ROCKY MOUNTAIN JOURNAL OF MATHEMATICS

TILING OF REGULAR POLYGONS WITH SIMILAR RIGHT TRIANGLES

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ABSTRACT. We prove that for every \( N \neq 4 \) there is only one right triangle that tiles the regular \( N \)-gon.

We say that a triangle \( T \) tiles a polygon \( A \), if \( A \) can be dissected into finitely many nonoverlapping triangles similar to \( T \).

Let \( R_N \) denote the regular \( N \)-gon. Connecting the center of \( R_N \) with the vertices of \( R_N \) we obtain a dissection of \( R_N \) into \( N \) congruent isosceles triangles. Bisecting each of these triangles into two right angled triangles, we get a dissection of \( R_N \) into \( 2N \) congruent right triangles with acute angles \( \pi/N \) and \( \left( \pi/2 \right) - \left( \pi/N \right) \). In particular, we find that the right triangle with acute angles \( \pi/N \) and \( \left( \pi/2 \right) - \left( \pi/N \right) \) tiles \( R_N \).

In this note we are concerned with the following question: are there other right triangles that tile \( R_N \)? If \( N = 4 \), then there are infinitely many such triangles. Indeed, the right triangle with legs 1 and \( 1/n \) tiles the unit square for every positive integer \( n \). In [1] and [4] it is proved that a right triangle with acute angle \( \alpha \) tiles the square if and only if \( \tan \alpha \) is a totally positive algebraic number; that is, if every real conjugate of \( \tan \alpha \) is positive.

It is also known that if \( N \geq 25 \) and \( N \neq 30, 42 \), then the only right triangle that tiles \( R_N \) has acute angles \( \pi/N \) and \( \left( \pi/2 \right) - \left( \pi/N \right) \) (see [2, Theorem 1.1]). The same was proved for \( N = 5 \) in [3]. We prove that this is true for every \( N \neq 4 \).

Theorem 1. For every \( N \neq 4 \) there is only one right triangle that tiles the regular \( N \)-gon.

Proof. Let \( T \) be a right triangle with acute angles \( \alpha, \beta \). We prove that if \( T \) tiles \( R_N \), where \( N \neq 4 \), then one of \( \alpha \) and \( \beta \) equals \( \pi/N \).

Let \( \delta_N \) denote the angle of \( R_N \); that is, \( \delta_N = (N - 2)\pi/N \). Suppose that \( T \) tiles \( R_N \), where \( N \neq 4 \), and fix such a tiling. Let \( V \) be a vertex of any of the tiles. We say that \( p\alpha + q\beta + r\gamma = \sigma \) is the equation at \( V \) if the number of triangles having \( V \) as a vertex and having angle \( \alpha \) (resp. \( \beta \) or \( \gamma = \pi/2 \)) at \( V \) equals \( p \) (resp. \( q \) or \( r \)). Here \( \sigma = \delta_N \) if \( V \) is one of the vertices of \( R_N \), and \( \sigma = \pi \) or \( 2\pi \) otherwise.

Let the equation at any of the vertices of \( R_N \) be \( p\alpha + q\beta + r\gamma = \delta_N \). Since \( N \neq 4 \), \( \delta_N \) is not an integer multiple of \( \pi/2 \), and thus \( p \neq q \). We obtain \( (p - q)\alpha + (q + r)\pi/2 = \delta_N = (N - 2)\pi/N \), showing that \( \alpha \) is a rational multiple of \( \pi \). Then so is \( \beta = (\pi/2) - \alpha \).

Suppose \( N \neq 6 \). By Theorem 1.2 of [2], each angle of \( R_N \) is packed with at most two tiles. Suppose \( N = 3 \). Then we have either \( \alpha = \delta_3 = \pi/3 \), or \( 2\alpha = \delta_3 = \pi/3 \), hence \( \alpha = \pi/6 \) and \( \beta = \pi/3 \). This proves the statement for \( N = 3 \).

The first author was supported by the Hungarian National Foundation for Scientific Research, Grant No. K124749.

2020 Mathematics Subject Classification. 52C20.

Key words and phrases. Tilings with right triangles, regular polygons.
Therefore, we may assume $N \geq 5$. Then $\delta_N > \pi/2$, and there must be exactly two tiles at each vertex of $R_N$. Then we have either $2\alpha = \delta_N = \pi - (2\pi/N)$ or $(\pi/2) + \alpha = \pi - (2\pi/N)$.

If $2\alpha = \delta_N$, then we get $\alpha = (\pi/2) - (\pi/N)$ and $\beta = \pi/N$, and we are done.

Suppose $(\pi/2) + \alpha = \pi - (2\pi/N)$. Then $\alpha = (\pi/2) - (\pi/N) = (N-4)\pi/(2N)$ and $\beta = 2\pi/N$.

By Lemma 1.10 of [2], we have

\[
\left\{ \frac{k(N-4)}{2N} \right\} + \left\{ \frac{2k}{N} \right\} + \left\{ \frac{k}{2} \right\} = 1
\]

whenever $\gcd(k,2N) = 1$ and $\{k/N\} < 1/2$. Let

\[
k = \begin{cases} 
(N/2) - 1 & \text{if } N \equiv 0 \pmod{4}, \\
N + (N-1)/2 & \text{if } N \equiv 1 \pmod{4}, \\
(N/2) - 2 & \text{if } N \equiv 2 \pmod{4}, \\
(N-1)/2 & \text{if } N \equiv 3 \pmod{4}.
\end{cases}
\]

Then $\gcd(k,2N) = 1$ and $\{k/N\} < 1/2$. We have $\{k/2\} = 1/2$, since $k$ is odd. Then (1) gives $\{2k/N\} \leq 1/2$. However, it is easy to check that $\{2k/N\} > 1/2$ for every $N \geq 5, N \neq 6$. Thus the case $\alpha = (\pi/2) - (\pi/N)$ is impossible. This proves the statement for every $N \neq 4, 6$.

Finally, suppose $N = 6$. In the following argument we assume $\alpha \leq \beta$. Put $b = 2\beta/\pi$; then we have $1/2 \leq b < 1$. If the equation at a vertex of $R_6$ is $p_0\alpha + q_0\beta + r_0\gamma = \delta_6 = 2\pi/3$, then we have

\[
p_0(1-b) + q_0b + r_0 = 4/3.
\]

Suppose $p_0 \leq q_0$. Then $(q_0 - p_0)b + p_0 + r_0 = 4/3$ gives $sb = 4/3$ or $sb = 1/3$, where $s = q_0 - p_0$ is a nonnegative integer. Since $1/2 \leq b < 1$, we have $s = 2$, $sb = 4/3$, $b = 2/3$, $\beta = \pi/3$ and $\alpha = \pi/6$.

That is, the statement of the theorem is true in this case. Therefore, we may assume that $p > q$ holds for the equation $p\alpha + q\beta + r\gamma = \delta_6$ at each vertex of $R_6$.

Then there must be an equation $p\alpha + q\beta + r\gamma = \sigma$ at a vertex different from the vertices of $R_6$ such that $p < q$. Since $\sigma = \pi$ or $2\pi$, we get $(q-p)b + p + r = p(1-b) + qb + r = 2$. Putting $s = q-p$, we get $sb = u$, where $u$ equals one of 1, 2, 3, 4, and $s$ is a positive integer. By $1/2 \leq b < 1$ we obtain $2 \leq s \leq 8$. Since $b = u/s$, the left hand side of (2) equals $\nu/s$, where $\nu$ is an integer. Then $\nu/s = 4/3$ gives $s = 3$ or $s = 6$.

If $s = 3$, then $u = 2$, $3b = 2$, $b = 2/3$, $\beta = \pi/3$, $\alpha = \pi/6$ and we are done. If $s = 6$, then $u = 3$ or 4.

If $u = 3$, then $6b = 3$ and $b = 1/2$. In this case, however, the left hand side of (2) equals $w/2$, where $w$ is an integer, which is impossible. If $u = 4$, then $6b = 4$, $b = 2/3$, and we obtain $\alpha = \pi/6$ again. This completes the proof. \[\square\]

References


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