On the local Kreiss resolvent condition

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Abstract

In this paper, we introduce a local version of the Kreiss resolvent condition for Banach space operators and we relate it to the local growth of powers. Also, we introduce a local Yosida approximation and we establish some local results.

Subject Class: 46XX, 47AXX, 47A10 and 47A11.

Keywords: Banach space, Kreiss resolvent condition, local resolvent, power boundedness, local Yosida approximation.

1 Introduction and Preliminaries

Let \( X \) be a complex Banach space and let \( \| \cdot \| \) be the operator norm induced by the vector norm in \( X \), and let \( B(\mathcal{X}) \) be the algebra of bounded linear operators on \( X \). We denote the spectrum of \( T \in B(\mathcal{X}) \) by \( \sigma(T) \), the identity operator on \( X \) by \( I \), and the resolvent of \( T \) by \( R(T,\lambda) = (\lambda I - T)^{-1}, \lambda \notin \sigma(T) \).

Let us recall (see, e.g., [10, 18]) that an operator \( T \) with spectrum in the unit disc is said to satisfy the strong Kreiss resolvent condition with constant \( M \geq 1 \) if

\[
\|R^n(T,\lambda)\| \leq \frac{M}{(|\lambda| - 1)^n} \quad \text{for all } |\lambda| > 1, \text{ and } n = 1, 2, \ldots \quad [SR]
\]

In the case, only \( n = 1 \) in [SR] above, \( T \) is said to satisfy the classical Kreiss resolvent condition [CR], which was observed in [11, 12, 18].

An operator \( T \in B(\mathcal{X}) \) is called power bounded, if there exists a constant \( M \geq 1 \) such that

\[
\|T^n\| \leq M, \quad \text{for all } n \in \mathbb{N}. \quad (1)
\]

Given a bounded linear operator \( T \) we define its Yosida approximation \( Y(\lambda, T) \) for \( \lambda \notin \sigma(T) \) by

\[
Y(\lambda, T) = \lambda TR(\lambda, T). \quad (2)
\]

In [15], Kreiss proved that for matrices, the power boundedness is equivalent to the resolvent condition [CR]. This equivalence works in the finite-dimensional case and in the case of algebraic operators, see
Assume that Proposition 1

**Analyticity of \( \hat{T} \)**

Assume that Lemma 1

**Write simply \( f \)**

The local spectrum \( \sigma_T(x) \) of \( T \) at \( x \in \mathcal{X} \) is defined as the set of all complex \( \lambda \in \mathbb{C} \) for which there exists an analytic \( \mathcal{X} \)-valued function \( w \) on some open neighborhood \( U \) of \( \lambda \) such that

\[
(\mu I - T)w(\mu) = x \quad \text{for all } \mu \in U.
\]

The local spectrum \( \sigma_T(x) \) of \( T \) at \( x \) is the complement in \( \mathbb{C} \) of \( \rho_T(x) \). It is well known that the resolvent mapping is unbounded. On the other hand, as observed in [13], the behavior of local resolvent functions may be quite different.

An operator \( T \in B(\mathcal{X}) \) is said to have the single-valued extension property (hereafter referred to as SVEP) if, for every open set \( U \subseteq \mathbb{C} \), the only analytic solution \( w : U \to \mathcal{X} \) of the equation

\[
(\lambda I - T)w(\lambda) = 0 \quad (\lambda \in U),
\]

is the constant function \( w \equiv 0 \).

If \( T \) has SVEP, then, for every \( x \in \mathcal{X} \), there exists a unique analytic function \( \hat{x}_T(\cdot) : \rho_T(x) \to \mathcal{X} \) such that

\[
(\lambda I - T)\hat{x}_T(\lambda) = x \quad \text{for all } \lambda \in \rho_T(x).
\]

The function \( \hat{x}_T(\cdot) \) is called the local resolvent function of \( T \) at \( x \) and satisfies

\[
\hat{x}_T(\lambda) = (\lambda I - T)^{-1}x \quad \text{for all } \lambda \in \rho(T).
\]

For \( T \in B(\mathcal{X}) \), the local spectral radius of \( T \) at \( x \) is defined by \( r_T(x) := \limsup_{n \to \infty} \|T^n x\|^{1/n} \).

In the following, we use the local functional calculus developed in [2, 3, 23] which extends, in several directions, the holomorphic functional calculus developed by D. Dunford and A. E. Taylor in [9, 22].

Let \( T \in B(\mathcal{X}) \) have the SVEP and let \( x \in \mathcal{X} \) such that \( \sigma_T(x) \subset K \), where \( K \) is a compact subset of \( \mathbb{C} \). For every holomorphic function \( f \) on a neighborhood of \( K \), the vector \( f[T]x \) is defined, in [4] (see also [2]), by

\[
f[T]x := \frac{1}{2\pi i} \int_{\Gamma} f(\mu)\hat{x}_T(\mu)d\mu.
\]

For every \( \lambda \in \mathbb{C} \), we denote by \( f_\lambda^n \) is the function given by \( f_\lambda^n(\mu) = (\lambda - \mu)^{-n} \), \( n = 1, 2, \ldots \). If \( n = 1 \), we write simply \( f_\lambda \) for \( f_\lambda^1 \).

**Lemma 1** [3] Assume that \( T \in B(\mathcal{X}) \) has the SVEP and let \( x \in \mathcal{X} \). If \( \lambda \in \rho_T(x) \), then \( \hat{x}_T(\lambda) = f_\lambda[T]x \).

Analyticity of \( \hat{x}_T(\cdot) \), Cauchy’s differentiation formula and the definitions yield the following.

**Proposition 1** [2] Assume that \( T \in B(\mathcal{X}) \) has the SVEP and let \( x \in \mathcal{X} \). For \( \lambda \in \rho_T(x) \), we have

\[
\frac{d^n \hat{x}_T(\lambda)}{d\lambda^n} = (-1)^n n! f_\lambda^{n+1}[T]x
\]
2 Main results

In this section, we will give local versions of some definitions and we will establish some results relating these notions.

**Definition 1** Let $T \in B(\mathcal{X})$ and $x \in \mathcal{X}$ such that $r_T(x) \leq 1$. We say that $T$ satisfies the local Kreiss resolvent condition at $x$ if there exists an analytic function $x_T(\cdot) : \mathbb{C} \setminus \sigma_T(x) \to \mathcal{X}$ such that $(\lambda I - T)x_T(\lambda) = x$ and
\[
\|x_T(\lambda)\| \leq \frac{M_x}{|\lambda| - 1} \quad \text{for all } |\lambda| > 1, \quad [LCR]
\]
for some constant $M_x \geq 0$.

**Example 1** Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Banach spaces. Let $T \in B(\mathcal{X}_1)$ be an operator satisfying the $[CR]$ condition and $S = 2I \in B(\mathcal{X}_2)$. Hence, the operator $L = T \oplus S \in B(\mathcal{X}_1 \oplus \mathcal{X}_2)$ does not satisfy the $[CR]$ condition. Indeed, $\sigma(L) = \sigma(T) \cup \sigma(S) \not\subset \mathbb{D}(0,1)$. But, for $x \in \mathcal{X}_1$, we set $f(\mu) = (L|x_1 - \mu I)^{-1} x$ for all $\mu \in \mathbb{C} \setminus \mathbb{D}$. Then, we get an analytic function satisfying $(L - \mu I)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \mathbb{D}$. Moreover, there exists $M > 0$ such that
\[
\|f(\mu)\| = \| (L|x_1 - \mu I)^{-1} x \| = \| (T - \mu I)^{-1} x \| \leq \frac{M}{|\mu| - 1}, \quad \text{for all } \mu \in \mathbb{C} \setminus \mathbb{D}.
\]

**Example 2** Let $\alpha = (\alpha_k)_{k \in \mathbb{N}}$ be a sequence in $\ell^\infty(\mathbb{N})$ such that $\alpha_k \in \mathbb{D}$ for all $k \neq 0$ and $|\alpha_0| > 1$. Define the operator $T_\alpha$ on $\ell^2(\mathbb{N})$ by
\[
T_\alpha : \mathcal{X} \rightarrow \mathcal{X}, \quad (x_k)_{k \in \mathbb{N}} \mapsto (\alpha_k x_0, \alpha_1 x_1, \alpha_2 x_2, \ldots).
\]

Since $\alpha_0 \in \sigma(T_\alpha)$, $T_\alpha$ does not satisfy the $[CR]$ condition. On the other hand, we choose $(e_k)_{k \in \mathbb{N}}$ such that $e_k$ be the element whose $k$-th entry is 1, while all others vanish. For $e_k$, $k \neq 0$, we have $T_\alpha e_k = \alpha_k e_k$, hence, by [16, Proposition 1.3.2(e)], $\sigma_{T_\alpha}(e_k) = \{\alpha_k\} \subset \mathbb{D}$. We set
\[
f(\mu) = \sum_{j=0}^{\infty} \frac{\alpha^j_k}{|\mu|^{j+1}} e_k \quad \text{for all } |\mu| > 1,
\]
thus $(\mu I - T_\alpha) f(\mu) = e_k$. Moreover,
\[
\|f(\mu)\| = \left\| \sum_{j=0}^{\infty} \frac{\alpha^j_k}{|\mu|^{j+1}} e_k \right\| \leq \sum_{j=0}^{\infty} \frac{|\alpha_k|^j}{|\mu|^{j+1}} = \frac{1}{|\mu| - |\alpha_k|}.
\]

Since $k \neq 0$, $\alpha_k \in \mathbb{D}$,
\[
\|f(\mu)\| \leq \frac{1}{|\mu| - 1} \quad \text{for all } \mu \in \mathbb{C} \setminus \mathbb{D}.
\]

Therefore, $T_\alpha$ satisfies the $[LCR]$ condition at each $e_k$ with $k \neq 0$.

The local power boundedness for an operator $T \in B(\mathcal{X})$ has been studied in many works, see e.g. [4, 5, 8].

**Definition 2** Let $T \in B(\mathcal{X})$ and $x \in \mathcal{X}$. $T$ is said to be locally power-bounded at $x$ if there exists a constant $M > 0$ such that
\[
\|T^n x\| \leq M \quad \text{for each } n \in \mathbb{N}.
\]
We obtain some local results using similar ideas to those of global results.

**Proposition 2** Let $T \in B(X)$ and let $x \in X$. Then

1. If $T$ have the SVEP and $r_T(x) < 1$, then $T$ is locally power-bounded at $x$.
2. If $T$ is locally power-bounded at $x$, then $r_T(x) \leq 1$.

**Proof.**

1. If $r_T(x) < 1$, then there exists $\delta > 0$ such that $r_T(x) < \delta < 1$, thus, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $\|T^n x\| \leq \delta^n < 1$. Therefore, $\{T^n x\}_n$ is bounded and $T$ is locally power-bounded at $x$.

2. Assume that there exists $M > 0$ such that $\|T^n x\| \leq M$ for every $n \in \mathbb{N}$. Then

$$\limsup_{n \to \infty} \|T^n x\|^{\frac{1}{n}} \leq \limsup_{n \to \infty} M^{\frac{1}{n}} = 1.$$ 

Therefore, from inequality $r_T(x) \leq \limsup_{n \to \infty} \|T^n x\|^{\frac{1}{n}}$ [8], we get $r_T(x) \leq 1$.

**Example 3** Consider Assani’s [1] matrix $T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$: it has $r(T) = 1$, $\|T^n\| = O(n)$, and $T$ is locally power bounded at $(1, 0)$, but not at $(0, 1)$ by direct computation of $T^n$. $T$ has SVEP since $X$ is finite-dimensional – see first paragraph on page 15 of [16]. By Proposition 2, $r_T((0, 1)) = 1$. As noted on page 2, for $\lambda \in \rho(T)$, in particular when $|\lambda| > 1$, $\hat{x}(\lambda) = R(\lambda, T)x$. By Theorem 1, $\|R(\lambda, T)(1, 0)\| \leq M_1/(|\lambda| - 1)$ for $|\lambda| > 1$. In view of Kreiss’s theorem, $T$ does not satisfy the local Kreiss resolvent condition at $(0, 1)$. This shows that linear growth of $\|T^n x\|$ does not imply [LCR], so the converse of Theorem 4 fails.

In [15], the so-called Kreiss matrix theorem relates, proportionally to the dimension of the space, the bounds for the powers to the bounds for the resolvent. Thus, this estimate has as such no generalization to the operators. However, in the literature, the estimate of the powers of operators under various resolvent conditions has been largely studied [10, 11, 12, 18, 19]. In the following Theorems, we give a local version of [18, Theorem 2.7.5].

**Theorem 1** Let $T \in B(X)$. Assume that $T$ has the SVEP and let $x \in X$. If there exists $M > 0$ such that $\|T^n x\| \leq M$ for every $n \in \mathbb{N}$, then

$$\|\hat{x}_T(\lambda)\| \leq \frac{M}{|\lambda| - 1} \text{ for all } |\lambda| > 1.$$ 

**Proof.** Assume that $\|T^n x\| \leq M$ for every $n \in \mathbb{N}$. Then, by Proposition 2, $r_T(x) \leq 1$. Thus, the series

$$f_\lambda(x) := \sum_{k=0}^{\infty} \lambda^{-k}T^k x, \quad \lambda \in \mathbb{C} \setminus \bar{D},$$

converges locally uniformly, so it defines an $X$-valued function on the set $\mathbb{C} \setminus \bar{D}$. Evidently,

$$(\lambda I - T)f_\lambda(x) = x \text{ for all } \lambda \in \mathbb{C} \setminus \bar{D}.$$ 

Since $T$ has the SVEP, $\hat{x}_T(\lambda) = f_\lambda(\lambda)$; hence

$$\|\hat{x}_T(\lambda)\| = \|\sum_{k=0}^{\infty} \lambda^{-k}T^k x\| \leq \frac{M}{|\lambda| - 1} \text{ for all } |\lambda| > 1.$$ 


Remark 1 As application of Theorem 1, we can consider the operator $T_\alpha$ defined in Example 2.

Suppose that $T \in B(\mathcal{X})$, and $T$ has the SVEP. In the following Theorem, we show that the local power boundedness of $T$ implies a local version of [SR].

**Theorem 2** Let $T \in B(\mathcal{X})$ have the SVEP and let $x \in \mathcal{X}$. If there exists $M > 0$ such that

$$\|T^n x\| \leq M \text{ for each } n \in \mathbb{N},$$

then

$$\|f_\lambda^n[T]x\| \leq \frac{M}{(|\lambda| - 1)^n} \text{ for all } |\lambda| > 1 \text{ and } n = 1, 2, \ldots.$$  

**Proof.** Assume that $\|T^n x\| \leq M$ for every $n \in \mathbb{N}$. Then, by Proposition 2, $r_T(x) \leq 1$. As in the proof of Theorem 1, for $|\lambda| > 1$, we have

$$\hat{x}_T(\lambda) = \sum_{k=0}^{\infty} \lambda^{-k-1}T^k x.$$  

Thus,

$$\frac{d^{n-1}\hat{x}_T(\lambda)}{d\lambda^{n-1}} = (-1)^{n-1} (n-1)! \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{T^k x}{\lambda^{n+k}}.$$  

(4)

Hence, by formulas (3) and (4), we deduce

$$\|f_\lambda^n[T]x\| = \frac{1}{(n-1)!} \left\| \frac{d^{n-1}\hat{x}_T(\lambda)}{d\lambda^{n-1}} \right\| \leq \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\|T^k x\|}{|\lambda|^{n+k}},$$

for any $n = 1, 2, \ldots$. Therefore, by hypothesis and by using the generalized binomial formula, we check that

$$\|f_\lambda^n[T]x\| \leq \frac{M}{|\lambda|} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1}{|\lambda|^k} \frac{M}{|\lambda|^n (1 - \frac{1}{|\lambda|})^n} = \frac{M}{|\lambda| - 1)^n},$$

for any $n = 1, 2, \ldots$, and all $|\lambda| > 1$. 

Let $\mathbb{C}^{s \times s}$ be the set of all $s \times s$ complex matrix, and $\mathbb{C}^s$ the set of column complex vectors. In the following Theorem, we show that in finite-dimensional spaces, the local Kreiss resolvent condition at $x$ implies local power-boundedness at $x$.

**Theorem 3** Let $s \geq 1$ and let $T \in \mathbb{C}^{s \times s}$ be a $s \times s$ complex matrix satisfying [LCR] at some column vector $x = (x_k)_{1 \leq k \leq s} \in \mathbb{C}^s$. Then, there exists $m \in \mathbb{N}^*$ such that

$$\|T^n x\| \leq Mms \sqrt{e}, \text{ for each } n \in \mathbb{N}.$$  

**Proof.** A corollary to the Hahn-Banach theorem (see, e.g., [20, Chapter 3] and [14, Chapter 5]) states that, for each normed vector space $\mathcal{X}$ and vector $y \in \mathcal{X}$, there exists a linear transformation $F : \mathcal{X} \rightarrow \mathbb{C}$ with

$$F(y) = \|y\|, \text{ and } |F(x)| \leq \|x\|, \text{ for all } x \in \mathcal{X}.$$
Applying this result with $\mathcal{X} = \mathbb{C}^s$, $y = T^n x$ we see that there exists a linear $F : \mathbb{C}^s \to \mathbb{C}$ such that
\[
F(T^n x) = \|T^n x\|, \text{ and } |F(X)| \leq \|X\|, \text{ for all } X \in \mathbb{C}^s.
\] (5)
There exists $m \in \mathbb{N}^*$ such that the integration path $\Gamma = \{ \lambda \in \mathbb{C} : |\lambda| = 1 + \frac{1}{mn} \} \subset \rho(T)$. In particular, if all eigenvalues of $T$ are lie in $\mathbb{D}$, we take $m = 1$. Then, by using the local functional calculus, we get
\[
T^n x = \frac{1}{2\pi i} \int_{\Gamma} \lambda^n \hat{x}_T(\lambda) d\lambda.
\]
Then
\[
\|T^n x\| = \frac{1}{2\pi i} \int_{\Gamma} \lambda^n F(\hat{x}_T(\lambda)) d\lambda.
\]
Set $q(\lambda) = F(\hat{x}_T(\lambda)) = F((\lambda I - T)^{-1} x)$. Integration by parts gives
\[
\|T^n x\| \leq \frac{1}{2\pi(n+1)} \left(1 + \frac{1}{mn}\right)^{n+1} \int_{\Gamma} |q'(\lambda)| d\lambda.
\] (6)
Let $E_j \in \mathbb{C}^s$ with jth row equal to 1, and all other entries 0, and $\hat{x}_T(\lambda) = \sum_{j=1}^s (\sum_{k=1}^s r_{jk}(\lambda) x_k) E_j$, where $r_{jk}(\lambda)$ are the entries of the matrix $(\lambda I - T)^{-1}$. Therefore,
\[
q(\lambda) = \sum_{j=1}^s \left(\sum_{k=1}^s r_{jk}(\lambda) x_k\right) F(E_j).
\]
Using Cramer’s rule, the entries $r_{jk}(\lambda)$ are rational functions of order $s$ (i.e. its numerator and denominator are polynomials of a degree not exceeding $s$) with the same denominator. By [6, Lemma 4.1],
\[
\int_{\Gamma} |q'(\lambda)| d\lambda \leq 2\pi s \max_{\lambda \in \Gamma} |q(\lambda)|.
\] (7)
The result now easily follows from $(1 + \frac{1}{mn})^{mn} \leq e$ and by combining (5), (6), (7) and $[\text{LCR}]$. 

**Theorem 4** Let $T \in B(\mathcal{X})$ have the SVEP and let $x \in \mathcal{X}$. If there exists $M > 0$ such that
\[
\|\hat{x}_T(\lambda)\| \leq \frac{M}{|\lambda| - 1} \text{ for all } |\lambda| > 1,
\]
then
\[
\|T^n x\| \leq M(n+1), \text{ for } n = 1, 2, \ldots.
\]

**Proof.** For $n \in \mathbb{N}^*$ fixed, by choosing the integration path $\Gamma = \{ \lambda \in \mathbb{C} : |\lambda| = 1 + \frac{1}{n} \}$ and using the local functional calculus, we get
\[
T^n x = \frac{1}{2\pi i} \int_{\Gamma} \lambda^n \hat{x}_T(\lambda) d\lambda.
\] (8)
Thus,
\[
\|T^n x\| \leq \frac{1}{2\pi} \int_{\Gamma} |\lambda|^n \|\hat{x}_T(\lambda)\| d\lambda
\leq \frac{1}{2\pi} \max_{\lambda \in \Gamma} |\lambda|^n \max_{\lambda \in \Gamma} \frac{M}{|\lambda| - 1} \int_{\Gamma} d\lambda
\leq \frac{1}{2\pi}(1 + \frac{1}{n})^n Mn.2\pi.(1 + \frac{1}{n})
\leq M(n+1).
Choosing \( \Gamma = \) then of the local resolvent, a local version of Yosida approximation. 

Let \( k \) be a natural number. Then, by using Proposition 1, we obtain

\[
\| T^n x \| \leq M \frac{n!e^n}{n^n} \leq M \sqrt{2\pi(n+1)} \quad \text{for every } n \in \mathbb{N}.
\]

**Proof.** Integrating (8) by parts \( k \) times, we get

\[
T^n x = (-1)^k \frac{1}{2\pi i} \int_\Gamma \frac{\lambda^{n+k-1}}{(n+1)(n+2)\ldots(n+k-1)} \frac{d^{k-1}f_T(\lambda)}{d\lambda^{k-1}} d\lambda.
\]

Then, by using Proposition 1, we obtain

\[
T^n x = \frac{1}{2\pi i} \int_\Gamma \frac{(k-1)!\lambda^{n+k-1}}{(n+1)(n+2)\ldots(n+k-1)} f^k_T[T] x d\lambda.
\]

Choosing \( \Gamma = \{ \lambda \in \mathbb{C} : |\lambda| = 1 + \frac{k}{n} \} \). Thus, using the hypothesis, we get

\[
\| T^n x \| \leq M \times \frac{(k-1)!{(1+\frac{k}{n})}^{n+k-1}{(\frac{n}{k})}^k{(1+\frac{k}{n})}}{(n+1)(n+2)\ldots(n+k-1)}
\]

\[
\leq M \times \frac{(k-1)!n!(1+\frac{k}{n})^n(1+\frac{n}{k})^k}{n!(n+1)(n+2)\ldots(n+k-1)}
\]

\[
\leq M \frac{1+\frac{k}{n}}{n^n} \times \frac{n!(k-1)!}{(k-1)!^k(n+k-1)!}
\]

\[
\leq M \frac{n!}{n^n} \times \frac{(k-1)!}{(k-1)!(n+k-1)!}
\]

Now, letting \( k \to \infty \), it follows by using Stirling’s approximation

\[
\| T^n x \| \leq M \frac{n!e^n}{n^n} \leq M \sqrt{2\pi(n+1)}, \quad \text{for every } n \in \mathbb{N}.
\]

Suppose that \( T \in B(X) \) has the SVEP and let \( x \in X \). In the following definition, we give, with the help of the local resolvent, a local version of Yosida approximation.

**Theorem 5** Let \( T \in B(X) \) have the SVEP and let \( x \in X \). If there exists \( M > 0 \) such that

\[
\| f^n_T[T]x \| \leq \frac{M}{(|\lambda| - 1)^n} \quad \text{for all } |\lambda| > 1, \ n = 1, 2, \ldots,
\]

then

Therefore, the result is verified.

**Corollary 1** Let \( T \) have SVEP and assume that the local Kreiss resolvent condition \([LCR]\) is satisfied at every \( x \in X \). Then \( \| T^n \| = O(n) \), \( r(T) \leq 1 \), and \( T \) satisfies the Kreiss resolvent condition \([CR]\).

**Proof.** By Theorem 3, \( \| (n+1)^{-1}T^n x \| \leq M e \) for every \( x \). Apply now the Banach-Steinhaus theorem to obtain \( \| T^n \| = O(n) \). This implies that \( r(T) \leq 1 \). Then for \( |\lambda| > 1 \) we have \( \hat{x}(\lambda) = R(\lambda, T)x \) for every \( x \) (see page 2), and \([LCR]\) yields \( \| R(\lambda, T)x \| \leq M / (|\lambda| - 1) \). Now apply the Banach-Steinhaus theorem to the family \( \{ (|\lambda| - 1)R(\lambda, T) : |\lambda| > 1 \} \).
**Definition 3** Assume that $T \in B(\mathcal{X})$ has the SVEP and let $x \in \mathcal{X}$. The local Yosida approximation $Y(\lambda, T, x)$ of $T$ at $x$, for $\lambda \in \rho_T(x)$, is defined by

$$Y(\lambda, T, x) = \lambda T \hat{x}_T(\lambda).$$

The following Theorem shows how an operator is related locally to its local Yosida approximation.

**Theorem 6** Assume that $T \in B(\mathcal{X})$ has the SVEP and let $x \in \mathcal{X}$. The local Yosida approximation $Y(\lambda, T, x)$ of $T$ at $x$ is, as a function of $\lambda$, analytic in $\rho_T(x)$ and the series representation

$$Y(\lambda, T, x) = \sum_{k=0}^{\infty} \frac{T^{k+1}}{\lambda^k} x$$

converges for $|\lambda| > r_T(x)$. Moreover,

1. $Y(\lambda, T, x) = \lambda^2 \hat{x}_T(\lambda) - \lambda x$;
2. If $r_T(x) < \|T\|$, then

$$\|Y(\lambda, T, x) - Tx\| \leq \frac{\|T^2 x\|}{|\lambda| - \|T\|}$$

for $|\lambda| > \|T\|$;
3. $Y(\lambda, T, x)$ approximates $Tx$ when $|\lambda| \to \infty$;
4. For the local resolvent we have

$$\rho_{Y(\lambda, T)}(x) = \left\{ \frac{z}{1 - \frac{T}{\lambda}}, z \in \rho_T(x) \right\}.$$

**Proof.**

1. Let $T \in B(\mathcal{X})$ and let $x \in \mathcal{X}$. If $|\lambda| > r_T(x)$, then, by [17, Proposition 1], the series $f(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^{-k}}{\lambda^k} T^k x$ is unconditionally convergent, so it defines an analytic $\mathcal{X}$-valued function on the set $\{ \lambda \in \mathbb{C} : |\lambda| > r_T(x) \}$. On the other hand, evidently we have

$$(\lambda I - T)f(\lambda) = x$$

for all $|\lambda| > r_T(x)$.

Since $T$ has the SVEP, thus $\hat{x}_T(\lambda) = \sum_{k=0}^{\infty} \lambda^{-k} T^k x$ for all $|\lambda| > r_T(x)$.

By evaluating $Y(\lambda, T, x)$ in terms of the local resolvent $\hat{x}_T(\lambda)$, for $|\lambda| > r_T(x)$, we have

$$Y(\lambda, T, x) = \lambda T \hat{x}_T(\lambda) = \lambda T \sum_{k=0}^{\infty} \frac{T^k x}{\lambda^k} = \sum_{k=0}^{\infty} \frac{T^{k+1}}{\lambda^k} x.$$

Therefore, the assertion of the theorem is true. Moreover,

$$Y(\lambda, T, x) = \sum_{k=0}^{\infty} \frac{T^{k+1}}{\lambda^k} x = \lambda x + T x + \frac{T^2}{\lambda} x + \ldots + \frac{T^{k+1}}{\lambda^k} x + \ldots - \lambda x$$

$$= \lambda^2 \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} x - \lambda x = \lambda^2 \hat{x}_T(\lambda) - \lambda x.$$
2. One can show that, from

\[ Y(\lambda, T, x) = \sum_{k=0}^{\infty} \frac{T^{k+1}}{\lambda^k} x, \]

we have

\[ Y(\lambda, T, x) - Tx = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^k} \frac{T^2 x}{\lambda} \]

for all |\lambda| > r_T(x). Hence,

\[ \|Y(\lambda, T, x) - Tx\| \leq \sum_{k=0}^{\infty} \left\| \frac{T^k}{\lambda^k} \right\| \frac{T^2 x}{\lambda} \leq \|T^2 x\| \sum_{k=0}^{\infty} \left\| \frac{T^k}{\lambda^k} \right\|. \]

Therefore,

\[ \|Y(\lambda, T, x) - Tx\| \leq \frac{\|T^2 x\|}{|\lambda| - \|T\|} \]

for \( \lambda > \|T\| \).

3. A direct consequence of (2).

4. \( \mu \in \rho_{Y(\lambda, T)}(x) \) if and only if there exists an open neighborhood \( U_\mu \) of \( \mu \) in \( \mathbb{C} \) and an analytic function \( f : U_\mu \to \mathcal{X} \) such that the equation \((\xi I - Y(\lambda, T))f(\xi) = x\) holds for all \( \xi \in U_\mu \). But,

\[ (\xi I - Y(\lambda, T))f(\xi) = x \iff (\xi I - \lambda T(\lambda I - T)^{-1})f(\xi) = x \iff (\xi + \lambda) \left( \frac{\lambda}{\xi + \lambda} I - T \right) (\lambda I - T)^{-1} f(\xi) = x. \]

Since \((\xi + \lambda)(\lambda I - T)^{-1} f(\xi)\) is analytic in \( U_\mu \). Thus, for all \( \xi \in U_\mu \), there exists \( z_\xi \in \rho_T(x) \) such that \( z_\xi = \frac{\lambda}{\xi + \lambda} \). Hence \( \xi = \frac{z_\xi}{1 - z_\xi} \). Therefore, in particular, for \( \mu \), we have the desired result.

**Remark 2** Note that the problem concerning the equivalence between the local strong Kreiss resolvent condition for an operator \( T \) at \( x \) and \( \|e^{zT}x\| \leq C_x e^{\|z\|} \) for every complex \( z \) remains open.

**Acknowledgments.** The authors would like to express their sincere gratitude to the referee for suggesting to study the problem in finite-dimensional spaces (Theorem 3), for his very helpful suggestions (Example 3 and Corollary 1) and many kind comments.

**References**


