ON THE FINITENESS OF EXT-INDICES OF RING EXTENSIONS

SAMIR BOUCHIBA AND SALAH KABBAJ

Abstract. The main goal of this paper is to investigate the finiteness of Ext-indices of ring extensions. We discuss some known related conjectures in the literature and observe the relationships among them within large classes of rings. This allows us to present interesting special cases verifying these conjectures. Finally, we tackle questions on the vanishing of Ext-index of trivial extensions of Artinian rings. We put the new results into use to construct new families of examples subject to the Gorenstein or Cohen-Macaulay conditions with finite Ext-index.

1. Introduction

Throughout, all rings considered are commutative with identity elements and all modules are unital. A Noetherian local ring \((R, m)\) is regular if its (Krull) dimension, \(\text{dim}(R)\), and embedding dimension, \(\text{embdim}(R)\), coincide, where \(\text{embdim}(R)\) denotes the dimension of \(m/m^2\) as an \(R/m\)-vector space. The ring \(R\) is a complete intersection if the completion \(\hat{R}\) of \(R\) with respect to the \(m\)-adic topology is the quotient ring of a local regular ring modulo an ideal generated by a regular sequence. The ring \(R\) is Gorenstein if its injective dimension (as an \(R\)-module) is finite; and Cohen-Macaulay if the grade and height coincide for every ideal of \(R\). All these notions are globalized by carrying over to localizations with respect to the prime ideals. For basic details on these notions, we refer the reader to 

In 1958, Auslander conjectured that every Artinian algebra \(R\) satisfies the following condition (called Auslander’s condition):

\[(\text{ac}) \text{ For every finite } R\text{-module } M, \text{ there exists an integer } n_M \geq 0 \text{ such that for every finite } R\text{-module } N:\]
\[\text{Ext}^i_R(M,N) = 0 \forall i \gg 0 \implies \text{Ext}^i_R(M,N) = 0 \forall i \geq n_M + 1.\]

This conjecture generated numerous research works and gave birth to several related homological problems and homologically defined classes of commutative rings in the literature over the last decades [3, p. 795] (see also [2, 8, 9, 18]). It’s reported in [18] that all commutative local rings, in which this conjecture is valid satisfy the following stronger condition (called the uniform Auslander condition):

\[(\text{uac}) \text{ There exists an integer } n \geq 0 \text{ such that for all finite } R\text{-modules } M \text{ and } N:\]
\[\text{Ext}^i_R(M,N) = 0 \forall i \gg 0 \implies \text{Ext}^i_R(M,N) = 0 \forall i \geq n + 1.\]

In 2000, Avramov & Buchweitz [4] developed geometric methods for the study of finite modules over locally complete intersection rings. Their results yielded unexpected and remarkable properties of finite modules over these rings [4].
considered, among others, the following symmetric condition for a Noetherian ring $R$

$$\text{(ee)} \quad \text{Ext}_{R}^{i}(M,N) = 0 \forall i \gg 0 \iff \text{Ext}_{R}^{i}(N,M) = 0 \forall i \gg 0$$

with $M,N$ ranging over all finite modules. They proved the following implications

locally complete intersection $\implies$ (ee) $\implies$ Gorenstein

and noted that, during the last four decades, research in commutative algebra has not produced a class of rings between locally complete intersection rings and Gorenstein rings. The question of whether the above implications are strict was left open.

In 2003, Huneke & Jorgensen \cite{16} investigated the symmetry in the vanishing of Ext for finite modules over local Gorenstein rings. Their interest in this problem came about, partly, from Avramov & Buchweitz’s symmetric property (ee) mentioned above. To this purpose, they defined the Ext-index of a ring $R$ by

$$\text{Ext-index}(R) = \sup \{ n \in \mathbb{N} \mid \text{Ext}_{R}^{n}(M,N) \neq 0 \},$$

where the supremum is taken over all pairs of finite $R$-modules $(M,N)$ with $\text{Ext}_{R}^{i}(M,N) = 0$ for all $i \gg 0$; and called a local Gorenstein ring with finite Ext-index an AB ring (where ‘AB’ stands for both Auslander-Bridger and Avramov-Buchweitz \cite{16} p. 162). Thus, an AB ring is a local Gorenstein ring which satisfies (uac) or, equivalently, (ac); recall at this point that the conditions (ac) and (uac) coincide in the class of Gorenstein rings \cite{24} Theorem 3.3] and \cite{8} Proposition 4.2]. A local regular ring is obviously an AB ring (since it has a finite global homological dimension). They proved that a complete intersection ring is an AB ring, constructed examples of AB rings which are not complete intersection rings, and proved that every AB ring satisfies (ee). The proof of the first fact (i.e., CI $\Rightarrow$ AB) was established via the notion of gap of length in the vanishing of Ext. Namely, $\text{Ext}_{R}^{n}(M,N)$ has a gap of length $t$ if for some $n \geq 0$, $\text{Ext}_{R}^{i}(M,N) = 0$ for $n + 1 \leq i \leq n + t$ with $\text{Ext}_{R}^{n}(M,N) \neq 0$ and $\text{Ext}_{R}^{n+i+1}(M,N) = 0$.

In 2004, Jorgensen and Sega \cite{18} solved Auslander’s long standing conjecture in the negative and, simultaneously, answered Avramov & Buchweitz’s open question on the existence of a class of rings intermediate between (locally) complete intersection rings and Gorenstein rings as well as Huneke & Jorgensen’s open question of whether all local Gorenstein rings are AB rings. Indeed, they proved that the class of AB rings lies strictly between the class of complete intersection rings and the class of local Gorenstein rings. Thus, for a Noetherian local ring, they refined the diagram of implications between these ring-theoretic properties as follows:

complete intersection $\implies$ AB $\implies$ (ee) $\implies$ Gorenstein

where it is still an open question of whether the second implication is reversible.

In 2010, Christensen and Holm \cite{8} introduced and studied AC rings; i.e., (left) Noetherian rings satisfying Auslander’s condition (ac). They noted (p. 28) that “Nagata’s regular ring of infinite Krull dimension \cite{25} Example 1, p. 203] is an example of a commutative Noetherian ring that satisfies (ac) but not (uac). However, in the realm of Artin algebras (or local rings) we do not know of such an example” and the same meaning is reiterated in Appendix A (p. 35). In 2012, in \cite{9}, they studied the transfer of (ac) and (uac) properties along local homomorphisms of Cohen-Macaulay rings. In particular, they showed that the (ac), (uac), and AB properties descend along homomorphisms of finite flat dimension \cite{9} Theorem 4.1]
and ascend along complete intersection homomorphisms [9, Theorem 4.3]. They also showed (in Section 3) how to construct new examples of Cohen-Macaulay rings subject to the (ac)/(uac) properties.

The literature abounds of papers dealing with geometric aspects of finite modules over trivial ring extensions (also called Nagata idealizations); including the transfer of the notions of Cohen-Macaulayness, Gorenstein ring, and complete intersection. Probably, the first results dealing with the vanishing of Ext for finite modules over trivial extensions appeared in [26], where Nasseh and Yoshino (2009) investigated the finiteness of Ext-indices for certain ring extensions. They proved that the trivial extension of an Artinian local ring by its residue field has finite Ext-index, and proved the Auslander-Reiten conjecture [30] for this type of rings. Precisely, let \((A, m)\) be an Artinian local ring and \(R := A \ltimes A/m\). Then \(\text{Ext-index}(R) = 0\) [26, Corollary 3.5]; and if \(M\) is a finite \(R\)-module with \(\text{Ext}_R^i(M, M \oplus R) = 0\) for some integer \(i \geq 3\), then \(M\) is a free \(R\)-module [26, Corollary 3.7].

The main goal of this paper is to investigate the finiteness of Ext-indices of ring extensions. We discuss some known related conjectures in the literature and observe the relationships among them within large classes of rings. This allows us to present interesting special cases verifying these conjectures. Our motivation stems from the following conjectures which seem to be still open (See [8]).

**Conjecture \((\text{L})\).** Let \(R\) be a ring. If \(\text{Ext-index}(R) < +\infty\), then \(\text{Ext-index}(R_p) < +\infty\) for each \(p \in \text{Spec}(R)\).

**Conjecture \((\text{P})\):** Let \(R\) be a ring. If \(\text{Ext-index}(R) < +\infty\), then \(\text{Ext-index}(R[X]) < +\infty\).

Section 2 features a preliminary result (Theorem 2.1) which will be used throughout this paper. It provides lower and upper bounds for the Ext-index of an arbitrary Noetherian ring, as well as improves and completes some known basic results on the Ext-index of Gorenstein rings.

Section 3 investigates the correlation between Conjectures \((\text{L})\) and \((\text{P})\). Precisely, the main result of Section 3, namely Theorem 3.1 proves that \((\text{P}) \Rightarrow (\text{L})\) in the class of semi-local Cohen-Macaulay rings \(R\) of dimension \(\leq 1\). Also, it asserts that the two conjectures \((\text{L})\) and \((\text{P})\) are equivalent for the rings \(R\) of this class having algebraically closed prime residue fields. Finally, it verifies the conjecture \((\text{P})\) for the reduced rings \(R\) of this class having only algebraically closed residue fields. As a consequence, Corollary 3.3 recovers and generalizes a known result on Artinian rings, due to Nasseh and Yoshino [26, Proposition 2.8], by dropping the Gorenstein assumption.

Section 4 handles a question on the vanishing of Ext-index of trivial extensions (also called Nagata idealizations) of Artinian rings. We put the new results into use to construct new families of examples subject to the Gorenstein or Cohen-Macaulay conditions with finite Ext-index.

Throughout, given a ring \(R\), we denote by \(\text{Max}(R)\) and \(\text{Spec}(R)\), respectively, the set of maximal ideals and the set of prime ideals of \(R\). The invariant \(\dim(R)\) denotes the Krull dimension of \(R\), \(\text{embdim}(R)\) the embedding dimension of \(R\) and \(\text{gl-dim}(R)\) the global dimension of \(R\). For \(p \in \text{Spec}(R)\), \(\kappa_R(p)\) denotes the residue field \(R_p/pR_p\) of \(R_p\). Finally, given an extension of fields \(k \subseteq K\), we denote by \(\text{t.d.}(K : k)\) the transcendence degree of \(K\) over \(k\).

22 Sep 2021 08:13:37 PDT
2. Locally AB rings

Recall that a local Gorenstein ring with finite Ext-index is said to be an AB ring. A ring $R$ is called locally AB if $R$ is Noetherian and the localizations $R_m$ are AB rings, for each $m \in \text{Max}(R)$. As mentioned in the introduction, a locally complete intersection ring is a locally AB ring [16, Corollary 3.5] and the following implications always hold

$$\text{locally complete intersection} \implies \text{locally AB} \implies \text{Gorenstein}$$

where the first implication is irreversible by [16, Theorem 3.6]. Following [2, 26], for nonzero modules $M$ and $N$ over a ring $R$, let

$$p_R(M,N) := \sup \{ i \in \mathbb{N} | \text{Ext}^i_R(M,N) \neq 0 \}.$$ 

It is easily seen that

$$\text{Ext-index}(R) = \sup \left\{ p_R(M,N) : M,N \text{ are finitely generated } R\text{-modules with } p_R(M,N) < +\infty \right\}.$$ 

We begin by announcing the first main theorem of this section. It provides lower and upper bounds for the Ext-index of an arbitrary Noetherian ring.

**Theorem 2.1.** Let $R$ be a Noetherian ring. Then:

$$\dim(R) \leq \text{Ext-index}(R) \leq \sup \left\{ \text{Ext-index}(R_m) : m \in \text{Max}(R) \right\}.$$ 

We need the following lemma.

**Lemma 2.2.** Let $R$ be a Noetherian ring and $M$ a finitely generated $R$-module. Let $n \geq 0$ be an integer. Then $\text{pd}_R(M) \leq n$ if and only if $\text{Ext}^{n+1}_R(M,N) = 0$ for each finitely generated $R$-module $N$.

**Proof.** Since $R$ is Noetherian, any finitely generated $R$-module admits a projective resolution each of whose terms is finitely generated. Then, it suffices to prove the lemma for the case where $n = 0$. Assume that $\text{Ext}^1_R(M,N) = 0$ for each finitely generated $R$-module $N$. Let $K$ be an arbitrary $R$-module. Since $M$ is finitely presented and $K = \lim\limits_{\rightarrow} N_i$ is a direct limit of its finitely generated submodules $N_i$, we get, by [32, Proposition 2.2], that

$$\text{Ext}^1_R(M,K) = \lim\limits_{\rightarrow} \text{Ext}^1_R(M,N_i) = 0.$$ 

Therefore $M$ is projective as claimed completing the proof.

Given a ring $R$, we denote by $\text{FPD}(R) := \sup \{ \text{pd}_R(M) : M \text{ is an } R\text{-module such that } \text{pd}_R(M) < +\infty \}$ the finitistic dimension of $R$.

**Proof of Theorem 2.1.** It is well known that, since $R$ is a commutative Noetherian ring, $\text{FPD}(R) = \dim(R)$ [14, Theorem 3.2.6]. It then suffices to prove that $\text{FPD}(R) \leq \text{Ext-index}(R)$. Let $M$ be a finitely generated $R$-module such that $\text{pd}_R(M) = n < +\infty$. Then, by Lemma 2.2, there exists a finitely generated $R$-module $N$ such that $\text{Ext}^n_R(M,N) \neq 0$ and

$$\text{Ext}^{n+k}_R(M,N) = 0$$

for each integer $k \geq 1$. Hence

$$p_R(M,N) = n.$$ 

That is, \( n \leq \text{Ext-index}(R) \). It follows that \( \text{FPD}(R) \leq \text{Ext-index}(R) \) and thus \( \dim(R) \leq \text{Ext-index}(R) \). For the second inequality, let

\[ d := \sup \{ \text{Ext-index}(R_m) \mid m \in \text{Max}(R) \} \]

Without loss of generality, we may assume that \( d \) is finite. Let \( M \) and \( N \) be finitely generated \( R \)-modules such that \( r := \text{Ext}^i_R(M, N) \) is finite. Notice that

\[ \text{Ext}^i_R(M, N)_m \cong \text{Ext}^i_{R_m}(M_m, N_m) = 0 \]

for each integer \( i \geq r + 1 \) and each \( m \in \text{Max}(R) \). Hence, as \( M_m \) and \( N_m \) are finitely generated \( R_m \)-modules, \( p_{R_m}(M_m, N_m) < \infty \). Therefore, \( p_{R_m}(M_m, N_m) \leq d \) for each \( m \in \text{Max}(R) \), yielding \( \text{Ext}^{d+k}_{R_m}(M, N)_m = 0 \) for each integer \( k \geq 1 \) and \( m \in \text{Max}(R) \). It follows that \( \text{Ext}^{d+k}_R(M, N) = 0 \) for each \( k \geq 1 \) so that \( p_R(M, N) \leq d \). Consequently, \( \text{Ext-index}(R) \leq d \), completing the proof of the first statement.

\[ \square \]

In the light of Theorem 2.1, our second main theorem of this section improves [16, Proposition 3.2] which states that if \( R \) is an AB-ring, then \( \text{Ext-index}(R) = \dim(R) \).

**Theorem 2.3.** Let \( R \) be a Noetherian ring. If \( R \) is a locally AB ring, then

\[ \text{Ext-index}(R) = \dim(R) \]

In particular, if \( R \) is a locally complete intersection, then

\[ \text{Ext-index}(R) = \dim(R) \]

**Proof.** Assume that \( R \) is locally AB. Let \( m \) be a maximal ideal of \( R \). Then \( R_m \) is an AB-ring and thus, by [26, Lemma 2.5], \( \text{Ext-index}(R_m) = \dim(R_m) \). Hence \( \sup \{ \text{Ext-index}(R_m) \mid M \in \text{Max}(R) \} \leq \sup \{ \dim(R_m) \mid M \in \text{Max}(R) \} = \dim(R) \). It follows by Theorem 2.1 that \( \text{Ext-index}(R) = \dim(R) \), as desired.

\[ \square \]

The next corollary characterizes finite-dimensional locally AB rings.

**Corollary 2.4.** Let \( R \) be a finite-dimensional Gorenstein ring. Then the following assertions are equivalent:

a) \( R \) is locally AB;

b) \( \text{Ext-index}(R) = \dim(R) \);

c) \( \text{Ext-index}(R) < \infty \).

**Proof.** Combine Theorem 2.3 with [26, Lemma 2.4].

Next, we show how one can easily build (non-local) locally AB rings of arbitrary finite Ext-index.

**Example 2.5.** Let \( m \geq n \geq 0 \) be two integers, \( k \) a field, and \( K, L \) two extension fields of \( k \) such that \( K \otimes_k L \) is Noetherian with \( t.d.(K : k) = n \) and \( t.d.(L : k) = m \). Then, \( R := K \otimes_k L \) is a (non-local) locally AB ring with \( \dim(R) = \text{Ext-index}(R) = n \). Indeed, \( R \) has dimension \( n \) by [31, Theorem 3.1]. Further, \( R \) is a locally complete intersection ring by [33, Proposition 5] and, a fortiori, a locally AB ring by [16, Theorem 3.5]. Corollary 2.4 forces \( \dim(R) = \text{Ext-index}(R) = n \). Notice that \( R \) is not local as both \( K \) and \( L \) are assumed to be transcendental over \( k \).
One of the legitimate questions one may ask is whether the Ext-index of a Cohen-Macaulay ring \( R \) always equals the upper bound in Theorem 2.1 in other words, is Ext-index\( (R) = \sup \{ \text{Ext-index}(R_m) \mid m \in \text{Max}(R) \} \)? Note that if \( R \) is a locally AB ring, then this equality holds. The next corollary records that the answer to this question is positive for Artinian rings.

**Corollary 2.6.** Let \( R \) be an Artinian ring. Then

\[
\text{Ext-index}(R) = \sup \{ \text{Ext-index}(R_m) : m \in \text{Max}(R) \}.
\]

**Proof.** Combine Theorem 2.1 and [26, Lemma 2.2]. \( \square \)

### 3. Conjectures (L) and (P)

Throughout, a ring \( R \) is said to have *algebraically closed residue fields* if \( R/m \) is algebraically closed, for each \( m \in \text{Max}(R) \); and \( R \) is said to have *algebraically closed prime residue fields* if \( \kappa_R(p) \) is algebraically closed, for each \( p \in \text{Spec}(R) \).

This section investigates the correlation between Conjectures (L) and (P) and provides classes of rings that verify these conjectures. Precisely, in the class of semi-local Cohen-Macaulay rings of dimension \( \leq 1 \), Theorem 3.1 proves that (P) \( \Rightarrow \) (L). Also, it asserts that the two conjectures (L) and (P) are equivalent for the rings \( R \) of this class having algebraically closed prime residue fields. Finally, it verifies the conjecture (P) for the reduced rings \( R \) of this class having only algebraically closed residue fields. As a consequence, Corollary 3.5 recovers and generalizes a known result on Artinian rings, due to Nasseh and Yoshino [26], by dropping the Gorenstein assumption.

We begin by announcing the main theorem of this section. It verifies, in particular, the conjecture (P) for reduced semi-local Cohen-Macaulay rings \( R \) of dimension \( \leq 1 \) having algebraically closed residue fields.

**Theorem 3.1.** Let \( R \) be a semi-local Cohen-Macaulay ring of dimension \( \leq 1 \) and \( n \geq 1 \) an integer. Then

1. If Ext-index\( (R[X]) \) is finite, then so is Ext-index\( (R_p) \), for each \( p \in \text{Spec}(R) \).
2. Assume that \( R \) has algebraically closed prime residue fields. Then, Ext-index\( (R[X_1, X_2, \ldots, X_n]) \) is finite if and only if Ext-index\( (R_p) \) is finite, for each \( p \in \text{Spec}(R) \).
3. Assume that \( R \) is a reduced ring with algebraically closed residue fields. If Ext-index\( (R) \) is finite, then Ext-index\( (R[X_1, X_2, \ldots, X_n]) \) is finite.

The proof requires the following lemmas. The first lemma extends [9, Theorem 2.3] to polynomial rings with several indeterminates.

**Lemma 3.2.** Let \( (R, m) \) be a local Cohen-Macaulay ring. Then Ext-index\( (R[X_1, X_2, \ldots, X_n]) \) is finite if and only if Ext-index\( (R) \) is finite.

**Proof.** The argument uses induction on \( n \). If \( n = 1 \), then we are done by [9, Theorem 2.3]. Assume that \( n \geq 2 \). Note that

\[
R[X_1, \cdots, X_n]_m = R[X_1]_m \cdot R[X_2, \cdots, X_n]_{(m, X_1)} = R[X_1]_m \cdot R[X_2, \cdots, X_n]_{(m, X_1)}.
\]

Then, by inductive assumptions, we get Ext-index\( (R[X_1, \cdots, X_n]) \) if and only if Ext-index\( (R[X_1]_m) \) if and only if Ext-index\( (R) \) if and only if Ext-index\( (R) \) as desired.
Lemma 3.3. Let $R$ be a Cohen-Macaulay ring and $p$ a prime ideal of $R$ such that $\kappa_R(p)$ is algebraically closed. Let $P$ be a prime ideal of $R[X_1, \cdots, X_n]$ such that $PR_p[X_1, \cdots, X_n]$ is a maximal ideal of $R_p[X_1, \cdots, X_n]$. Then

1. $\operatorname{Ext-index}(R[X_1, X_2, \cdots, X_n]_{\mathfrak{p}}) = \operatorname{Ext-index}(R_p[X_1, X_2, \cdots, X_n]_{(pR_p, X_1, X_2, \cdots, X_n)})$.
2. $\operatorname{Ext-index}(R[X_1, X_2, \cdots, X_n]_{\mathfrak{p}})$ is finite if and only if $\operatorname{Ext-index}(R_p)$ is so.

Proof. Let $X$ denote $X_1, \ldots, X_n$. As $PR_p[X]$ is maximal in $R_p[X]$ and $\kappa_R(p)$ is algebraically closed, we have $PR_p[X] = (pR_p, X_1-a_1, \ldots, X_n-a_n)$, for some $a_1, \ldots, a_n \in R_p$. Consider the natural ring isomorphism $\theta : R_p[X] \to R_p[X]$ defined by $\theta(X_i) = X_i + a_i$, for $i = 1, \ldots, n$. Clearly, $\theta(PR_p[X]) = (pR_p, X)$ and so

$$\operatorname{Ext-index}(R[X]_{\mathfrak{p}}) = \operatorname{Ext-index}(R_p[X]_{pR_p[X]})$$

This proves (1). It follows, by Lemma 3.2, that $\operatorname{Ext-index}(R[X]_{\mathfrak{p}})$ is finite if and only if $\operatorname{Ext-index}(R_p[X]_{pR_p[X]})$ is finite if and only if so is $\operatorname{Ext-index}(R_p)$, establishing (2). This completes the proof of the lemma.

Proof of Theorem 3.1 (1) We mimic the proof of Assertion (1) in [26, Theorem 2.6]. Assume that $\operatorname{Ext-index}(R[X])$ is finite. Let $p$ be a prime ideal of $R$. If $p$ is maximal, then $\operatorname{Ext-index}(R_p)$ is finite by [26, Lemma 2.4]. Now, assume that $\dim(R/p) = 1$. As $R$ is semi-local, there exists an element $\alpha$ in the Jacobson radical $J(R)$ such that $\alpha$ is a non zero-divisor of $R$, that is, $\alpha$ does not belong to any minimal prime ideal of $R$. Then $R_\alpha$ is Artinian and $R_\alpha \cong \frac{R[X]}{(\alpha X - 1)}$ with $R_\alpha$ is the localization of $R$ by the multiplicative subset $S = \{\alpha^n : n \geq 1\}$ of $R$. It is clear that $\alpha X - 1$ is a non zero-divisor of $R[X]$. Hence, as $\operatorname{Ext-index}(R[X]) < +\infty$, we get, by [26, Lemma 2.1(4)], that $\operatorname{Ext-index}(R_\alpha) = \operatorname{Ext-index}\left(\frac{R[X]}{(\alpha X - 1)}\right) < +\infty$. Now, since $\alpha \notin p$, $(R_\alpha)_p = R_p$. It follows, by [26, Corollary 2.3], that $\operatorname{Ext-index}(R_p) < +\infty$ since $R_\alpha$ is Artinian.

(2) Assume that $\operatorname{Ext-index}(R_p)$ is finite, for each $p \in \operatorname{Spec}(R)$. Then, by Lemma 3.3, $\operatorname{Ext-index}(R[X]_{\mathfrak{p}})$ is finite for each prime ideal $P$ of $R[X]$ such that $PR_p[X]$ is a maximal ideal of $R_p[X]$. Hence $\operatorname{Ext-index}(R[X]_{\mathfrak{p}})$ is finite for each maximal ideal $P$ of $R[X]$. Moreover, $\operatorname{Spec}(R)$ is finite. It follows, by Theorem 2.1 and Lemma 3.3(1), that

$$\operatorname{Ext-index}(R[X]) \leq \sup \left\{ \operatorname{Ext-index}(R[X]_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Max}(R[X]) \right\} \leq \sup \left\{ \operatorname{Ext-index}(R_p[X]_{pR_p[X], \mathfrak{p}}) \mid p \in \operatorname{Spec}(R) \right\} < +\infty.$$

The converse is ensured by (1).

(3) Assume that $R$ is reduced and $\operatorname{Ext-index}(R) < +\infty$. Let $M$ be a maximal ideal of $R[X_1, \cdots, X_n]$ and let $p = M \cap R$. If $p$ is a maximal ideal of $R$, then, as $\kappa_R(p)$ is algebraically closed, we get, by Lemma 3.3, that

$$\operatorname{Ext-index}(R[X_1, \cdots, X_n]_M) = \operatorname{Ext-index}\left(\frac{R[X_1, \cdots, X_n]}{(pR_p, X_1, \cdots, X_n)}\right).$$
Now, assume that \( p \) is a minimal prime ideal of \( R \). Then, as \( R \) is reduced, \( R_p \) is a field and thus
\[
\text{Ext-index}(R[X_1, \ldots, X_n]_{\mathfrak{m}}) = \text{Ext-index}(R_p[X_1, \ldots, X_n]_{\mathfrak{m}R_p[X_1, \ldots, X_n]}) \\
\leq \text{gl-dim}(R_p[X_1, \ldots, X_n]) \\
= n.
\]
Therefore, by Theorem \ref{thm:main},
\[
\text{Ext-index}(R[X_1, \ldots, X_n]) \leq \sup \{ n, \text{Ext-index} \left( R_p[X_1, \ldots, X_n]_{(pR_p,X_1,\ldots,X_n)} \right) : p \in \text{Max}(R) \}.
\]
Since \( \text{Ext-index}(R) < +\infty \), we get, by \cite{26} Lemma 2.4, that \( \text{Ext-index}(R_p) < +\infty \) for each maximal ideal \( p \) of \( R \). Hence, by Lemma \ref{lem:finite-index}, \( \text{Ext-index}(R_p[X_1, \ldots, X_n]_{(pR_p,X_1,\ldots,X_n)}) < +\infty \) for each maximal ideal \( p \) of \( R \). Since \( R \) is semi-local, it follows that \( \text{Ext-index}(R[X_1, \ldots, X_n]) < +\infty \) completing the proof of the theorem.
\( \square \)

By virtue of Theorem \ref{thm:finite-index}, we next provide a non regular one-dimensional semi-local Cohen-Macaulay ring which verifies the conjecture (\( \mathbb{P} \)).

**Example 3.4.** Let \( k \) be an algebraically closed field and \( R = \frac{k[X,Y,Z]}{(XY, XZ, YZ)(X,Y,Z)} \).
Note that \( I = (XY, XZ, YZ) \) is a radical ideal of \( k[X,Y,Z] \) and thus \( R \) is a reduced local Cohen-Macaulay ring of dimension 1. Also, observe that \( XY, XZ, YZ \) is not a regular sequence of \( k[X,Y,Z] \) as \( I \subseteq (X,Y) \), that is, \( \text{grade}(I) \leq 2 \). Moreover, note that \( \text{embdim}(R) = 3 \) so that \( \text{embdim}(R) - \dim(R) = 2 \). Hence, by \cite{18} Proposition 1.1, we get \( \text{Ext-index}(R) \) is finite. Also, note that the residue field of \( R \) is \( k \) and it is algebraically closed. Now, applying Theorem \ref{thm:finite-index}, it follows that \( R \) verifies the conjecture (\( \mathbb{P} \)), that is, \( \text{Ext-index}(R[X_1, X_2, \ldots, X_n]) < +\infty \) for each integer \( n \geq 1 \).

The next corollary presents a class of rings which verify (\( \mathbb{P} \)). It recovers and generalizes \cite{26} Proposition 2.8 by dropping the Gorenstein assumption.

**Corollary 3.5.** Let \( R \) be an Artinian ring with algebraically closed residue fields. If \( \text{Ext-index}(R) \) is finite, then so is \( \text{Ext-index}(R[X_1, X_2, \ldots, X_n]) \), for any integer \( n \geq 1 \).

**Proof.** Combine Theorem \ref{thm:main} Theorem \ref{thm:finite-index}(a) and \cite{26} Corollary 2.3. \( \square \)

**Example 3.6.** It is easy to construct an Artinian ring \( R \) which is not Gorenstein such that the residue fields of \( R \) are algebraically closed. It suffices, for this sake, to take \( R = \frac{k[X,Y]}{(X^2,XY,Y^3)} \), where \( k \) is an algebraically closed field (see \cite{?}, Example 1.8)). It turns out that \( R \) is an Artinian local ring which is not Gorenstein and with residue field \( k \) which is algebraically closed, as desired. Thus \( R \) satisfies Conjecture (\( \mathbb{P} \)).

The last result of this section handles the finiteness of the Ext-index of the polynomial ring \( R[X_1, X_2, \ldots, X_n] \) over a Hilbert Cohen-Macaulay ring \( R \) having algebraically closed residue fields. It verifies the conjecture (\( \mathbb{P} \)) for finite dimensional Gorenstein Hilbert rings with algebraically closed residue fields. It also represents a generalization of \cite{26} Proposition 2.8. It is worth recalling that a ring \( R \) is called a Hilbert ring if any maximal ideal \( M \) of the polynomial ring \( R[X] \) contracts to a maximal ideal \( m \) of \( R \), i.e., \( M \cap R = m \in \text{Max}(R) \) \cite{20} Definition and Theorem 27, page 18].
Theorem 3.7. Let $R$ be a Hilbert Cohen-Macaulay ring having algebraically closed residue fields. Let $n \geq 1$ be an integer and $X_1, X_2, \ldots, X_n$ be indeterminates over $R$. Then

1. Ext-index($R[X_1, X_2, \ldots, X_n]_M$) is finite, for each maximal ideal $M$ of $R[X_1, X_2, \ldots, X_n]$ if and only if Ext-index($R_m$) is finite, for each maximal ideal $m$ of $R$.

2. Moreover, if $R$ is a Gorenstein ring of finite Krull dimension, then Ext-index($R[X_1, X_2, \ldots, X_n]$) is finite if and only if so is Ext-index($R$).

Proof. (1) Note that, as $R$ is Hilbert, for each maximal ideal $M$ of $R[X]$, we have $m = M \cap R$ is a maximal ideal of $R$. Then, since $R[X_1, \ldots, X_r]$ is also a Hilbert ring for any integer $r \geq 1$, we get, for each maximal ideal $M$ of $R[X_1, \ldots, X_n]$, $m = M \cap R$ is a maximal ideal of $R$. Hence (1) holds by applying Lemma 3.3(2).

(2) It follows from (1) and the combination of Lemma 2.4 and Lemma 2.5 of [26], as desired.

\[ \square \]

4. Trivial ring extensions of Artinian rings

For a ring $A$ and an $A$-module $E$, the trivial ring extension of $A$ by $E$ is the commutative ring $R := A \ltimes E$, where the underlying group is $A \times E$ and the multiplication is given by

\[(a, e)(b, f) = (ab, af + be).\]

The ring $R$ is also called the idealization of $E$ over $A$ and is denoted by $A(+)E$. This construction was first introduced by Nagata in his 1962 book “Local Rings” [25] in order to embed the module $E$ as an ideal in the ring $R$ which contains $A$ as a subring. Throughout, we will be using the “trivial (ring) extension” terminology.

Recall, for convenience, that for an arbitrary ideal $I$ of $A$ and a submodule $E'$ of $E$ with $IE \subseteq E'$, $J := I \ltimes E'$ is always an ideal of the trivial extension $R := A \ltimes E$; whereas, not all ideals of $R$ are of this form [19, Example 2.5]. However, the prime ideals (resp., maximal ideals) of $R$ have the form $p \ltimes E$, where $p$ is a prime (resp., maximal) ideal of $A$; and hence dim$(A \ltimes E) = \dim(A)$ [15, Theorem 25.1]. For more background on commutative trivial extensions (or idealizations), we refer the reader to Glaz’s and Huckaba’s respective books [12, 15], and also Olberding’s recent work [27] and D. D. Anderson-Winders’ comprehensive (survey) paper [1].

At this point, it is worthwhile recalling that the trivial extension $R := A \ltimes E$ is Noetherian (resp., Artinian) if and only if $A$ is Noetherian (resp., Artinian) and $E$ is finitely generated (cf. [1, Theorem 4.8], [28, Proposition 1], [34, Proposition 2.7]).

This section investigates the vanishing of Ext-index of trivial ring extensions of Artinian rings. Throughout, given a ring $A$, $A_{\text{red}}$ will denote its reduced ring (i.e., $A_{\text{red}} := \frac{A}{\text{Nil}(A)}$). The first result handles the case of trivial extensions of Artinian rings.

Proposition 4.1. Let $A$ be an Artinian ring and let $R := A \ltimes A_{\text{red}}$. Then, Ext-index($R$) = 0.

Proof. Let $m$ be a maximal ideal of $A$. Then

\[ R_{mA_{\text{red}}} \cong A_m \ltimes (A_{\text{red}})_m. \]

But, we have

\[ A_{\text{red}} \cong \frac{A}{m_1} \times \frac{A}{m_2} \times \cdots \times \frac{A}{m_r}. \]
where the \( m_i \)'s are the maximal ideals of \( A \). Therefore
\[
(A_{\text{red}})_{m_i} \cong \frac{A}{m_i}
\]
for each \( i = 1, \ldots, r \). It follows that
\[
R_{m_i \otimes A_{\text{red}}} \cong A_{m_i} \otimes k
\]
with \( k \) being the residue field of \( m \). By \cite{25} Corollary 3.5, we obtain
\[
\text{Ext-index}\left(R_{m_i \otimes A_{\text{red}}}ight) = \text{Ext-index}(A_{m_i} \otimes k) = 0
\]
for each maximal ideal \( m \) of \( A \). Applying Theorem \ref{t1}, we get \( \text{Ext-index}(R) = 0 \), as desired. \( \square \)

The next result handles the case of trivial extensions of von Neumann regular rings.

**Theorem 4.2.** Let \( A \) be a reduced Artinian ring, \( M \) a finitely generated \( A \)-module, and \( R := A \ltimes M \). Then, \( \text{Ext-index}(R) = 0 \).

We need the following lemma. It is a special case of \cite{18} Proposition 1.4] which asserts that, if \( R \) is a Golod ring, then \( \text{Ext-index}(R) = \dim(R) \). For the sake of self-containment, we give an explicit proof of this lemma.

**Lemma 4.3.** Let \( (R, m) \) be a local Noetherian ring with \( m^2 = 0 \). Then, \( \text{Ext-index}(R) = 0 \).

**Proof of Lemma 4.3.** Let \( M \) and \( N \) be finitely generated \( R \)-modules such that \( n := p_R(M, N) < \infty \). Assume, by way of contradiction, that \( n \geq 1 \). In the proof of \cite{18} Proposition 1.1(1)], it is showed, concisely, that \( N \) has finite injective dimension. For the sake of completeness, we offer a detailed proof of this fact. We have \( \text{pd}_R(M) \geq 1 \). Let
\[
\cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow M \longrightarrow 0
\]
be a minimal free resolution of \( M \) over \( R \) and let \( H \) be its first syzygy \( R \)-module. Then \( H \subseteq mL_0 \) and thus \( mH = 0 \). Hence, \( H \) can be viewed as a \( k \)-vector space, where \( k := R/m \). It follows that
\[
\text{Ext}_R^t(H, N) = 0
\]
for each \( t \geq n \); that is, \( p_R(H, N) = n - 1 \). Now, notice that
\[
\text{Hom}_R(H, N) \cong \text{Hom}_k(H, \text{Hom}_R(k, N)).
\]
So, as the functor \( \text{Hom}_k(H, \cdot) \) is exact, we get
\[
\text{Ext}_R^t(H, N) \cong \text{Hom}_k(H, \text{Ext}_R^t(k, N))
\]
for each positive integer \( t \). Hence
\[
\text{Ext}_R^t(k, N) = 0
\]
for each integer \( t \geq n \). Since \( R \) is local with residue field \( k \), then \( \text{id}_R(N) \leq n \) by \cite{20} Theorem 212]; that is, the injective dimension of \( N \) over \( R \) is finite. Therefore, by Bass formula, we obtain
\[
\text{id}_R(N) = \text{depth}(R) = 0.
\]
That is, \( N \) is an injective \( R \)-module, which contradicts the assumption that \( p_R(M, N) \geq 1 \). Consequently, \( p_R(M, N) = 0 \) whenever \( p_R(M, N) \) is finite so that \( \text{Ext-index}(R) = 0 \), completing the proof of the lemma. \( \square \)
Proof of Theorem 4.2. Let $k$ be a field, $E$ a finitely generated $k$-vector space, $S := k \ltimes E$, and $\mathfrak{M} := 0 \ltimes E$ its maximal ideal. Clearly, $(S, \mathfrak{M})$ is Noetherian with $\mathfrak{M}^2 = 0$ and, hence, $\text{Ext-index}(S) = 0$ by Lemma 4.3. Next, let $\mathfrak{m}$ be a maximal ideal of $A$. Then, $A_\mathfrak{m}$ is a field so that

$$\text{Ext-index}(R_{\mathfrak{m} \ltimes M}) = \text{Ext-index}(A_{\mathfrak{m}} \ltimes M_{\mathfrak{m}}) = 0.$$ 

It follows, by Theorem 2.1, that $\text{Ext-index}(R) = 0$, completing the proof of the theorem. \qed

As an application of the above results, one can use trivial ring extensions (of Artinian rings) to build (non-local) locally AB rings as well as (non-local) non locally AB rings with finite Ext-index, as shown below.

For the reader’s convenience, recall that Valtonen (1989) summarized most well-known results on trivial ring extensions subject to the notion of Cohen-Macaulay and related conditions in [34]: namely, for a Noetherian local ring $A$ and a finitely generated $A$-module $E$, the trivial extension $R = A \ltimes E$ is never regular, and $R$ is complete intersection if and only if so is $A$ with $E \cong A$. The ring $R$ is Cohen-Macaulay if and only if so is $A$ and $E$ is a maximal Cohen-Macaulay module (i.e., $\text{Gr}(E) = \dim(E) = \dim(A)$); and $R$ is Gorenstein if and only if $A$ is Cohen-Macaulay and $E$ is a canonical module. Recall that, for a local Cohen-Macaulay $d$-dimensional ring $(A, \mathfrak{m}, k)$, an $A$-module $\omega_A$ is called canonical if it is finitely generated with $\text{Ext}^i_A(k, \omega) = k$ and $\text{Ext}^i_A(k, \omega) = 0$ for $i \neq d$ [11]. Any two canonical $A$-modules are isomorphic and every complete local Cohen-Macaulay ring has a canonical module. Also, a local ring $A$ is Gorenstein if and only if $A$ is Cohen-Macaulay with $A \cong \omega_A$.

Example 4.4. Let $A$ be a reduced Artinian ring (e.g., any finite product of fields) and $M$ a finitely generated $A$-module. Let $R := A \ltimes M$.

1. Assume that $M \cong A$. Then, $R$ is a non-regular locally AB ring (indeed, a locally complete intersection ring) with $\text{Ext-index}(R) = 0$ by Proposition 4.1 or Theorem 4.2.

2. Assume that $M \not\cong A$. Then, $R$ is a non-Gorenstein Cohen-Macaulay ring with $\text{Ext-index}(R) = 0$ by Theorem 4.2. Notice that $A$ is a (Noetherian von Neumann) regular ring.

Observe that in the following two examples the base ring $A$ is a non-reduced Artinian ring and so one cannot appeal to Theorem 4.2.

Example 4.5. Let $k$ be a field and $x, y, z$ indeterminates over $k$. Let

$$A := \frac{k[x, y, z]}{(x^2, xz, yz)}; \quad M := \frac{k[x, y, z]}{(x, yz)}; \quad R := A \ltimes M.$$ 

So, $R$ is a non-Gorenstein Cohen-Macaulay ring with $\text{Ext-index}(R) = 0$. Indeed, clearly, $A$ is Cohen-Macaulay since Artinian; and from [5] Example 6.10] $A$ is not Gorenstein. It follows that $R$ is Cohen-Macaulay but not Gorenstein. Finally, one can easily check that $\text{Nil}(A) = (x, yz)$ and so $A_{\text{red}} \cong M$. By Proposition 4.1, $\text{Ext-index}(R) = 0$ as desired.

Here is a more simple example.

Example 4.6. Let $R := \frac{\mathbb{Z}}{12\mathbb{Z}} \ltimes \frac{\mathbb{Z}}{6\mathbb{Z}}$. Then, $R$ is a non-Gorenstein Cohen-Macaulay ring with $\text{Ext-index}(R) = 0$ by Proposition 4.1.
References


S. Bouchiba, Department of Mathematics, Moulay Ismail University, Meknes 50000, Morocco
E-mail address: s.bouchiba@fs.umi.ac.ma

S. Kabbaj, Department of Mathematics, KFUPM, Dhahran 31261, KSA
E-mail address: kabbaj@kfupm.edu.sa