MODIFIED LUPAȘ-KANTOROVICH OPERATORS WITH PÓLYA DISTRIBUTION

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ABSTRACT. In this paper, we present the modification of the Kantorovich variant of the operators due to Lupaș and Lupaș which are connected with Pólya distribution and we provide quantitative Voronovskaya-kind theorem involving modulus of continuity. Also, we observe that the better approximation can be achieved for our modified form of operators. We also estimate the difference of the original operator with its modified form. In the end, we depict the convergence of the modified operators to some functions using graphical representation.

1. Introduction

Stancu [22] in the year 1968 proposed the alteration of the Bernstein polynomials, based on Pólya distribution. Almost twenty years later Lupaș and Lupaș [18] examined the following form of their operators by considering the parameter as \( n^{-1} \) in the following way:

\[
(P_n^{1/n} f)(x) = \frac{2(n!)^2}{(2n)!} \sum_{j=0}^{n} q_{n,j}^{(1/n)}(x) f\left( \frac{j}{n} \right), \quad (0 \leq x \leq 1)
\]

where

\[
q_{n,j}^{(1/n)}(x) = \frac{2(n!)^2}{(2n)!} \binom{n}{j} (nx)^j (n - nx)^{n-j}
\]

and \((nx)_j = \prod_{i=0}^{j-1} (nx + i)\). The approximation results of the operators (1) were further studied by Miclăuș [20]. Very recently, the Durrmeyer type integral form of the operators (1) was introduced by Gupta and Rassias [15]. For detailed work, one may refer [1], [3], [12] etc. Different types of modifications of the Bernstein-Durrmeyer operators were introduced by Gupta [7] and Gupta and Maheshwari [13] and their approximation properties were investigated. Further modifications were done by Gupta and Duman [11] so as to preserve the linear functions and better error estimations were shown on certain subintervals of \([0, 1]\). Also, certain \(q\)-Durrmeyer type operators in complex setting were introduced and studied by Agrawal and Gupta [4]. Additionally, Agrawal et al. [2] proposed and studied the Kantorovich form of the operators, defined by (1), as follows:

\[
(Q_n^{1/n} f)(x) = (1 + n) \sum_{j=0}^{n} q_{n,j}^{(1/n)}(x) \int_{I_{j,n}} f(t) dt
\]

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where \( q_{n,j}^{(1/n)}(x) \) is defined in (1) above and \( I_{j,n} = \left[ \frac{j}{n+1}, \frac{j+1}{n+1} \right] \). The operators \( (Q_n^{(1/n)}f) \) defined in [2] have no direct link with the original discrete operators based on Pólya distribution. Also, these operators do not preserve the linear functions, this motivated us to study in this direction and here, we propose the modification of these operators as: Let us consider \( r_n(x) = \frac{2x(n+1) - 1}{2n} \), then the modified form of (2), for \( f \in C[0, 1] \), can be defined as follows:

\[
(\tilde{Q}_n^{(1/n)}f)(x) = (1 + n) \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x)) \int_{I_{j,n}} f(t) dt
\]

where \( r_n(x) \in [0, 1] \) and

\[
q_{n,j}^{(1/n)}(r_n(x)) = \frac{2(n!)^2}{(2n)!} \binom{n}{j} \left( \frac{2x(n+1) - 1}{2} \right) \binom{2n(1-x) - 2x + 1}{j} n^{-j}.
\]

These modified operators preserve the linear functions along with the constants. These modified operators are well defined on the interval \([0.25, 0.75] \). In the present article, we first provide moments, establish some direct results and combine our estimates with the original operators (2). We observe that a better approximation can be achieved for modified form but on compact interval. We also find difference estimates and in the end some graphical description is indicated.

### 2. Moments

In this section, we deal with the moment estimations.

**Lemma 2.1.** [2] For \( x \in [0, 1] \) with \( e_k(t) = t^k, k = 0, 1, 2, \ldots \), we immediately have

\[
(Q_n^{(1/n)}e_0)(x) = 1, \quad (Q_n^{(1/n)}e_1)(x) = \frac{2nx + 1}{2(1+n)},
\]

\[
(Q_n^{(1/n)}e_2)(x) = \frac{3n^3x^2 - 3n^2x + 9nx + 3nx + n + 1}{3(1+n)^3},
\]

\[
(Q_n^{(1/n)}e_3)(x) = \frac{1}{4(1+n)^4(2+n)} \left( 4(n^3 + 9n^2 + 2n)x + 4(n^5 + 3n^4 + 2n^3)x^3 + 6(n^4 + n^3 - 2n^2)x^2 + 1n(2+n) \right),
\]

\[
(Q_n^{(1/n)}e_4)(x) = \frac{1}{(1+n)^5(2+n)(n+3)} \left( nx \left( 2n^6x^3 + 2n^5x(-3x^2 + 6x + 1) + n^4(11x^2 - 48x + 62) + x^3(-6x^3 - 12x^2 + 32x + 27) + 4n^2(5x + 1) + n(11 - 36x) + 6 \right) \right).
\]

**Lemma 2.2.** For \( x \in [0, 1] \) and \( n = 1, 2, 3, \ldots \), we have

\[
(Q_n^{(1/n)}(e_1 - xe_0))(x) = \frac{1 - 2x}{2(1+n)},
\]

\[
(Q_n^{(1/n)}(e_1 - xe_0)^2)(x) = \frac{3x(1-x)(2n^2 - n - 1) + (1+n)}{3(1+n)^3}.
\]
The following lemmas follow immediately from Lemma 2.1 and Lemma 2.2.

**Lemma 2.3.** For the operators defined in (3), there hold:

\[
\begin{align*}
\widetilde{Q}_n^{(1/n)} e_0(x) &= 1, \\
\widetilde{Q}_n^{(1/n)} e_1(x) &= x, \\
\widetilde{Q}_n^{(1/n)} e_2(x) &= \frac{1}{12(1+n)^3} \left\{ 12n^3 x^2 + 12n^2 x(x+2) + n(-12x^2 + 48x - 11) -12x^2 + 24x - 5 \right\}, \\
\widetilde{Q}_n^{(1/n)} e_3(x) &= \frac{1}{4(1+n)^4(2+n)} \left\{ 4n^5 x^3 + 24n^4 x^3 + n^3 x(56x^2 - 12x + 1) + 8n^2 x(8x^2 - 6x + 5) + n(36x^3 - 60x^2 + 65x - 15) + 8x^3 - 24x^2 + 26x - 6 \right\}, \\
\widetilde{Q}_n^{(1/n)} e_4(x) &= \frac{1}{16(1+n)^5(2+n)(n+3)} \left\{ \left( 2(1+n)x - 1 \right) \left( 8n^6 x^3 + 4n^5 x(-6x^2 + 21x + 4) - 2n^4(16x^2 + 72x^2 - 211x + 4) + n^3(80x^3 - 768x^2 + 1010x - 9) + 2n^2(36x^3 - 348x^2 + 463x - 93) - n(56x^3 + 84x^2 + 2x + 27) + 6(-8x^3 + 12x^2 - 54x + 33) \right) \right\}.
\end{align*}
\]

**Lemma 2.4.** For the operators defined in (3), there hold:

\[
\begin{align*}
\widetilde{Q}_n^{(1/n)} (e_1 - xe_0)(x) &= 0, \\
\widetilde{Q}_n^{(1/n)} (e_1 - xe_0)^2(x) &= \frac{1}{12(1+n)^3} \left\{ -24n^2(x-1)x + n(-48x^2 + 48x - 11) -24x^2 + 24x - 5 \right\}, \\
\widetilde{Q}_n^{(1/n)} (e_1 - xe_0)^3(x) &= \frac{1}{4(1+n)^4(2+n)} \left\{ 3 \left( 8n^4(x-1)x^2 + 4n^3x(10x^2 - 11x + 1) + 2n^2x(36x^2 - 44x + 13) + n(14x - 5)(1 - 2x)^2 + 2(2x - 1)^3 \right) \right\}, \\
\widetilde{Q}_n^{(1/n)} (e_1 - xe_0)^4(x) &= \frac{1}{16(1+n)^5(2+n)(n+3)} \left\{ 32n^6 x^2(-12x^2 + 11x + 1) -16n^5 x(156x^3 - 114x^2 - 43x + 2) -8n^4(936x^4 - 736x^3 - 206x^2 + 57x - 1) + n^3(-12672x^4 + 12736x^3 - 784x^2 - 1160x + 9) -2n^2(6144x^3 - 7920x^2 + 3064x^2 + 148x - 93) + n(-6336x^4 + 10016x^3 - 6560x^2 + 1448x + 27) -6(224x^4 - 416x^3 + 368x^2 - 168x + 33) \right\}.
\end{align*}
\]
3. Main Results

Theorem 3.1. For \( f \in C[0,1] \), we have
\[
\lim_{n \to \infty} (\tilde{Q}_n^{(1/n)} f)(x) = f(x).
\]

Proof. Using Lemma 2.3, we obtain
\[
\lim_{n \to \infty} (\tilde{Q}_n^{(1/n)} e_0)(x) = 1, \quad \lim_{n \to \infty} (\tilde{Q}_n^{(1/n)} e_1)(x) = x
\]
and \( \lim_{n \to \infty} (\tilde{Q}_n^{(1/n)} e_2)(x) = x^2 \). Thus, the result follows by applying the Korovkin theorem. \(\square\)

Theorem 3.2. Let \( f \in C[0,1] \), then
\[
| (\tilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq 2 \omega_1(f, \eta),
\]
where \( \eta = \sqrt{(\tilde{Q}_n^{(1/n)} (e_1 - xe_0)^2)(x)} \).

Proof. From the familiar property of modulus of continuity, we have
\[
| f(t) - f(x) | \leq \omega_1(f, \eta) (|t-x|^2 \eta^{-2} + 1), \text{ for } \eta > 0.
\]
Hence
\[
| (\tilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq (\tilde{Q}_n^{(1/n)} |f(t) - f(x)|, x)
\leq \left( 1 + \eta^{-2} (\tilde{Q}_n^{(1/n)} (e_1 - xe_0)^2)(x) \right) \omega_1(f, \eta).
\]
Using Lemma 2.4, it follows that
\[
\lim_{n \to \infty} (\tilde{Q}_n^{(1/n)} (e_1 - xe_0)^2)(x) = 0.
\]
Thus, by choosing \( \eta = \sqrt{(\tilde{Q}_n^{(1/n)} (e_1 - xe_0)^2)(x)} \), we are led to the required result. \(\square\)

Theorem 3.3. For \( f \in C[0,1] \), we have
\[
| (\tilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq 2 \omega_1(f, \gamma),
\]
where \( \gamma = \sqrt{\frac{3(x-1)(2n^2-n-1)+(1+n)}{3(1+n)^3}} \).

The proof follows on the same lines as the above theorem.

Remark 3.4. The error estimate in Theorem 3.2, for our modified operator is finer than the one considered for the original operator as indicated in Theorem 3.3 above, because of the fact that if we
Further consider

\[
\eta \leq \gamma
\]

\[
\Leftrightarrow -24n^2(x-1)x + n(-48x^2 + 48x - 11) - 24x^2 + 24x - 5
\]

\[
\leq \frac{3x(1-x)(2n^2-n-1) + (1+n)}{3(1+n)^3}
\]

\[
\Leftrightarrow \frac{(-6n^2 - 17n - 9)x^2}{4(1+n)^3} + \frac{(6n^2 + 17n + 9)x}{4(1+n)^3} + \frac{-4n - 2}{4(1+n)^3} \leq 0
\]

\[
\Leftrightarrow x(1-x) \leq \frac{4(1+n)^3 + 2(1+2n)}{9 + 17n + 6n^2}
\]

which holds good for \( x \in [\frac{1}{4}, \frac{3}{4}] \) and for any \( n \in \mathbb{N} \). Hence, we observe that our modified operator has better approximation in the interval \([0.25, 0.75]\).

Following [5], we obtain the following lemmas:

**Lemma 3.5.** Let \( i \in \mathbb{N} \). For \( f \in C^i[a,b] \) and \( g \in C^{i+2}[a,b] \), the remainder term in Taylor’s expansion is given as

\[
(R_i f)(x) = f(t) - \sum_{s=0}^{i} \frac{1}{s!} f^{(s)}(x)(t-x)^s
\]

and it can be expressed as

\[
(R_i f)(x) = (R_i (f - g))(x) + ((R_i - R_{i+1})g)(x) + (R_{i+1}g)(x),
\]

where \( x \in [a,b] \). Further

\[
| (R_i f)(x) | \leq 2 \frac{|x-t|^i}{i!} || f^{(i)} ||,
\]

\[
((R_i - R_{i+1})g)(x) = \frac{1}{(i+1)!} (t-x)^{i+1} g^{(i+1)}(x)
\]

and

\[
| (R_{i+1}g)(x) | \leq \frac{|x-t|^{i+2}}{(i+2)!} || g^{(i+2)} ||.
\]

**Lemma 3.6.** If \( f \in C^2[0,1] \) and \( 0 < \beta \leq \frac{1}{2} \), then for \( \varepsilon > 0 \) there exist polynomials \( q \) such that

\[
|| f'' - q'' || \leq \frac{3}{4} \omega_2(f'', \beta) + \varepsilon,
\]

\[
|| q'' || \leq \frac{5}{\beta} \omega_1(f'', \beta)
\]

and

\[
|| q'' || \leq \frac{3}{2\beta^2} \omega_2(f'', \beta).
\]
Let us denote $\varphi_r = (\tilde{Q}_n^{(1/n)}(e_1 - xe_0)^r)(x)$, $r = 0, 1, 2, ...$

**Theorem 3.7.** For the operator $\tilde{Q}_n^{(1/n)}$ and for $f \in C^2[0, 1]$, the following result holds:

$$\lim_{n \to \infty} |n((\tilde{Q}_n^{(1/n)} f)(x) - f(x)) + x(x-1)f''(x)|$$

$$\leq 2x(x-1)\left\{\frac{5x}{2\beta} \omega_1(f'', \beta) - \left(\frac{3}{4} + \frac{1}{16\beta^2}x(1+12x)\right)\omega_2(f'', \beta)\right\}, \ 0 < \beta \leq \frac{1}{2}.$$

**Proof.** Using Lemma 3.5 and Lemma 3.6, we have

$$|\tilde{Q}_n^{(1/n)}(R_2 f)(x)| = |(\tilde{Q}_n^{(1/n)} f)(x) - f(x) - \frac{1}{2} \varphi_2 f''(x)|$$

$$\leq |(\tilde{Q}_n^{(1/n)} R_2 (f-q))(x)| + |(\tilde{Q}_n^{(1/n)} (R_2 - R_3)q)(x)|$$

$$+ |(\tilde{Q}_n^{(1/n)} R_3 q)(x)|$$

$$\leq |(f-q)'| \varphi_2 + \frac{1}{3!} ||q''''|| \varphi_3 + \frac{1}{4!} ||q''''|| \varphi_4$$

$$\leq \varphi_2 \left\{\frac{3}{4} \omega_2(f'', \beta) + \epsilon\right\} + |\varphi_3| \frac{5}{6\beta} \omega_1(f'', \beta) + |\varphi_4| \frac{1}{16\beta^2} \omega_2(f'', \beta).$$

Taking $\epsilon \to 0$, the following inequality is obtained:

$$|(\tilde{Q}_n^{(1/n)} f)(x) - f(x) - \frac{1}{2} \varphi_2 f''(x)|$$

$$\leq \varphi_2 \left\{\frac{|\varphi_3|}{\varphi_2} \frac{5}{6\beta} \omega_1(f'', \beta) + \left(\frac{3}{4} + \frac{|\varphi_4|}{\varphi_2} \frac{1}{16\beta^2}\right) \omega_2(f'', \beta)\right\}.$$

Now, from Lemma 2.3, we get

$$\lim_{n \to \infty} n \varphi_2 = -2x(x-1), \lim_{n \to \infty} \frac{\varphi_3}{\varphi_2} = -3x \text{ and } \lim_{n \to \infty} \frac{\varphi_4}{\varphi_2} = x(1+12x).$$

Hence, on multiplying the above obtained inequality by $n$ and taking limit $n \to \infty$, we get the desired result. \hfill \Box

**Theorem 3.8.** Let $f \in C^1[0, 1]$ and $x \in [0, 1]$, then

$$|(\tilde{Q}_n^{(1/n)} f)(x) - f(x)| \leq 2\sqrt{\varphi_2} \omega_1(f', \sqrt{\varphi_2}).$$

where $\varphi_2 = (\tilde{Q}_n^{(1/n)}(e_1 - xe_0)^2)(x)$

**Proof.** For $x \in [0, 1]$ and $t \in [0, 1]$, we have

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (f'(s) - f'(x)) ds.$$
Applying the operator \( \widetilde{Q}_n^{(1/n)} \) on both the sides and using the characteristic of modulus of continuity, we obtain

\[
| (\widetilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq |f'(x)| |(\widetilde{Q}_n^{(1/n)} (e_1 - xe_0))(x)| \\
+ \omega_1(f', \eta) \left( \frac{1}{\eta} \varphi_2 + |\widetilde{Q}_n^{(1/n)} (e_1 - xe_0))(x)| \right).
\]

From Lemma 2.4 and Cauchy-Schwarz inequality, we have

\[
| (\widetilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq \omega_1(f', \eta) \left( \frac{1}{\eta} \sqrt{\varphi_2} + 1 \right) \sqrt{\varphi_2}.
\]

Taking \( \eta = \sqrt{\varphi_2} \), we get the desired result. \( \square \)

For an interval \( I \), we consider the Lipschitz-type space \([21]\) defined as:

\[
\text{Lip}_M^*(k) = \{ f \in C[I] : |f(t) - f(x)| \leq M|t - x|^k(t + x)^{-\frac{k}{2}} \},
\]

where \( M > 0 \) and \( 0 < k, x, t < 1 \).

**Theorem 3.9.** Let every \( x \in (0, 1) \) and \( f \in \text{Lip}_M^*(k) \), we have

\[
| (\widetilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq M\eta^k x^{-\frac{k}{2}},
\]

where \( \eta = \sqrt{\frac{-24n^2(x-1)x+n(-48x^2+48x-11)-24x^2+24x-5}{12(1+n)^3}} \).

**Proof.** Let \( f \in \text{Lip}_M^*(k) \) and \( 0 < k < 1 \), then on applying Hölder’s inequality, we obtain

\[
| (\widetilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq (1 + n) \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x)) \int_{I_{j,n}} |f(t) - f(x)| dt \\
\leq \left\{ (1 + n) \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x)) \int_{I_{j,n}} |f(t) - f(x)| \frac{1}{t} dt \right\}^k.
\]

By the definition of Lipschitz-type space, we have

\[
| (\widetilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq M \left\{ (1 + n) \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x)) \int_{I_{j,n}} |t - x|(t + x)^{-\frac{k}{2}} dt \right\}^k.
\]

Using Cauchy-Schwarz inequality and the fact that \( (t+x)^{-\frac{k}{2}} < x^{-\frac{k}{2}} \), it follows that

\[
| (\widetilde{Q}_n^{(1/n)} f)(x) - f(x) | \leq Mx^{-\frac{k}{2}} \left\{ (1 + n) \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x)) \int_{I_{j,n}} |t - x| dt \right\}^k \\
\leq Mx^{-\frac{k}{2}} \left( (\widetilde{Q}_n^{(1/n)} (e_1 - xe_0)^2)(x) \right)^k.
\]

Applying Lemma 2.4 completes the proof. \( \square \)
4. Difference of Operators

The study of differences of linear positive operators became an active area of research after the problem introduced by Lupaş in [19]. The solution was provided by Gonska et al. [6] in the form of moduli of continuity. Various interesting outcomes on difference estimates are presented in [8, 9, 14, 16, 17].

Very recently, Gupta et al. [10] obtained some general results for differences of operators having different basis functions. Inspired by such work, we provide the difference estimates of few operators with different basis.

If we denote $F_{n,j}(f) := (1 + n) \int_{t_j} t_j f(t) \, dt$, then the operators defined by (2) and (3) can be represented as

$$(Q_n^{(1/n)} f)(x) = \sum_{j=0}^{n} q_{n,j}^{(1/n)}(x) F_{n,j}(f)$$

and

$$(\hat{Q}_n^{(1/n)} f)(x) = \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x)) F_{n,j}(f)$$

respectively.

**Lemma 4.1.** If we set $v_i^{F_{n,j}} = F_{n,j}(e_1 - \psi^{F_{n,j}} e_0)^i$, where $\psi^{F_{n,j}} = F_{n,j}(e_1)$, then

$$v_2^{F_{n,j}} = \frac{1}{12(1+n)^2}, \quad v_3^{F_{n,j}} = 0$$

and

$$v_4^{F_{n,j}} = \frac{1}{80(1+n)^4}.$$

**Proof.** Since

$$F_{n,j}(e_r) = (1+n) \int_{t_j} t^{r} \, dt = \frac{1}{(r+1)(1+n)^r} [(j+1)^{r+1} - j^{r+1}],$$

we obtain

$$\psi^{F_{n,j}} = F_{n,j}(e_1) = \frac{(2j+1)}{2(1+n)}.$$
Hence

\[
V_2^{F_{n,j}} = F_{n,j}(e_1 - \psi_{F_{n,j}}e_0)^2
= F_{n,j}(e_1) + \left(\frac{(2j+1)}{2(1+n)}\right)^2 - 2F_{n,j}(e_1)\left(\frac{(2j+1)}{2(1+n)}\right)
= \frac{1}{12(1+n)^2},
\]

\[
V_3^{F_{n,j}} = F_{n,j}(e_1 - \psi_{F_{n,j}}e_0)^3
= F_{n,j}(e_1) - 3F_{n,j}(e_2)\left(\frac{(2j+1)}{2(1+n)}\right)^3 + 3F_{n,j}(e_1)\left(\frac{(2j+1)}{2(1+n)}\right)^2
- F_{n,j}(e_0)\left(\frac{(2j+1)}{2(1+n)}\right)^3
= 0
\]

and

\[
V_4^{F_{n,j}} = F_{n,j}(e_1 - \psi_{F_{n,j}}e_0)^4
= F_{n,j}(e_1) - 4F_{n,j}(e_2)\left(\frac{(2j+1)}{2(1+n)}\right)^3 + 6F_{n,j}(e_2)\left(\frac{(2j+1)}{2(1+n)}\right)^2
- 4F_{n,j}(e_1)\left(\frac{(2j+1)}{2(1+n)}\right)^3 + F_{n,j}(e_0)\left(\frac{(2j+1)}{2(1+n)}\right)^4
= \frac{1}{80(1+n)^4}.
\]

\[\square\]

Let \( C_B[0, \infty) \) consist of all bounded and uniformly continuous functions on \([0, \infty)\). Proceeding along the lines of [10], we obtain the quantitative estimate for the difference of the operators \((Q_n^{(1/n)}f)(x)\) and \((\tilde{Q}_n^{(1/n)}f)(x)\).

**Theorem 4.2.** Let \( f \in C_B[0, 1] \). Then for the operators \((Q_n^{(1/n)}f)(x)\) and for the modified operators \((\tilde{Q}_n^{(1/n)}f)(x)\), we have

\[|((Q_n^{(1/n)} - \tilde{Q}_n^{(1/n)}))f)(x)| \leq \frac{A(x)}{2}||f'|| + 2\omega_1(f, \alpha_1) + 2\omega_1(f, \alpha_2),\]

with \(||.|| = \sup_{x \in [0, \infty)} |f(x)| < \infty.\)

The values of \( A(x), \alpha_1 \) and \( \alpha_2 \) are indicated in the proof of the theorem.
Proof. Using Lemma 4.1, we obtain

\[
A(x) = \sum_{j=0}^{n} q_{n,j}^{(1/n)}(x) v_2^{F_{n,j}} + \sum_{j=0}^{\infty} q_{n,j}^{(1/n)}(r_n(x)) v_2^{F_{n,j}} = \frac{1}{6(1+n)^2}.
\]

Also, using Lemma 2.1 and Lemma 2.3, we have

\[
\alpha_1^2 = \sum_{j=0}^{n} q_{n,j}^{(1/n)}(x) [F_{n,j}(e_1) - x]^2 = \frac{1}{12(1+n)^5} \left\{-24n^4(x-1)x + n^3 \left(48x^2 - 48x + 13\right) + n^2 \left(96x^2 - 96x + 25\right) + 15n(1 - 2x)^2 + 3(1 - 2x)^2\right\}
\]

and

\[
\alpha_2^2 = \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x)) [F_{n,j}(e_1) - x]^2 = \frac{1}{12(1+n)^5} \left\{-24n^4(x-1)x + 4n^3 \left(9x^2 - 9x + 2\right) + 4n^2 \left(3x^2 - 3x + 1\right) + 9n(1 - 2x)^2 + 3(1 - 2x)^2\right\}.
\]

This completes the proof.

Following the result provided by Gupta et al. [14], we present the difference estimate as follows:

**Theorem 4.3.** Let \( f \in C_B[0,1] \). Then for the operators \( (Q_n^{(1/n)} f)(x) \) and for the modified operators \( (\tilde{Q}_n^{(1/n)} f)(x) \), we have

\[
|(Q_n^{(1/n)} f)(x) - (\tilde{Q}_n^{(1/n)} f)(x)| \leq \zeta_1(x) ||f^iv|| + \zeta_2(x) ||f^iv'|| + \zeta_3(x) ||f''|| + 2\omega_1(f, \alpha_1) + 2\omega_1(f, \alpha_2),
\]

where the estimates of \( \zeta_1(x) \), \( \zeta_2(x) \), \( \zeta_3(x) \), \( \delta_1 \) and \( \delta_2 \) are indicated in the proof.
Proof. Using Lemma 2.1, 2.3 and 4.1, we obtain

\begin{align*}
\zeta_1(x) &= \frac{1}{4!} \sum_{j=0}^{n} (q_{n,j}^{(1/n)}(x) + q_{n,j}^{(1/n)}(r_n(x))) F_{n,j}^4 \\
&= \frac{1}{960(1+n)^4}, \\
\zeta_2(x) &= \frac{1}{3!} \left| \sum_{j=0}^{n} (q_{n,j}^{(1/n)}(x) - q_{n,j}^{(1/n)}(r_n(x))) F_{n,j}^3 \right| \\
&= 0, \\
\zeta_3(x) &= \frac{1}{2!} \left| \sum_{j=0}^{n} (q_{n,j}^{(1/n)}(x) - q_{n,j}^{(1/n)}(r_n(x))) F_{n,j}^2 \right| \\
&= \frac{1}{12(1+n)^4}, \\
\alpha_1^2 &= \sum_{j=0}^{n} q_{n,j}^{(1/n)}(x)(F_{n,j}(e_1) - x)^2 \\
&= \frac{1}{12(1+n)^5} \left\{ -24n^4(x-1)x + n^3 (48x^2 - 48x + 13) \\
&+ n^2 (96x^2 - 96x + 25) + 15n(1 - 2x)^2 + 3(1 - 2x)^2 \right\}, \\
\alpha_2^2 &= \sum_{j=0}^{n} q_{n,j}^{(1/n)}(r_n(x))(F_{n,j}(e_1) - x)^2 \\
&= \frac{1}{12(1+n)^5} \left\{ -24n^4(x-1)x - 4n^3 (9x^2 - 9x + 2) \\
&+ 4n^2 (3x^2 - 3x + 1) + 9n(1 - 2x)^2 + 3(1 - 2x)^2 \right\}.
\end{align*}
5. Graphical Depiction

The graphical representations of \((\tilde{Q}_n^{(1/n)} f)(x)\) to \(f_1(x) = -x^3 + 4x^2 - 4x + 1\) and \(f_2(x) = x^4 - x^3 - 2x^2 + x + 1\) are illustrated in Fig. 1 and Fig. 2 respectively. We observe here that the approximation of both the functions by \((\tilde{Q}_n^{(1/n)} f)(x)\) becomes better as the value of \(n\) increases.

![Graphical Depiction](image)

**Figure 1.** The convergence of \((\tilde{Q}_n^{(1/n)} f)(x)\) to \(f_1(x) = -x^3 + 4x^2 - 4x + 1\) for \(n = 30, n = 50, n = 100\) and \(n = 200\).

**Figure 2.** The convergence of \((\tilde{Q}_n^{(1/n)} f)(x)\) to \(f_2(x) = x^4 - x^3 - 2x^2 + x + 1\) for \(n = 30, n = 50, n = 100\) and \(n = 200\).

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References


