ON A REMARK OF SIERPİŃSKI

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ABSTRACT. In a remark on page 80 of his classical book 250 Problems in Elementary Number Theory, Sierpinski stated that it was not known if the equation \( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4 \) has solutions in positive integers. Bondarenko [Investigation of one class of Diophantine equations, Ukrainian Math. J. 52 (6) (2000), 953-959] gave a negative answer to Sierpinski’s remark by showing that the equation \( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4k^2 \) does not have solutions in positive integers if \( 3 \nmid k \). However, Garaev [Diophantine equations of third degree, Proc. Steklov Inst. Math. 218 (1997), 94-103] had already proved that the equation \( x^3 + y^3 + z^3 = nxyz \) has no positive integer solutions if \( n = 4k, n = 8k - 1, \) or \( n = 2^{2m+1}(2k - 1) + 3 \), where \( m, k \in \mathbb{Z}^+ \), which Bondarenko’s result is a consequence of. In this paper, we shall partially extend Garaev’s result by showing that the equation \( \frac{x}{y} + \frac{y}{z} + m \cdot \frac{z}{x} = mn \) does not have solutions in positive integers if \( m \) is odd and \( 4|n \) or \( 8|n + 1 \). Our method is different from Garaev’s method and has been successfully applied to several situations.

1. Introduction

Let \( n \) be a positive integer. Integer solutions to the equation

\[
\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = n
\]

(1.1)

has been studied by several authors. The earliest results were stated without proof by Sylvester [13] in 1847. Cassels [4], Sansone and Cassels [9] showed that (1.1) does not have integer solutions if \( n = 1 \). Dofs [6] studied the equation \( x^3 + y^3 + z^3 = nxyz \), where several solutions of (1.1) when \(-81 \leq n \leq 0\) could be deduced from his Table 4. Bremner and Guy [2] extensively studied (1.1) when \( n \) is in the range \( |n| \leq 200 \). The question becomes more interesting if we ask for positive integer solutions to (1.1). Sierpinski remarked on page 80 of his book 250 Problems in Elementary Number Theory [11] that it was not known if equation (1.1) has positive integers when \( n = 4 \). Bondarenko [1] gave a negative answer to this question by showing that (1.1) does not have positive integer solutions if \( n = 4k^2 \), where \( k \in \mathbb{Z}^+ \) and \( 3 \nmid k \). However, it had been known from Garaev’s work [8, Theorem 1] that equation \( x^3 + y^3 + z^3 = nxyz \) does not have positive solutions if \( n \) is one of following forms: \( n = 4k, n = 8k - 1, \) or \( n = 2^{2m+1}(2k - 1) + 3 \) where \( k, m \in \mathbb{Z}^+ \). A consequence of this result and the Sylvester transformation ([8, Lemma 1]), by taking \( a = \alpha = \beta = 1 \) and \( A = x/y, B = y/z, \) and \( C = z/x \), we see that (1.1) has no positive integer solutions if \( n \) is \( n = 4k, \) or \( n = 8k - 1, \) or \( n = 2^{2m+1}(2k - 1) + 3 \) where \( k, m \in \mathbb{Z}^+ \). Garaev’s result could be reformulated in the following form.

Theorem 1. There do not exist positive rationals \( x, y, z \) such that \( xyz = 1 \) and \( x + y + z = n \), where \( n = 4k, 8k - 1, \) or \( 2^{2m+1}(2k - 1) + 3 \) with \( k, m \in \mathbb{Z}^+ \).

In this paper, we will partially extend Theorem 1 as the following.

2020 Mathematics Subject Classification. 11D68, 11D72, 11D25.

Key words and phrases. Sums of positive rationals, elliptic curves, Hilbert symbols.
Theorem 2. Let \( m \) be an odd positive integer. Then there do not exist positive rationals \( x, y, z \) such that \( xyz = m \) and \( x + y + z = nm \), where \( 4|n \) or \( 8|n + 1 \).

Garaev’s method is classical and uses the quadratic reciprocity law. Our method was based on an idea of Stoll’s [12] and uses Hilbert symbols and elliptic curves. The main idea is the following:

Assume that we want to show that a rational number \( X \) is positive. The trick is to find a rational number \( D < 0 \) such that \((X, D)_p = 1\) for all prime numbers \( p \), where \((X, D)_p\) denotes the Hilbert symbol. Then the product formula for the Hilbert symbol (Serre [10, Theorem 3, p. 23]) forces \((X, D)_\infty = 1\). Since \( D < 0 \), we must have \( X > 0 \). Our experience shows that when \( X \) is the \( x \)-coordinate of a rational point on an elliptic curve of the form

\[
y^2 = f(x),
\]

where \( f \) is a cubic polynomial with rational coefficients, quantity \( D \) usually is a factor of the discriminant of \( f(x) \). This idea can be applied to different problems. For the problem on the presentation of positive integers \( n \) in the form \( n = (x + y + z)/(1/x + 1/y + 1/z) \) or \( n = (x + y + z + w)/(1/x + 1/y + 1/z + 1/w) \), where \( x, y, z, w \in \mathbb{Z}^+ \) see [3, 15]. For the problem on the presentation of positive integers \( n \) in the form \( n = x/y + dy/z + z/w + dw/x \), where \( x, y, z, w, d \in \mathbb{Z}^+ \), see [7, 14].

By replacing \( x, y, z \) by \( x/y, y/z, \) and \( z/x \) respectively in Theorem 2, we get the following theorem.

Theorem 3. Let \( m \) be a positive odd integer. Then the equation

\[
\frac{x}{y} + \frac{y}{z} + m \cdot \frac{z}{x} = nm
\]

does not have solutions in positive integers if \( 4|n \) or \( 8|n + 1 \).

Let \( m = 1 \) and \( n = 4 \) in Theorem 3, we also have the answer to Sierpiński’s remark.

2. Preliminaries

Let \( p \) be a prime number. Let \( \mathbb{Q}_p \) be the \( p \)-adic completion of \( \mathbb{Q} \) with respect to \( p \)-adic topology, and let \( \mathbb{Z}_p \) be the ring of \( p \)-adic integers in \( \mathbb{Q}_p \). Let \( \mathbb{Q}_p^3 = \{(x, y, z) : x, y, z \in \mathbb{Q}_p\} \) and \( \mathbb{Z}_p^2 = \{x^2 : x \in \mathbb{Z}_p\} \). For \( a \in \mathbb{Q}_p^* \), denote \( v_p(a) \) the highest power of \( p \) dividing \( a \). For \( a \) and \( b \) in \( \mathbb{Q}_p^* \), the Hilbert symbol \((a, b)_p\) is defined by

\[
(a, b)_p = \begin{cases} 
1 & \text{if } ax^2 + by^2 = z^2 \text{ has a solution } (x, y, z) \neq (0, 0, 0) \text{ in } \mathbb{Q}_p^3, \\
-1 & \text{otherwise}.
\end{cases}
\]

For \( a, b \in \mathbb{R} \), the symbol \((a, b)_\infty\) is +1 if \( a > 0 \) or \( b > 0 \), and −1 otherwise. The following properties of the Hilbert symbol are true, see Serre [10, pp. 19-26]:

(i) For all \( a, b, c \in \mathbb{Q}_p^* \), then

\[
(a, b^2)_p = 1, \\
(a, bc)_p = (a, b)_p(a, c)_p
\]

(ii) For all \( a, b \in \mathbb{Q}^* \), then

\[
(a, b)_\infty \prod_{p \text{ prime}} (a, b)_p = 1.
\]
(iii) Let \(a, b \in \mathbb{Q}^*\). Write \(a = p^\alpha u, b = p^\beta v\), where \(\alpha = v_p(a)\) and \(\beta = v_p(b)\). Then

\[
(a, b)_p = (-1)^{\alpha \beta (p - 1)/2} \left( \frac{u}{p} \right)^\alpha \left( \frac{v}{p} \right)^\beta \text{ if } p \neq 2,
\]

\[
(a, b)_p = (-1)^{\frac{\alpha (\alpha - 1)}{2} \cdot \frac{\beta (\beta - 1)}{2}} \frac{G}{\beta (\beta - 1)} \frac{G}{\beta (\beta - 1)} \text{ if } p = 2,
\]

where \(\left( \frac{u}{p} \right)\) denotes the Legendre symbol.

3. Proof of Theorem 2

We prove Theorem 2 by contradiction. Assume that there exist positive rationals \(x, y, z\) satisfying \(xyz = m\) and \(x + y + z = nm\). Then \(z = \frac{m}{xy}\). Hence

\[
(x + y) = mn - z = m(n - \frac{1}{xy}).
\]

Therefore

\[
(x - y)^2 = m^2(n - \frac{1}{xy})^2 - 4xy.
\]

Hence

\[
(xy(x - y))^2 = -4(xy)^3 + m^2(nxy - 1)^2.
\]

Let \(Y = xy(x - y)\) and \(X = -xy\). Then

\[
Y^2 = 4X^3 + m^2(1 + nX)^2.
\]

Since \(x, y > 0\), from (3.1) we have \(1 - nxy < 0\). So in (3.2) we have

\[
1 + nX < 0.
\]

But we will show that (3.3) is impossible by means of the following lemmas:

Lemma 1. Let \(p\) be an odd prime. Then in (3.2) we have

\[
-n, 1 + nX\rangle_p = 1.
\]

Proof. Case 1: \(v_p(nX) > 0\). Then \(1 + nX \equiv 1 \pmod{p}\). Hence \(1 + nX \in \mathbb{Z}_p^*\). Therefore \((-n, 1 + nX)_p = 1\).

Case 2: \(v_p(nX) = 0\). Then \(v_p(X) = -v_p(n)\). Let \(n = p^s n_1\) and \(X = p^-s X_1\), where \(s \geq 0\), \(p \nmid n_1\) and \(p \nmid X_1\).

Case 2.1: \(s = 0\). Then \(-n = -n_1\) and \(1 + nX = 1 + n_1 X_1\).

If \(p \nmid 1 + n_1 X_1\), then both \(-n_1\) and \(1 + n_1 X_1\) are units in \(\mathbb{Z}_p\). Hence

\[
(-n, 1 + nX)_p = (-n_1, 1 + n_1 X_1)_p = 1.
\]

If \(p|1 + n_1 X_1\), then taking (3.2) \(\pmod{p}\) gives

\[
Y^2 \equiv 4X_1^3 \pmod{p}.
\]

Hence

\[
X_1 \equiv \left( \frac{Y}{2X_1} \right)^2 \pmod{p}.
\]

Therefore

\[
-n_1 \equiv \frac{1}{X_1} \equiv \left( \frac{2X_1}{Y} \right)^2 \pmod{p}.
\]
Thus \(-n_1 \in \mathbb{Z}_p^2\). Therefore

\[(-n, 1+nX) \equiv (-n_1, 1+n_1X_1) \equiv 1.\]

**Case 2.2:** \(s > 0\). From (3.2) we have

\[Y^2 = \frac{4X_1^3}{p^{3s}} + m^2(1+n_1X_1)^2.\]

Hence

\[p^{3s}Y^2 = 4X_1^3 + p^{3s}m^2(1+n_1X_1)^2.\]

Since \(p \nmid X_1\), we have \(3s + 2v_p(Y) = 0\). Therefore \(2|s\). Taking (3.5) mod \(p\) gives \(X_1 \equiv \square \) (mod \(p\)). Hence \(X_1 \in \mathbb{Z}_p^2\). Let \(X_1 = \alpha^2\), where \(\alpha \in \mathbb{Z}_p\), \(p \nmid n_1\).

If \(p|1+n_1X_1\), then

\[-n_1 \equiv \frac{1}{X_1} \equiv \left(\frac{1}{\alpha}\right)^2 \pmod{p}.\]

Hence \(-n_1 \in \mathbb{Z}_p^2\). Since \(-n = -p^sn_1\) and \(2|s\), we have \(-n \in \mathbb{Z}_p^2\). Therefore

\[(-n, 1+nX) \equiv 1.\]

If \(p \nmid 1+n_1X_1\), then both \(-n_1, 1+n_1X_1\) are units in \(\mathbb{Z}_p\). Since \(2|s\), we have

\[(-n, 1+nX) \equiv (-p^sn_1, 1+n_1X_1) \equiv (-n_1, 1+n_1X_1) \equiv 1.\]

**Case 3:** \(v_p(nX) < 0\). Let \(n = p^sn_1\) and \(X = p^{-t}X_1\), where \(t > s \geq 0\), \(p \nmid n_1\), and \(p \nmid X_1\). From (3.2) we have

\[Y^2 = \frac{4X_1^3}{p^{3s}} + m^2(1 + X_1/p^{t-s})^2.\]

Therefore

\[p^{3s}Y^2 = 4X_1^3 + m^2p^{t+2s}(p^{t-s} + X_1)^2.\]

Since \(p \nmid 4X_1^3\) and \(t + 2s > 0\), from (3.6) we have \(3s + 2v_2(Y) = 0\). Taking (3.6) mod \(p\) gives \(X_1 \equiv \square \) (mod \(p\)). Therefore \(X_1 \in \mathbb{Z}_p^2\).

**Case 3.1:** \(2|s\). Since \(-n_1\) and \(p^{t-s} + n_1X_1\) are units in \(\mathbb{Z}_p\), we have \((-n_1, p^{t-s} + n_1X_1) \equiv 1\). Therefore

\[(-n, 1+nX) \equiv (-p^sn_1, p^{s-t}(p^{t-s} + n_1X_1)) \equiv 1.\]
Case 3.2: \(2 \nmid s\). Then
\[
\begin{align*}
(-n, 1+nX)_p &= (-p^s n_1, p^{s-t}(p^{1-s}+n_1 X_1))_p \\
&= (-p n_1, p(p^{1-s}+n_1 X_1))_p \\
&= (p, p)_p(p, p^{1-s}+n_1 X_1)_p (-n_1, p)_p(-n_1, p^{1-s}+n_1 X_1)_p \\
&= (-1) \frac{p-1}{2} \left( \frac{p^{1-s}+n_1 X_1}{p} \right) \left( \frac{-n_1}{p} \right) \\
&= (-1) \frac{p-1}{2} \left( \frac{n_1 X_1}{p} \right) \left( \frac{-n_1}{p} \right) \\
&= (-1) \frac{p-1}{2} \left( \frac{n_1}{p} \right) \left( \frac{-n_1}{p} \right) \\
&= (-1) \frac{p-1}{2} \left( \frac{-1}{p} \right)^2 \left( \frac{n_1}{p} \right)^2 \\
&= 1.
\end{align*}
\]

\[\square\]

Lemma 2. If \(4 \mid n\), then in (3.2) we have
\[
(-n, 1+nX)_2 = 1.
\]

Proof. Let \(n = 4k\), where \(k \in \mathbb{Z}^+\). Then (3.2) becomes
\[
Y^2 = 4X^3 + m^2(1+4kX)^2
\]

Case 1: \(v_2(kX) \geq 1\). Then \(1+nX \equiv 1 \pmod{8}\). Hence \(1+nX \in \mathbb{Z}_2^2\). Therefore
\[
(-n, 1+nX)_2 = 1.
\]

Case 2: \(v_2(kX) = 0\). Then \(v_2(X) = -v_2(k)\). Let \(k = 2^s k_1\) and \(X = 2^{-s}X_1\), where \(s \geq 0\), \(2 \nmid k_1\), and \(2 \nmid X_1\).

Case 2.1: \(s = 0\). From (3.8) we have
\[
Y^2 = 4X_1^3 + m^2(1+4k_1X_1)^2,
\]
impossible mod 8 since \(1 \not\equiv 4X_1 + 1 \pmod{8}\).

Case 2.2: \(s = 1\). From (3.8) we have
\[
Y^2 = \frac{X_1^3}{2} + m^2(1+4k_1X_1)^2.
\]
Hence
\[
2Y^2 = X_1^3 + 2m^2(1+4k_1X_1)^2,
\]
impossible mod 2.

Case 2.3: \(s \geq 2\). From (3.8) we have
\[
Y^2 = \frac{X_1^3}{2^{3s-2}} + m^2(1+4k_1X_1)^2.
\]
Hence
\[
2^{3s-2}Y^2 = X_1^3 + 2^{3s-2}m^2(1+4k_1X_1)^2.
\]

(3.9)
Since $s \geq 2$, we have $3s - 2 \geq 4$. Since $2 \nmid X_1$, in (3.9) we must have $3s - 2 + 2\nu_2(Y) = 0$. Hence $2\mid s$. Since $n = 2^{s+2}k_1, 1 + 4nX = 1 + 2^2k_1X_1$, we have

\[(−n, 1 + nX)_2 = (−2^{s+2}k_1, 1 + 4k_1X_1)_2 = (−k_1, 1 + 4k_1X_1)_2 = (−1)^{−k_1−1} \frac{1 + 4k_1X_1−1}{2} = 1.\]

**Case 3:** $\nu_2(kX) < 0$. Let $k = 2^s k_1, X = 2^{-t} X_1$, where $t > s \geq 0, 2 \nmid k_1$, and $2 \nmid X_1$. From (3.8) we have

\[Y^2 = 2^{2−3r}X_1^3 + m^2(1 + 2^{2s−t}k_1X_1)^2.\]

Therefore

\[(3.10) \quad 2^{3r−2}Y^2 = X_1^3 + 2^{3r−2}m^2(1 + 2^{2s−t}k_1X_1)^2.\]

**Case 3.1:** $t = s + 1$. Then from (3.10) we have

\[(3.11) \quad 2^{3r−2}Y^2 = X_1^3 + 2^{3r−2}m^2(1 + 2k_1X_1)^2.\]

Since $3t - 2 > 0$ and $2 \nmid X_1$, from (3.11) we have $3t - 2 + 2\nu_2(Y) = 0$. Hence $2\mid t$. Therefore $3t - 2 \geq 4$.

Taking (3.11) mod 4 gives $1 \equiv X_1 \pmod{4}$. Since $s = t - 1$ and $2 \mid t$, we have $2 \nmid s$. Therefore

\[\begin{align*}
(-n, 1 + nX)_2 &= (−2^{s+2}k_1, 1 + 2k_1X_1)_2 \\
&= (−2k_1, 1 + 2k_1X_1)_2 \\
&= (−1)^{−k_1−1} \frac{(1 + 2k_1X_1−1)}{4} = 1.
\end{align*}\]

**Case 3.2:** $t = s + 2$. From (3.10) we have

\[(3.12) \quad 2^{3r−2}Y^2 = X_1^3 + 2^{3r−2}k^2(1 + k_1X_1)^2.\]

Since $2 \nmid X_1$, we have $3t - 2 + \nu_2(Y) = 0$. Hence $2\mid 3t - 2$. Therefore $2\mid t$. Since $s = t - 2$, we have $2\mid s$. Since $3t - 2 \geq 4$, taking (3.12) mod 8 gives $X_1 \equiv 1 \pmod{8}$. Let $1 + k_1X_1 = 2^r A$, where $r \geq 1$ and $2 \nmid A$.

**Case 3.2.1:** $r \geq 3$. Then $1 + k_1X_1 \equiv 0 \pmod{8}$. Since $X_1 \equiv 1 \pmod{8}$, we have $−k_1 \equiv 1 \pmod{8}$. Hence $−k_1 \in \mathbb{Z}_2^\times$. Therefore

\[\begin{align*}
(-n, 1 + nX)_2 &= (−2^{2s}k_1, 1 + k_1X_1)_2 \\
&= (−k_1, 1 + k_1X_1)_2 \\
&= 1.
\end{align*}\]

**Case 3.2.2:** $r = 2$. Then $1 + k_1X_1 = 4A$, where $2 \nmid A$. Since $X_1 \equiv 1 \pmod{8}$, we have $k_1 \equiv −1 \pmod{4}$.

Therefore

\[\begin{align*}
(-n, 1 + nX)_2 &= (−2^{2s}k_1, 4A)_2 \\
&= (−k_1, A)_2 \\
&= (−1)^{−k_1−1} \frac{(A−1)}{4} = 1.
\end{align*}\]
Case 3.2.3: \( r = 1 \). Then \( 1 + k_1 X_1 = 2A \). Since \( 2 \nmid A \), we have \( 2A \equiv 2 \pmod{4} \). Since \( X_1 \equiv 1 \pmod{4} \), we have \( k_1 \equiv 1 \pmod{4} \). Then

\[
(-n, 1 + nX)_2 = (-2^{2^s} k_1, 2A)_2 = (-k_1, 2A)_2 = (-1) \frac{(-k_1-1)(A-1)}{4} + \frac{k_1^2-1}{8}
\]

If \( k_1 \equiv 1 \pmod{8} \), then \( 2A = 1 + k_1 X_1 \equiv 2 \pmod{8} \). Hence \( A \equiv 1 \pmod{4} \). Therefore \( \frac{(-k_1-1)(A-1)}{4} \equiv 0 \pmod{2} \) and \( \frac{k_1^2-1}{8} \equiv 0 \pmod{2} \). Hence

\[
(-1) \frac{(-k_1-1)(A-1)}{4} + \frac{k_1^2-1}{8} = 1.
\]

If \( k_1 \equiv 5 \pmod{8} \), then \( 2A = 1 + k_1 X_1 \equiv 6 \pmod{8} \). Hence \( A \equiv 3 \pmod{4} \). Therefore \( \frac{(-k_1-1)(A-1)}{4} \equiv 1 \pmod{2} \) and \( \frac{k_1^2-1}{8} \equiv 1 \pmod{2} \). Hence

\[
(-1) \frac{(-k_1-1)(A-1)}{4} + \frac{k_1^2-1}{8} = 1.
\]

So in (3.13) we have

\[
(-n, 1 + nX)_2 = 1.
\]

Case 3.3: \( t > s + 2 \). Then from (3.10) we have

\[
2^{3t-2}y^2 = X_1^3 + 2^{t+s+2} m^2 (2^{t-s-2} + k_1 X_1)^2.
\]

Since \( 2 \nmid X_1 \), from (3.14) we have \( 3t - 2 + 2v_2(Y) = 0 \). Hence \( 2 \mid t \). Since \( t + 2s + 2 \geq 3 \), taking (3.14) mod 8 gives \( X_1 \equiv 1 \pmod{8} \).

Case 3.3.1: \( t = s + 3 \). Since \( 2 \nmid t \), we have \( 2 \nmid s \). Then

\[
(-n, 1 + nX)_2 = (-2^{s+2} k_1, 2^{-1}(2 + k_1 X_1))_2 = (-2k_1, 2^{-1}(2 + k_1 X_1))_2 = (-1) \frac{(-k_1-1)(2+2X_1-1)}{4} + \frac{k_1^2-1}{8} + \frac{(k_1 X_1 + 2)^2 - 1}{8}
\]

If \( k_1 \equiv 1 \pmod{8} \), then \( k_1 X_1 \equiv 1 \pmod{8} \). Hence \( \frac{(-k_1-1)(2+2X_1-1)}{4} \equiv 1 \pmod{2} \), \( \frac{k_1^2-1}{8} \equiv 0 \pmod{2} \), and \( \frac{(k_1 X_1 + 2)^2 - 1}{8} \equiv 1 \pmod{2} \). Therefore

\[
(-1) \frac{(-k_1-1)(2+2X_1-1)}{4} + \frac{k_1^2-1}{8} + \frac{(k_1 X_1 + 2)^2 - 1}{8} = 1.
\]

If \( k_1 \equiv 3 \pmod{8} \), then \( k_1 X_1 \equiv 3 \pmod{8} \). Hence \( \frac{(-k_1-1)(2+2X_1-1)}{4} \equiv 0 \pmod{2} \), \( \frac{k_1^2-1}{8} \equiv 1 \pmod{2} \), and \( \frac{(k_1 X_1 + 2)^2 - 1}{8} \equiv 1 \pmod{2} \). Therefore

\[
(-1) \frac{(-k_1-1)(2+2X_1-1)}{4} + \frac{k_1^2-1}{8} + \frac{(k_1 X_1 + 2)^2 - 1}{8} = 1.
\]
If $k_1 \equiv 5 \pmod{8}$, then $k_1 X_1 \equiv 5 \pmod{8}$. Hence $\frac{(-k_1-1)(2+k_1 X_1-1)}{4} \equiv 1 \pmod{2}$, $\frac{k_1^2-1}{8} \equiv 1 \pmod{2}$, and $\frac{(k_1X_1+2)^2-1}{8} \equiv 0 \pmod{2}$. Therefore

\[ (-1) \frac{(-k_1-1)(2+k_1 X_1-1)}{4} - \frac{k_1^2-1}{8} + \frac{(k_1X_1+2)^2-1}{8} = 1. \]

If $k_1 \equiv 7 \pmod{8}$, then $k_1 X_1 \equiv 7 \pmod{8}$. Hence $\frac{(-k_1-1)(2+k_1 X_1-1)}{4} \equiv 0 \pmod{2}$, $\frac{k_1^2-1}{8} \equiv 0 \pmod{2}$, and $\frac{(k_1X_1+2)^2-1}{8} \equiv 0 \pmod{2}$. Therefore

\[ (-1) \frac{(-k_1-1)(2+k_1 X_1-1)}{4} - \frac{k_1^2-1}{8} + \frac{(k_1X_1+2)^2-1}{8} = 1. \]

So in (3.14), we have

\[ (-n, 1+nX) = 1. \]

**Case 3.3.2: $t = s + 4$.** Since $2|t$, we have $2|s$. Then

\[ (-n, 1+nX) = (-2^{s+2}k_1, 2^{-2}(4+k_1 X_1)) \]

\[ = (-k_1, 4+k_1 X_1) \]

\[ = (-1) \frac{(-k_1-1)(4+k_1 X_1)}{4} \]

\[ = (-1) \frac{-(k_1+1)(3+k_1 X_1)}{4} \]

Since $X_1 \equiv 1 \pmod{8}$, we have $3+k_1 X_1 \equiv k_1 - 1 \pmod{4}$. Therefore

\[ (-1) \frac{-(k_1+1)(3+k_1 X_1)}{4} = (-1) \frac{-(k_1+1)(k_1+1)}{4} = (-1) \frac{k_1^2-1}{4} = 1, \]

since $8|k_1^2 - 1$. So in (3.16), we have $(-n, 1+nX) = 1$.

**Case 3.3.3: $t \geq s + 5$.**

- If $2 \nmid s$, then $2|s + 1 - t$. Therefore

\[ (-n, 1+nX) = (-2k_1, 2^{s+2-t}(2^{s-s-2} + k_1 X_1)) \]

\[ = (-2k_1, 2(2^{s-s-2} + k_1 X_1)) \]

\[ = (-1) \frac{(-k_1-1)(2^{s-s-2}+k_1 X_1-1)}{4} - \frac{k_1^2-1}{8} + \frac{(2^{s-s-2}+k_1 X_1)^2-1}{8} \]

If $k_1 \equiv 1 \pmod{8}$, then $k_1 X_1 + 2^{s-s-2} \equiv 1 \pmod{8}$. Hence $\frac{(-k_1-1)(2^{s-s-2}+k_1 X_1-1)}{4} \equiv 0 \pmod{2}$, $\frac{k_1^2-1}{8} \equiv 0 \pmod{2}$, and $\frac{(2^{s-s-2}+k_1 X_1)^2-1}{8} \equiv 0 \pmod{2}$. Therefore

\[ (-1) \frac{(-k_1-1)(2^{s-s-2}+k_1 X_1-1)}{4} - \frac{k_1^2-1}{8} + \frac{(2^{s-s-2}+k_1 X_1)^2-1}{8} = 1. \]

If $k_1 \equiv 3 \pmod{8}$, then $k_1 X_1 + 2^{s-s-2} \equiv 3 \pmod{8}$. Hence $\frac{(-k_1-1)(2^{s-s-2}+k_1 X_1-1)}{4} \equiv 0 \pmod{2}$, $\frac{k_1^2-1}{8} \equiv 1 \pmod{2}$, and $\frac{(2^{s-s-2}+k_1 X_1)^2-1}{8} \equiv 1 \pmod{2}$. Therefore

\[ (-1) \frac{(-k_1-1)(2^{s-s-2}+k_1 X_1-1)}{4} - \frac{k_1^2-1}{8} + \frac{(2^{s-s-2}+k_1 X_1)^2-1}{8} = 1. \]
If \( k_1 \equiv 5 \) (mod 8), then \( k_1 X_1 + 2^{s-2} \equiv 5 \) (mod 8). Hence \( (\frac{(-k_1-1)(2^{r-s-2}+k_1 X_1)}{4}) \equiv 0 \) (mod 2), \( \frac{k_1^2-1}{8} \equiv 1 \) (mod 2), and \( \frac{(2^{r-s-2}+k_1 X_1)^2-1}{8} \equiv 1 \) (mod 2). Therefore

\[
(-1)^\left(\frac{(-k_1-1)(2^{r-s-2}+k_1 X_1)}{4}\right) \cdot \frac{k_1^2-1}{8} \cdot \frac{(2^{r-s-2}+k_1 X_1)^2-1}{8} = 1.
\]

If \( k_1 \equiv 7 \) (mod 8), then \( k_1 X_1 + 2^{s-2} \equiv 7 \) (mod 8). Hence \( (\frac{(-k_1-1)(2^{r-s-2}+k_1 X_1)}{4}) \equiv 0 \) (mod 2), \( \frac{k_1^2-1}{8} \equiv 0 \) (mod 2), and \( \frac{(2^{r-s-2}+k_1 X_1)^2-1}{8} \equiv 0 \) (mod 2). Therefore

\[
(-1)^\left(\frac{(-k_1-1)(2^{r-s-2}+k_1 X_1)}{4}\right) \cdot \frac{k_1^2-1}{8} \cdot \frac{(2^{r-s-2}+k_1 X_1)^2-1}{8} = 1.
\]

So in (3.17) we have

\[ (-n, 1+nX)_2 = 1. \]

- If \( 2 \mid s \), then \( 2 \mid s + 2 \) and \( 2 \mid s + 2 - t \). Therefore

\[
(-n, 1+nX)_2 = (-2^{s+2}k_1, 2^{s+2-t}(2^{r-s-2}+k_1 X_1))_2
\]

(3.18)

\[
= (-k_1, 2^{r-s-2}+k_1 X_1)_2
\]

\[
= (-1)^\left(\frac{(-k_1-1)(2^{r-s-2}+k_1 X_1)}{4}\right) \cdot \frac{k_1^2-1}{8} \cdot \frac{(2^{r-s-2}+k_1 X_1)^2-1}{8}.
\]

Since \( 2 \nmid k_1 \), \( X_1 \equiv 1 \) (mod 8), and \( t-s-2 \geq 3 \), we have

\[
(-k_1-1)(2^{r-s-2}+k_1 X_1-1) \equiv -(k_1+1)(k_1-1) \equiv 0 \quad \text{(mod 8)}.
\]

So in (3.18) we have

\[ (-n, 1+nX)_2 = 1. \]

Lemma 2 is proved. \( \square \)

**Lemma 3.** If \( 8 \mid n+1 \), then in (3.2) we have

\[ (-n, 1+nX)_2 = 1 \]

**Proof.** Since \( 8 \mid n+1 \), we have \( -n \equiv 1 \) (mod 8). Therefore \( -n \in \mathbb{Z}_2^2 \). Hence

\[ (-n, 1+nX)_2 = 1. \]

\( \square \)

**Lemma 4.** If \( 4 \mid n \) or \( 8 \mid n+1 \), then in (3.2) we have

\[ (-n, 1+nX)_\infty = 1. \]

**Proof.** By Lemmas 1, 2, and 3 we have

\[ (-n, 1+nX)_p = 1 \]

for all prime numbers \( p \). Since

\[ (-n, 1+nX)_\infty \cdot \prod_{\text{prime } p} (-n, 1+nX)_p = 1. \]

Therefore

\[ (-n, 1+nX)_\infty = 1. \]

\( \square \)
Lemma 5. If $4|n$ or $8|n+1$, then in (3.2) we have 

$$1 + nX > 0.$$ 

Proof. By Lemma 4, we have $(-n, 1+nX)_x = 1$. Hence the equation $-nC^2 + (1+nX)D^2 = 1$ has real solutions. Therefore $1+nX > 0$. 

Now Lemma 5 shows that $1+nX > 0$, contradicting (3.3). Therefore there do not exist positive rational numbers $x, y, z$ such that $xyz = m$ and $x + y + z = mn$. Theorem 2 is proved.

Acknowledgement

The author would like to thank Professor Erik Dofs for the Russian version of the reference [8]. Professor Erik Dofs also showed to the insolvability of the equation $x/y + y/z + z/x = 4$ in integers follows from his work in [5]. The author would like to thank the referee for pointing out the English version of the reference [8] and for many valuable comments improving the presentation of this paper. Part of this work was completed during the author’s stay at Vietnam Institute of Advanced Study in Mathematics (VIASM). The author would like to thank the Institute for their support and funding.

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