ON UNICOHERENCE AND CONTRACTIBILITY OF HYPERSPACES OF NON-METRIZABLE CONTINUA

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ABSTRACT. Let $X$ be a Hausdorff continuum (a nondegenerate, compact, connected, Hausdorff space). Let $C(X)$ denote the hyperspace of its subcontinua, endowed with the Vietoris topology. In this paper we extend some results of the metric case about unicoherence and the existence of selections for $C(X)$. We also introduce two definitions of contractibility of $C(X)$ and discuss their relation with some properties of $X$. In particular, we show that both definitions are equivalent in the metrizable case, but one of them is more general in the Hausdorff continuum case.

1. INTRODUCTION

A Hausdorff continuum is a nondegenerate compact connected Hausdorff space. A continuum is a metrizable Hausdorff continuum.

Given a Hausdorff continuum $X$ we consider the following hyperspaces:

$$2^X = \{ A \subset X : A \text{ is closed and nonempty} \},$$

$$C(X) = \{ A \in 2^X : A \text{ is connected} \},$$

$$F_n(X) = \{ A \in 2^X : A \text{ has at most } n \text{ points} \}.$$ 

These hyperspaces are considered with the Vietoris topology [12, Definition 1.1 and Theorem 1.2].

In the research on hyperspaces of continua many topics have been considered. The interested reader can see many of them in the books [15] and [12]. In this paper we are interested in the following general and natural problem.

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Problem 1. What tools and results for hyperspaces of continua can be used and/or developed for studying hyperspaces of Hausdorff continua?

The following topics have been developed in the direction of Problem 1.

- Existence of order arcs (see [12, Theorem 15.3]).
- Contractibility (see [17]).
- Local connectedness (see [3]).
- Existence of Whitney levels (see [8]).
- Uniqueness of hyperspaces (see [10]).

Continuing the work we made in [8], in this paper we work on Problem 1, studying the following topics: unicoherence, selections, contractibility and property of Kelley.

As a generalization of the result stated by S. B. Nadler, Jr. in [15], we show in this paper:

Theorem 14. If $X$ is a Hausdorff continuum, then $C(X)$ is unicoherent.

Also, we present two kinds of contractibility of $C(X)$: contractibility in itself and contractibility of $F_1(X)$ by order arcs in $C(X)$. We show:

Theorem 27. Let $X$ be a Hausdorff continuum. If $C(X)$ is contractible (in itself), then $F_1(X)$ is contractible by order arcs in $C(X)$.

Theorem 29. Let $X$ be a continuum. Then $C(X)$ is contractible (in itself), if and only if $F_1(X)$ is contractible by order arcs in $C(X)$.

Furthermore, we present an example of a Hausdorff continuum where $F_1(X)$ is contractible by order arcs in $C(X)$, but $C(X)$ is not contractible in itself.

Additionally, we show that the only continua admitting selections for $C(X)$ are generalized dendroids, and that if $X$ is a hereditarily indecomposable Hausdorff continuum, then $F_1(X)$ is contractible by order arcs in $C(X)$. 
2. PRELIMINARIES

Let X be a Hausdorff continuum. Recall that the Vietoris topology has the following basis:

\[ \{ \langle U_1, \ldots, U_n \rangle : n \in \mathbb{N} \text{ and } U_i \text{ is open for each } i \in \{1, \ldots, n\} \} \]

where

\[ \langle U_1, \ldots, U_n \rangle = \{ A \in 2^X : A \subset U_1 \cup \ldots \cup U_n \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, n\} \} \]

A generalized arc is a Hausdorff continuum T with a linear order and the topology induced by its order. That is, the topology on T is given by the subbasis:

\[ \{(\leftarrow, x) : x \in T\} \cup \{(x, \rightarrow) : x \in T\}, \]

where \( (\leftarrow, x) = \{ y \in T : y < x \} \) and \( (x, \rightarrow) = \{ y \in T : x < y \} \). Every generalized arc T contains the minimal element (denoted by min(T)) and the maximal element (denoted by max(T)). Given \( x, y \in T \) such that \( x \leq y \), we define \( [x, y]_T = \{ z \in T : x \leq z \leq y \} \). Notice that every generalized arc is locally connected.

An arc is a space homeomorphic to the unit interval [0, 1]. A generalized circle is the quotient space obtained by identifying the end points of a generalized arc. Notice that every generalized circle is locally connected.

Given a Hausdorff continuum X and \( A, B \in C(X) \) such that \( A \subsetneq B \), an order arc from A to B in \( C(X) \) is a subcontinuum \( \mathcal{A} \) of \( C(X) \) such that \( A = \bigcap_{E \in \mathcal{A}} E, B = \bigcup_{E \in \mathcal{A}} E \) and, for every \( C, D \in \mathcal{A} \), either \( C \subset D \) or \( D \subset C \).

The fundamental theorem for order arcs in \( C(X) \) is the next one. The interested reader can see the proof in [7, Theorem 2.23].

**Theorem 2.** Let X be a Hausdorff continuum and let \( A, B \in C(X) \). Then there exists an order arc from A to B in \( C(X) \) if and only if \( A \subsetneq B \).

**Lemma 3** [8, Lemma 8]. Let X be a Hausdorff continuum and let \( \mathcal{A} \) be an order arc from A to B in \( C(X) \). Then the order in \( \mathcal{A} \) given by the inclusion is a linear order and \( \mathcal{A} \) has the topology induced by this order.
Corollary 4. Let $X$ be a Hausdorff continuum and $A \in C(X)$. For any $x \in A$, there exists an order arc $\mathcal{A}$ in $C(X)$ from $\{x\}$ to $X$ such that $A \in \mathcal{A}$.

Lemma 5. Let $X$ be a Hausdorff normal space and let $A$ be a closed subset of $X$. Given $n \in \mathbb{N}$ and given $\{C_i \subset X : i \in \{1, \ldots, n\}\}$ a closed cover of $A$, if $\{O_i \subset X : i \in \{1, \ldots, n\}\}$ is an open cover of $A$ such that $C_i \subset O_i$ for each $i \in \{1, \ldots, n\}$, then there exists an open cover $\{U_i \subset X : i \in \{1, \ldots, n\}\}$ of $A$ such that, if $U_i \cap U_j \neq \emptyset$, then $C_i \cap C_j \neq \emptyset$ and $C_i \subset U_i \subset O_i$ for each $i \in \{1, \ldots, n\}$.

Proof. Given $i \in \{1, \ldots, n\}$, $K_i = \bigcup \{C_j : C_i \cap C_j = \emptyset\}$ is a closed subset of $X$. Since $C_i \cap K_i = \emptyset$ and $X$ is a normal space, there exist two open subsets $V_i$ and $W_i$ of $X$ such that $C_i \subset V_i$, $K_i \subset W_i$ and $V_i \cap W_i = \emptyset$.

We define $U_i = V_i \cap (\bigcap \{W_j : C_i \subset W_j\}) \cap O_i$. Notice that $C_i \subset U_i \subset O_i$ for each $i \in \{1, \ldots, n\}$. Hence, $\{U_i : i \in \{1, \ldots, n\}\}$ is an open cover of $A$. Let $i_1, i_2 \in \{1, \ldots, n\}$. If $C_{i_1} \cap C_{i_2} = \emptyset$, then $C_{i_2} \subset K_{i_1} \subset W_{i_1}$.

Therefore, $U_{i_2} \subset \bigcap \{W_j : C_{i_2} \subset W_j\} \subset W_{i_1}$. Since $U_{i_1} \subset V_{i_1}$, we have that $U_{i_1} \cap U_{i_2} \subset V_{i_1} \cap W_{i_1} = \emptyset$. Thus if $U_{i_1} \cap U_{i_2} \neq \emptyset$, then $C_{i_1} \cap C_{i_2} \neq \emptyset$. ■

3. UNICOHERENCE

A Hausdorff continuum $X$ is said to be unicoherent if for any two subcontinua $A$ and $B$ of $X$ with $X = A \cup B$, the intersection $A \cap B$ is connected. In [15, Corollary 1.176] S. B. Nadler, Jr. stated that the hyperspaces $C(X)$ and $2^X$ are unicoherent for every metric continuum $X$. A detailed proof can be read in [1, Chapter 2.2] which follows a different approach from that proposed by Nadler. In the following section we generalize this result for the hyperspace of subcontinua of a Hausdorff continuum.

A map is a continuous function. A map $f : Z \to S^1$, where $Z$ is a topological connected space and $S^1$ is the unit circle in the Euclidean plane $\mathbb{R}^2$, has a lifting if there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$, where $\exp$ is the map of $\mathbb{R}$ on $S^1$ defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A connected topological space $Z$ has property (b) if each map $f : Z \to S^1$ has a lifting. It is known [20, Theorem 7.3, p. 227] that if $Z$ is...
normal and it has property (b), then $Z$ is unicoherent. Since Hausdorff continuum is also a normal space, we will show that $C(X)$ is unicoherent by showing that it has property (b).

We will use some auxiliary results. Proposition 7 is stated in [2, Lemma 12] for metric continua. The proof presented there can be easily extended to the Hausdorff continua case.

Let $X$ be a Hausdorff continuum. Given $A \in C(X)$, $M(A)$ will denote the set $\{C \in C(X) : A \subset C\}$. Using the fact that $X$ is a normal space, it is easy to show that $M(A)$ is a closed subset of $C(X)$. In fact, $M(A)$ is a subcontinuum of $C(X)$.

**Proposition 6** (see [9, Theorem 5.1]). If $Z$ is a connected space and $f, g : Z \to \mathbb{R}$ are two maps such that $\exp f = \exp g$ and $f(z) = g(z)$, for some $z \in Z$, then $f = g$.

**Proposition 7** (cf. [2, Lemma 12]). Let $X$ be a Hausdorff continuum, $A \in C(X)$ and $f : M(A) \to S^1$ a map. Consider two order arcs $\mathcal{A}$ and $\mathcal{B}$ from $A$ to $X$ in $C(X)$. If $h_{\mathcal{A}} : \mathcal{A} \to \mathbb{R}$ and $h_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}$ are liftings of $f|_{\mathcal{A}}$ and $f|_{\mathcal{B}}$, respectively, such that $h_{\mathcal{A}}(X) = h_{\mathcal{B}}(X)$, then $h_{\mathcal{A}}(A) = h_{\mathcal{B}}(A)$.

**Proposition 8.** Let $Z$ be a generalized arc, $f : Z \to S^1$ a map, $M$ the maximum of $Z$ and $t_0 \in \mathbb{R}$ such that $\exp(t_0) = f(M)$. Then, there exists a lifting $g$ of $f$ such that $g(M) = t_0$.

**Proof.** Since $f$ is a continuous function, given $x \in Z$, there is an open basic subset $U_x$ of $Z$ such that $x \in U_x$ and diameter of $f(U_x)$ is less than $\frac{1}{4}$. By [6, 6.3.1, p. 372], there exist a subset $\{U_1, \ldots, U_n\}$ of $\{U_x \mid x \in Z\}$ such that $M \in U_1$, $\min(Z) \in U_n$ and $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Since $U_i$ is connected for each $i \in \{1, \ldots, n\}$, we have that $U = U_1 \cup \ldots \cup U_n$ is a connected subset of $Z$ such that $\{\min(Z), M\} \subset U$. Thus, $Z = U$ and $\{U_1, \ldots, U_n\}$ is a cover of $Z$.

**Claim 1.** For each $i \in \{1, \ldots, n\}$, given $x_i \in U_i$ and $s_i \in \mathbb{R}$ such that $\exp(s_i) = f(x_i)$, there exists a lifting $h_i$ of $f|_{U_i}$ such that $h_i(x_i) = s_i$. 
Take \( z \in S^1 \) such that \( z \not\in f(U_i) \) and \( r \in \exp^{-1}(z) \) such that \( r < s_i < r + 1 \). We define \( h_i : U_i \to \mathbb{R} \) by
\[
h_i = [\exp|_{(r,r+1)}]^{-1} \circ f|_{U_i}.
\]
Hence, \( h_i \) is a lifting of \( f|_{U_i} \) and \( h_i(x_i) = s_i \). This completes the proof of Claim 1.

By Claim 1, there exists a lifting \( h_1 \) of \( f|_{U_1} \) such that \( h_1(M) = t_0 \). Take \( x_1 \in U_1 \cap U_2 \), by Claim 1 there exists a lifting \( h_2 \) of \( f|_{U_2} \) such that \( h_2(x_1) = h_1(x_1) \). Notice that \( I_1 = U_1 \cap U_2 \) is connected. Since \( h_1|_{I_1} \) and \( h_2|_{I_1} \) are liftings of \( f|_{I_1} \) such that \( h_1|_{I_1}(x_1) = h_2|_{I_1}(x_1) \), by Proposition 6 we have that \( h_1|_{I_1} = h_2|_{I_1} \). We define \( g_1 : U_1 \cup U_2 \to \mathbb{R} \) by \( g_1(x) = h_i(x) \) for \( x \in U_i \). Thus, \( g_1 \) is a lifting of \( f|_{U_1 \cup U_2} \) such that \( g_1(M) = t_0 \).

Notice that \( U_1 \cup \ldots \cup U_i \) is an open interval in \( Z \) for each \( i \in \{1, \ldots, n\} \) and \( U_{i+1} \cap U_i \) is connected for all \( i \in \{1, \ldots, n-1\} \). Inductively, we can define a lifting \( g \) of \( f \) such that \( g(M) = t_0 \). ■

Lemma 9. If \( X \) is a Hausdorff continuum and \( x \in X \), then \( M(\{x\}) \) has property (b).

Proof. Let \( f : M(\{x\}) \to S^1 \) be a map. We choose \( t_0 \in \mathbb{R} \) such that \( \exp(t_0) = f(X) \). Given \( A \in M(\{x\}) \), by Corollary 4 we can consider an order arc \( \mathcal{A} \) from \( \{x\} \) to \( X \) such that \( A \in \mathcal{A} \). Notice that \( \mathcal{A} \subset M(\{x\}) \).

By Proposition 8, \( f|_{\mathcal{A}} \) has a lifting \( h_{\mathcal{A}} \) such that \( h_{\mathcal{A}}(X) = t_0 \). We define \( h : M(\{x\}) \to \mathbb{R} \) by \( h(A) = h_{\mathcal{A}}(A) \).

Using Lemma 3, it is easy to see that \( [A,X]_{\mathcal{A}} \) is an order arc in \( C(X) \). Since \( M(A) \subset M(\{x\}) \) and \( h_{\mathcal{A}}|_{[A,X]_{\mathcal{A}}} \) is a lifting of \( f|_{[A,X]_{\mathcal{A}}} \) such that \( h_{\mathcal{A}}|_{[A,X]_{\mathcal{A}}}(X) = t_0 \), Proposition 7 implies that \( h \) is well defined.

We will show that \( h \) is a continuous function. Let \( A_0 \in M(\{x\}) \). Let \( U \) be an open basic neighborhood of \( h(A_0) \) in \( \mathbb{R} \) such that \( |r-s| < 1 \) for each \( r, s \in U \). Since \( \exp \) is an open map, \( \exp(U) \) is a proper open subset of \( S^1 \). Let \( \mathcal{W} \) be an open basic neighborhood of \( A_0 \) in \( C(X) \) such that \( f(\mathcal{W}) \subset \exp(U) \). Notice that, for each \( A \in \mathcal{W} \cap M(\{x\}) \), \( x \in A \cap A_0 \) and \( A_0 \cup A \in \mathcal{W} \cap M(\{x\}) \). By Corollary 4, there exist order arcs \( \mathcal{A}_{A_0} \) and \( \mathcal{A} \) from \( \{x\} \) to \( X \) such that \( \{A_0,A_0 \cup A\} \subset \mathcal{A}_{A_0} \) and \( \{A,A_0 \cup A\} \subset \mathcal{A} \). Clearly, \( \mathcal{A}_{A_0} \subset M(\{x\}) \) and \( \mathcal{A} \subset M(\{x\}) \).

In the case that \( A \neq A_0 \), we define \( \mathcal{B}_{A_0} = [A_0,A_0 \cup A]_{\mathcal{A}_{A_0}} \) and \( \mathcal{B}_A = [A,A_0 \cup A]_{\mathcal{A}_A} \). Using Lemma 3, it is easy to see that \( \mathcal{B}_{A_0} \) and \( \mathcal{B}_A \) are order arcs in \( C(X) \). By Proposition 8, \( f|_{\mathcal{A}_{A_0}} \) has a lifting \( h_{\mathcal{A}_{A_0}} \) such
that \( h_{\mathcal{A}_0}(X) = t_0 \). For each \( B \in \mathcal{A}_0 \), we define \( \mathcal{B}_B = [B, X]_{\mathcal{A}_0} \). By Lemma 3, \( \mathcal{B}_B \) is an order arc in \( C(X) \). Notice that \( \mathcal{B}_B \subset M(B) \subset M(\{x\}) \) and \( h_{\mathcal{A}_0} | \mathcal{B}_B \) is a lifting of \( f | \mathcal{B}_B \) such that \( h_{\mathcal{A}_0} | B_0 \)(\( X \)) = t_0 \). By definition of \( h \) and Proposition 7, we have that \( h_{\mathcal{A}_0} | B_0 \) = \( h_{\mathcal{A}_0} \). Similarly, \( f | \mathcal{A}_0 \) has a lifting \( h_{\mathcal{A}_0} \) such that \( h_{\mathcal{A}_0} | \mathcal{B}_B | \mathcal{B}_B = h | \mathcal{B}_B \). Notice that \( \mathcal{B}_B \cup \mathcal{B}_B \subset \mathcal{Y} \). Therefore, since \( h | \mathcal{A}_0 \) is a lifting of \( f | \mathcal{B}_B \) and \( \mathcal{A}_0 \) is a lifting of \( f | \mathcal{B}_B \), we have that \( h(\mathcal{B}_B \cup \mathcal{B}_B) \subset \exp^{-1}(\exp(U)) \).

Since \( h(\mathcal{B}_B) \) and \( h(\mathcal{A}_B) \) are connected subsets of \( \mathbb{R} \) and \( \mathcal{A}_0 \cap \mathcal{A} \subset \mathcal{A}_0 \cap \mathcal{B}_B \), then \( h(\mathcal{A}_0) \cup \mathcal{B}_B \) is a connected subset of \( \exp^{-1}(\exp(U)) \). Notice that \( U \) is a component of \( \exp^{-1}(\exp(U)) \). Since \( h(\mathcal{A}_0) \cup h(\mathcal{A}_B) \) is connected and \( \mathcal{A}_0 \subset U \cap (h(\mathcal{A}_0) \cup h(\mathcal{A}_B)) \), we conclude that \( h(\mathcal{A}_0) \cup h(\mathcal{A}_B) \subset U \) and, in particular, \( h(\mathcal{A}) \subset U \). We have shown that \( h(\mathcal{Y} \cap M(\{x\})) \subset U \). Hence, \( h \) is a continuous function and we conclude that \( h \) is a lifting of \( f \) in \( M(\{x\}) \).

Proposition 10 is easy to prove using Proposition 7.

**Proposition 10.** Let \( A \subset C(X) \) and let \( f : C(X) \to S^1 \) be a map. Given \( x, y \in X \) such that \( A \in M(\{x\}) \cap M(\{y\}) \), if \( h_x : M(\{x\}) \to \mathbb{R} \) is a lifting of \( f | M(\{x\}) \), \( h_y : M(\{y\}) \to \mathbb{R} \) is a lifting of \( f | M(\{y\}) \) and \( h_x(x) = h_y(x) \), then \( h_x(A) = h_y(A) \).

**Lemma 11.** Let \( X \) be a Hausdorff continuum and let \( f : X \to S^1 \) be a map. If \( A \) is a closed subset of \( X \) and \( h : A \to \mathbb{R} \) is a lifting of \( f | A \), then there exists an open subset \( O \) of \( X \) such that \( A \subset O \) and \( f | O \) has a lifting \( g \) such that \( g | A = h \).

**Proof.** Given \( x \in A \), let \( U_x = (h(x) - \frac{1}{8}, h(x) + \frac{1}{8}) \). Notice that \( \exp(U_x) \) is open in \( S^1 \) and \( \exp | U_x \) is a homeomorphism. Let \( C_x = h^{-1}([h(x) - \frac{1}{8}, h(x) + \frac{1}{8}]) \subset A \). Therefore, \( C_x \) is closed in \( X \) and \( x \in \text{int}_A(C_x) \).

Since \( h \) is a lifting of \( f | A \), \( C_x \subset f^{-1}(\exp(U_x)) \).

Consider the open cover of \( A \) given by \( \{\text{int}_A(C_x) : x \in A\} \). Since \( A \) is compact, there exist \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in A \) such that \( \{\text{int}_A(C_{x_1}), \ldots, \text{int}_A(C_{x_n})\} \) is a cover for \( A \). Notice that \( \{C_{x_1}, \ldots, C_{x_n}\} \) is a cover of \( A \) by closed subsets of \( X \). Additionally, \( \{f^{-1}(\exp(U_{x_1})), \ldots, f^{-1}(\exp(U_{x_n}))\} \) is an open cover of \( A \).
by open subsets of $X$, such that $C_{x_i} \subset f^{-1}(\exp(U_{x_i}))$ for each $i \in \{1, \ldots, n\}$. By Lemma 5, there exists
\[ \{W_i : i \in \{1, \ldots, n\}\}, \] a cover of $A$ by open subsets of $X$, such that if $W_i \cap W_j \neq \emptyset$, then $C_{x_i} \cap C_{x_j} \neq \emptyset$ and
\[ x_i \in C_{x_i} \subset W_i \subset f^{-1}(\exp(U_{x_i})) \] for each $i \in \{1, \ldots, n\}$.

For each $i \in \{1, \ldots, n\}$ we define $h_i : W_i \to \mathbb{R}$ by $h_i(x) := [\exp|U_{x_i}|]^{-1}(f(x))$. Notice that each $h_i$ is a map. Also, if $u, v \in W_i$, then $|h_i(u) - h_i(v)| < \frac{1}{4}$. We will use the Pastcing Lemma to get a map $g$ that extends each of the maps $h_i$. To do that, we will show that given $i, j \in \{1, \ldots, n\}$ such that $W_i \cap W_j \neq \emptyset$, we have that $h_i|_{W_i \cap W_j} = h_j|_{W_i \cap W_j}$.

**Claim 1.** Given $i \in \{1, \ldots, n\}$ and $x \in W_i \cap A$, we have that $h_i(x) = h(x)$. In particular, $h_i(x_i) = h(x_i)$.

We only need to consider two cases:

1. If $x \in C_{x_i}$, then

\[ h_i(x) = [\exp|U_{x_i}|]^{-1}(f(x)) = [\exp|U_{x_i}|]^{-1}(\exp(h(x))). \]

Since $h(C_{x_i}) \subset U_{x_i}$, we have that $h(x) \in U_{x_i}$. Hence, $h_i(x) = h(x)$.

2. If $x \in W_i \setminus C_{x_i}$, then there exists $j \in \{1, \ldots, n\}$ such that $x \in C_{x_j} \subset W_j$. Since $x \in W_i \cap W_j$, then $C_{x_i} \cap C_{x_j} \neq \emptyset$. Consider $y \in C_{x_i} \cap C_{x_j}$. Since $h(C_a) \subset U_a$ for each $a \in A$, we have that $|h(x) - h_i(x)| \leq |h(x) - h(y)| + |h(y) - h_i(x)| = |h(x) - h(y)| + |h_i(y) - h_i(x)| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Therefore, $h_i(x) = h(x)$.

This completes the proof of Claim 1.

Consider $i, j \in \{1, \ldots, n\}$ such that $W_i \cap W_j \neq \emptyset$. Thus, $C_{x_i} \cap C_{x_j} \neq \emptyset$. Let $y \in C_{x_i} \cap C_{x_j}$. Therefore,
\[ |h(x_i) - h(x_j)| \leq |h(x_i) - h(y)| + |h(y) - h(x_j)| \leq \frac{1}{6} \leq \frac{1}{4}. \] Let $x \in W_i \cap W_j$. By Claim 1, we have that
\[ |h_i(x) - h_j(x)| \leq |h_i(x) - h_i(x)| + |h_i(x) - h_j(x)| + |h_j(x) - h_j(x)| < \frac{1}{8} + \frac{1}{4} + \frac{1}{8} < 1. \]

We have shown that $|h_i(x) - h_j(x)| < 1$. Since $\exp(h_i(x)) = f(x) = \exp(h_j(x))$, we have that $|h_i(x) - h_j(x)|$ is an integer. Therefore, $h_i(x) = h_j(x)$. 


The set $O = \bigcup_{i=1}^{n} W_i$ is open in $X$. By the Pasting Lemma, there is a map $g : O \to \mathbb{R}$ that extends each of the maps $h_i$. Since $\exp \circ h_i = f$ for each $i \in \{1, \ldots, n\}$, we have that $h_i$ is a lifting of $f|_{W_i}$. Thus, $g$ is a lifting of $f|_O$. Since $g$ extends each $h_i$ and $A \subset O$, Claim 1 implies that $g|_A = h$. ■

Let $X$ be a Hausdorff continuum. Given $U \subset X$, we define $I(U) = \bigcup\{M(\{x\}) : x \in \text{cl}_X(U)\}$. Since $C(X) \setminus I(U) = \langle X \setminus \text{cl}_X(U) \rangle$ is an open set in $C(X)$, we have that $I(U)$ is a closed subset of $C(X)$.

**Lemma 12.** Let $X$ be a Hausdorff continuum and let $x \in X$. If $\mathcal{U}$ is an open subset of $C(X)$ such that $M(\{x\}) \subset \mathcal{U}$, then there exists an open subset $U$ of $X$ such that $x \in U$ and $I(U) \subset \mathcal{U}$.

**Proof.** Suppose, to the contrary, that for each open neighborhood $U$ of $x$ in $X$, there exists $A \in I(U) \setminus \mathcal{U}$. Consider the family of sets $\mathcal{I} = \{I(U) \setminus \mathcal{U} : U \text{ is open in } X \text{ and } x \in U\}$. Since $\mathcal{I}$ has the finite intersection property and $C(X)$ is compact, there exist $B \subset C(X)$ such that $B \cap \mathcal{I}$. We consider two cases. If $x \in B$, then $B \subset M(\{x\}) \subset \mathcal{U}$, which contradicts $B \cap \mathcal{I}$. If $x \notin B$, then there exists an open neighborhood $V$ of $x$ in $X$ such that $\text{cl}_X(V) \cap B = \emptyset$. Then, $B \cap \mathcal{I}$ implies that $B \subset I(V)$, and this contradicts the fact that $\text{cl}_X(V) \cap B = \emptyset$. ■

**Theorem 13.** If $X$ is a Hausdorff continuum, then $C(X)$ has property (b).

**Proof.** Let $f : C(X) \to S^1$ be a map. We will show that $f$ has a lifting. Fix a number $t_0 \in \exp^{-1}(f(X))$.

For each $A \subset C(X)$, we choose $x_A \in A$. By Lemma 9, $f|_{M(\{x_A\})}$ has a lifting $h_{x_A} : M(\{x_A\}) \to \mathbb{R}$ such that $h_{x_A}(X) = t_0$. We define $h : C(X) \to \mathbb{R}$ by $h(A) = h_{x_A}(A)$. By Proposition 10, $h$ is well defined. Next, we prove that $h$ is continuous.

Given $A \subset C(X)$, let $O$ an open neighborhood of $h(A)$ in $\mathbb{R}$. By Lemma 11, there is an open set $\mathcal{O}_{x_A}$ in $C(X)$ such that $M(\{x_A\}) \subset \mathcal{O}_{x_A}$ and $f|_{\mathcal{O}_{x_A}}$ has a lifting $g_{x_A}$ such that $g_{x_A}|_{M(\{x_A\})} = h|_{M(\{x_A\})}$. Additionally, by Lemma 12, there exists an open neighborhood $U_{x_A}$ of $x_A$ in $X$ such that $I(U_{x_A}) \subset \mathcal{O}_{x_A}$. Given $B \subset I(U_{x_A})$, we choose $\mathcal{A}$ an order arc from $B$ to $X$. We have that $\mathcal{A} \subset I(U_{x_A})$, so $\mathcal{A} \subset \mathcal{O}_{x_A}$. Since $g_{x_A}|_{\mathcal{A}}(X) = h|_{\mathcal{A}}(X)$, it follows by Proposition 6 that $g_{x_A}|_{\mathcal{A}}(B) = h|_{\mathcal{A}}(B)$. Thus, $g_{x_A}|_{I(U_{x_A})} = h|_{I(U_{x_A})}$. 


Let $\mathcal{U} = \langle U_x, X \rangle$. Notice that $\mathcal{U}$ is an open neighborhood of $A$ in $C(X)$. Given $B \in \mathcal{U}$, since $B \cap U_x \neq \emptyset$, we have that $B \in I(U_x)$, so $g_{U_x}(B) = h(B)$. Therefore, $h|_{\mathcal{U}} = g_{U_x}|_{\mathcal{U}}$. Since $g_{U_x}$ is a continuous function, there exists $Y$, an open neighborhood of $A$ in $\mathcal{U}$, such that $g_{U_x}(Y) \subset O$. Since $Y$ is open in $C(X)$ and $h(Y) \subset O$, we conclude that $h$ is continuous. ■

The following result is a consequence of Theorem 13 and [20, Theorem 7.3, p. 227].

**Theorem 14.** If $X$ is a Hausdorff continuum, then $C(X)$ is unicoherent.

J. G. Anaya proves in [1] that for any continuum $X$ the hyperspace $2^X$ is also unicoherent, using strongly the metrizability of the hyperspace $2^X$. We emphasize that the proof of Theorem 13 presented here deeply depends on the existence of order arcs as stated in Theorem 2. It cannot be easily extended because this is not true for $2^X$. Therefore, the following question remains open.

**Question 15.** Is the hyperspace $2^X$ unicoherent if $X$ is a non-metrizable Hausdorff continuum?

### 4. SELECTIONS

A Hausdorff continuum $X$ is said to be *connected by generalized arcs* if every pair of points in $X$ is the boundary of some generalized arc embedded in $X$. A Hausdorff continuum $X$ is said to be a *generalized dendroid* (also an *arboroid*) if it is connected by generalized arcs and every subcontinuum of $X$ is unicoherent.

Given the hyperspace $C(X)$ of a Hausdorff continuum $X$, a *selection* for $C(X)$ is a continuous function $s : C(X) \rightarrow X$, such that $s(A) \in A$ for each $A \in C(X)$.

In the case that $X$ is a metric continuum, it is known that if $C(X)$ admits a selection, then $X$ is a dendroid, which is a metrizable generalized dendroid (see [16, Lemma 3]). In this section, we generalize this result for Hausdorff continua. In order to do this, we will use the following results.

**Lemma 16** [14, Theorem 2.3]. Let $X$ be a Hausdorff continuum which is locally connected and unicoherent. If $r$ is a retraction of $X$, then $r(X)$ is unicoherent.
Lemma 17 [15, 1.208.1]. If $X$ is locally connected, then $C(X)$ is locally connected.

Proposition 18 is easy to prove.

**Proposition 18.** If $Z$ a generalized circle, $Z$ is not unicoherent.

**Theorem 19.** If $Z$ is a generalized circle, $C(Z)$ does not admit a selection.

**Proof.** Suppose, to the contrary, that $s : C(Z) \to Z$ is a selection for $C(Z)$. Thus, $(s|_{F_1(Z)})^{-1} \circ s : C(Z) \to F_1(Z)$ is a retraction. By Lemma 16 and Lemma 17, $C(Z)$ is locally connected. Therefore, by Lemma 16 and Theorem 14, $F_1(Z)$ is unicoherent. Since $s|_{F_1(Z)} : F_1(Z) \to Z$ is a homeomorphism, we have that $Z$ is unicoherent. This is a contradiction to Proposition 18. ■

The following result is an immediate consequence of the unique theorem in [18, p. 879].

**Lemma 20.** Let $X$ be a Hausdorff continuum and $Y$ a generalized arc. If $\psi : Y \to X$ is a map, then there exists a subcontinuum $T$ of $X$ such that $T$ is homeomorphic to a generalized arc with end points $\{\psi(\text{min}(Y)), \psi(\text{max}(Y))\}$.

**Lemma 21.** Let $X$ be a Hausdorff continuum. If $s : C(X) \to X$ is a selection, then:

(a) $X$ is connected by generalized arcs.

(b) For each $A \in C(X)$, $A$ is connected by generalized arcs.

**Proof.** (a) Given $x$ and $y \in X$, by Corollary 4 there are order arcs $\mathcal{A}$ and $\mathcal{B}$ in $C(X)$ from $\{x\}$ to $X$ and from $\{y\}$ to $X$, respectively. By Lemma 3, $\mathcal{A}$ and $\mathcal{B}$ are generalized arcs. Consider $\mathcal{A}$ with the order given by the inclusion and $\mathcal{B}$ with the reverse order. Notice that $\mathcal{A} \cap \mathcal{B}$ is a closed nonempty subset of $\mathcal{A}$. Let $m = \min(\mathcal{A} \cap \mathcal{B})$. Thus, $C = [\{x\}, m]_{\mathcal{A}} \cup [m, \{y\}]_{\mathcal{B}}$, with the order induced by the order in $\mathcal{A}$ and $\mathcal{B}$, is a generalized arc. Since $s$ is a map, by Lemma 20, there exists a subcontinuum $T$ of $X$ such that $T$ is homeomorphic to a generalized arc with endpoints $\{s(\{x\}), s(\{y\})\} = \{x, y\}$.

(b) Follows from (a) and the fact that $s|_{A}$ is a selection for $C(A)$. ■
**Theorem 22.** Let $X$ be a Hausdorff continuum. If $C(X)$ admits a selection, then $X$ is a generalized dendroid.

**Proof.** Let $s$ be a selection for $C(X)$. By Lemma 21 (a), $X$ is connected by generalized arcs. We are going to prove that every subcontinuum of $X$ is unicoherent. Suppose, to the contrary, that there are subcontinua $A, B$ and $C \in C(X)$ such that $A \cup B = C$ and $A \cap B$ is not connected. Let $H$ and $K$ be disjoint closed nonempty subsets of $X$ such that $H \cup K = A \cap B$. Let $p \in H$ and $q \in K$. By Lemma 21 (b), there exist $I \in C(A)$ and $J \in C(B)$ such that $I$ and $J$ are generalized arcs with end points \{p, q\}. Notice that $I \not\subset A \cap B$.

Since $I \in C(A)$, there exists $r_0 \in I \setminus B$. We may assume that $p = \min(I)$. Thus, we have that $p < r_0 < q$. Let

$$p' = \max([x \in [p, r_0]) \cap J)$$

and

$$q' = \min([x \in [r_0, q]) \cap J).$$

Notice that $p' \in [p, r_0] \cap J$ and $q' \in [r_0, q) \cap J$. Therefore, $r_0 \not\in J$ implies that $p' < r_0 < q'$. Let $I' = [p', q']_I$. Let $J'$ be the subarc of $J$ that has \{p', q'\} as its end points. Since $I'$ and $J'$ are generalized arcs and $I' \cap J' = \{p', q'\}$, is easy to see that $I' \cup J'$ is a generalized circle.

Since $s|_{C([p', q'])}$ is a selection for $C(I' \cup J')$, we have a contradiction to Theorem 19. ■

### 5. Contractibility of $C(X)$

Two maps $f, g : X \to Y$ are said to be homotopic if there exist a generalized arc $T$ with minimum element $m$ and maximum element $M$, and a map $h : X \times T \to Y$ such that, for each $x \in X$, $h(x, m) = f(x)$ and $h(x, M) = g(x)$. Let $X \subset Y$, $X$ nonempty. The subspace $X$ is said to be contractible in $Y$ if there is a constant map $r : X \to Y$ which is homotopic to the inclusion map $e : X \to Y$.

Let $X$ be a Hausdorff continuum. Let $\mathcal{X}$ and $\mathcal{Y}$ be subcontinua of $C(X)$ such that $\mathcal{X} \subset \mathcal{Y}$ and $X \in \mathcal{Y}$.

Then $\mathcal{X}$ is said to be contractible using an order preserving homotopy in $\mathcal{Y}$ if there exist a generalized arc $T$ with minimum element $m$ and maximum element $M$, and a map $h : \mathcal{X} \times T \to \mathcal{Y}$ such that, for
each $A \in \mathcal{X}$, $h(A, m) = A$ and $h(A, M) = X$, and for all $A, B \in \mathcal{X}$ such that $A \subset B$ and all $s \leq t \in T$ the
inclusion $h(A, s) \subset h(B, t)$ holds.

Notice that, if $\mathcal{X}$ is contractible using an order preserving homotopy in $\mathcal{X}$, then $\mathcal{X}$ is contractible in
itself.

The following result is due to D. G. Paulowich.

**Theorem 23** [17, Theorem 4]. Let $X$ be a Hausdorff continuum. The following three statements are

(1) $F_1(X)$ is contractible in $2^X$.

(2) $2^X$ is contractible (in itself) using an order preserving homotopy.

(3) $C(X)$ is contractible (in itself) using an order preserving homotopy.

Given a Hausdorff continuum, we say that a subspace $\mathcal{C} \subset C(X)$ is *contractible by order arcs* in $C(X)$
if there exist a map $F : \mathcal{C} \rightarrow C(C(X))$ such that, for each $A \in \mathcal{C}$, $F(A)$ is an order arc from $A$ to $X$.

In the first part of this section we will show that if $X$ is a Hausdorff continuum and $C(X)$ is contractible
(in itself), then $F_1(X)$ is contractible by order arcs in $C(X)$. In order to do this, we will use some auxiliary
results. Propositions 24 and 25 are easy to prove. For the proof of Proposition 26 use Proposition 25.

**Proposition 24.** Let $X$ be a Hausdorff continuum and let $T$ be a generalized arc. Given a continuous
map $H : T \rightarrow C(X)$ which is order preserving, we have that $H(T)$ is an order arc from $H(\min(T))$ to
$H(\max(T))$.

**Proposition 25.** Let $X$ be a Hausdorff continuum. Let $\mathcal{U}$ be an open basic subset of $2^X$ and let
$A, B, C \in 2^X$. If $A \subset B \subset C$ and $\{A, C\} \subset \mathcal{U}$, then $B \in \mathcal{U}$.

**Proposition 26.** Let $X$ be a Hausdorff continuum, $\mathcal{A}$ an order arc in $C(X)$ from $\{x\}$ to $X$ and
$\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ an open cover of $\mathcal{A}$ with $\mathcal{U}_i$ an open subset of $C(X)$ for each $i \in \{1, \ldots, n\}$. Then, there exist
Let \( X \) be a Hausdorff continuum. If \( C(X) \) is contractible (in itself), then \( F_1(X) \) is contractible by order arcs in \( C(X) \).

**Proof.** Since \( C(X) \) is contractible (in itself), there is a constant map \( r : C(X) \to C(X) \) which is homotopic to the identity map \( e : C(X) \to C(X) \). Since \( F_1(X) \subset C(X) \), we have that \( r|_{F_1(X)} : F_1(X) \to C(X) \) is a constant map which is homotopic to \( e|_{F_1(X)} : F_1(X) \to C(X) \). Hence, \( F_1(X) \) is contractible in \( C(X) \).

Therefore, \( C(X) \subset 2^X \) implies that \( F_1(X) \) is contractible in \( 2^X \). By Theorem 23, \( C(X) \) is contractible (in itself) using an order preserving homotopy. Thus, there exist a generalized arc \( T \) and a homotopy \( h : C(X) \times T \to C(X) \) which is order preserving.

For each \( x \in X \), since \( \{x\} \times T \) is homeomorphic to \( T \), we have that \( \{x\} \times T \) is a generalized arc. Since \( h|_{\{x\} \times T} \) is an order preserving map, it follows by Proposition 24 that \( h(\{x\} \times T) \) is an order arc in \( C(X) \) from \( \{x\} \) to \( X \).

We define \( H : F_1(X) \to C(C(X)) \) by \( H(\{x\}) = h(\{x\} \times T) \). We will show that \( H \) is a continuous function. Let \( \{x_0\} \in F_1(X) \) and \( \mathcal{U} \) be an open basic neighborhood of \( H(\{x_0\}) \) in \( C(C(X)) \). We may assume that \( \mathcal{U} = \{\mathcal{V}_1, \ldots, \mathcal{V}_n\} \) where \( n \in \mathbb{N} \) and \( \mathcal{V}_i \) is a basic open subset of \( C(X) \) for each \( i \in \{1, \ldots, n\} \). By Proposition 26, there is an open finite cover \( \{\mathcal{V}_1, \ldots, \mathcal{V}_m\} \) of \( H(\{x_0\}) \), given by open basic subsets of \( C(X) \), that satisfy \( \bigcup_{j=1}^m \mathcal{V}_j \subset \bigcup_{i=1}^n \mathcal{U}_i \), \( \{x_0\} \in \mathcal{V}_1 \), \( X \in \mathcal{V}_m \) and

\[
(\mathcal{V}_i \cap H(\{x_0\})) \cap (\mathcal{V}_j \cap H(\{x_0\})) \neq \emptyset
\]

if and only if \( |i - j| \leq 1 \).

It is clear that for each \( i \in \{1, \ldots, m - 1\} \), \( H(\{x_0\}) \cap (\mathcal{V}_i \cap \mathcal{V}_{i+1}) \neq \emptyset \).

We define \( t_0 = \min(T) \), \( t_m = \max(T) \), and, for each \( i \in \{1, \ldots, m - 1\} \), we take \( t_i \in T \) such that \( h(\{x_0\}, t_i) \in \mathcal{V}_i \cap \mathcal{V}_{i+1} \). For each \( i \in \{1, \ldots, n\} \), we take \( t_{m+i} \in T \) such that \( h(\{x_0\}, t_{m+i}) \in H(\{x_0\}) \cap \mathcal{U}_i \).
For each \( i \in \{1, \ldots, n\} \) we take an open neighborhood \( \mathcal{V}_{m+i} \) of \( h(\{x_0\}, t_{m+i}) \) such that \( \mathcal{V}_{m+i} \subset U_i \). Given \( i \in \{1, \ldots, m+n\} \), since \( h \) is a map, there exists an open basic neighborhood \( \mathcal{W}_i \times V_i \) of \( (\{x_0\}, t_i) \) in \( \mathcal{F}_i(X) \times T \) such that \( h(\mathcal{W}_i \times V_i) \subset \mathcal{V}_i \). If \( i \in \{1, \ldots, m-1\} \), we can choose \( \mathcal{W}_i \times V_i \) such that \( h(\mathcal{W}_i \times V_i) \subset \mathcal{V}_i \cap \mathcal{V}_{i+1} \). We may assume that \( \mathcal{W}_i = \langle W_i \rangle \cap \mathcal{F}_i(X) \), with \( W_i \) an open subset of \( X \). Let \( W = \cap_{i=1}^{m+n} W_i \). Notice that \( x_0 \in W \). We will show that \( H(\langle W \rangle \cap \mathcal{F}_i(X)) \subset \Omega \).

Take \( x \in W \). Given \( i \in \{1, \ldots, n\} \), since \( (\{x\}, t_{m+i}) \in \mathcal{W}_{m+i} \times V_{m+i} \), we have that \( h(\{x\}, t_{m+i}) \in \mathcal{V}_{m+i} \subset U_i \). Thus, \( h(\{x\}, t_{m+i}) \in H(\{x\}) \cap U_i \).

On the other hand, given \( h(\{x\}, t) \in H(\{x\}) \), since \( T \) is a generalized arc, \( t \) is comparable with each one of the elements of \( \{t_0, \ldots, t_{m-1}\} \). Let \( i_0 = \max\{i \in \{0, \ldots, m-1\} : t_i \leq t\} \). Hence, \( t \in [t_{i_0}, t_{i_0+1}] \). By the choices of \( t_0 \) and \( t_{i_0+1} \), we have that \( h(\{x\}, t_{i_0}), h(\{x\}, t_{i_0+1}) \) \( \subset \mathcal{V}_{i_0+1} \). Since \( h(\{x\}, t_{i_0}) \subset h(\{x\}, t) \subset h(\{x\}, t_{i_0+1}) \), Proposition 25 implies that \( h(\{x\}, t) \in \mathcal{V}_{i_0+1} \subset \bigcup_{i=1}^{m+n} U_i \).

We have shown that \( H(\{x\}) \subset \Omega \). ■

Now, we will present a main result for metrizable continua. We will show that if \( X \) is a continuum, then \( C(X) \) is contractible (in itself) if and only if \( \mathcal{F}_i(X) \) is contractible by order arcs in \( C(X) \). In the last part of this section we will show that this does not hold in the case of Hausdorff continua that are not metrizable.

In order to do this we will use an auxiliary result. Given a continuum \( X \), a \textit{Whitney map} for \( C(X) \) is a continuous function \( \mu : C(X) \to [0, 1] \) such that \( \mu(\{x\}) = 0 \) for each \( x \in X \), and, if \( A, B \in C(X) \) and \( A \subset B \), then \( \mu(A) < \mu(B) \). It is known that when \( X \) is a continuum, \( C(X) \) is metric and always has a Whitney map. A Whitney level in \( C(X) \) is a set of the form \( \mu^{-1}(t) \subset C(X) \), where \( \mu : C(X) \to [0, 1] \) is a Whitney map and \( t \in [0, 1] \).

The following result is a direct consequence of [11, Theorem 2].

\textbf{Lemma 28.} Let \( X \) be a continuum. Given \( x \in X \) and \( \mathcal{W} \) a Whitney level in \( C(X) \), if \( \mathcal{A} \) is an order arc in \( C(X) \) from \( \{x\} \) to \( X \), then there exists \( A \in \mathcal{A} \) such that \( \mathcal{A} \cap \mathcal{W} = \{A\} \).
Theorem 29. Let $X$ be a continuum. Then $C(X)$ is contractible (in itself), if and only if $F_1(X)$ is contractible by order arcs in $C(X)$.

Proof. By Theorem 27, it is sufficient to show that if $F_1(X)$ is contractible by order arcs in $C(X)$ then $C(X)$ is contractible (in itself).

Suppose that $F_1(X)$ is contractible by order arcs in $C(X)$. Let $F : F_1(X) \rightarrow C(C(X))$ be a map such that $F(\{x\}) = \mathcal{A}_x$ for each $\{x\} \in F_1(X)$, where $\mathcal{A}_x$ is an order arc in $C(X)$ from $\{x\}$ to $X$. Given $x \in X$, by Lemma 3, $\mathcal{A}_x$ is a generalized arc. Hence, $\mathcal{A}_x$ is homeomorphic to $[0, 1]$. Since $X$ is a continuum, there exists a Whitney map $\mu : C(X) \rightarrow [0, 1]$. We may assume that $\mu(X) = 1$.

Given $x \in X$ and $t \in [0, 1]$, by Lemma 28, there exists $C_{xt} \in X$ such that $\mu^{-1}(t) \cap \mathcal{A}_x = \{C_{xt}\}$. We define $G : F_1(X) \times [0, 1] \rightarrow C(X)$ by $G(\{x\}, t) = C_{xt}$. Since $\mu$ is a Whitney map, we have that $G(\{x\}, 0) = \{x\}$ and $G(\{x\}, 1) = X$. We will show that $G$ is a continuous function.

Let $(\{x_0\}, t_0) \in F_1(X) \times [0, 1]$ and let $\{(x_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $F_1(X) \times [0, 1]$ such that $\lim(x_n, t_n) = (x_0, t_0)$. We have that $\lim x_n = x_0$ and $\lim t_n = t_0$. Since $F$ is a map, $\lim \mathcal{A}_{x_n} = \mathcal{A}_{x_0}$. Since $C(X)$ is compact, we may assume that there exists $A \in C(X)$ such that $\lim G(\{x_n\}, t_n) = A$. It is sufficient to show that $G(\{x_0\}, t_0) = A$. Since $G(\{x_n\}, t_n) \in \mathcal{A}_{x_n}$ for each $n \in \mathbb{N}$, we have that $A \in \mathcal{A}_{x_0}$. Additionally, $\mu(G(\{x_n\}, t_n)) = t_n$ for each $n \in \mathbb{N}$. Since $\mu$ is a map, we have that $\mu(A) = t_0$. Therefore, $G(\{x_0\}, t_0) = A$ and $G$ is a map.

We have shown that $F_1(X)$ is contractible in $C(X)$. It follows by Theorem 23 that $C(X)$ is contractible (in itself). $\blacksquare$

Finally, we will show that in general, for non-metric continua, the converse of Theorem 27 does not hold. D. G. Paulowich showed [17, Theorem 5] that there exists a generalized circle $Z$ such that $C(Z)$ is not contractible (in itself). We will show that, if $Z$ is a generalized circle, then $F_1(Z)$ is contractible by order arcs in $C(Z)$. 


The following result is a direct consequence of [8, Lemma 16]:

**Lemma 30.** Let $T$ be a generalized arc with maximum $M$. Given $x \in T$, define $\mathcal{A}_x = \{[x,y] \in C(T) : y \in [x,M]\}$. Then $\mathcal{A}_x$ is an order arc in $C(T)$ from $\{x\}$ to $[x,M]$.

In the proof of Theorem 32 we will use Lemma 31.

**Lemma 31** [12, Lemma 13.3]. Let $X$ and $Y$ be Hausdorff continua. If $f : X \to Y$ is a map we will define $f^* : 2^X \to 2^Y$ by $f^*(A) = \{f(x) : x \in A\}$ for each $A \in 2^X$. Then, $f^*$ is a map.

**Theorem 32.** Let $Z$ be a generalized circle. The hyperspace $F_1(Z)$ is contractible by order arcs in $C(Z)$.

**Proof.** Since $Z$ is a generalized circle, there exist a generalized arc $T$ and a surjective map $p : T \to Z$ such that $p(m) = p(M)$ ($m = \min(T)$ and $M = \max(T)$), $p|_{(m,M)} : (m,M) \to Z$ is injective and $p((m,M)) \cap p(\{m,M\}) = \emptyset$.

We define $g : T \to 2^{2^Z}$ by $g(x) = \{p([x,y]) \in 2^Z : y \in [x,M]\} \cup \{p([x,M]) \cup p([m,y]) \in 2^Z : y \in [m,M]\}$.

We will show that $g$ is continuous.

Consider $F : T \to C(C(T))$ given by $F(x) = \mathcal{A}_x$, where $\mathcal{A}_x$ is defined as in Lemma 30.

**Claim 1.** $F$ is continuous. Let $R = \{[x,y] \in T \times T : x \leq y\}$, where $[x,y]$ is an ordered pair. Since $k : R \to C(T)$ given by $k(x,y) = [x,y]$ is continuous, Lemma 31 implies that $k^* : 2^R \to 2^{C(T)}$ is a map.

Notice that the image of $k^*|_{C(R)}$ is a subset of $C(C(T))$ and $C(R) \subset C(T \times T)$.

We define $h : T \to C(R)$ by $h(x) = \{x\} \times [x,M]$. We will show that $h$ is a continuous function. Given $x \in T$ and $U$ an open subset of $C(T \times T)$ such that $h(x) \in U$, there exist $n \in \mathbb{N}$ and open subsets $U_1, \ldots, U_n$ of $T$ such that $\{x\} \times [x,M] = h(x) \in \langle U \times V_1, \ldots, U \times V_n \rangle \subset U$. Given $i \in \{1, \ldots, n\}$, there exists $[x,y_i] \in \{\{x\} \times [x,M]\} \cap (U \times V_i)$. Therefore, $[x,M] \cap V_i \neq \emptyset$. Additionally, $\{x\} \times [x,M] \subset (U \times V_1) \cup \ldots \cup (U \times V_n)$ implies that $[x,M] \subset V_1 \cup \ldots \cup V_n$. Thus, $k(x,M) = [x,M] \in \langle V_1, \ldots, V_n \rangle$. Since $k$
is a map, there are open subsets $P$ and $Q$ of $T$ such that $x \in P$, $M \in Q$ and $k(R \cap (P \times Q)) \subset \langle V_1, \ldots, V_n \rangle$.

We define $O = P \cap U$. Notice that $O$ is an open subset of $T$ and $x \in O$.

We will show that $h(O) \subset \mathcal{W}$. Let $y \in O$. Since $[y, M] \in (P \times Q) \cap R$, we have that $[y, M] = k([y, M]) \in \langle V_1, \ldots, V_n \rangle$. Therefore, $[y, M] \subset V_1 \cup \ldots \cup V_n$ and $[y, M] \cap V_i \neq \emptyset$ for each $i \in \{1, \ldots, n\}$. Hence, $\{y\} \times [y, M] \subset (U \times V_1) \cup \ldots \cup (U \times V_n)$. Given $i \in \{1, \ldots, n\}$, there exists $z_i \in [y, M] \cap V_i$. Thus, $[y, z_i] \in (U \times V_i) \cap (\{y\} \times [y, M])$. We have shown that $h(y) = \{y\} \times [y, M] \in (U \times V_1, \ldots, U \times V_n) \subset \mathcal{W}$. Therefore, $h$ is continuous.

Given $x \in T$, $k^*(h(x)) = k^*(\{x\} \times [x, M]) = \{k(x, y) \in C(T) : y \in [x, M]\} = \{[x, y] \in C(T) : y \in [x, M]\} = F(x)$. Hence, $F$ is continuous. This completes the proof of Claim 1.

Now, we define $f : T \to C(T)$ by $f(x) = [x, M]$.

**Claim 2.** $f$ is continuous.

Consider $R = \{[x, y] \in T \times T : x \leq y\}$. We define $e : T \to R$ by $e(x) = [x, M]$. We will show that $e$ is continuous. Given $x \in X$ and $(U \times V) \cap R$ an open basic neighborhood of $e(x) = [x, M]$ in $R$, such that $U$ and $V$ are open intervals in $T$, $x \in U$ and $M \in V$. Thus, $e(U) = U \times \{M\} \subset (U \times V) \cap R$. Hence, $e$ is continuous.

Consider $j : R \to C(T)$ given by $j([x, y]) = [x, y]$. Since $j$ is a map, we have that $f = j \circ e$ is continuous.

This completes the proof of Claim 2.

Consider $u : 2^Z \times 2^Z \to 2^Z$ given by $u(\mathcal{A}, \mathcal{B}) = \mathcal{A} \cup \mathcal{B}$. Consider $v : F_1(2^Z) \times 2^Z \to 2^Z$ given by $v(\{A\}, \mathcal{B}) = \{A \cup B : B \in \mathcal{B}\}$. Notice that $u$ and $v$ are maps. Since the image of $F$ is a subset of $2^Z$, $(p^*)^* : 2^Z \to 2^Z$, the image of $f^*|_{F_1(T)}$ is a subset of $F_1(2^T)$, the image of $(p^*)^*|_{F_1(2^T)}$ is a subset of $F_1(2^Z)$, we can define

$$g(x) = u\left((p^*)^*(F(x)), v((p^*)^*(f^*(\{x\})), (p^*)^*(F(m)))\right).$$
We have shown that $F, f^*$ and $(p^*)^*$ are continuous, $g$ is a map.

We will calculate $g(m)$ and $g(M)$.

$$g(m) = \{ p([m, y]) \in 2^Z : y \in [m, M] \} \cup \{ p([m, M]) \cup p([m, y]) \in 2^Z : y \in [m, M] \}$$

$$= \{ p([m, y]) \in 2^Z : y \in [m, M] \} \cup \{ Z \}$$

$$= \{ p([m, y]) \in 2^Z : y \in [m, M] \}.$$

$$g(M) = \{ p(\{ M \}) \} \cup \{ p(\{ M \}) \cup p([m, y]) \in 2^Z : y \in [m, M] \}.$$

Since $p(\{ M \}) = \{ p(m) \}$, we conclude that $g(M) = \{ p([m, y]) \in 2^Z : y \in [m, M] \} = g(m)$.

By the Transgression Theorem (see [5, Theorem 3.2, p. 123]), there exist a map $G : Z \to 2^Z$ such that the following diagram is commutative:

$$\begin{array}{ccc}
T & \overset{p}{\rightarrow} & Z \\
\downarrow{g} & & \downarrow{G} \\
2^Z & &
\end{array}$$

Given $x \in T$ such that $x \neq M$, $\{ [x, y] \in 2^T : y \in [x, M] \}$ is an order arc from $\{ x \}$ to $[x, M]$. Thus, $\{ p([x, y]) \in 2^Z : y \in [x, M] \}$ is an order arc from $\{ p(x) \}$ to $p([x, M])$. Likewise, given $x \in T$, $\{ p([x, M]) \cup p([m, y]) \in 2^Z : y \in [m, M] \}$ is an order arc from $p([x, M]) \cup p(\{ m \}) = p([x, M]) \cup p([m, M]) = Z$. Since the ending point of the first arc is the beginning point of the second, we conclude that $g(x)$ is an order arc from $\{ p(x) \}$ to $Z$.

Given $z \in Z$, since $p$ is surjective, there exist $x \in T$ such that $p(x) = z$. Hence, $G(z) = G(p(x)) = g(x)$.

We have shown that $G(z)$ is an order arc from $\{ p(x) \} = \{ z \}$ to $Z$. ■
6. HEREDITARILY INDECOMPOSABLE CONTINUA AND CONTRACTIBILITY OF $C(X)$

A Hausdorff continuum is **hereditarily indecomposable** if, given $A, B \in C(X)$ such that $A \cap B \neq \emptyset$, we have that $A \subset B$ or $B \subset A$.

Recall that $M(\{p\})$ will denote the set $\{C \in C(X) : p \in C\}$. A Hausdorff continuum $X$ has the property of **Kelley at a point** $p \in X$ if for any $K \in M(\{p\})$ and for any open neighborhood $\mathcal{U}$ of $K$ in $C(X)$ there is a neighborhood $U$ of $p$ in $X$ such that, if $q \in U$, then there is $L \in C(X)$ with $q \in L \in \mathcal{U}$. A Hausdorff continuum $X$ has the property of **Kelley** if it has the property of Kelley at each of its points.

For the metric case, it is known that if $X$ is hereditarily indecomposable then it has the property of Kelley (see [15, Theorem 16.27]), which is a sufficient condition for $C(X)$ to be contractible (in itself).

For the Hausdorff continuum case, we will show that, if $X$ is hereditarily indecomposable, then $C(X)$ is contractible by order arcs.

The following Lemma is an immediate consequence of [10, Lemma 4.1].

**Lemma 33.** Let $X$ be a hereditarily indecomposable Hausdorff continuum. Given $x \in X$ and $\mathcal{A}$ an order arc in $C(X)$ from $\{x\}$ to $X$, if $B \in M(\{x\})$, then $B \in \mathcal{A}$.

**Theorem 34** [10, Theorem 3.1]. Let $X$ be a hereditarily indecomposable Hausdorff continuum. Then $X$ has the property of Kelley.

**Theorem 35.** Let $X$ be a hereditarily indecomposable Hausdorff continuum. Then $F_1(X)$ is contractible by order arcs in $C(X)$.

**Proof.** By Theorem 2 and Lemma 33, given $x \in X$ there is a unique order arc $\mathcal{A}_x$ in $C(X)$ from $\{x\}$ to $X$. We define $H : F_1(X) \to C(C(X))$ by $H(\{x\}) = \mathcal{A}_x$. We will show that $H$ is continuous.

Let $\{x_0\} \in F_1(X)$ and $\mathcal{U}$ be an open basic neighborhood of $\mathcal{A}_{x_0}$ in $C(C(X))$. We may assume that $\mathcal{U} = \{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ where $n \in \mathbb{N}$ and $\mathcal{U}_i$ is an open basic of $C(X)$ for each $i \in \{1, \ldots, n\}$.
Claim 1. There exists an open neighborhood $U$ of $x_0$ such that $H(\{x\}) \cap \mathcal{W}_i \neq \emptyset$ for each $i \in \{1, \ldots, n\}$ and for each $x \in U$.

Given $i \in \{1, \ldots, n\}$, since $H(\{x_0\}) \in \mathcal{U}$ there is $A \in H(\{x_0\}) \cap \mathcal{W}_i$. Since $X$ has the property of Kelley, for each $i \in \{1, \ldots, n\}$ there exists an open neighborhood $U_i$ of $x_0$ such that for each $x \in U_i$, there is $B \in \mathcal{C}(X)$ such that $x \in B \in \mathcal{W}_i$. Since $x \in B$, by Lemma 32, $B \in H(\{x\})$. Thus, we conclude that $H(\{x\}) \cap \mathcal{W}_i \neq \emptyset$. We define $U = U_1 \cap \ldots \cap U_n$. This completes the proof of Claim 1.

Claim 2. There exist an open basic neighborhood $V$ of $x_0$ such that $H(\{x\}) \subset \bigcup_{i=1}^n \mathcal{W}_i$ for each $x \in V$.

By Proposition 26 we know that there is an open cover $\{\mathcal{V}_1, \ldots, \mathcal{V}_m\}$ of $H(\{x_0\})$ open basic subsets of $C(X)$, that satisfy $\bigcup_{j=1}^m \mathcal{V}_j \subset \bigcup_{i=1}^n \mathcal{W}_i$, $\{x_0\} \in \mathcal{V}_1$, $X \in \mathcal{V}_m$ and

$$\left(\mathcal{V}_i \cap H(\{x_0\})\right) \cap \left(\mathcal{V}_j \cap H(\{x_0\})\right) \neq \emptyset$$

if and only if $|i - j| \leq 1$.

By Theorem 34, $X$ has the property of Kelley. Given $j \in \{1, \ldots, m - 1\}$, we take $C_j \in H(\{x_0\}) \cap \mathcal{V}_j \cap \mathcal{V}_{j+1}$. Since $x_0 \in C_j$ and $C_j \in \mathcal{V}_j \cap \mathcal{V}_{j+1}$, there is an open neighborhood $V_j$ of $x_0$ in $X$ such that if $x \in V_j$, then there exists $B \in \mathcal{C}(X)$ such that $x \in B \in \mathcal{V}_j \cap \mathcal{V}_{j+1}$. Since $\mathcal{V}_1$ is an open neighborhood of $\{x_0\}$, we take an open neighborhood $V_0$ of $x_0$ in $X$ such that $\langle V_0 \rangle \subset \mathcal{V}_1$. Let $V = \bigcap_{j=0}^{m-1} V_j$. We will show that $H(\{x\}) \subset \bigcup_{i=1}^n \mathcal{W}_i$ for each $x \in V$.

Let $x \in V$. For each $j \in \{1, \ldots, m-1\}$, since $x \in V_j$, there exists $B_j \in \mathcal{C}(X)$ such that $x \in B_j \in \mathcal{V}_j \cap \mathcal{V}_{j+1}$.

We define $B_0 = \{x\}$ and $B_m = X$. Notice that $B_0 \in \langle V_0 \rangle \subset \mathcal{V}_1$ and $B_m \in \mathcal{V}_m$. Since $x \in B_j$ for each $j \in \{1, \ldots, m\}$, by Lemma 33, $B_j \in H(\{x\})$ for each $j \in \{0, \ldots, m\}$.

Given $B \in H(\{x\})$, since $H(\{x\})$ is an order arc, $B$ is comparable with any element of $\{B_0, \ldots, B_{m-1}\}$.

Let $j_0 = \min\{j \in \{1, \ldots, m\} : B \subset B_j\}$. Hence, $B \in [B_{j_0-1}, B_{j_0}]_{H(\{x\})}$. By the choice of $B_{j_0-1}$ and $B_{j_0}$, we know that $\{B_{j_0-1}, B_{j_0}\} \subset \mathcal{V}_{j_0}$. By Proposition 25, we have that $B \in \mathcal{V}_{j_0} \subset \bigcup_{i=1}^n \mathcal{W}_i$. This completes the proof of Claim 2.
The set $U \cap V$ is an open neighborhood of $x_0$. By Claim 1 and 2, we can conclude that $H((U \cap V) \cap F_1(X)) \subset \mathcal{Y}$. We have proved that $H$ is a continuous function, so $F_1(X)$ is contractible by order arcs in $C(X)$.

7. HOMOGENEITY, PROPERTY OF KELLEY AND CONTRACTIBILITY OF $C(X)$

A Hausdorff continuum is homogeneous if, for each pair of points $x$ and $y$, there exists an homeomorphism $h : X \to X$ such that $h(x) = y$.

J. L. Kelley proved in 1942 that if $X$ is a continuum with the property of Kelley, then $C(X)$ is contractible (in itself) (see [13, Theorem 3.3]). In 1977, R. W. Wardle showed that if $X$ is an homogeneous continuum, then $X$ has the property of Kelley (see [19, Theorem 2.7]). As a consequence, if $X$ is an homogeneous continuum, then $C(X)$ is contractible (in itself).

In the non-metric case the results of Kelley and Wardle do not hold. In fact, in [4] W. J. Charatonik constructed an homogeneous Hausdorff continuum without the property of Kelley, showing that Wardle’s theorem does not hold in the non-metric case. This example has the form of the product $X \times X$, where $X$ is a Hausdorff continuum. The space $X$ of W. J. Charatonik, that we will denote as the circle of circles, is interesting on its own.

In [4] W. J. Charatonik showed that the circle of circles is an homogeneous Hausdorff continuum. A technical proof (that we will omit) shows that the circle of circles has the property of Kelly and that the hyperspace of its subcontinua is not contractible (in itself). The interested reader can find the proof in [7, Theorems 6.38 and 6.40].

References


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