frac{1}{h(z)^2} \right),$$

that is,

$$(6.6) \quad i \left( h(z + c) + \frac{1}{h(z + c)} \right) = \left( \alpha \frac{\partial h(z + c)}{\partial z_1} + \beta \frac{\partial h(z + c)}{\partial z_2} \right) \left( 1 + \frac{1}{h(z)^2} \right).$$

Assume that $c_1 = c_2 = 0$. Thus, it follows from (6.6) that

$$(6.7) \quad \left( ih(z_1, z_2) - \alpha \frac{\partial h(z_1, z_2)}{\partial z_1} - \beta \frac{\partial h(z_1, z_2)}{\partial z_2} \right) \left( 1 + \frac{1}{h(z_1, z_2)^2} \right) = 0.$$
If $1 + \frac{1}{h(z_1, z_2)^n} = 0$, then $h(z) = \pm i$. Thus, we can conclude from (6.4) that $f(z)$ is a constant, a contradiction. Thus, it follows that

$$ih(z_1, z_2) - \frac{\partial h(z_1, z_2)}{\partial z_1} - \beta \frac{\partial h(z_1, z_2)}{\partial z_2} = 0,$$

that is,

$$\frac{\alpha \partial h(z_1, z_2)}{\partial z_1} + \frac{\beta \partial h(z_1, z_2)}{\partial z_2} = ih(z_1, z_2). \quad (6.8)$$

The characteristic equations for equation (6.8) are

$$\frac{dz_1}{dt} = \alpha, \quad \frac{dz_2}{dt} = \beta, \quad \frac{dh}{dt} = ih.$$

Using the initial conditions: $z_1 = 0, z_2 = s_2$, and $h = h(0, s_2) := g(s_2)$ with a parameter $s_2$, where $g(s_2)$ is a meromorphic function in $s_2$. Thus, we obtain the following parametric representation for the solutions of the characteristic equations:

$$h(t, s_2) = e^{it + g(s_2)}, \quad z_1 = \alpha t, \quad z_2 = \beta t + s_2,$$

that is,

$$h(z_1, z_2) = h(t, s_2) = e^{i(z_1/\alpha)} + g(\frac{\alpha z_2 - \beta z_1}{\alpha}). \quad (6.9)$$

Combining (6.9) with (6.4), we have

$$f(z_1, z_2) = \frac{e^{i(z_1/\alpha - ig(\frac{\alpha z_2 - \beta z_1}{\alpha}))} - e^{-i(z_1/\alpha - ig(\frac{\alpha z_2 - \beta z_1}{\alpha}))}}{2i} = \sin \left( \frac{z_1}{\alpha} - ig(\frac{\alpha z_2 - \beta z_1}{\alpha}) \right).$$

Therefore, this completes the proof of Theorem 2.4 \(\square\)

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The author declares that none of the authors have any competing interests in the manuscript.

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