MERCER TYPE VARIANTS OF THE JENSEN–STEFFENSEN INEQUALITY

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Abstract. An integral Jensen–Mercer’s inequality for weights satisfying conditions as for the reversed Jensen–Steffensen inequality is proved in this article. Several integral inequalities involving more than one monotonic functions with reversed Jensen–Steffensen conditions are proved as well. Furthermore, a couple of general companion inequalities related to the integral Jensen Mercer inequality with reversed Jensen-Steffensen conditions are presented as well. Applications for the generalization of weighted Ky Fan’s inequality, classical Power Mean and classical Arithmetic, Geometric and Harmonic Mean inequalities involving bounded variation are given as well.

1. Introduction and Preliminaries

One of the most important concepts in the theory of inequalities is the notion of convex functions and note that the definition of a convex function is, in fact, an inequality. The theory of convex functions has experienced a rapid development. This can be attributed to several causes: first, a great many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities, and many important inequalities are consequences of the applications of convex functions. Let us start with convex functions definition extracted from [16]. Throughout the article $I$ denotes an interval in $\mathbb{R}$.

Definition 1.1. A function $\zeta : I \to \mathbb{R}$ is said to be convex if the inequality

\[ \zeta(\sigma \alpha + (1 - \sigma) \beta) \leq \sigma \zeta(\alpha) + (1 - \sigma) \zeta(\beta) \]  \hspace{1cm} (1.1)

holds for each $\alpha, \beta \in I$ and $\sigma \in [0, 1]$. If the inequality (1.1) is reversed, then $\zeta$ is said to be concave.

In our article, we need the following characteristics of convex functions which are proved in [18, pp. 4-12]. An important property of convex functions is the existence of the left and right derivatives on the interior $I^0$ of $I$. If $\zeta : I \to \mathbb{R}$ is convex then for any $x \in I^0$ the left derivative $\zeta_-'(x)$ and the right derivative $\zeta_+'(x)$ are increasing on $I^0$ and

\[ \zeta_-'(x) \leq \zeta_+'(x) \] for all $x \in I^0$.

It can also be proved that for any convex function $\zeta : I \to \mathbb{R}$ the inequalities

\[ \zeta(z) + c(z)(y - z) \leq \zeta(y), \quad c(z) \in [\zeta_-'(z), \zeta_+'(z)] \]  \hspace{1cm} (1.2)

\[ \zeta(y) \leq \zeta(z) + c(y)(y - z), \quad c(y) \in [\zeta_-'(z), \zeta_+'(z)] \]  \hspace{1cm} (1.3)

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for all $y, z \in I^0$.

One consequence of (1.2) and (1.3) is that $\zeta : I \to \mathbb{R}$ is convex if and only if there is at least one line of support for $\zeta$ at each $x_o \in I^0$. Furthermore, $\zeta$ is differentiable if and only if the line of support at $x_o \in I^0$ is unique. In this case, the line of support is

$$A(x) = \zeta(x_o) + \zeta'(x_o)(x - x_o).$$

There are many known inequalities for convex functions, but without any doubts one of the most important of them is Jensen’s inequality. In its integral form, it is stated as follows [16, p. 45] (see also [2]).

**Proposition 1.2.** Let $(\Omega, \mathcal{A}, \varrho)$ be a measure space with $0 < \varrho(\Omega) < \infty$, and let $x : \Omega \to I$, be a function from $L^1(\varrho)$. Then for any convex function $\zeta : I \to \mathbb{R},$

$$\zeta \left( \frac{1}{\varrho(\Omega)} \int_{\Omega} x \, d\varrho \right) \leq \frac{1}{\varrho(\Omega)} \int_{\Omega} (\zeta \circ x) \, d\varrho$$

(1.4)

holds.

There are many versions, variants and generalization of Jensen inequality, see for example [7]–[13].

The following integral version of Jensen–Steffensen inequality was first proved by Steffensen in [19], but here we consider a variant given by Boas [5] (see also [16, p. 58]). Here, and in the rest of the article, $[\mu, \nu]$ is an interval in $\mathbb{R}$, and $\mu < \nu$.

**Proposition 1.3.** Let $x : [\mu, \nu] \to I$ be a continuous and monotonic function.

Let $\varrho : [\mu, \nu] \to \mathbb{R}$ be either continuous or of bounded variation satisfying

$$\varrho(\mu) \leq \varrho(t) \leq \varrho(\nu) \quad \text{for all } t \in [\mu, \nu], \quad \varrho(\nu) - \varrho(\mu) > 0. \quad (1.5)$$

Then for any continuous convex function $\zeta : I \to \mathbb{R},$

$$\zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) \, d\varrho(t) \right) \leq \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\varrho(t)$$

(1.6)

holds.

The following integral version of “Jensen–Mercer inequality” was proved by Cheung et. al. in [6], but we consider here a special case from [3]. Here, and in the rest of this section, $[m, M] \subseteq I$, and $m < M$.

**Proposition 1.4.** Let $(\Omega, \mathcal{A}, \varrho)$ be a measure space with $0 < \varrho(\Omega) < \infty$, and let $x : \Omega \to [m, M]$ be a measurable function. Then for any continuous convex function $\zeta : [m, M] \to \mathbb{R},$

$$\zeta \left( m + M - \frac{1}{\varrho(\Omega)} \int_{\Omega} x \, d\varrho \right) \leq \zeta(m) + \zeta(M) - \frac{1}{\varrho(\Omega)} \int_{\Omega} (\zeta \circ x) \, d\varrho$$

(1.7)

holds.

Barić and Matković [3] proved the following integral variant of Jensen–Mercer inequality under the conditions of Jensen–Steffensen inequality.

**Proposition 1.5.** Let $x : [\mu, \nu] \to [m, M]$ be a continuous and monotonic function, and let $\varrho : [\mu, \nu] \to \mathbb{R}$ be either continuous or of bounded variation satisfying
(1.5). Then for any continuous and convex function $\zeta : [m, M] \to \mathbb{R}$,

$$
\zeta \left( m + M - \frac{1}{\int_{\mu}^{\nu} x(t) \, d\varrho(t)} \int_{\mu}^{\nu} x(t) \, d\varrho(t) \right) \leq \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} \zeta(x(t)) \, d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\varrho(t)
$$

holds.

Pečarić proved the following Reversed Jensen–Steffensen inequality in [17], which is a generalization of an inequality of Barlow et al. in [4]. Main results in this paper will be stated for functions which satisfy its assumptions, i.e., for functions for which the reverse of Jensen–Steffensen inequality (1.6) holds.

**Proposition 1.6.** Let $x : [\mu, \nu] \to I$ be a continuous and monotonic function, and let $\varrho : [\mu, \nu] \to \mathbb{R}$ be a function of bounded variation such that

$$
\int_{\mu}^{\nu} x(t) \, d\varrho(t) \in [m, M], \quad \text{and} \quad \varrho(\nu) > \varrho(\mu).
$$

Then for any continuous convex function $\zeta : I \to \mathbb{R}$ the reverse inequality holds in (1.6).

**Remark 1.7.** In Proposition 1.6, the interval $[\mu, \vartheta]$ and $(\vartheta, \nu]$ can be replaced by $[\mu, \vartheta)$ and $[\vartheta, \nu]$ respectively.

The goal of the article is to treat reversed Jensen–Steffensen type inequality in a variety of ways. The article is organized in the following way: In Section 1 introduction and preliminaries are provided; In Section 2 we prove Mercer’s inequality under reversed Jensen–Steffensen conditions (our Theorem 2.2); In Section 3 further generalization of Theorem 2.2 involving two or more functions are given in Theorem 3.2–3.6; In Section 4 we prove a couple of general companion inequalities related to the Theorem 2.2; The last section covers some of the applications of our proven results.

2. **Mercer’s Inequality with Reversed Jensen–Steffensen Conditions**

Before we further proceed here we need an important property of convex function proved by Mercer in [13] (see also [3, Theorem 1]). Here, and in the rest of this section $[m, M] \subseteq I$, and $m < M$.

**Lemma 2.1.** If $\zeta : [m, M] \to \mathbb{R}$ is convex function, then $\forall \, x \in [m, M]$ we have

$$
\zeta(m + M - x) \leq \zeta(m) + \zeta(M) - \zeta(x).
$$

Now we are ready to state and prove Jensen–Mercer’s inequality for weights satisfying conditions as for the reversed Jensen–Steffensen inequality. We prove Theorem 2.2 by using the techniques of Theorem 2.1 of [12].

**Theorem 2.2.** Let $x : [\mu, \nu] \to [m, M]$ be a continuous and monotonic function. Let $\varrho : [\mu, \nu] \to \mathbb{R}$ be a function of bounded variation such that

$$
\frac{1}{\int_{\mu}^{\nu} \varrho(t) \, \varrho(t)} \int_{\mu}^{\nu} x(t) \, d\varrho(t) \in [m, M], \quad \text{and} \quad \varrho(\nu) > \varrho(\mu).
$$
We further choose a $\vartheta \in [\mu, \nu]$ such that the condition (1.9) holds. Then for any continuous convex function $\zeta : [m, M] \rightarrow \mathbb{R}$,

$$
\zeta \left( m + M - \frac{1}{\int_\mu^\nu d\varrho(t)} \int_\mu^\nu x(t) d\varrho(t) \right) \leq \zeta(m) + \zeta(M) - \frac{1}{\int_\mu^\nu d\varrho(t)} \int_\mu^\nu \zeta(x(t)) d\varrho(t)
$$

holds.

Proof. Let $\zeta : [m, M] \rightarrow \mathbb{R}$ be a convex function. From Lemma 2.1, $\forall \in [m, M]$ we have

$$
\zeta(m + M - x) \leq \zeta(m) + \zeta(M) - \zeta(x).
$$

Further for $\int_\mu^\nu x(t) d\varrho(t) \in [m, M]$ we have

$$
\zeta \left( m + M - \frac{1}{\int_\mu^\nu d\varrho(t)} \int_\mu^\nu x(t) d\varrho(t) \right) \\
\leq \zeta(m) + \zeta(M) - \zeta \left( \frac{1}{\int_\mu^\nu d\varrho(t)} \int_\mu^\nu x(t) d\varrho(t) \right) \\
\leq \zeta(m) + \zeta(M) - \frac{1}{\int_\mu^\nu d\varrho(t)} \int_\mu^\nu \zeta(x(t)) d\varrho(t),
$$

last inequality follows from Proposition 1.6.\qed

3. Reversed Jensen–Steffensen and Mercer’s Type Inequalities Involving Two or More Functions

In paper [1], Abramovich et. al. proved Jensen–Steffensen type inequality for a sequence of functions $\eta_0, \eta_1, \ldots, \eta_r$, $r \in \mathbb{N}$ where each $\eta_i$ is either convex or concave for $i \in \{0, 1, \ldots, r\}$ and $\eta_i$ is monotonic for $i \in \mathbb{N}$. The goal of this section is to prove analogous results for integral reversed Jensen–Steffensen type inequality. Here, we need the following important property of convex function which we would use repeatedly in our proofs (see [18, p. 16]). Throughout this section $J$ denotes an interval in $\mathbb{R}$.

**Lemma 3.1.** Let $\zeta : I \rightarrow \mathbb{R}$ and $\eta : J \rightarrow \mathbb{R}$ where range $(\zeta) \subseteq J$. If $\zeta$ and $\eta$ are convex and $\eta$ is increasing, then the composite function $\eta \circ \zeta$ is convex on $I$.

Now we can present the following result concerning two functions.

**Theorem 3.2.** Let $\zeta : I \rightarrow J$ be a continuous function, and $\eta : J \rightarrow \mathbb{R}$ be a continuous function. Let $x : [\mu, \nu] \rightarrow I$ be a continuous function, and let $\varrho : [\mu, \nu] \rightarrow \mathbb{R}$ be a function of bounded variation such that $\frac{1}{\int_\mu^\nu d\varrho(t)} \int_\mu^\nu x(t) d\varrho(t) \in I$ and $\varrho(\nu) > \varrho(\mu)$. We further choose a $\vartheta \in [\mu, \nu]$ such that the condition (1.9) holds.

(i) If either $\zeta$ is convex on $I$ and $\eta$ is increasing and convex on $J$ and
\[
\frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} x(t) \, d\mu(t)} \int_{\mu}^{\nu} x(t) \, d\mu(t) \in J, \text{ or } \zeta \text{ is concave on } I \text{ and } \eta \text{ is decreasing and convex on } J \text{ and } \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \in J, \text{ then following inequalities hold.}
\]
\[
\eta \circ \zeta \left( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} x(t) \, d\mu(t)} \int_{\mu}^{\nu} x(t) \, d\mu(t) \right) \geq \eta \circ \zeta \left( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \right)
\]
\[
\geq \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} (\eta \circ \zeta)(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} (\eta \circ \zeta)(x(t)) \, d\mu(t). \quad (3.1)
\]

(ii) If either \( \zeta \) is convex on \( I \) and \( \eta \) is decreasing and concave on \( J \) and \( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \in J, \) or \( \zeta \) is concave on \( I \) and \( \eta \) is increasing and concave on \( J \) and \( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \in J, \) then the reverse inequalities hold in (3.1).

Proof. If \( \zeta \) is continuous and convex, then by Proposition 1.6 we have
\[
\zeta \left( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} x(t) \, d\mu(t)} \int_{\mu}^{\nu} x(t) \, d\mu(t) \right) \geq \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t).
\]
Since \( \zeta \left( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} x(t) \, d\mu(t)} \int_{\mu}^{\nu} x(t) \, d\mu(t) \right) \in J \) and \( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \in J \). Therefore, if \( \eta \) is increasing, then we have
\[
\eta \circ \zeta \left( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} x(t) \, d\mu(t)} \int_{\mu}^{\nu} x(t) \, d\mu(t) \right) \geq \eta \circ \zeta \left( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \right).
\]
Since \( \zeta \) is assumed to be monotonic on \( I \) we have \( \zeta \circ x : [\mu, \nu] \to J \) be a monotonic function, i.e., \( \zeta \circ x(t_1) \leq \zeta \circ x(t_2) \) for \( t_1 \leq t_2 \), or \( \zeta \circ x(t_1) \leq \zeta \circ x(t_2) \) for \( t_1 \geq t_2 \), where \( t_1, t_2 \in [\mu, \nu] \). Since \( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \in J \) and \( \vartheta \in [\mu, \nu] \) which satisfies condition (1.9), therefore, by using Proposition 1.6 for convex function \( \eta \) we obtain the following inequality
\[
\eta \circ \left( \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, d\mu(t) \right) \geq \frac{1}{\int_{\mu}^{\nu} \int_{\mu}^{\nu} (\eta \circ \zeta)(x(t)) \, d\mu(t)} \int_{\mu}^{\nu} (\eta \circ \zeta)(x(t)) \, d\mu(t).
\]
By transitive property of inequalities we will get our required result.

The first inequality in (3.1) also follows from the fact that \( \eta \) is decreasing in case when \( \zeta \) is concave and then the second inequality in (3.1) follows from the convexity of \( \eta \).

On the other hand, analogous argument give the reverse of (3.1), i.e., the reverse of first inequality in (3.1) follows from the fact that \( \eta \) is decreasing in case when \( \zeta \) is convex and the reverse of second inequality of (3.1) follows from the fact that \( \eta \) is concave, or the reverse of first inequality in (3.1) follows from the fact that \( \eta \) is increasing in case when \( \zeta \) is concave and the reverse of second inequality of (3.1) follows from the fact that \( \eta \) is concave.

For our next result which is Mercer’s type inequalities for more than one function, we assume that \([m, M] \subseteq I, m < M\). Further we assume that \( m_1 = \min \{\zeta(m), \zeta(M)\}, M_1 = \max \{\zeta(m), \zeta(M)\} \) where \([m_1, M_1] \subseteq J\).

**Theorem 3.3.** Let \( \zeta : [m, M] \to [m_1, M_1] \) be a continuous \( \to \) monotonic function, and \( \eta : [m_1, M_1] \to \mathbb{R} \) be a continuous function. Let \( x : [\mu, \nu] \to [m, M] \) be
a continuous and monotonic function, and let \( \varrho : [\mu, \nu] \to \mathbb{R} \) be a function of bounded variation such that \( \frac{1}{\varrho(t)} \int_{\mu}^{\nu} x(t) \, dg(t) \in [m, M] \), and \( \varrho(\nu) < \varrho(\mu) \). We further choose \( \vartheta \in [\mu, \nu] \) such that the condition (1.9) holds.

(i) If either \( \zeta \) is convex on \([m, M]\) and \( \eta \) is increasing and concave on \([m_1, M_1]\) and \( \frac{1}{\varrho(t)} \int_{\mu}^{\nu} \zeta(t) \, dg(t) \in [m_1, M_1] \), or \( \zeta \) is concave on \([m, M]\) and \( \eta \) is decreasing and convex on \([m_1, M_1]\) and \( \vartheta \in [\mu, \nu] \), then

\[
\eta \circ \zeta \left( m + M - \frac{1}{\varrho(t)} \int_{\mu}^{\nu} x(t) \, dg(t) \right) 
\leq \eta \circ \left( \zeta(m) + \zeta(M) - \frac{1}{\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) \, dg(t) \right) 
\leq \eta \circ \zeta(m) + \eta \circ \zeta(M) - \frac{1}{\varrho(t)} \int_{\mu}^{\nu} (\zeta \circ \eta)(x(t)) \, dg(t). \tag{3.2}
\]

(ii) If either \( \zeta \) is convex on \([m, M]\) and \( \eta \) is decreasing and concave on \([m_1, M_1]\) and \( \frac{1}{\varrho(t)} \int_{\mu}^{\nu} \zeta(t) \, dg(t) \in [m_1, M_1] \), or \( \zeta \) is concave on \([m, M]\) and \( \eta \) is increasing and convex on \([m_1, M_1]\) and \( \vartheta \in [\mu, \nu] \), then reverse inequalities hold in (3.2).

Proof. Now we follow our proof of "Theorem 3.2" but instead of applying "Proposition 1.6" we apply "Theorem 2.2". Thus we obtain (3.2).

It is interesting that these results concerning two functions, each of which is either convex or concave, can be generalized by induction to any set of functions satisfying certain conditions. Consider a set of \( r + 1 \) functions where \( r \in \mathbb{N} \),

\[
\zeta : I \to \mathbb{R}, \quad \eta_1 : I_1 \to \mathbb{R}, \ldots, \eta_r : I_r \to \mathbb{R},
\]

where \( I, I_1, \ldots, I_r \) are intervals in \( \mathbb{R} \) such that

\[
\zeta(I) \subseteq I_1, \quad \eta_k(I_k) \subseteq I_{k+1}, \quad k \in \{1, \ldots, r-1\}. \tag{3.3}
\]

Then the following two sets of auxiliary functions \( \Phi_k, \Psi_k, k = 1, \ldots, r \), such that \( \Phi_k : I \to \mathbb{R}, \Psi_k : I_k \to \mathbb{R} \), \( \Phi_k, k = 1, \ldots, r \), are well defined as

\[
\Phi_k : I \to \mathbb{R}, \quad \Psi_k : I_k \to \mathbb{R}, \quad \Phi_k = \eta_k \circ \eta_{k-1} \circ \cdots \circ \eta_1 \circ \zeta, \tag{3.4}
\]

\[
\Psi_k : I_k \to \mathbb{R}, \quad \Psi_k = \eta_r \circ \eta_{r-1} \circ \cdots \circ \eta_k, \tag{3.5}
\]

and \( k \in \{1, \ldots, r\} \), \( [m_k, M_k] \subseteq I_k \), where \( m_k = \min \{\phi_{k-1}(m), \phi_{k-1}(M)\} \) and \( M_k = \max \{\phi_{k-1}(m), \phi_{k-1}(M)\} \) with \( \phi_0 = \zeta \), and

\[
\zeta([m, M]) \subseteq [m_1, M_1], \quad \text{and} \quad \eta_k([m_k, M_k]) \subseteq [m_{k+1}, M_{k+1}], \quad \forall k \in \{1, \ldots, r-1\}. \tag{3.6}
\]

Now, we assume that each of the considered functions of the set \( \{\zeta, \eta_1, \ldots, \eta_r\} \) is either convex or concave and that the following monotonicity condition is fulfilled. The following condition is taken from [1].

**Monotonicity condition.**

Denote \( \eta_0 = \zeta \). We say that a set of functions \( \eta_0, \eta_1, \ldots, \eta_r \) \( (r \in \mathbb{N}) \) satisfies the Monotonicity condition MC, if for all \( k \in \{0, 1, \ldots, r-1\} \) and all pairs \((\eta_k, \eta_{k+1})\) satisfy the following conditions:
(1) when both functions $\eta_k$ and $\eta_{k+1}$ are either convex or concave, then $\eta_{k+1}$ is increasing;
(2) when either $\eta_k$ is convex and $\eta_{k+1}$ is concave, or $\eta_k+1$ is convex and $\eta_k$ is concave, then $\eta_{k+1}$ is decreasing.

Note that when the functions $\eta_0, \eta_1, \ldots, \eta_r$ satisfy MC, then all of them except possibly $\eta_0$ are monotonic. Also, we have:

**Proposition 3.4.** Let the functions $\zeta, \eta_1, \ldots, \eta_r$ be as above and satisfy MC. Let $\Phi_k, k = 1, \ldots, r$ be defined by (3.4). Then for all $k \in \{1, \ldots, r\}$, we have:

(i) if $\eta_k$ is convex on $I_k$, then $\Phi_k$ is convex on $I$;
(ii) if $\eta_k$ is concave on $I_k$, then $\Phi_k$ is concave on $I$.

**Proof.** We prove it by induction. For $k = 1$ we set $\eta = \eta_1$ and it is easily seen that the proposed statement is in fact a statement of Lemma 3.1. By induction hypothesis, the result holds for $k = r - 1$ and we check for $k = r$. As the result holds for $k = r - 1$, so for $\eta_{r-1}$ is convex we have $\Phi_{r-1}$ is convex, i.e., from (1.2) we have

$$
\Phi_{r-1}(\sigma \alpha + (1 - \sigma) \beta) \leq \sigma \Phi_{r-1}(\alpha) + (1 - \sigma) \Phi_{r-1}(\beta). \quad (3.7)
$$

Let $\eta_r$ is convex and by induction hypothesis we have $\eta_{r-1}$ is also convex so from MC $\eta_r : I_r \longrightarrow \mathbb{R}$ is an increasing function, thus we have

$$
\eta_r \circ \Phi_{r-1}(\sigma \alpha + (1 - \sigma) \beta) \leq \eta_r(\sigma \Phi_{r-1}(\alpha) + (1 - \sigma) \Phi_{r-1}(\beta)) \\
\leq \sigma \eta_r \circ \Phi_{r-1}(\alpha) + (1 - \sigma) \eta_r \circ \Phi_{r-1}(\beta). \quad (3.8)
$$

Hence, $\Phi_r = \eta_r \circ \Phi_{r-1}$ is convex.

Similarly analogous arguments give that $\Phi_r$ is concave, i.e., the reverse of (3.7) follows by induction hypothesis i.e., $\eta_{r-1}$ is concave implies that $\Phi_{r-1}$ is concave. Therefore if $\eta_r$ is concave, then from MC and from the reverse of (3.7), the reverse of (3.8) follows.

It is now clear that we can extend Theorem 3.2 to the following general result.

**Theorem 3.5.** Let $\zeta : I \longrightarrow \mathbb{R}$ and $\eta_k : I_k \longrightarrow \mathbb{R}, k \in \{1, \ldots, r\}$, be either continuous convex or continuous concave, satisfying (3.3). Assume that $\zeta$ and $\eta_k, k \in \{1, \ldots, r\}$, satisfy MC and additionally assume $\zeta$ to be monotonic. Let $x : [\mu, \nu] \longrightarrow I$ be a continuous and monotonic function, and let $\varphi : [\mu, \nu] \longrightarrow \mathbb{R}$ be a function of bounded variation such that $\frac{1}{\mu - \varphi(\mu)} \int_{\mu}^{\nu} x(t) d\varphi(t) \in I$, $\frac{1}{\mu - \varphi(\mu)} \int_{\mu}^{\nu} \zeta(x(t)) d\varphi(t) \in I_1$, and $\varphi(\nu) > \varphi(\mu)$. We further choose a $\vartheta \in [\mu, \nu]$ such that the condition (1.9) holds. Define the auxiliary functions $\Phi_k$ by (3.4) such that

$$
\frac{1}{\mu - \varphi(\mu)} \int_{\mu}^{\nu} \Phi_k(x(t)) d\varphi(t) \in I_{k+1} \quad \text{for} \quad k \in \{1, \ldots, r - 1\}.
$$
(i) If \( \eta_r \) is convex on \( I_r \), then

\[
\Phi_r \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! x(t) \! d\varphi(t) \right) \geq \Psi_1 \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \zeta(x(t)) \! d\varphi(t) \right) \\
\geq \Psi_2 \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \Phi_1(x(t)) \! d\varphi(t) \right) \geq \cdots \geq \Psi_r \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \Phi_{r-1}(x(t)) \! d\varphi(t) \right) \\
\geq \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \Phi_r(x(t)) \! d\varphi(t),
\]

where \( \Psi_k, \forall \ k \in \{1, \ldots, r\} \) be defined by (3.5).

(ii) If \( \eta_r \) is concave on \( I_r \), then reverse inequalities hold in (3.9).

Proof. We prove it by induction. For \( r = 1 \) we set \( \eta = \eta_1 \) and it is easily seen that the proposed statement is in fact a statement of Theorem 3.2. Suppose the result holds for \( r - 1 \), now we check for \( r \), as the result holds for \( r - 1 \), so for \( \eta_{r-1} \) is convex we have

\[
\Phi_{r-1} \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! x(t) \! d\varphi(t) \right) \geq \Psi_1 \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \zeta(x(t)) \! d\varphi(t) \right) \\
\geq \Psi_2 \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \Phi_1(x(t)) \! d\varphi(t) \right) \geq \cdots \\
\geq \Psi_{r-1} \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \Phi_{r-2}(x(t)) \! d\varphi(t) \right) \\
\geq \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! \Phi_{r-1}(x(t)) \! d\varphi(t).
\]

We claim that all the terms in the series of inequalities (3.10) belong to \( I_r \).

Firstly, we claim

\[
\Phi_{r-1} \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! x(t) \! d\varphi(t) \right) \in I_r.
\]

Given that

\[
\frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! x(t) \! d\varphi(t) \in I.
\]

Using (3.3) we have

\[
\zeta \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! x(t) \! d\varphi(t) \right) \in I_1
\]

and

\[
\eta_1 \circ \zeta \left( \frac{1}{\int_\mu^\nu \! d\varphi(t)} \int_\mu^\nu \! x(t) \! d\varphi(t) \right) \in I_2.
\]
by continuing in a similar manner we obtain
\[
\eta_{r-1} \circ \eta_{r-2} \circ \cdots \circ \eta_1 \circ \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} x(t) d\varrho(t) \right)
= \Phi_{r-1} \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \in I_r.
\]

**Secondly**, we claim
\[
\Psi_1 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \right) \in I_r.
\]

Given that
\[
\frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \in I_1.
\]

Using (3.3) we have
\[
\eta_1 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \right) \in I_2
\]
and
\[
\eta_2 \circ \eta_1 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \right) \in I_3,
\]
by continuing in a similar manner we have
\[
\eta_{r-1} \circ \eta_{r-2} \circ \cdots \circ \eta_1 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \right)
= \Psi_1 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \right) \in I_r.
\]

**Finally**, From the given condition we have
\[
\frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \Phi_{r-1}(x(t)) d\varrho(t) \in I_r.
\]

From the previous observations we conclude that all the terms in (3.10) belong to
\( I_r \). Now let \( \eta_r \) is convex and from induction hypothesis \( \eta_{r-1} \) is also convex, so from
MC \( \eta_r : I_r \rightarrow \mathbb{R} \) is increasing, i.e.,
\[
\eta_r \circ \Phi_{r-1} \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \geq \eta_r \circ \Psi_1 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \right)
\geq \eta_r \circ \Psi_2 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \Phi_1(x(t)) d\varrho(t) \right) \geq \cdots
\geq \eta_r \circ \Psi_{r-1} \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \Phi_{r-2}(x(t)) d\varrho(t) \right)
\geq \eta_r \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t) } \int_{\mu}^{\nu} \Phi_{r-1}(x(t)) d\varrho(t) \right) . \tag{3.11}
\]
By using MC, (3.3), (3.4) and (3.5) we have $\Phi_{r-1} \circ x : [\mu, \nu] \rightarrow \mathbb{R}$ be a monotonic function such that $\Phi_{r-1} \circ x(t) \in I_r$, i.e., $\Phi_{r-1} \circ x(t_1) \leq \Phi_{r-1} \circ x(t_2)$ for $t_1 \leq t_2$, or $\Phi_{r-1} \circ x(t_1) \geq \Phi_{r-1} \circ x(t_2)$ for $t_1 \geq t_2$, where $t_1, t_2 \in [\mu, \nu]$. Since 

$$
\int_{\mu}^{\nu} \Phi_{r-1}(x(t))d\varrho(t) \in I_r \quad \forall \varrho \in [\mu, \nu]
$$

Therefore, by using Proposition 1.6 on (3.11) for convex function $\eta_r$ we obtain,

$$
\Phi_r \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \right) \geq \Psi_1 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t))d\varrho(t) \right)
$$

$$
\geq \Psi_2 \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \Phi_1(x(t))d\varrho(t) \right) \geq \cdots \geq \Psi_r \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \Phi_{r-1}(x(t))d\varrho(t) \right)
$$

$$
\geq \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \Phi_r(x(t))d\varrho(t). \quad (3.12)
$$

It is now clear that we can extend Theorem 3.3 to the following general result.

**Theorem 3.6.** Let $\zeta : [m, M] \rightarrow \mathbb{R}$ and $\eta_k : [m_k, M_k] \rightarrow \mathbb{R}$, $k \in \{1, \ldots, r\}$, $(r \in \mathbb{N})$ be either continuous convex or continuous concave, satisfying (3.3) and (3.6). Assume that $\zeta$ and $\eta_k$, $k \in \{1, \ldots, r\}$, satisfy MC and additionally assume $\zeta$ to be monotonic. Let $x : [\mu, \nu] \rightarrow [m, M]$ be a continuous and monotonic function. Let $g : [\mu, \nu] \rightarrow \mathbb{R}$ be a function of bounded variation such that $\frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \in [m, M]$, $\frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t))d\varrho(t) \in [m_1, M_1]$, and $g(\nu) > g(\mu)$. We further choose a $\varrho \in [\mu, \nu]$ such that the condition (1.9) holds. Define the auxiliary functions $\Phi_k : [m, M] \rightarrow \mathbb{R}$ by (3.4) such that $\frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \Phi_k(x(t))d\varrho(t) \in [m_{k+1}, M_{k+1}]$ for $k \in \{1, \ldots, r-1\}$.

(i) If $\eta_r$ is convex on $[m_r, M_r]$, then

$$
\Phi_r \left( m + M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \right) \leq \Psi_1 \left( \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t))d\varrho(t) \right)
$$

$$
\leq \Psi_2 \left( \Phi_1(m) + \Phi_1(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \Phi_1(x(t))d\varrho(t) \right) \leq \cdots
$$

$$
\leq \Psi_r \left( \Phi_{r-1}(m) + \Phi_{r-1}(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \Phi_{r-1}(x(t))d\varrho(t) \right)
$$

$$
\leq \Phi_r(m) + \Phi_r(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \Phi_r(x(t))d\varrho(t), \quad (3.13)
$$

where $\Psi_k$, $\forall k \in \{1, \ldots, r\}$ be defined by (3.5).

(ii) If $\eta_r$ is concave on $[m_r, M_r]$, then reverse inequalities hold in (3.13).

**Proof.** By following the proof of Theorem 3.5, we obtain Theorem 3.6 by simply replacing “Proposition 1.6” and “Theorem 3.2” by “Theorem 2.2” and “Theorem 3.3” respectively. \qed
4. Companion Inequalities Related to the Mercer’s Inequality

In this section we consider a convex function \( \zeta : (a, b) \to \mathbb{R} \), where \(-\infty \leq a < m < M < b \leq +\infty \). For \( \zeta'(x) \), where \( x \in (a, b) \), we may take any element of \([\zeta'_-(x), \zeta'_+(x)]\), but without any loss of generality we can set \( \zeta'(x) = \zeta'_+(x) \)(of course, if \( \zeta \) is differentiable then \( \zeta'(x) = \zeta'_+(x) = \zeta'_-(x) \)).

In [2] a couple of companion inequalities to Mercer’s inequality were proved under Jensen and Jensen–Steffensen condition. In the following theorem, we give companion inequalities to Mercer’s inequality under reversed Jensen–Steffensen condition.

**Theorem 4.1.** Let \( x : [\mu, \nu] \to [m, M] \) be a continuous and monotonic function. Let \( \varrho : [\mu, \nu] \to \mathbb{R} \) be a function of bounded variation such that

\[
\frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \in [m, M], \quad \text{and} \quad \varrho(\nu) > \varrho(\mu).
\]

We further choose a \( \vartheta \in [\mu, \nu] \) such that the condition (1.9) holds. Then for any convex function \( \zeta : (a, b) \to \mathbb{R} \) the following inequalities hold.

\[
\zeta(c) + \zeta'(c) \left( m + M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) - c \right) \\
\leq \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t))d\varrho(t) \\
\leq \zeta(d) + \zeta'(m)(m-d) + \zeta' \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \right) \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) - d \right) \\
+ \zeta'(M)(M-d) + \zeta' \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t))d\varrho(t) \quad (4.1)
\]

holds for all \( c, d \in [m, M] \).

**Proof.** First, we prove the left hand inequality in (4.1).

If in (1.2) we choose \( z = c \) and \( y = m + M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \), then we obtain

\[
\zeta(c) + \zeta'(c) \left( m + M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) - c \right) \\
\leq \zeta \left( m + M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \right) \\
\leq \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t))d\varrho(t). \quad (4.2)
\]

The last inequality is a consequence of Theorem 2.2. Now it remains to prove the second inequality of (4.1). Let \( d, \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t) \in [m, M] \).

We consider two cases.
Case 1. \[ \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \geq d. \] From (1.3) we have

\[ \zeta(m) - \zeta(d) \leq \zeta'(m)(m - d) \]

\[ \zeta(M) - \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \leq \zeta'(M) \left( M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right). \]

Consider

\[ \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \]

\[ = \zeta(d) + \zeta(m) - \zeta(d) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \]

\[ \leq \zeta(d) + \zeta'(m)(m - d) + \zeta'(M) \left( M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \]

\[ + \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \]

\[ = \zeta(d) + \zeta'(m)(m - d) + \zeta'(M)(M - d) - \zeta'(M) \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) - d \right) \]

\[ + \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \] (4.3)

Since \( \zeta \) is convex, the derivative \( \zeta' \) is nondecreasing. Thus \( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \leq M \) implies that \( \zeta' \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \leq \zeta'(M) \), hence (4.3) implies

\[ \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \]

\[ \leq \zeta(d) + \zeta'(m)(m - d) - \zeta' \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) - d \right) \]

\[ + \zeta'(M)(M - d) + \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) \] (4.4)
Case 2. \( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \leq d \). Similarly, as in the previous case, we can write
\[
\zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t)
\]
\[
= \zeta(d) + \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t)
\]
\[
\leq \zeta(d) + \zeta'(m) \left( m - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) + \zeta'(M)(M - d)
\]
\[
+ \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t)
\]
\[
= \zeta(d) + \zeta'(m)(m - d) + \zeta'(M)(M - d) + \zeta'(m) \left( d - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right)
\]
\[
+ \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t).
\]
(4.5)

Since \( \zeta \) is convex, the derivative \( \zeta' \) is nondecreasing and we know that from \( m \leq \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \) follows \( \zeta'(m) \leq \zeta' \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \), hence (4.5) implies
\[
\zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t)
\]
\[
\leq \zeta(d) + \zeta'(m)(m - d) + \zeta' \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) \left( d - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right)
\]
\[
+ \zeta'(M)(M - d) + \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t).
\]
(4.6)

which is again (4.4).

In other words, for any \( d \geq \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \in [m, M] \) the inequality in (4.6) holds. The proof is complete. \( \square \)

**Corollary 4.2.** The following inequalities are valid under the assumption of Theorem 4.1.
\[
0 \leq \zeta(m) + \zeta(M) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t) - \zeta(\bar{x})
\]
\[
\leq \zeta'(m)(m - M) + \zeta'(M)(M - m)
\]
\[
+ \zeta \left( \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \right) - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} \zeta(x(t)) d\varrho(t),
\]
(4.7)

where \( \bar{x} = m + M - \frac{1}{\int_{\mu}^{\nu} d\varrho(t)} \int_{\mu}^{\nu} x(t) d\varrho(t) \).

**Proof.** Follows from (4.1) for \( c = d = \bar{x} \). \( \square \)

Note that the first inequality in (4.7) is the Jensen-Mercer’s inequality.
5. Applications

We define the following notations.

**Generalized Arithmetic Mean**

\[
A(x, \varrho) = m + M - \frac{1}{\int_t^{\infty} d\varrho(t)} \int_t^{\infty} x(t) d\varrho(t)
\]

\[
A'(x, \varrho) = 1 - A(x, \varrho).
\]

**Generalized Geometric Mean**

\[
G(x, \varrho) = \frac{mM}{\exp \left\{ \frac{1}{\int_t^{\infty} d\varrho(t)} \int_t^{\infty} \ln(x(t)) d\varrho(t) \right\}}
\]

\[
G'(x, \varrho) = \frac{(1 - m)(1 - M)}{\exp \left\{ \frac{1}{\int_t^{\infty} d\varrho(t)} \int_t^{\infty} \ln(1 - (x(t))) d\varrho(t) \right\}}.
\]

**Generalized Harmonic Mean**

\[
H(x, \varrho) = \left( \frac{1}{m} + \frac{1}{M} - \frac{1}{\int_t^{\infty} d\varrho(t)} \int_t^{\infty} \frac{1}{x(t)} d\varrho(t) \right)^{-1}
\]

\[
H'(x, \varrho) = 1 - H(x, \varrho).
\]

**Generalized Power Mean**

\[
M^r(x, \varrho) = \left( m^r + M^r - \frac{1}{\int_t^{\infty} d\varrho(t)} \int_t^{\infty} x^r(t) d\varrho(t) \right)^{\frac{1}{r}}, \quad r \in \mathbb{R} \setminus \{0\}
\]

**Theorem 5.1.** Under the assumptions of Theorem 3.3 with \( \zeta : (0, 1) \to \mathbb{R} \), \( \eta : \mathbb{R} \to \mathbb{R} \) and \( \eta \circ \zeta : (0, 1) \to \mathbb{R} \) we have,

(i) For \( 0 < r < 1 \),

\[
\left[ \left( \frac{1}{m} - 1 \right)^r + \left( \frac{1}{M} - 1 \right)^r \right] - \ln \left( \exp \left\{ \frac{1}{\int_t^{\infty} d\varrho(t)} \int_t^{\infty} \left( \frac{1}{x(t)} - 1 \right)^r d\varrho(t) \right\} \right) \quad \leq \quad \frac{G(x, \varrho)}{G'(x, \varrho)} \leq \frac{A(x, \varrho)}{A'(x, \varrho)}
\]

(ii) For \( r = 1 \),

\[
\frac{H(x, \varrho)}{H'(x, \varrho)} \leq \frac{G(x, \varrho)}{G'(x, \varrho)} \leq \frac{A(x, \varrho)}{A'(x, \varrho)}.
\]

**Proof.** (i) Let \( 0 < r < 1 \), \( \zeta : (0, 1) \to \mathbb{R} \), \( \eta : \mathbb{R} \to \mathbb{R} \) and \( \eta \circ \zeta : (0, 1) \to \mathbb{R} \) be defined as

\[
\zeta(x) = \ln \frac{x}{1 - x}, \quad \eta(x) = \exp^{-rx}
\]

\[
\eta \circ \zeta(x) = \left( \frac{1 - x}{x} \right)^r.
\]
Then $\zeta$ is strictly concave and strictly increasing on $[m,M] = (0, \frac{1}{2}]$, while $\eta$ is strictly convex and strictly decreasing on $[m_1,M_1] = \mathbb{R}$ so (3.2) becomes
\[
\left(1 - \frac{m - M}{m + M - \frac{1}{\int_{\mu} x(t) d\varrho(t)}} \right)^r
\leq \exp -r \left\{ \ln \frac{m}{1-m} + \ln \frac{M}{1-M} - \ln \left( \exp \left\{ \frac{1}{\int_{\mu} x(t) d\varrho(t)} \int_{\mu} x(t) d\varrho(t) \right\} \right) \right\}
\]

Applying property of ln and exp we have
\[
\left(1 - \frac{m - M}{m + M - \frac{1}{\int_{\mu} x(t) d\varrho(t)}} \right)^r
\leq \left\{ \frac{1}{\int_{\mu} x(t) d\varrho(t)} \int_{\mu} x(t) d\varrho(t) \right\}
\]

Using Generalized Arithmetic mean and Generalized Geometric mean
\[
\left(\frac{A'(x,\varrho)}{A(x,\varrho)}\right)^r \leq \left\{ \frac{G'(x,\varrho)}{G(x,\varrho)} \right\}^r
\]

Since $\phi(t) = t^{\frac{1}{r}}$, $t > 0$ is strictly decreasing the above inequalities can be written in equivalent form as
\[
\left[ \left( \frac{1}{m} - 1 \right) + \left( \frac{1}{M} - 1 \right) \right]^{\frac{1}{r}} - \ln \left( \exp \left\{ \frac{1}{\int_{\mu} x(t) d\varrho(t)} \int_{\mu} x(t) d\varrho(t) \right\} \right)^r
\]

(ii) Choosing $r = 1$ in (5.1) and also using Generalized Harmonic mean we have,
\[
\frac{H(x,\varrho)}{H'(x,\varrho)} \leq \frac{G(x,\varrho)}{G'(x,\varrho)} \leq \frac{A(x,\varrho)}{A'(x,\varrho)}.
\]

Remark 5.2. The second inequality in (5.2) is a generalized variant of weighted Ky Fan's inequality (see, for example, [15, pp. 25-28]).

We show now how some monotonicity properties of power means proved in [14] can be obtained as a special case of Theorem 3.3.

Theorem 5.3. Under the assumptions of Theorem 3.3 with $[m,M] \subseteq (0, \infty)$, $\zeta : [m,M] \rightarrow [m_1,M_1]$ , $\eta : [m_1,M_1] \rightarrow \mathbb{R}$ and $\eta \circ \zeta : [m,M] \rightarrow \mathbb{R}$ we have,
(i) For $r < s < 0$

\[ M[r](x, \varrho) \leq M[s](x, \varrho) \leq A(x, \varrho). \]

(ii) For $r < 0 < s \leq 1$

\[ M[r](x, \varrho) \leq G(x, \varrho) \leq A(x, \varrho). \]

(iii) For $0 < r < s \leq 1$

\[ M[r](x, \varrho) \leq M[s](x, \varrho) \leq A(x, \varrho). \]

(iv) For $1 \leq r < s$

\[ A(x, \varrho) \leq M[r](x, \varrho) \leq M[s](x, \varrho). \]

Proof. (i) Let $r < s < 0$, $\zeta : [m, M] \rightarrow [m_1, M_1]$, $\eta : [m_1, M_1] \rightarrow \mathbb{R}$ and $\eta \circ \zeta : [m, M] \rightarrow \mathbb{R}$ we have,

\[ \zeta(x) = x^s, \quad \eta(x) = x^r, \quad \eta \circ \zeta(x) = x^r. \]

Then $\zeta$ is strictly convex and strictly decreasing while $\eta$ is convex and increasing so (3.2) becomes

\[
\left( m + M - \frac{1}{\int_{\mu} d\varrho(t)} \int_{\mu} x(t) d\varrho(t) \right)^r \leq \left( m^s + M^s - \frac{1}{\int_{\mu} d\varrho(t)} \int_{\mu} x^s(t) d\varrho(t) \right)^s \leq \left( m^r + M^r - \frac{1}{\int_{\mu} d\varrho(t)} \int_{\mu} x^r(t) d\varrho(t) \right)
\]

Since $\phi(t) = t^2$, $t > 0$ is strictly decreasing the above inequalities can be written in equivalent form as

\[
\left( m^r + M^r - \frac{1}{\int_{\mu} d\varrho(t)} \int_{\mu} x^r(t) d\varrho(t) \right)^\frac{1}{2} \leq \left( m^s + M^s - \frac{1}{\int_{\mu} d\varrho(t)} \int_{\mu} x^s(t) d\varrho(t) \right)^\frac{1}{2} \leq \left( m + M - \frac{1}{\int_{\mu} d\varrho(t)} \int_{\mu} x(t) d\varrho(t) \right).
\]

Using Generalized Arithmetic Mean and Generalized Power Mean we get

\[ M[r](x, \varrho) \leq M[s](x, \varrho) \leq A(x, \varrho). \]

(ii) Let $r < 0 < s \leq 1$, $\zeta : [m, M] \rightarrow [m_1, M_1]$, $\eta : [m_1, M_1] \rightarrow \mathbb{R}$ and $\eta \circ \zeta : [m, M] \rightarrow \mathbb{R}$ we have,

\[ \zeta(x) = \ln x^s, \quad \eta(x) = \exp^\frac{5}{2}x, \quad \eta \circ \zeta(x) = x^r. \]
Then $\zeta$ is strictly concave and strictly increasing while $\eta$ is strictly convex and decreasing so (3.2) becomes

$$
\left(m + M - \frac{1}{\int_{\mu}^{\nu} x(t)d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t)\right)^r
\leq \exp \left(\ln m^s + \ln M^s - \frac{1}{\int_{\mu}^{\nu} \ln x^s(t)d\varrho(t)} \int_{\mu}^{\nu} \ln x^s(t)d\varrho(t)\right)
\leq \left(m^r + M^r - \frac{1}{\int_{\mu}^{\nu} x^r(t)d\varrho(t)} \int_{\mu}^{\nu} x^r(t)d\varrho(t)\right).
$$

Applying property of $\ln$ and $\exp$ we get

$$
\left(m + M - \frac{1}{\int_{\mu}^{\nu} x(t)d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t)\right)^r
\leq \left(m M \exp \left\{\frac{1}{\int_{\mu}^{\nu} \ln x^s(t)d\varrho(t)} \int_{\mu}^{\nu} \ln x^s(t)d\varrho(t)\right\}\right)^r
\leq \left(m^r + M^r - \frac{1}{\int_{\mu}^{\nu} x^r(t)d\varrho(t)} \int_{\mu}^{\nu} x^r(t)d\varrho(t)\right).
$$

Since $\phi(t) = t^\frac{1}{r}$, $t > 0$ is strictly decreasing the above inequalities can be written in equivalent form as

$$
\left(m^r + M^r - \frac{1}{\int_{\mu}^{\nu} x^r(t)d\varrho(t)} \int_{\mu}^{\nu} x^r(t)d\varrho(t)\right)^{\frac{1}{r}}
\leq \left(m M \exp \left\{\frac{1}{\int_{\mu}^{\nu} \ln x^s(t)d\varrho(t)} \int_{\mu}^{\nu} \ln x^s(t)d\varrho(t)\right\}\right)
\leq \left(m + M - \frac{1}{\int_{\mu}^{\nu} x(t)d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t)\right).
$$

Using Generalized Arithmetic Mean, Generalized Geometric Mean and Generalized Power Mean we get

$$
M^{[r]}(x, \varrho) \leq G(x, \varrho) \leq A(x, \varrho).
$$

(iii) Let $0 < r < s \leq 1$, $\zeta : [m, M] \rightarrow [m_1, M_1]$, $\eta : [m_1, M_1] \rightarrow \mathbb{R}$ and $\eta \circ \zeta : [m, M] \rightarrow \mathbb{R}$ we have,

$$
\zeta(x) = x^s, \quad \eta(x) = x^\frac{s}{r}, \quad \eta \circ \zeta(x) = x^r.
$$

Then $\zeta$ is strictly concave and strictly increasing while $\eta$ is strictly concave and strictly increasing so reverse of (3.2) becomes

$$
\left(m^r + M^r - \frac{1}{\int_{\mu}^{\nu} x^r(t)d\varrho(t)} \int_{\mu}^{\nu} x^r(t)d\varrho(t)\right)^{\frac{1}{r}}
\leq \left(m^s + M^s - \frac{1}{\int_{\mu}^{\nu} x^s(t)d\varrho(t)} \int_{\mu}^{\nu} x^s(t)d\varrho(t)\right)^{\frac{1}{s}}
\leq \left(m + M - \frac{1}{\int_{\mu}^{\nu} x(t)d\varrho(t)} \int_{\mu}^{\nu} x(t)d\varrho(t)\right)^{\frac{1}{s}}.
$$
Since \( \phi(t) = t^{\frac{1}{r}} \), \( t > 0 \) is strictly increasing the above inequalities can be written in equivalent form as
\[
\left( m^r + M^r - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x^r(t) d\varrho(t) \right)^\frac{1}{r} \\
\leq \left( m^s + M^s - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x^s(t) d\varrho(t) \right)^\frac{1}{s} \\
\leq \left( m + M - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x(t) d\varrho(t) \right)^\frac{1}{r}.
\]
Using Generalized Arithmetic Mean and Generalized Power Mean we get
\[
M^{\mu r}(x, \varrho) \leq M^{\mu s}(x, \varrho) \leq A(x, \varrho).
\]
(iv) Let \( 1 \leq r < s \), \( \zeta : [m, M] \rightarrow [m_1, M_1] \), \( \eta : [m_1, M_1] \rightarrow \mathbb{R} \) and \( \eta \circ \zeta : [m, M] \rightarrow \mathbb{R} \) we have,
\[
\zeta(x) = x^r, \quad \eta(x) = x^s, \quad \eta \circ \zeta(x) = x^s.
\]
Then \( \zeta \) is strictly convex and strictly increasing while \( \eta \) is strictly convex and increasing so (3.2) becomes
\[
\left( m + M - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x(t) d\varrho(t) \right)^s \\
\leq \left( m^r + M^r - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x^r(t) d\varrho(t) \right)^\frac{s}{r} \\
\leq \left( m^s + M^s - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x^s(t) d\varrho(t) \right).\]
Since \( \phi(t) = t^{\frac{1}{r}} \), \( t > 0 \) is strictly increasing the above inequalities can be written in equivalent form as
\[
\left( m + M - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x(t) d\varrho(t) \right) \\
\leq \left( m^r + M^r - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x^r(t) d\varrho(t) \right)^\frac{1}{r} \\
\leq \left( m^s + M^s - \frac{1}{\int_\mu d\varrho(t)} \int_\mu x^s(t) d\varrho(t) \right)^\frac{1}{s}.
\]
Using Generalized Arithmetic Mean and Generalized Power Mean we get
\[
A(x, \varrho) \leq M^{\mu r}(x, \varrho) \leq M^{\mu s}(x, \varrho).
\]

\[\square\]

**Corollary 5.4.** Under the assumptions of Theorem 3.3 with \( [m, M] \subseteq (0, \infty) \), \( \zeta : [m, M] \rightarrow [m_1, M_1] \), \( \eta : [m_1, M_1] \rightarrow \mathbb{R} \) and \( \eta \circ \zeta : [m, M] \rightarrow \mathbb{R} \) we have,
\[
H(x, \varrho) \leq G(x, \varrho) \leq A(x, \varrho). \tag{5.3}
\]

**Proof.** Choosing \( r = -1 \) and \( s = 1 \) in the proof of (ii) case of Theorem 5.3 we get inequalities (5.3). \( \square \)
Remark 5.5. Inequalities (5.3) are the generalization of classical Arithmetic, Geometric and Harmonic Mean inequalities.

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