SOLUTIONS TO NONLINEAR RECURRENCE EQUATIONS

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ABSTRACT. Let $F(z)$ be any function. Suppose that $w$ is a fixed point of $F(z)$, that is, $F(w) = w$. Then the recurrence equation

$$x_{n+1} = F(x_n)$$

for $n = 0, 1, 2, \ldots$ has a solution of the form

$$x_n(w) = w + \sum_{i=1}^{\infty} a_i A_i F_i(w)^n,$$

where $F_i(z) = dF(z)/dz$. So, for each $w$ there is a set of complex $x_0$ such that $x_0(w) = x_0$. We assume that $F(z)$ is analytic at $w$. This solution appears to be new, even for such famous examples like the logistic map and the Mandelbrot equation.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. It is well known that the linear recurrence equation in $\mathbb{C}$,

$$x_n = \sum_{j=0}^{p} c_j x_{n-j}$$

has a solution of the form

$$x_n = \sum_{i=1}^{p} a_i F_i^n.$$ 

Less known is its solution and the solution of its non-homogeneous form, in terms of the Bell polynomials below, as given in [7]. In contrast there appears to be very little available in the way of solutions to non-linear recurrence equations. The ones we are aware of are [5], [6]. Both these papers considered particular cases of the solutions given in this paper.

The videos https://plus.maths.org/content/unveiling-mandelbrot-set, https://www.youtube.com/watch?v=ovJcsL7vyrk, https://www.youtube.com/watch?v=a3XDry3EwiU give remarkable visual accounts of the recurrence equations in $\mathbb{C}$,

$$x_{n+1} = c + x_n^2,$$

$$x_{n+1} = cx_n(1 - x_n),$$

and their relation to Mandelbrot sets and chaos theory. See also https://www.youtube.com/watch?v=NGMRB4O922I and Mandelbrot’s TED talk https://www.youtube.com/watch?2020 Mathematics Subject Classification. 65H20.

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Solutions to (1) and (2) are given in Examples 2.2 and 2.4. Section 2 gives solutions to the general recurrence equation

\[ x_{n+1} = F(x_n) \]

for \( F(z) \) a quadratic. Section 3 gives a solution to (3) for any function \( F(z) \) that is analytic at \( w \). For \( r \) in \( \mathcal{C} \) and \( a = (a_1, a_2, \ldots) \) any sequence in \( \mathcal{C} \), set

\[ S(r, a) = \sum_{j=1}^{\infty} a_j r^j. \]

Then the solutions all have the form

\[ x_n(w) = w + z_n, \]

where

\[ z_n = S(r^n, a) = \sum_{j=1}^{\infty} a_j r^{jn}, \quad r = F_1(w), \]

where \( F_j(z) \) denotes the \( j \)th derivative of \( F(z) \), and \( w \) is any fixed point of \( F \). That is, \( w = F(w) \). This simple class of solutions includes \( z_n = r^n \) and \( z_n = \exp(r^n) \). If \( x_0 = w \) then \( x_n \equiv w \). By (5) this holds if and only if \( F_1(w) = 0 \). We exclude this case. We also exclude \( w \) for which \( F_1(w) = 1 \); these have another class of solutions. So, if there are \( q \) such \( w \), we have \( q \) solutions, each with a different range of \( x_0 \). When there is a choice of \( w \), we can choose that closest to \( x_0 \), or to minimize \( |r| = |F_1(w)| \). \( a_i \) is given in terms of \( w \) and \( a_1 \) by a recurrence equation.

Set \( f_2 = F_2(w)/2 \). Transform from \( a_i \) to \( s_i \) via

\[ a_i = d_1 \left( f_2/r \right)^{i-1} s_i, \]

implying

\[ x_n(w) = w + z_n = w + rf_2^{-1} G \left( a_1 f_2 r^{n-1} \right) \quad \text{if} \quad f_2 \neq 0, \]

\[ = \sum_{j=1}^{\infty} a_1 \left( f_2/r \right)^{i-1} s_i r^{jn} \quad \text{for} \quad G(z) = G_r(z) = S(z, s) = \sum_{j=1}^{\infty} s_j z^j, \]

where \( s_i \) is given by a simpler recurrence equation. Theorem 3.1 gives \( G(x) \) and \( s_i \) as functions of \( w \) via \( r, v_3, v_4, \ldots \), where \( v_j \) is a standardised form of \( F_j(w) \).

We call \( x_0 \) solvable with respect to \( w \) if there exists \( a_1 = a_1(x_0) \) such that \( x_0 - w = S(1, a) \). If \( r = 0 \) then only \( x_0 = w \) is solvable with respect to \( w \). Otherwise \( a_1 \) is given as a series in \( x_0 - w \) by Theorem 4.1 using Lagrange inversion. One can check for convergence within a desired limit by plotting the partial sums of the series.

By (6), for \( f_2 \neq 0 \), \( x_0 \) is solvable with respect to \( w \) if and only if \( y_0 \in G(\mathcal{C}) \) for \( y_0 = (x_0 - w) f_2/r \), where \( G(\mathcal{C}) = \{ G(z) : z \in \mathcal{C} \} \). So, if \( G(\mathcal{C}) = \mathcal{C} \), all \( x_0 \) in \( \mathcal{C} \) are solvable with respect to \( w \). This is true for Example 2.1 and the 3 cases of Theorem 2.2 and Corollary 2.3.

Of particular interest are intervals of the real numbers, \( \mathcal{R} \), in which \( x_0 \) is solvable. Corollary 2.3 gives 3 cases, where (2) is solvable with explicit forms for \( a_i \). For example, for (2) with \( c = -2 \), \( x_0 \) in \([-1/2, 3/2]\) is solvable with respect to 0, and \( x_0 \) in \([1, 3]\) is solvable with respect to 3/2. For (2) with \( c = 2 \), \( x_0 < 1/2 \) is solvable with respect to 0, and \( x_0 = 1/2 \) is solvable with respect to 1/2. For (2) with
Taking the coefficient of $r^j$ for $1 \leq i \leq 10$ and defined by

\begin{equation}
F(z) = \sum_{j=0}^{p} c_j z^j,
\end{equation}

then $v_j = 0$ for $j > p$. So, for all $F(x)$ of the form (7), the solution (6) is given by $w$ and the same function $G(z)$ but with the different values of its arguments $r, v_3, \ldots, v_p$ determined by $w$.

If $p = 2$ then $f_2 = c_2$, and $G(z)$ and $s_i$ of (6) are determined by $r$ alone: Theorem 2.2 then gives closed forms of $G_r(z)$ for $r = -2, 2, 4$.

Theorem 3.2 gives a solution when $f_2 = 0$. Section 5 gives some results on bifurcation points.

We make use of the partial ordinary Bell polynomial $B_{i,k}(a)$. They are tabled on page 309 of [1] for $1 \leq i \leq 10$ and defined by

\begin{equation}
S(r, a)^j = \sum_{i=0}^{\infty} \hat{B}_{i,j}(a) r^i
\end{equation}

for $j = 0, 1, \ldots$, implying

\begin{align*}
z_n^j &= \sum_{i=0}^{\infty} \hat{B}_{i,j}(a) r^{ni} \\
\end{align*}

for $S(r, a)$ of (4) and $z_n$ of (5). They satisfy

\begin{equation}
\hat{B}_{i,j}(a) = 0 \text{ for } i < j, \quad \hat{B}_{i,0}(a) = \delta_{i,0}, \quad \hat{B}_{i,1}(a) = a_i, \quad \hat{B}_{i,i}(a) = a_i.
\end{equation}

Furthermore, $y_i \equiv b' c a_i$ implies

\begin{equation}
\hat{B}_{i,j}(y) = b' c^j \hat{B}_{i,j}(a).
\end{equation}

Taking the coefficient of $r^j$ in $S(r, a)^j = S(r, a)^{j-1} S(r, a)$ gives

\begin{equation}
\hat{B}_{i,j}(a) = \sum_{k=j-1}^{i-1} \hat{B}_{k,j-1}(a) a_{i-k}
\end{equation}

for $i \geq j \geq 1$. An alternative is to work with

\begin{equation}
T(r, b) = \sum_{i=1}^{\infty} b_i r^i / i! = S(r, a)
\end{equation}

for $b_i = i! a_i$ and the partial exponential Bell polynomial $B_{i,k}(b)$. They are tabled on page 307 of [1] for $1 \leq i \leq 12$, and defined by

\begin{equation}
T(r, b)^i / j! = \sum_{i=0}^{\infty} B_{i,j}(b) r^j / i!.
\end{equation}

implying

\begin{equation}
z_n^j / j! = \sum_{i=j}^{\infty} B_{i,j}(b) r^{ni} / i!.
\end{equation}
They satisfy (9) and (10) with $\hat{B}_{i,j}$ replaced by $B_{i,j}$. Taking the coefficient of $r^i$ in $T(r,b)^j = T(r,b)^{j-1}T(r,b)$ gives

$$B_{i,j}(b) = \sum_{k=j-1}^{i-1} \binom{i}{k} B_{k,j-1}(b) b_{i-k}$$

for $i \geq j \geq 1$.

Bell polynomials are available in Matlab as IncompleteBellPoly, in Mathematica as BellY, in Maple as IncompleteBellB, and in Wolfram as BellB[n, x]. See also

http://freesourcecode.net/matlabprojects/70276/bell-polynomials-of-the-second-kind-in-matlab,
https://www.mathworks.com/matlabcentral/fileexchange/14483-bell-polynomials-of-the-second-kind,
https://en.wikipedia.org/wiki/Bell_polynomials

Throughout, we set

$$x_j = x(x-1)\cdots(x-j+1)$$

and $I = \sqrt{-1}$.

2. Solutions for quadratic recurrence

We assume that $c_2 \neq 0$. The fixed points are the roots of

$$w = \sum_{j=0}^{2} c_j w^j,$$

that is,

$$w = \left(1 - c_1 \pm \delta^{1/2}\right) / (2c_2) = w_1, w_2$$

say, for $\delta = (c_1 - 1)^2 - 4c_0 c_2$. That is, $w_1$ takes the $+$ sign, and $w_2$ the $-$ sign. For (5) to work we need

$$S(r^{n+1}, a) - w = z_{n+1} = c_0 + c_1 (w + z_n) + c_2 \left(w^2 + 2wz_n + z_n^2\right) - w = (c_1 + 2c_2 w) z_n + c_2 z_n^2.$$

If $a_1 \neq 0$, the coefficient of $r^n$ is $a_1 r = (c_1 + 2c_2 w) a_1$, implying

$$r = c_1 + 2c_2 w = F_1(w) = 1 \pm \delta^{1/2}.$$

Assume that $r = F_1(w) \neq 0$ or 1. By (8), for $i \geq 2$, the coefficient of $r^i$ is $a_i r^i = ra_i + c_2 \hat{B}_{i,2}(a)$, implying

$$a_i = c_2 \hat{B}_{i,2}(a) / R_i,$$

where

$$R_i = r^i - r, \hat{B}_{i,2}(a) = \sum_{j+k=i} a_j a_k = \sum_{j=1}^{i-1} a_j a_{i-j}.$$
To remove $c_2$ and $a_1$ in this recurrence equation, we transform to $A_i$ defined by $A_1 = 1$ and $a_i = c_2^{-1}a_1^{i-1}A_i$ implies $z_n = c_2^{-1}S(a_1 c_2 r^n, A)$ for $A = (A_1, A_2, \ldots)$. By (10) the recurrence equation becomes

$$A_1 = 1, A_i = \hat{B}_i(A)/R_i$$

for $i \geq 2$, and the solution (5) can be written

$$x_n(w) - w = z_n = c_2^{-1} \sum_{i=1}^{\infty} (a_1 c_2 r^n)^i A_i$$

for $r = c_1 + 2 c_2 w$. To simplify further, we transform again to

$$U_i = R_i/r = r^{i-1} - 1, s_i = r^{i-1} A_i.$$ 

The recurrence equation now becomes

$$s_1 = 1, s_i(r) = s_i = \hat{B}_i(s)/U_i$$

for $i \geq 2$. and so on. This proves

**Theorem 2.1.** Take $w_i$, $\delta$ of (13) and $r$ of (14). Assume that $\delta \neq 0$, and that if $\delta = 1$ then $w = w_1$. If $\delta \neq 1$, $w = w_1$ or $w = w_2$. Then a solution of the recurrence equation (3) for $F(z)$ of (7) with $p = 2$, is given by (5) with $a_i = a_i(r)$ of (15), or equivalently by (17) with $A_i = A_i(r)$ of (16), or by (6) with $s_i = s_i(r)$ of (19).

So, if $x_0$ is solvable with respect to $w$, $a_1 = a_1(w)$ is given by

$$a_1 = r c_2^{-1} G_r^{-1}(y_0)$$

at $r = c_1 + 2 c_2 w$ and $y_0 = (x_0 - w) c_2/r$, where $G_r^{-1}(y)$ is any inverse of $G_r(z) = G(z)$ of (6). If $\delta = 1$ and $w = w_2$, then $r = 0$ and $x_n = w_2 = -c_1/(2c_2)$.

**Corollary 2.1.** For $U_i$ of (18) and $i \geq 2$, $s_i$ is given by

$$s_i = N_i/D_i,$$

where

$$D_i = \prod_{j=2}^{i} U_j.$$ 

If $\{c_j\}$ are real and $\delta < 0$, then $w$ is not real, yet Theorem 5.1 gives $a_1$ such that $x_0(w) = x_0$ when $x_0$ is solvable. By (13) and (14), $r - c_1 = 2 c_2 w = 1 - c_1 \pm \delta^{1/2}$ and $(r - 1)^2 = \delta = (c_1 - 1)^2 - 4 c_0 c_2$. So, $r \neq 0$ if and only if $4 c_0 c_2 \neq c_1^2 - 2 c_1$, $r \neq 1$ if and only if $4 c_0 c_2 \neq (c_1 - 1)^2$.

**Example 2.1.** Take $F(z) = c_2 z^2$, that is, $c_0 = c_1 = 0$, $c_2 = c$. There are 2 fixed points, $w_2 = 0$ and

$$w_1 = c^{-1}. w = 0 \text{ implies } r = 0, x_n = 0 \text{ so that as noted only } x_0 = 0 \text{ is solvable with respect to } w = 0.$$

$$w = c^{-1} \text{ implies } r = 2, \text{ and by (18),}$$

$$x_n = c^{-1} + c^{-1} \sum_{i=1}^{\infty} (a_1 c_2^{2^n})^i A_i = c^{-1} + 2 c^{-1} \sum_{i=1}^{\infty} (a_1 c_2^{2^n-1})^i s_i.$$
But \( x_{n+1} = cx_n^2 \) implies
\[
x_n = c^{-1} (cx_0)^2^n = c^{-1} \sum_{i=0}^{\infty} \gamma^2^{ni}/i!,
\]
where \( \gamma = \ln cx_0 \). Taking the coefficient of \( 2^n \) gives \( a_1 = \gamma/c \). So, all \( x_0 \) are solvable with respect to \( w_1 \).

Taking the coefficient of \( 2^ni \) gives \( A_i = 1/i! \), \( s_i = 2^{i-1}/i! \).

**Corollary 2.2.** If \( r = 2 \) then (16) has solution \( A_i = 1/i! \) and (19) has solution \( s_i = 2^{i-1}/i! \).

Corollary 2.1 confirms this. For an extension, see Example 4.1.

Consider the linear case \( c_2 = 0 \) when \( c_1 \neq 1 \). \( x_{n+1} = c_0 + c_1x_n \) implies \( x_n = c_0(c^n_2 - 1)/(c_1 - 1) + c^n_2x_0 \). Our method gives \( w = -c_0/(c_1 - 1) \), \( r = c_1 \), \( a_i = 0 \) for \( i \geq 2 \), that is, \( x_n(w) = w + a_1c^n_2 \) in agreement with (5). So, \( a_1 = x_0 - w \). If \( c_1 = 1 \) then \( x_n = nc_0 + x_0 \), but \( r = 1 \) so that our method only applies by letting \( r \to 1 \).

Write (19) as \( U_{ij} = B_{i,2}(s) \) for \( i \geq 1 \). Multiplying by \( z^i \) and summing gives the functional relation
\[
G(rz)/r = G(z) + G(z)^2.
\]
In Example 2.4 we prove

**Theorem 2.2.** For \( z \in \mathcal{C} \) and \( G_r(z) = G(z) \),
\[
G_{-2}(z) = \cos \left( \pi/3 - 2z/\sqrt{3} \right) - 1/2
\]
\[
= \sum_{i=1}^{\infty} \left[ (-4/3)^{i-1} z^{2i-1}/(2i-1)! + 2^{-1} \left( -4z^2/3 \right)^i/(2i)! \right]
\]
\[
= z - z^2/3 - 2z^3/9 + z^4/27 = 2z^5/135 - 2z^6/1215 + \cdots,
\]
\[
G_2(z) = 2^{-1} \exp(2z) - 1/2 = z + z^2 + 2z^3/3 + z^4/4 + z^5/315 + \cdots,
\]
\[
G_4(z) = 2^{-1} \cosh \left( 2z^{1/2} \right) - 1/2 = z + z^2/3 + 2z^3/45 + z^4/315 + 2z^5/14175 + \cdots.
\]

So, for \( r = -2, 2, 4 \), \( G(\mathcal{C}) = \mathcal{C} \) so that all \( x_0 \in \mathcal{C} \) are solvable with respect to \( w \). Also for \( z \in \mathcal{R} \), \( G_{-2}(z) \in [-3/2, 1/2] \), \( G_2(z) > -1/2 \) for \( z \geq 0 \), \( G_4(z) > -1/2 \) for \( z \geq 0 \), \( G_4(z) = 2^{-1} \cos \left( 2(-z)^{1/2} \right) - 1/2 \in [-1, 0] \) for \( z \leq 0 \).

**Corollary 2.3.** Take \( w_i, r, \delta \) as in Theorem 2.1. For \( r = -2 \), \( x_0 \in w_i + [-1, 3]/c_2 \) is solvable with respect to \( w_i \). For \( r = 2 \), \( x_0 \in w_i + (-1/2, \infty)/c_2r = -2 \) implies \( w = w_2 \) and \( \delta \) of (13) is 1, that is, \( c_1^2 - 2c_1 = 4c_0c_2 = 0 \). \( r = 2 \) implies \( w = w_1 \) and \( \delta = 1, \) that is, \( c_1^2 - 2c_1 = 4c_0c_2 = 0 \). \( r = 4 \) implies \( w = w_1 \) and \( \delta = 9, \) that is, \( c_1^2 - 2c_1 = 8 + 4c_0c_2 = 0 \).

For example, \([-1, 3]/(-2) = [-3/2, 1/2] \).

**Example 2.2.** Solution of (1): \( F(z) = c + z^2 \). \( w = F(w) \) implies \( w = c + w^2 \) which implies \( r = 2w = 1 \pm \delta^{1/2} \), where \( \delta = 1 - 4c \). Call these \( w_1 \) and \( w_2 \). By Theorem 2.1 with \( c_0 = c, c_1 = 0, c_2 = 1 \),
\[
\sum_{i=1}^{\infty} r^{ni}a_i = rG(a_1r^{n-1})
\]
for
\[ G(z) = \sum_{i=1}^{\infty} z^i s_i. \]

for \( s_i = s_i(r) \) of (19), and \( r = 2w \neq 1 \), that is, \( c \neq 1/4 \). For \( c > 1/4 \), \( w \) is not real. Yet Theorem 5.1 gives \( a_1 \) such that (23) holds when \( x_0(w) = x_0 \) is solvable.

For \( c = -2, r = 2w_1 = -2 \) with \( G(z) \) of (20), or \( r = 2w_2 = 4 \) with \( G(z) \) of (22). For \( c = 0, r = 2w_1 = 0 \) or \( r = 2w_2 = 2 \) with \( G(z) \) of (21).

The Mandelbrot set, \( M \), is the set of \( c \in \mathcal{C} \) such that \( x_0 = 0 \) and \( x_n \) remain bounded as \( n \to \infty \). Given \( c \in \mathcal{C} \), the (filled) Julia set with respect to \( c \) is the set of \( x_0 \in \mathcal{C} \) such that \( x_n \) remains bounded as \( n \to \infty \). We do not pursue these ideas here.

Example 2.3. Take \( F(z) = cz(1 - \alpha z) \). Set \( \gamma = (1 - c^{-1}) \alpha^{-1} \). Then \( r = F_1 = c(1 - 2\alpha w), f_2 = -c\alpha, (w,r) = (0, c) \) or \( (w,r) = (\gamma, 2 - c) \). By Theorem 2.1,
\[
x_n(0) = -\alpha^{-1}G_c(-a_1\alpha^n) \text{ if } c \neq 1,
\]
\[
x_n(\gamma) = \gamma + (1 - 2c^{-1})\alpha^{-1}G_{2-c}(-a_1\alpha(2-c)^{n-1}) \text{ if } c \neq 1.
\]

If \( c = -2 \) then \( x_n(0) = -\alpha^{-1}G_{-2}(-a_1\alpha(-2)^n) \) of (20), \( \gamma = 3\alpha^{-1}/2 \) and \( x_n(\gamma) = \alpha^{-1}/2 + \alpha^{-1} \cos((a_1\alpha)\sqrt{2^{n+1}}/2) \).

If \( c = 2 \) then \( x_n(0) = (2\alpha)^{-1}[1 - \exp(-a_1\alpha2^n)] \), \( \gamma = \alpha^{-1}/2 \), \( x_n(\gamma) = \gamma \) and \( x_0 \) is solvable with respect to \( \gamma \) if and only if \( x_0 = \alpha^{-1}/2 \). If \( c = 4 \) then \( x_n(0) = \alpha^{-1}[1 - \cos((a_1\alpha)^{1/2}2^n)] \), and for \( \alpha \in \mathbb{R}, x_0 \) is solvable with respect to \( 0 \) if and only if \( (\alpha > 0, x_0 \in [0,1/\alpha]) \) or \( (\alpha > 0, x_0 < 1/\alpha) \), or \( (\alpha < 0, x_0 \in [2/\alpha,0]) \) or \( (\alpha < 0, x_0 > 1/\alpha) \). Furthermore, \( \gamma = 3\alpha^{-1}/4 \) and
\[
x_n(\gamma) = \gamma + (2\alpha)^{-1}G_{-2}((-1)^n a_1\alpha^{2n+1}) \text{ of (20)}
\]
\[
= (2\alpha)^{-1}\left\{1 + \cos\left[\pi/3 + (-1)^n a_1\alpha^{2n+2}/\sqrt{3}\right]\right\}.
\]

Example 2.4. Solution of (2): \( F(z) = c(z(1 - z)) \) for \( c \neq 1 \). This is called the logistic map and is the simplest model of population with negative feedback and growth rate \( c \). Another video of its behaviour is https://www.youtube.com/watch?v=ETrYE4MdoLQ&t=23s The fixed points are given by \( w = cw(1-w) \), implying \( w = 0 \) or \( w = 1 - c^{-1} = w_0 \) say. By Theorem 2.1,
\[
x_n(w) - w = -rc^{-1}G_r(-a_1cr^{n-1})
\]
for
\[ G_r(x) = G(x) = \sum_{i=1}^{\infty} x^i s_i, \]
So, for
\[ x_n(0) = -G_c(-a_1c^n), \]
\[ x_n(w_0) = 1 - c^{-1} + (1 - 2c^{-1}) G_{2-c}(-a_1c(2-c)^{n-1}). \]

If \(|c| < 1\) then \(x_n(0) \to 0\) as \(n \to \infty\). If \(1 < c < 3\), then \(|2-c| < 1\) and \(x_n(w_1) \to w_1\). \(c = 3\) is the first bifurcation point. If \(c > 3\) then \(x_n(w_1)\) will not converge since \(|r| > 1\).

If we transform to \(g_i = (-1)^{i-1}s_i, a_1 = a^2_0\), then
\[ x_n(0) = -G_c(-a_1c^n) = F_c\left(c^{n/2}a_0\right), \]
where
\[ F_c(x) = -G_c\left(-x^2\right) = \sum_{i=1}^{\infty} x^{2i} g_i, \]
and \(g_1 = 1, g_i = -\hat{B}_{i/2}(g)/U_i\) at \(r = c\) for \(i \geq 2\), \(a_1 = -G_c^{-1}(-x_0)\), \(a = F_c^{-1}(x_0)\). If there is more than one inverse, we may take that closest to 0. The \(F_c\) form of solution in (24) is given by (8) of [2] for (2) on \((0,1)\): it only applies if \(a_1 \geq 0\). He gives dramatic 3D pictures of \(F_c(x)\), some interesting properties, and suggestions for its truncation and inversion. His Table 1 gives \(g_i\) as \(n_i/D_i\) with \(n_i = (-1)^{i-1}N_i\) for \(2 \leq i \leq 8\) with an expanded form of \(D_i\). He has kindly provided us with figures of \(G_c(y)\) truncated to 8 and 10 terms for \(c = 2,3,4\). For \(c = 2,\) there is no discernible difference for \(y\) in \((-2,2)\) but a slight difference at \(y = -2.1\). For \(c = 3\) and 4, there is no discernible difference for \(y\) in \((-4,4)\). He notes that for \(c\) in \((-1,1)\), they only agree for \(y\) in \((-0.2,0.2)\).

https://mathworld.wolfram.com/LogisticMap.html gives 3 values of \(c\) with known exact solutions. They are all of the form \(x_n = 1/2 - f_c((c^n)/2,\) where \(b = f_c^{-1}(1-2x_0)\).

For these to agree with \(x_n(0) = -G_c(-a_1c^n)\), we need \(-G_c(-a_1y) = 1/2 - f_c(by)/2\), that is, \(G_c(z) = f_c(vz)/2 - 1/2\), where \(v = -b/a_1\) is given by \(1 = s_1 = G_{c,1}(0) = v f_c,0(0)\). So, \(G_c(z)\) is given by Theorem 2.2 for \(c = -2,2,4\) with \(v = 12,2,1\), respectively, in agreement with (19) at \(r = 4\).

So, for \(c = -2,2,4, x_n(0) = -G_c(-a_1c^n)\) and all \(x_0 \in C\) are solvable with respect to 0. (21) gives another proof of Corollary 2.2. \(c = -2\) implies
\[ x_n(3/2) = 3/2 + 2G_4(a_12^{2n-1}) = 1/2 + \cosh\left(a_1^{1/2}2^{n+1/2}\right) = 1/2 + \cosh\left((-a_1)^{1/2}2^{n+1/2}\right). \]
\(c = 4\) implies \(x_n(3/4) = 3/4 - G_{-2}(2a(-2^n)/2 = 1 - 2^{-1} \cos\left(\frac{\pi}{3} - 4a(-2^n)/\sqrt{3}\right).\)

As \(r \to 1, N_i \to (i-1)!, U_i/(r-1) \to i-1, D_i/(r-1)^{i-1} \to (i-1)!, (r-1)^{i-1}s_i \to 1\) and
\[ G\left((r-1)z\right)/(r-1) \to \sum_{i=1}^{\infty} z^i z/(1-z) \]
for \(|z| < 1\). Reparameterize from \(a_1\) to \(a_0 = a_1/(r-1)\). By (6) for fixed \(a_0\) and \(z = a_0f_2^{n-1}, x_n(w) = w = rf_2^{-1}G\left((r-1)z\right) \approx f_2^{-1}(r-1)z/(1-z)\) for \(|a_0f_2| < 1\), so that \(x_n(w) \approx w = (1-c_1)/(2c_2) = -2c_0/(c_1 + 1).\)
For a specific example with \( r = 1 \), take \( F(z) = z - z^2 \). Then

\[
x_n = \sum_{i=1}^{2^n} a_n x_0,
\]

where \( a_{n1} = 1, a_{n2} = -n, a_{n3} = n(n-1), a_{n4} = -n(n-1)(2n-3) \) and so on.

3. Solutions for general \( F(z) \)

Given \( w = F(w) \), suppose that \( F(z) \) is analytic at \( w \), and set

\[
F_j = F_j(w), \quad r = F_1, \quad R_i = r^i - r, \quad U_i = R_i/r = r^{i-1} - 1, \quad D_i = \prod_{j=2}^{i} U_j,
\]

\[
f_j = F_j/j!, \quad u_j = f_j/f_2^{-1}, \quad v_j = r^{j-2}u_j, \quad z_n = x_n(w) - w \text{ of (5)}.
\]

So, \( v_2 = u_2 = 1 \). By (8) and Taylor’s expansion, since \( f_0 = w \),

\[
\sum_{i=1}^{\infty} a_i r^{i+n} = z_{n+1} = F(w + z_n) - F(w) = \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} \hat{B}_{i,j}(a) r^n = \sum_{i=1}^{\infty} r^{i} C_i,
\]

where

\[
C_i = \sum_{j=1}^{i} \hat{B}_{i,j}(a) f_j = a_if_1 + E_i, \quad E_i = \sum_{j=2}^{i} \hat{B}_{i,j}(a) f_j
\]

if \( i \geq 2 \). The coefficient of \( r^n \) in (27) is \( a_1 r = C_1 = a_1 f_1 \), implying \( r = f_1 \). We assume that

\[
r = F_1 \neq 0 \text{ or } 1, \quad \text{and } F_2 \neq 0.
\]

For \( i \geq 2 \), the coefficient of \( r^n \) in (27) is \( a_i r^i = C_i = a_i r + E_i \), implying

\[
a_i = E_i/R_i = R_i^{-1} \sum_{j=2}^{i} \hat{B}_{i,j}(a) f_j.
\]

This recurrence equation for \( a_i \) extends (15). It gives

\[
a_2 = a_1^2 f_2 / R_2, \quad a_3 = (2a_1 a_2 f_2 + a_1^3 f_3) / R_3, \quad \ldots.
\]

Transforming to \( a_i = a_1^i f_2^{i-1} A_i \) then to \( s_i = r^{i-1} A_i \), where \( A_1 = s_1 = 1 \) gives

\[
a_i = a_1^i (f_2/r)^{i-1} s_i, \quad \hat{B}_{i,j}(a) = (a_1 f_2 / r)^j (f_2/r)^{-j} \hat{B}_{i,j}(s)
\]

by (10) so that the recurrence equation (29) for \( a_i \) transforms to

\[
s_1 = 1, \quad s_i = U_i^{-1} \sum_{j=2}^{i} \hat{B}_{i,j}(s) v_j
\]

for \( i \geq 2 \), a recurrence equation for \( s_i \). This proves
Theorem 3.1. Let $F(z)$ be any analytic function. Let $w$ be any solution of $w = F(w)$. Suppose that (28) holds. In the notation of (25) and (26), solutions to the recurrence equation (3) are

\[ x_n(w) - w = \sum_{i=1}^{\infty} r^i a_i = f_2^{-1} \sum_{i=1}^{\infty} (a_1 f_2 r^n) i A_i \]

\[ = \sum_{i=1}^{\infty} a_i (f_2/r)^i s_i r^{ni} = (f_2/r)^{i-1} G(\alpha (a_1 f_2 r^{n-I})), \]

where

\[ G(x) = G(x : r, v_3, v_4, \ldots) = \sum_{i=1}^{\infty} s_i x^i. \]

For $i \geq 2$, $a_i$ and $A_i$ are given iteratively by

\[ a_i = R_i^{-1} \sum_{j=2}^{i} \tilde{B}_{i,j}(a) f_j, \quad A_i = R_i^{-1} \sum_{j=2}^{i} \tilde{B}_{i,j}(A) f_j f_2^{1-i}, \]

respectively, and $s_i$ is given iteratively by (30).

If $F(z)$ is the polynomial (7), then

\[ f_k = \sum_{i=0}^{n-k} c_{k+i} \binom{k+i}{k} w^i \]

for $j \leq p$, $f_p = c_p$, $f_j = v_j = 0$ for $j > p$, and we can replace the upper limit $i$ in the sum of (30) by $\min(i, p)$, because $\tilde{B}_{i,j}(x) = 0$ for $i < j$. So, $s_j$ and $G(x)$ depend only on $r, v_3, \ldots, v_p$. For $F(z)$ of (7) with $p = 2$, they are only functions of $r$; for $p = 3$, $s_j$ and $G(x)$ are functions of $r$ and $v_3$; for $p = 4$, they are functions of $r, v_3$ and $v_4$; and so on.

Example 3.1. For cubic recurrence, that is, (3) for $F(z)$ of (7) with $p = 3$,

\[ r = F_1 = c_1 + 2c_2 w + 3c_3 w^2, \quad f_2 = F_2/2 = c_2 + 3c_3 w; \]

\[ f_3 = c_3, \quad v_3 = c_3 r/(c_2 + 3c_3 w), \]

for $j > 3$. Take $F(z) = c x (1 - \alpha x^2)$. Set $\gamma = (1 - c^{-1}) \alpha^{-1}$. The fixed points are $w = 0$ or $w = \pm w_1$, where $w_1 = \gamma^{1/2}$ is real if and only if $(1 - c^{-1}) \alpha > 0$.

If $w = 0$ then $r = c$, $f_2 = 0$ so that the method does not apply as $a_i = 0$ for $i > 1$, giving $x_n(0) = 0$. See Example 3.6 below for a solution. If $w = \pm w_1$ then $r = 3 - 2c$, $f_2 = \mp 3c\alpha w_1$, $f_3 = -c\alpha$, $v_3 = -(3 - 2c)^{-1}/(c-1)$.

Example 3.2. For quartic recurrence, that is, (3), (7), $p = 4$,

\[ r = F_1 = \sum_{j=1}^{4} c_j j w^{j-1}, \quad f_2 = c_2 + 3c_3 w + 6c_4 w^2, \quad f_3 = c_3 + 4c_4 w, \]

\[ f_4 = c_4, \quad v_3 = r f_3 / f_2^2, \quad v_4 = r^2 f_4 / f_2^3, \]

for $j > 4$.
for $j > 4$. Take $F(z) = cz(1 - \alpha z^2)$. Set $\gamma = (1 - c^{-1}) \alpha^{-1}$. Then $w = 0$ or $w^3 = \gamma$, that is, $w = w_k$
for $k = 0, 1, 2$, where

$$w_k = \exp(2\pi ik/3) \gamma^{1/3} \text{ if } \gamma > 0,$$
$$w_k = -\exp(2\pi ik/3) (-\gamma)^{1/3} \text{ if } \gamma < 0.$$

If $w = 0$ then $r = c$, $f_2 = 0$: see Example 3.7 below for a solution. If $w = w_k$ then $r = 4 - 3c$, $f_2 = -6c\alpha w_k^2$, $f_3 = -4c\alpha w_k$, $f_4 = -c\alpha$, $v_3 = (4 - 3c)(c - 1)^{-1}/9$, $v_4 = (4 - 3c)^2(c - 1)^{-2}/216$.

**Example 3.3.** Suppose that (7) holds with $p = 5$, $c_5 \neq 0$. Then

$$r = \sum_{j=1}^5 c_j jw_j^{j-1}, f_2 = c_2 + 3c_3 w + 6c_4 w^2 + 10c_5 w^3, f_3 = c_3 + 4c_4 w + 10c_5 w^2,$$

and $s_i = N_i/D_i$ of Theorem 3.1. Take $F(z) = cz(1 - \alpha z^4)$. Set $\gamma = (1 - c^{-1}) \alpha^{-1}$. Then $w = 0$ or

$w^4 = \gamma$, so that $w = w_k$ for $k = 0, 1, 2, 3$, where $w_k = \exp(2\pi ik/4) \gamma^{1/4}$. If $w = 0$ then $r = c$, $f_2 = 0$:
see Example 3.8 below for a solution. If $w = w_k$ then $r = 5 - 4c$, $f_2 = -10c\alpha w^3$, $f_3 = -10c\alpha w^2$, $f_4 = -5c\alpha w$, $f_5 = -c\alpha$, $v_3 = r(1 - c)^{-1}/10$, $v_4 = -r^2(1 - c)^{-3}/200$, $v_5 = r^3(1 - c)^{-3}/10^4$.

**Example 3.4.** Take $F(z) = cz^{N+1}$ for some integer $N \geq 1$. The fixed points are $w = 0$ and $w_k = c^{-1/N} \exp[2\pi ik/N]$ for $k = 0, 1, \ldots, N - 1$. Set $y_0 = c^{1/N}x_0$, $\gamma = \ln y_0$. An exact solution is

$$x_n = c^{-1/N} y_0^{(N+1)^n} = c^{-1/N} \exp[\gamma(N+1)^n] = c^{-1/N} \sum_{i=0}^\infty \gamma(N+1)^{ni}/i!.$$

Also $F_j(x) = c(N + 1) j x^{N+1-j}$. So, for $w = w_k$,

$$F_j = c(j-1)/(N+1) \exp[2\pi ik(1-j)/N], r = F_1 = N + 1,$$

$$x_n(w_k) = w_k + \sum_{i=1}^\infty a_i(N+1)^{ni},$$
in agreement if $a_i = c^{-1/N}\gamma^{i}/i!$ and $c^{-1/N} = w_k$, that is, $k = 0$.

**Example 3.5.** $F(z) = z\exp(c - cz)$ implies $F_j(z) = (j - cz)(-c)^{j-1} \exp(c - cz)$. Its fixed points are $w = 0$ and $w = 1$. If $w = 0$ then $r = e^c$, $F_j = j(-c)^{j-1}$, $f_j = (c-1)^{-1}r^j(1 - j)!$, $v_j = 1/(j - 1)!$.
So, by Theorem 3.1, $x_0(0) = -c^{-1}G(-a_1 e^{cn})$ for $G(x)$ given by (33). If $w = 1$ then $r = 1 - c$,

$$F_j = (j - c)(-c)^{j-1}, f_j = (1 - c/j)(-c)^{j-1}/j!, v_j = (1 - c/j)(1 - c)^{-1}(1 - c/2)^{-j}/j!.$$ By Theorem 3.1, for $G(x)$ given by (33), $x_0(1) = 1 + (1 - c)C^{-1}G(-a_1 C(1 - c)^{n-1})$ for $C = c(1/2 - 1)$. For a video on this example see https://www.youtube.com/watch?v=ETrYE4MdOLQ&t=23s. Table 4 of [3] catalogues its stable cycles with periods up to 6. [4] discuss and plot the solution in Figure 4.

The assumption in Theorem 3.1 that $F_2 \neq 0$, was just a convenience to give the solution with the minimum number of parameters. We transform from $a_i$ to $A_i = a_i/a_1$. Set $C(ab) = a/R_b + b/R_a$ if $a \neq b$, $C(a^2) = a/R_a$ and $f(3^a 4^b \ldots) = f_3^a f_4^b \ldots$. 


Theorem 3.2. Let \( F(z) \) be any analytic function. Let \( w \) be any solution of \( w = F(w) \). Suppose that \( F_2 = 0 \) but \( r = F_1 \neq 0 \) or \( 1 \). Then in the notation of (25) and (26), a solution to the recurrence equation (3) is

\[
x_n(w) = w + \sum_{i=1}^{\infty} (r^n a_i) A_i,
\]

where \( A_1 = 1, A_2 = 0 \) and for \( i \geq 3, A_i = N_i/R_i, \) where

\[
N_i = \sum_{j=3}^{i} \tilde{B}_{i,j}(A) f_j.
\]

Theorem 3.3. Suppose that \( F_i \neq 0 \) or \( 1 \), and that for some \( p \geq 3, F_p \neq 0 \) but \( F_j = 0 \) for \( 1 < j < p \) and \( j > p \). Set \( q = p - 1, r_k = R_k q \). Then \( N_i \) of Theorem 3.1 are 0 except for \( N_{kq+1} = C_k f_p^k \) for \( C_k \) a function of \( r \) given iteratively by \( C_1 = e_1 = 1, e_{k+1} = C_k/r_k \) for \( k \) and so on. Take \( F(z) = \alpha(1 - \alpha z^2) \) and \( w = 0 \). Then

\[
r = c, F_2 = 0, f_3 = -c, A_{2k+1} = C_k(-c) k/r_k.
\]

Example 3.6. Suppose that \( F(z) \) is a cubic, \( F_1 \neq 0 \) or \( 1 \), and \( F_2 = 0 \). Set \( r_k = R_{2k+1} \). Then

\[
A_1 = 1, A_{2k} = 0, A_{2k+1} = C_k f_3^k/r_k
\]

for \( C_1 = 1, C_2 = 3/r_1 \) and so on. Take \( F(z) = \alpha(1 - \alpha z^2) \) and \( w = 0 \). Then

\[
r = c, F_2 = 0, f_3 = -c, A_{2k+1} = C_k(-c) k/r_k.
\]

Example 3.7. Suppose that \( F(z) \) is a quartic, \( F_1 \neq 0 \) or \( 1 \), and \( F_2 = 0 \). Set \( r_k = R_{3k+1} \). Then \( A_1 = 1, A_j = 0 \) unless \( j = 3k+1, A_{3k+1} = C_k f_4^k/r_k \) for \( C_1 = 1, C_2 = 4/r_1 \) and so on. Take \( F(z) = \alpha(1 - \alpha z^3) \) and \( w = 0 \). Then

\[
r = c, F_2 = 0, f_4 = -c, A_{3k+1} = C_k(-c) k/r_k.
\]

Example 3.8. Suppose that (7) holds with \( p = 5, F_1 \neq 0 \) or \( 1 \), and \( F_j = 0 \) for \( j = 2, 3, 4 \). Set \( r_k = R_{4k+1} \). Then \( A_1 = 1, A_j = 0 \) unless \( j = 4k+1, A_{4k+1} = C_k f_5^k/r_k \) is given by \( C_k = \tilde{B}_{k+4,5}(e) \). Take \( F(z) = \alpha(1 - \alpha z^4) \) and \( w = 0 \). Then

\[
r = c, F_j = 0 \text{ for } j = 2, 3, 4, f_5 = -c, \text{ and } A_{4k+1} = C_k(-c) k/r_k.
\]

We now present an alternative framework. Let us replace \( a_i \) in (5) by \( b_i/i! \). This is a more natural framework for many functions \( F(z) \). By (11), (27) becomes

\[
x_n+1(w) = w + \sum_{i=1}^{\infty} r^i C_i / i!
\]

for

\[
C_i = \sum_{j=1}^{i} B_{i,j}(b) F_j = b_i F_1 + E_i, \quad E_i = \sum_{j=2}^{i} B_{i,j}(b) F_j, \quad i \geq 2.
\]
implying
\[ B_{i,j}(b) = (b_1 F_2/r)^i (F_2/r)^{-j} B_{i,j}(f) \]

by the hatless forms of (9) and (10). This gives

**Theorem 3.4.** Let \( F(z) \) be any analytic function. Let \( w \) be any solution of \( w = F(w) \). Suppose that (28) holds. A solution to (3) is

\[ x_n(w) = w + \sum_{i=1}^{\infty} r^i b_i / i! = w + (F_2/r)^{-1} G(b_1 F_2 r^{n-1}) , \]

where
\[ G(x) = G(x : r, v_3, v_4, \ldots) = \sum_{i=1}^{\infty} x^i s_i / i! . \]

For \( i \geq 2, s_i \) is given iteratively in terms of \( U_i = r^{i-1} - 1 \) by

\[ s_1 = 1, \quad s_i = U_i^{-1} \sum_{j=2}^{i} B_{i,j}(s)v_j \]

for \( v_j = F_j F_2^{1-j} r^{2-2} \). Also \( s_i = N_i / D_i \), where \( s'_i = U_i s_i = N_i / D_i - 1 \).

\( G, s_i, N_i, v_j, A_i \) have different meanings from those of Theorem 3.1. But viewed as polynomials in \( r \), the new \( A_i, C_5 \) are just constant multiples of the old. To compare them, for \( j = 1, 2 \), write \( s_i, N_i, G \) of Theorems 3.1 and 3.4 as \( s_{i,1}, N_{i,1}, G_1 \) and \( s_{i,2}, N_{i,2} G_2 \). Comparing (31) and (34) gives

\[ 2G_1(x/2) = G_2(x), \quad d_i = s_{i,2} / s_{i,1} = N_{i,2} / N_{i,1} = i! / 2^{i-1} = id_{i-1} / 2. \]

For \( F(z) \) of (7) with \( p = 2 \), if \( r = 0 \), then \( N_{i,1} = (-1)^{i-1} s_{i,1} = (2i - 2) / (i - 1)! \) and \( N_{i,2} = (-1)^{i-1} s_{i,2} = 2^{1-i}(2i - 2) / (i - 1)! \).

For \( r = F_1 \neq 0 \) or 1, \( b_1 \) of (34) is given by Theorem 5.1 when \( x_0 \) is solvable. For \( i \geq 2, b_i \) is given iteratively in terms of \( R_i = r^i - r \) by

\[ b_1 = 1, \quad b_i = R_i^{-1} \sum_{j=2}^{i} B_{i,j}(b) F_j . \]

Now set \( c_j = d_j / j! \) so that the polynomial \( F(z) \) in (7) is

\[ F(z) = \sum_{j=0}^{p} d_j x^j / j! , \]

implying

\[ F_i = \sum_{j=1}^{p} d_j w^{j-i} / (j-i)! = \sum_{k=0}^{p-i} d_{i+k} w^k / k!, \quad F_p = d_p . \]

We can replace the upper limit \( i \) in the sum of (35) by \( \min(i, p) \), and \( F_j = v_j = 0 \) for \( j > p \). \( s_j \) and \( G(x) \) depend only on \( r, v_3, \ldots, v_p \). For \( p = 2 \) they are only functions of \( r, N_{i,2} = N_i \) of Theorem 3.4 reduces to \( d_i N_{i,1} \) for \( d_i \) of (36) and \( N_{i,1} = N_i \) of Corollary 2.1.
Since \( v_1 = 1 \) we can write (30) as
\[
r^{i-1}s_i = \sum_{j=1}^{i} \hat{B}_{i,j}(s)v_j
\]
for \( i \geq 1 \). That is,
\[
r^is_i = r^{-1}f_2 \sum_{j=1}^{i} \hat{B}_{i,j}(s)(r/f_2)^jf_j.
\]
Multiplying by \( x^i \) and summing gives the odd functional relation
\[
G(rx)/r = \sum_{j=1}^{\infty} v_j G(x)^j = r^{-2}f_2[F(w+y) - w]
\]
at \( y = rf_2^{-1}G(x) \). Since \( v_1 = v_2 = 1 \), if \( F(z) \) is quadratic, then the right hand side is \( G(x) + G(x)^2 \). If \( F(z) \) is the polynomial (7), then the right hand side is
\[
\sum_{j=1}^{p} v_j G(x)^j.
\]

4. Lagrange inversion

In this section, we deal with the issue of how to choose \( a_1 = b_1 \) in Theorems 3.1 and 3.4 for a given \( x_0 \). Maritz [2] has done this numerically for (24). Here, we give \( a_1 \) as an expansion in \( x_0 - w \). \( w \) can be chosen to minimise \( |x_0 - w| \). We use the notation (12). By (31) and (34) at \( n = 0 \),
\[
x_0 - w = \sum_{i=1}^{\infty} a_i d_i/i!,
\]
where \( d_i = i!(f_2/r)^{i-1} s_{i,1} = (F_2/r)^{i-1} s_{i,2} s_{i,1} = s_i \) of Theorem 3.1 and \( s_{i,2} = s_i \) of Theorem 3.4.

**Theorem 4.1.** Set \( g_i = d_{i+1}/(i+1) \). If
\[
x_0 - w = \sum_{i=1}^{\infty} a_i d_i/i!
\]
then
\[
a_1 = b_1 = \sum_{i=1}^{\infty} (x_0 - w)^i E_i/i!
\]
for
\[
E_i = \sum_{k=1}^{i-1} (-i)_k B_{i-1,k}(g).
\]

**Proof:** The inverse of
\[
x_0 - w = \sum_{i=1}^{\infty} e_i b_i^i/i!
\]
is given by applying Theorem E, page 150 of [1]. □
If \( x_0 \) is not solvable with respect to \( w \), then (37) diverges. We can judge convergence by plotting the partial sums
\[
\sum_{i=1}^{p} (x_0 - w)^i E_i / i!
\]
against \( p \).

For Section 2 and \( N_i \) of Corollary 2.1, \( d_i \) reduce to
\[
d_1 = 1, \quad d_i = e_i D_i^{-1} (F_2/r)^{i-1} N_i
\]
for \( i \geq 2 \).

**Example 4.1.** For Example 2.1,
\[
f_2 = c, \quad r = 2, \quad s_i,1 = 2^{i-1} / i!, \quad g_i = c^i / (i+1),
\]
\[
E_i = (i-1)!(c^i - 1), \quad a_1 = c^{-1} \ln(1-y)
\]
for \( y = (x_0 - 1/c) (-c) = 1 - cx_0 \). The expansion
\[
\ln(1-y) = -\sum_{i=1}^{\infty} y^i / i
\]
holds for \( y < 1 \), that is for \( cx_0 > 0 \).

5. Bifurcation points

Given \( F_1(z) = F(z) \), for \( N = 1, 2, \ldots \), set \( F_{N+1}(z) = F_N(F(z)) \). So, if \( F(z) \) is a polynomial of degree \( p \), then \( F_N(z) \) is a polynomial of degree \( NP \). Set \( S_N(z) = (F_N(z) - z) / (F(z) - z) \).

**Theorem 5.1.** Suppose that \( w = F(w) \). Then \( w = F_N(w) \). So, if \( w = F_N(w_N) \), then for \( M = 1, 2, \ldots, \)
\( w = F_{MN}(w_N) \). If \( F(z) \) is a polynomial of degree \( p \), then \( S_N(z) \) is a polynomial of degree \( (N-1)p \).
So, the fixed points of \( F_N(x) \) are those of \( F(z) \) and the roots of \( S_N(z) \). Also, \( x_{n+1} = F(x_n), \) \( n = 0, 1, \ldots \)
implies \( x_{n+N} = F_N(x_n), \) \( N = 0, 1, \ldots \) and
\[
(38) \quad F_{N,1}(w) = r^N,
\]
where \( r = F_1(w) \).

**Proof:** \( w = F(w) \) implies \( w = F_2(w) = F_3(w) = \cdots \). The last result follows from
\[
F_{N,1}(z) = \prod_{i=0}^{N-1} F_i(F_i(z)) ,
\]
where \( F_0(z) = z \). \( \square \)

For \( N = 2 \), (38) was given by (10) of [3] for the case \( F(z) = c(z-z^2) \) and by (7) of [4]. Table 3 of [3] catalogues the stable cycles of (2) with periods up to 6.

**Corollary 5.1.** A solution to \( x_{n+1} = F_N(x_n) \) is (31) or (32) with \( r \) replaced by \( r^N \).
Example 5.1. Suppose as in Section 2 that

\[ F(z) = \sum_{j=0}^{2} c_j z^j. \]

So,

\[ F_2(z) = \sum_{j=0}^{4} c_j z^j \]

and

\[ S_2(z) = \sum_{j=0}^{2} s_j z^j, \]

where

\[ c_{20} = c_0 s_0, \quad c_{21} - 1 = c_0 s_1 + (c_1 - 1) s_0, \quad c_{22} = c_0 s_2 + (c_1 - 1) s_1 + c_2 s_0, \]

\[ c_{23} = 2 c_1 c_2 = (c_1 - 1) s_2 + c_2 s_1, \quad c_{24} = c_2^3 = c_2 s_2 \]

since \( 0 = F_2(z) - z = (F(z) - z) S_2(z) \). So, if \( c_0 \neq 0 \), then

\[ s_0 = c_{20}/c_0, \quad s_1 = (c_{21} - 1 - (c_1 - 1) s_0)/c_0, \quad s_2 = (c_{22} - (c_1 - 1) s_1 - c_2 s_0)/c_0. \]

If \( c_0 = 0 \) and \( c_1 \neq 1 \), then

\[ s_0 = (c_{21} - 1)/c_1, \quad s_1 = (c_{22} - c_2 s_0)/(c_1 - 1), \quad s_2 = (c_{23} - c_2 s_1)/(c_1 - 1). \]

Bifurcation from period 2 to period 4 will occur when the 2 roots of \( S_2(w) \) become real. That is when

\[ s_1^2 = 4 s_0 s_2. \]

Consider the logistic map, \( F(z) = c (z - z^2) \) of Example 2.4. So, \( c_0 = 0 \), \( c_1 = c \), \( c_2 = -c \), and for \( c \neq 1 \), \( c_{20} = 0, c_{21} = c^2, c_{22} = -c^2 (1 + c), c_{23} = 2 c^3, c_{24} = -c^3 \), \( s_0 = c + 1, s_1 = -c (c + 1), s_2 = c^2 \), implying \( S_2(z) = (c + 1) (1 - c^2) + c^2 z^2 \) with roots

\[ w_2, w_3 = \frac{c + 1 \pm \delta^{1/2}}{2c}, \]

where \( \delta = (c + 1) (c - 3) \). That \( F_2(z) - z \) factorizes as \( S_2(z) (F(z) - z) \) is not obvious! Bifurcation will occur when (39) holds, that is at \( c = -1 \) and \( c = 3 \). We now give the solutions of \( x_{n+1} = F_2(x_n) \) corresponding to the 4 fixed points of \( F_2(x) \): those of \( F(z) \), that is \( w = 0 \) and \( w = w_1 = 1 - c^{-1} \), and the roots of \( S_2(z) \), (40). Apply Example 3.2 to \( F(z) = F_2(z) \). (25), (26) and (30) hold with

\[ r = F_{2,1}(w) = c^2 (1 - 2w) \left[ 1 - 2c (w - w^2) \right], \quad f_2 = -c^2 (c + 1) + 6 c^3 (w - w^2), \]

\[ f_3 = 2 c^3 (1 - 2w), \quad f_4 = -c^3, \quad U_i = c^{2i-2} - 1, \quad D_i = \prod_{j=1}^{i} U_j, \quad G(z) = \sum_{i=1}^{\infty} z^i s_i, \]

\[ s_1 = 1, \quad s_i = N_i/D_i, \quad N_2 = 1, \quad v_3 = 2 c (c + 1)^{-1}, \quad v_4 = -c (c + 1)^{-3}, \]

\[ N_3 = 2 (c^2 + 1) (c + 1)^{-1}, \quad N_4 = \left( c^3 - 8 c^4 + 7 c^3 - 13 c^2 + 6 c - 5 \right) (c + 1)^{-1}, \]

\[ x_0(0) = -(c + 1)^{-1} G(-a_1 (c + 1) c^2 n). \]
If \( w = w_1 = 1 - c^{-1} \) then \( r = (2 - c)^2, f_2 = -c(c - 2)(c - 3), f_3 = -2c^2(c - 2), v_3 = -2(c - 2)(c - 3)^{-2}, v_4 = (c - 2)(c - 3)^{-3}, U_l = (2 - c)^{2i-2} - 1, U_2 = (c - 1)(c - 3), N_3 = -2\left(c^2 - 4c + 5\right)(c - 3)^{-1} \) and \( N_4 = \left(c^7 - 8c^6 + 41c^5 - 136c^4 + 414c^3 - 1184c^2 + 1491c - 702\right)(c - 3)^{-3}. \)

Appendix A of [4] gives some other theory.

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References


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