VARIATIONAL INEQUALITIES FOR HYPERSINGULAR INTEGRALS WITH VARIABLE KERNELS

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ABSTRACT. In this paper, we systematically study variational inequalities for hypersingular integral operators. More precisely, we show the variational inequalities for the families $\mathcal{T}_\alpha := \{T_{\alpha,\varepsilon}\}_{\varepsilon > 0}$ of truncated hypersingular integrals with variable kernels, which are defined by

$$T_{\alpha,\varepsilon}f(x) = \int_{|x-y| > \varepsilon} \frac{\Omega(x, x-y) f(y)}{|x-y|^{n+\alpha}} dy,$$

where $\alpha \geq 0$ and the kernel $\Omega$ belongs to $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > \max\{1, \frac{2(n-1)}{n+2\alpha}\}$ and satisfies some cancellation condition in its second variable. The result is sharp in the sense that the $(\dot{L}^2_\alpha, L^2)$ boundedness of $T_\alpha$ fails if $q \leq \frac{2(n-1)}{n+2\alpha}$. If strengthen the smoothness of $\Omega(x, z')$ in its second variable, the authors give the weighted boundedness of the variation of the hypersingular integrals with smooth variable kernels from $\dot{L}^p_\alpha(w)$ to $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$. Finally, we extend the result to the Sobolev-Morrey spaces.

1. INTRODUCTION

A function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q \geq 1$, if it satisfies the following conditions:

$$\Omega(x, \lambda z) = \Omega(x, z), \quad \text{for all } x, z \in \mathbb{R}^n \text{ and all } \lambda > 0,$$

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there exists a constant $C$ such that the conclusion of Theorem B still holds for $q > 2(n - 1)/n$. The same conclusion was also obtained...

\begin{equation}
||\Omega||_{L^q(S^{n-1})} := \sup_{x \in \mathbb{R}^n} \left( \int_{S^{n-1}} |\Omega(x, z')|^q d\sigma(z') \right)^{\frac{1}{q}} < \infty,
\end{equation}

where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the unit sphere with Lebesgue measure $d\sigma$ on $\mathbb{R}^n$ ($n \geq 2$) and $z' = \frac{z}{|z|}$ for every $z \in \mathbb{R}^n \setminus \{0\}$.

For $\alpha \geq 0$, define the hypersingular integral $T_\alpha$ with variable kernel by

$$T_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x, y)}{|y|^{n+\alpha}} f(x - y) dy,$$

where $\Omega \in L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})$ satisfies

\begin{equation}
\int_{S^{n-1}} \Omega(x, y) Y_m(y) d\sigma(y) = 0,
\end{equation}

for all spherical harmonic polynomials $Y_m$ with degree $\leq [\alpha]$. Define the maximal hypersingular integral operator $T_{\alpha,0}$ with variable kernel by

$$T_{\alpha,0} f(x) = \sup_{\epsilon > 0} \int_{|x - y| > \epsilon} \frac{\Omega(x, y)}{|y|^{n+\alpha}} f(y) dy = \sup_{\epsilon > 0} |T_{\alpha,\epsilon} f(x)|,$$

where

\begin{equation}
T_{\alpha,\epsilon} f(x) = \int_{|x - y| > \epsilon} \frac{\Omega(x, y)}{|y|^{n+\alpha}} f(y) dy.
\end{equation}

For $\alpha = 0$, $T_0$ is the singular integral operator with variable kernels which was first studied by Calderón and Zygmund [5]. They [5] investigated the $L^2$ boundedness of the singular integral $T_0$ with variable kernel and found that these operators connect closely with the problem about the second order linear elliptic equations with variable coefficients (see also [35], [13], [19] and [20]).

**Theorem A.** ([5]) If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 2(n - 1)/n$ and satisfies

\begin{equation}
\int_{S^{n-1}} \Omega(x, y') d\sigma(y') = 0,
\end{equation}

then there exists a constant $C > 0$ such that

$$||T_0 f||_{L^2} \leq C ||\Omega||_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} ||f||_{L^2}.$$

In [5], Calderón and Zygmund showed that the the condition $q > 2(n - 1)/n$ is optimal in the sense that the $L^2$ boundedness of $T_0$ fails if $q \leq 2(n - 1)/n$.

It is well known that maximal singular integral operators play a key role in studying the convergence of the singular integral operators almost everywhere. The mapping properties of the maximal singular integrals with convolution kernels have been extensively studied (see [21] and [22], for example). In 1980, Aguilera and Harboeur [1] considered the $L^2$ boundedness of the maximal singular integral operator $T_0$ with variable kernel.

**Theorem B.** ([5]) If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 4(n - 1)/(2n - 1)$ and satisfies (1.5), then there exists a constant $C > 0$ such that

$$||T_{0,\epsilon} f||_{L^2} \leq C ||\Omega||_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} ||f||_{L^2}.$$

In 1985, using spherical harmonic expansions of the kernel, Cowling and Mauceri [16] proved that the conclusion of Theorem B still holds for $q > 2(n - 1)/n$. The same conclusion was also obtained...

**Theorem C.** ([16] or [15]) If \( \Omega \in L^{\infty}(\mathbb{R}^n) \times L^{q}(\mathbb{R}^{n-1}) \) for \( q > 2(n-1)/n \) and satisfies (1.5), then there exists a constant \( C > 0 \) such that
\[
\|T^* f\|_{L^2} \leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^{q}(\mathbb{R}^{n-1})}\|f\|_{L^2}.
\]

Obviously, the range of \( q \) in Theorem C is also optimal.

For \( \alpha > 0 \), in 2003, Chen, Fan and Ying [10] gave the boundedness of \( T_\alpha \) from the homogeneous Sobolev space \( \dot{L}^2_\alpha \) to the Lebesgue space \( L^2 \) (see [23] for the definition of \( \dot{L}^2_\alpha \)).

**Theorem D.** ([10]) Let \( \alpha > 0 \). If \( \Omega \in L^{\infty}(\mathbb{R}^n) \times L^{q}(\mathbb{R}^{n-1}) \) for \( q > \max\{1, 2(n-1)/n+2\alpha\} \) and satisfies (1.3), then there is a constant \( C > 0 \) such that
\[
\|T_\alpha f\|_{L^2} \leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^{q}(\mathbb{R}^{n-1})}\|f\|_{L^2_\alpha}.
\]

In the present paper, we would like to study the variational inequalities for hypersingular integrals with variable kernels. Let \( \mathcal{F} = \{F_t : t \in I \subset \mathbb{R}\} \) be a family of Lebesgue measurable functions defined on \( \mathbb{R}^n \). For \( x \) in \( \mathbb{R}^n \), the value of the \( q \)-variation function \( V_q(\mathcal{F}) \) of the family \( \mathcal{F} \) at \( x \) is defined by
\[
V_q(\mathcal{F})(x) := \sup \left( \sum_{k \geq 1} |F_{t_k}(x) - F_{t_{k-1}}(x)|^q \right)^{\frac{1}{q}}, \quad q \geq 1,
\]
where the supremum runs over all increasing subsequences \( \{t_k : k \geq 0\} \subset I \). Suppose \( \mathcal{A} = \{A_t\}_{t \in I} \) is a family of operators on \( L^p(\mathbb{R}^n) \) \((1 \leq p \leq \infty)\), the associated strong \( q \)-variation operator \( V_q(\mathcal{A}) \) is defined as
\[ V_q(\mathcal{A})(f)(x) = V_q(\{A_t f(x)\}_{t \in I}). \]

There are two elementary but important observations that motivate the development of variational inequalities in ergodic theory and harmonic analysis. The first one is that from the fact that \( V_q(\mathcal{F})(x) < \infty \) with finite \( q \) implies the convergence of \( F_t(x) \) as \( t \to t_0 \) whenever \( t_0 \) is a fixed point for \( I \), it is easy to observe that \( A_t(f) \) converges a.e. for \( f \in L^p \) whenever \( V_q(\mathcal{A}) \) for finite \( q \) is of weak type \((p, p)\) with \( p < \infty \). The second one is that \( q \)-variation function dominate pointwisely the maximal function: for any \( q \geq 1 \),
\[ A^*(f)(x) \leq A_{t_0}f(x) + V_q(\mathcal{A})(f)(x), \]
where \( A^* \) is the maximal operator defined by \( A^*(f)(x) := \sup_{t \in I} |A_t(f)(x)| \) and \( t_0 \in I \) is any fixed number.

The first variational inequalities was proved by Lépingle [29] for general martingale in 1976. Later, Bourgain [3] used Lépingle’s result to obtain corresponding variational estimates for ergodic counterpart which considerably improves the classical Dunford-Schwartz maximal ergodic inequality. Since then, many works on the variational inequality have appeared in harmonic analysis and ergodic theory (cf. e.g. [14], [4] [8], [9], [17], [18], [24], [25], [28], [31], [32], [33], [34], [39] and [40]).

For \( \alpha \geq 0 \), let \( T_{\alpha,\varepsilon} \) be defined as in (1.4). For \( \rho \geq 1 \), the \( \rho \)-variation operator \( V_\rho(T_{\alpha,f}) \) generated by the family of the hypersingular integrals with variable kernels \( T_{\alpha} = \{T_{\alpha,\varepsilon}\}_{\varepsilon > 0} \) is defined by
\[
V_\rho(T_{\alpha,f})(x) = \|\{T_{\alpha,\varepsilon}(f)(x)\}_{\varepsilon > 0}\|_{V_\rho} = \sup \left( \sum_{k \geq 1} |T_{\alpha,\varepsilon_k}(f)(x) - T_{\alpha,\varepsilon_{k-1}}(f)(x)|^\rho \right)^{\frac{1}{\rho}},
\]
where the supremum takes over all finite increasing sequences \( \{\varepsilon_k : k \geq 1\} \) of positive number.
When $\alpha = 0$. If $\Omega(x, y') = \Omega(y') \in L\log^k L(\mathbb{S}^{n-1})$ and satisfies (1.3) for $\rho > 2$, Campbell et al [9] gave the $L^p(1 < p < \infty)$ boundedness of the $\rho$-variation of the operator of $\mathcal{T}_0$. In 2008, using the Fourier transform and the square function estimates given in [21], Jones, Seeger and Wright [25] developed a general method, which allows one to obtain some jump inequalities for families of the truncated singular integral operators $\mathcal{T}_0$ when $\Omega(x, y') = \Omega(y')$ and of other integral operators arising from harmonic analysis.

Motivated by Theorem C, it is natural to ask whether the $\rho$-variation of the hypersingular integral with variable kernel is still from the homogeneous Sobolev space $L^2_\alpha$ to the Lebesgue space $L^2$ with the same condition in Theorem D? In this paper, we will give a positive answer to this problem.

**Theorem 1.1.** For $\alpha \geq 0$, let $\mathcal{T}_\alpha$ be the family of the hypersingular integrals with variable kernels. Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^p(\mathbb{S}^{n-1})$ for $q > \max\{1, \frac{2(n-1)}{n+2\alpha}\}$ and satisfies (1.3), then there exists a constant $C > 0$ such that

$$\|V_\rho(\mathcal{T}_\alpha f)\|_{L^2} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^p(\mathbb{S}^{n-1})}\|f\|_{L^2},$$

holds for all $\rho > 2$.

**Remark 1.2.** In Section 2 of this paper (see Proposition 2.3) we shall show the sharpness in the sense that for no $n$ can we replace the exponent $\frac{2(n-1)}{n+2\alpha}$ of Theorem 1.1 by a smaller one.

**Remark 1.3.** It is not difficult to find that Theorem 1.1 is an improvement and extension of Theorem C and Theorem D, because of the pointwise estimate

$$T^*_\alpha f(x) \leq V_\rho(\mathcal{T}_\alpha f)(x) \text{ for } \rho > 2,$$

and even for $\alpha > 0$

$$\|T^*_\alpha f\|_{L^2} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^p(\mathbb{S}^{n-1})}\|f\|_{L^2},$$

is also a new result.

If $\Omega(x, y')$ was assumed to be a very stronger smoothness on $\mathbb{S}^{n-1}$ in its second variates, we will establish the $(L^p_\alpha(w), L^p(w))$ boundedness of $V_\rho(\mathcal{T}_\alpha f)$ for $1 < p < \infty$ and $w \in A_p$. For $1 < p < \infty$, we say that non-negative locally integrable function $w$ belong to $A_p$ if there exists a constant $C > 0$ such that for any cube $Q$

$$\sup_{Q} \left(\frac{1}{|Q|} \int_{Q} w(x) dx\right)\left(\frac{1}{|Q|} \int_{Q} w(x)^{1-p'} dx\right)^{p-1} \leq C,$$

where $p' = p/(p-1)$ denotes the dual exponent of $p$. For $\alpha \geq 0$, $1 < p < \infty$, $w \in A_p$, $L^p_\alpha(w)$ is the homogeneous weighted Sobolev space, which is defined by

$$\|f\|_{L^p_\alpha(w)} = \left(\int_{\mathbb{R}^n} |D^\alpha f(x)|^p w(x) dx\right)^{1/p},$$

where $D^\alpha$ is a fractional order differentiation operator defined by $D^\alpha f(\xi) = |\xi|^\alpha \hat{f}(\xi)$. When $\alpha = 0$, $L^p_\alpha(w) = L^p(w)$.

The second result of this paper is stated as follows.

**Theorem 1.4.** For $\alpha \geq 0$, let $\mathcal{T}_\alpha$ be the family of the hypersingular integral with variable kernels. Suppose that $\Omega$ satisfies (1.3) and

$$\max_{|\beta| \leq 2n} \|\partial^{\beta}/\partial y^{\beta}\Omega(x, y')\|_{L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})} < \infty.$$
For all $\rho > 2$, there exists a constant $C > 0$ such that
\begin{equation}
(1.10) \quad \|V_\rho(T_\alpha f)\|_{L^p(w)} \leq C\|f\|_{L^p_\infty(w)}
\end{equation}
for $1 < p < \infty$ and $w \in A_\rho$.

The result of Theorem 1.4 can also be extended to the homogeneous Sobolev-Morrey spaces. In 1938, the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey (see [30]), which is defined by
\[
\|f\|_{L^{p,\lambda}} = \left( \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \int_{Q(x,r)} |f(y)|^p dy \right)^{1/p},
\]
where $Q(x, r)$ stands for any cube of radius $r$ and centered at $x$, $\lambda \in (0, n)$. For $\alpha \geq 0$, $L^{p,\lambda}_\alpha(\mathbb{R}^n)$ is the homogeneous Sobolev-Morrey spaces, which is defined by
\[
\|f\|_{L^{p,\lambda}_\alpha} = \left( \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \int_{Q(x,r)} |D^\alpha f(y)|^p dy \right)^{1/p}.
\]
When $\alpha = 0$, $L^{p,\lambda}_\alpha = L^{p,\lambda}$. It is well known that the Morrey space is connected to certain problems in elliptic PDEs. Later, the Morrey spaces were found to have many important applications to the Navier-Stokes equations, the Schrödinger equations, elliptic equations and potential analysis ([20], [27], [37], [19]).

We state our the third result of this paper as follows.

**Theorem 1.5.** For $\alpha \geq 0$, let $T_\alpha$ be the family of the hypersingular integral with variable kernels. Suppose that $\Omega$ satisfies (1.3) and (1.9). For all $\rho > 2$, there exists a constant $C > 0$ such that
\begin{equation}
(1.11) \quad \|V_\rho(T_\alpha f)\|_{L^p_\infty} \leq C\|f\|_{L^p_\infty},
\end{equation}
for $1 < p < \infty$ and $\lambda \in (0, n)$.

The rest of this paper is organized as follows. In Section 2, we will show Theorem 1.1. We establish the proof of Theorem 1.4 and Theorem 1.5 in Section 3.

The notations "$\wedge$" and "$\vee$" denote the Fourier transform and the inverse Fourier transform, respectively. The letter $C$ stands for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence.

2. **Proof of Theorem 1.1 and Sharpness of Theorem 1.1**

As in [25], to obtain the $\rho$-variation estimate by separately proving the long and short variational estimates. That is, we are reduced to proving for $\rho > 2$,
\begin{equation}
(2.1) \quad \|V_\rho((T_{\alpha,2^k} f)_{k \in \mathbb{Z}})\|_{L^2} \leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^q(S^{n-1})}\|f\|_{L^2},
\end{equation}
where
\[
T_{\alpha, 2^k} f(x) = \int_{|y| > 2^k} \frac{\Omega(x, y')}{|y|^{n+\alpha}} f(x - y) dy,
\]
and
\begin{equation}
(2.2) \quad \|S_2(T_\alpha f)\|_{L^2} \leq C\|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^q(S^{n-1})}\|f\|_{L^2},
\end{equation}
where
\[
S_2(T_\alpha f)(x) = \left( \sum_{k \in \mathbb{Z}} [V_{2^k}(T_\alpha f)(x)]^2 \right)^{1/2}
\]
with
\[ V_{2,k}(T_\alpha f)(x) = \left( \sup_{t_1 < \cdots < t_j} \sum_{l=1}^{j-1} |T_{\alpha,t_{l+1}} f(x) - T_{\alpha,t_l} f(x)|^2 \right)^{1/2}. \]

2.1. **Proof of (2.1).** Denote \( \mathcal{H}_m \) by the space of spherical harmonics of degree \( m \), and \( \mathcal{H}_m \) is a finite-dimensional vector space, let \( \dim \mathcal{H}_m = D_m \). As in [7], by a limit argument we may reduce the proof of Theorem 1.1 to the case that \( f \in C_0^\infty \) and
\[ \Omega(x, z') = \sum_{m \geq 0} \sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(z') \]
is a finite sum, where each \( Y_{m,j} \) is a spherical harmonic polynomial of degree \( m \) and
\[ a_{m,j}(x) = \int_{S^{n-1}} \Omega(x, y') \tilde{Y}_{m,j}(y') d\sigma(y'). \]
Notice that \( \Omega \) satisfies (1.3) so \( a_{m,j} \equiv 0 \) for \( m = 0, \ldots, [\alpha] \). Therefore, we get actually
\[ \Omega(x, z') = \sum_{m = [\alpha] + 1}^{\infty} \sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(z'). \]
Denote by
\[ T_{\alpha,m,j} f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{Y_{m,j}(y')}{|y'|^{n+\alpha}} f(x - y) dy \]
and
\[ T_{\alpha,m,j,x} f(x) = \int_{|y| < \epsilon} \frac{Y_{m,j}(y')}{|y'|^{n+\alpha}} f(x - y) dy, \]
for \( m \geq [\alpha] + 1, j = 1, 2, \ldots, D_m \). Then we can write for \( \epsilon > 0, \)
\[ T_{\alpha,x} f(x) = \sum_{m = [\alpha] + 1}^{\infty} \sum_{j=1}^{D_m} T_{\alpha,m,j,x} f(x). \]
By the definition of the strong \( \rho \)-variation function, we have
\[ V_{\rho}(\{T_{\alpha,2^k} f\}_{k \in \mathbb{Z}})(x) = \sup \left( \sum_{l \geq 1} |T_{\alpha,2^l} f(x) - T_{\alpha,2^{l-1}} f(x)|^\rho \right)^{1/\rho}, \]
where the supremum runs over all increasing sequences \( \{2^k : t \geq 1\} \). Then by the Minkowski inequality and the Hölder inequality twice, we get for \( 0 < \theta < 1, \)
\[ \left( V_{\rho}(\{T_{\alpha,2^k} f\}_{k \in \mathbb{Z}})(x) \right)^2 \leq C \left( \sum_{m = [\alpha] + 1}^{\infty} \sum_{j=1}^{D_m} a_{m,j}^2(x) m^{-\theta(1+2\alpha)} \right) \sum_{m = [\alpha] + 1}^{\infty} m^{\theta(1+2\alpha)} \sum_{j=1}^{D_m} \left( \sup_{l \geq 1} |T_{\alpha,m,j,2^l} f(x) - T_{\alpha,m,j,2^{l-1}} f(x)|^\rho \right)^{1/\rho} \right)^2. \]
Let \( \phi_k \) show that for some \( \eta \) where \( q \equiv \| \varphi \| \leq 1 \) and close 1 sufficiently, then,

\[
\sum_{m=|\alpha|+1}^{\infty} \sum_{j=1}^{D_m} a_{m,j}^2(x)m^{-\theta(1+2\alpha)} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^1(\mathbb{S}^{n-1})}.
\]

By (2.5) and (2.6), we have

\[
\|V_p((T_{\alpha,m}^j f)_{k \in \mathbb{N}})\|_{L^2}^2 \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^1(\mathbb{S}^{n-1})} \sum_{m=|\alpha|+1}^{\infty} \sum_{j=1}^{D_m} m^{\theta(1+2\alpha)} \left\| \left( \sum_{j=1}^{D_m} (V_p((T_{\alpha,m}^j f)_{k \in \mathbb{N}}))^2 \right)^{1/2} \right\|_{L^2}^2.
\]

Let \( \eta = \theta(1 + 2\alpha) \). Taking 0 < \( \theta < 1 \) and close 1 sufficiently such that \( 1 - \eta + 2\alpha < 1 \). We need to show that for some 0 < \( \beta < 1 - \eta + 2\alpha \) such that

\[
\left\| \left( \sum_{j=1}^{D_m} (V_p((T_{\alpha,m}^j f)_{k \in \mathbb{N}}))^2 \right)^{1/2} \right\|_{L^2}^2 \leq Cm^{-1+\beta/2-\alpha}\|f\|_{L^2}^2.
\]

For \( k \in \mathbb{Z}, m \geq |\alpha| + 1 \) and \( j = 1, 2, \ldots, D_m \), set

\[
\sigma_{\alpha,k,m,j} \ast f(x) = \int_{|y| \leq m+1} \frac{Y_{m,j}(y')}{|y|^{\alpha+1}} f(x-y)dy.
\]

Let \( \phi(x) = \phi(|x|) \in \mathcal{S}(\mathbb{R}^n) \) such that \( \phi(\xi) = 1 \) for \( |\xi| \leq 2, \phi(\xi) = 0 \) for \( |\xi| > 4 \). We decompose

\[
T_{\alpha,m}^j f(x) = \phi_k \ast T_{\alpha,m}^j f(x) + \sum_{s \geq 0} (\delta_0 - \phi_s) \ast \sigma_{\alpha,k,s,m,j} \ast f(x) - \phi_k \ast \sum_{s \geq 0} \sigma_{\alpha,k,s,m,j} \ast f(x)
\]

\[
= T_{k,m,j}^1 f(x) + T_{k,m,j}^2 f(x) - T_{k,m,j}^3 f(x),
\]

where \( \phi_k(x) = 2^{-k\alpha} \phi(2^{-k}x) \) with \( \phi(\xi) = \phi(2^k \xi) \), \( \delta_0 \) is the Dirac measure at 0. Let \( T_{m,j}^i \) be the family \( \{T_{k,m,j}^i\}_{k \in \mathbb{N}} \) for \( i = 1, 2, 3 \). To prove (2.7), it suffices to prove that for 0 < \( \beta < 1 - \eta + 2\alpha \),

\[
\left\| \left( \sum_{j=1}^{D_m} (V_p((T_{m,j}^i f))^2) \right)^{1/2} \right\|_{L^2}^2 \leq Cm^{-1+\beta/2-\alpha}\|f\|_{L^2}^2, \quad i = 1, 2, 3.
\]

**Estimate of (2.8) for \( i = 1 \).** By \( \|V_p((\phi_k \ast f)_{k \in \mathbb{N}})\|_{L^2} \leq C\|f\|_{L^2} \) (see [25]) and

\[
\sum_{j=1}^{D_m} \|T_{\alpha,m,j} f\|^2_{L^2} \leq C m^{-2-2\alpha}\|f\|_{L^2}^2
\]

(see [10, (5.9)]), we have

\[
\left\| \left( \sum_{j=1}^{D_m} (V_p((T_{m,j}^1 f))^2) \right)^{1/2} \right\|_{L^2}^2 \leq C \sum_{j=1}^{D_m} \|V_p((T_{m,j}^1 f))\|_{L^2}^2 \leq C m^{-2-2\alpha}\|f\|_{L^2}^2.
\]
Estimate of (2.8) for \( i = 2 \). By the Minkowski inequality, we get

\[
\left( \sum_{j=1}^{D_n} (V_p(T_{m,j}^2f)(x))^2 \right)^{1/2} \leq C \left( \sum_{j=1}^{D_n} \sum_{k \in \mathbb{Z}} \left| \sum_{s \geq 0} (\delta_0 - \phi_k) * \sigma_{\alpha,k+s,m,j} * f(x) \right|^2 \right)^{1/2}.
\]

Let \( \psi(\xi) = \psi(|\xi|) \in C_c^\infty \) such that \( 0 \leq \psi \leq 1 \), \( \text{supp} \psi \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \) and \( \sum_{k \in \mathbb{Z}} \psi^2(-i\xi) = 1 \) for \( |\xi| \neq 0 \). Define the multiplier \( \Delta_l \) by \( \Delta_l f(\xi) = \psi(2^{-l}\xi) \hat{f}(\xi) \). Then by the Minkowski inequality,

\[
(2.9) \quad \left( \sum_{j=1}^{D_n} (V_p(T_{m,j}^2f)(x))^2 \right)^{1/2} \leq C \sum_{l \geq 0} \left( \sum_{j=1}^{D_n} \sum_{k \in \mathbb{Z}} \left| (\delta_0 - \phi_k) * \sigma_{\alpha,k+s,m,j} * \Delta_{l-k}^2 f(x) \right|^2 \right)^{1/2}.
\]

We are reduced to establishing the estimate of \( \|G_{s,l}f\|_{L^2} \) for \( l \in \mathbb{Z} \) and \( s \geq 0 \). However, we have the following lemmas.

**Lemma 2.1.** (see [12]) Suppose that \( \alpha \geq 0, 0 < \beta < 1, k \in \mathbb{Z} \). Let

\[
\sigma_{\alpha,k,m,j}(x) = \frac{Y_{m,j}(x')}{|x|^\alpha \chi_{[2^k \leq |x| \leq 2^{k+1}]}(x)}.
\]

Then

\[
|\sigma_{\alpha,k,m,j}(\xi)| \leq C|\xi|^{m-\alpha^1-\alpha}|2^k\xi|^{m-\alpha^1}|2^{-s}Y_{m,j}(\xi)|, \quad |2^k\xi| \leq 1,
\]

(2.10)

\[
|\sigma_{\alpha,k,m,j}(\xi)| \leq C|\xi|^{m-\alpha^1+\beta/2-\alpha^2}|2^k\xi|^{-\beta/2}|2^{-s}Y_{m,j}(\xi)|, \quad |2^k\xi| > 1,
\]

(2.11)

where \( \varrho = (n-2)/2 \).

Since

\[
\text{supp}(1 - \hat{\phi}_k) \subset \{ \xi : |2^k\xi| > 1 \},
\]

then combined with (2.11) by taking \( 0 < \beta < 1 - \eta + 2\alpha \), we have for \( \varrho = \frac{n-\vartheta}{2} \),

\[
|1 - \hat{\phi}_k(\xi)||\sigma_{\alpha,k,m,j+s}(\xi)| \leq C|\xi|^{m-\alpha^1-\alpha^2}|2^{-\alpha^2}|2^{-s}|2^{-\beta/2}|2^{-s}Y_{m,j}(\xi)|.
\]

(2.12)

Since \( \text{supp} \psi \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \), it is easy to see that

\[
|\psi(2^{-l}\xi)||1 - \hat{\phi}_k(\xi)||\sigma_{\alpha,k,m,j+s}(\xi)| \leq C|\xi|^{m-\alpha^1-\alpha^2}|2^{-\alpha^2}|2^{-s}2^{-\beta/2}|2^{-s}Y_{m,j}(\xi)|.
\]

(2.13)

Applying the Plancherel theorem, (2.13), \( \sum_{j=1}^{D_n} |Y_{m,j}(\xi)|^2 = m^{2\varrho} \) and Littlewood-Paley theory, we get

\[
\|G_{s,l}f\|_{L^2}^2 \leq C \sum_{j=1}^{D_n} \sum_{k \in \mathbb{Z}} \|2^s \sigma_{\alpha,k,j+s} \Delta_{l-k}^2 f(\xi)\|^2_{L^2} \leq Cm^{-2+\beta-2\alpha}2^{-s}2^{|s|} \|f\|_{L^2}^2.
\]

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Then by (2.9) we get
\[ \left\| \left( \sum_{j=1}^{D_n} (V_p(T_{m,j}^2 f))^2 \right)^{1/2} \right\|_{L^2} \leq C \left( m^{1+\beta/2-a} \sum_{s \geq 0} \sum_{k \in \mathbb{Z}} 2^{-\beta s/2} \min\{2^{-\beta/2}, 2^l\} \|f\|_{L^2} \right) \]
\[ \leq C m^{1+\beta/2-a} \|f\|_{L^2}. \]

**Estimate of (2.8) for i = 3.** Similarly, we have
\[ \left( \sum_{j=1}^{D_n} (V_p(T_{m,j}^3 f)(x))^2 \right)^{1/2} \leq C \left( \sum_{j=1}^{D_n} \left( \sum_{k \in \mathbb{Z}} |\phi_k \ast \sigma_{\alpha,k} \ast f(x)|^2 \right) \right)^{1/2} \]
By the Minkowski inequality and \( \sum_{t \in \mathbb{Z}} \Delta_t^2 f = f \), we get
\[ \left( \sum_{j=1}^{D_n} (V_p(T_{m,j}^3 f)(x))^2 \right)^{1/2} \leq C \sum_{s \leq \beta} \left( \sum_{j=1}^{D_n} \left( \sum_{k \in \mathbb{Z}} |\phi_k \ast \sigma_{\alpha,k} \ast f(x)|^2 \right) \right)^{1/2} \]
\[ \leq C \sum_{s \leq \beta} \left( \sum_{j=1}^{D_n} \left( \sum_{k \in \mathbb{Z}} |\phi_k \ast \sigma_{\alpha,k} \ast f(x)|^2 \right) \right)^{1/2} =: \sum_{s \leq \beta} \sum_{j=1}^{D_n} H_{s,j} f(x). \]

Therefore, we need to take about the \( L^2(\mathbb{R}^n) \) boundedness of \( H_{s,j} f \) for \( l \in \mathbb{Z} \) and \( s < 0 \). Since \( \text{supp} \tilde{\phi}_k \subset \{ \xi : |2^k \xi| < 1 \} \), by (2.10), we get for \( \varrho = \frac{\alpha}{2} \),
\[ |\tilde{\phi}_k(\xi)| |\sigma_{\alpha,k} \ast f(\xi)| \leq C |\xi|^{\alpha} m^{-1-\varrho-\alpha} 2^{s(|\alpha|+1-\alpha)} \min\{2^k, 2^{k+1}\} \|f\|_{L^2}. \]

Since \( \text{supp} \varphi \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \), then (2.14)
\[ |\varphi(2^{-l}\xi)||\tilde{\phi}_k(\xi)| |\sigma_{\alpha,k} \ast f(\xi)| \leq C |\xi|^{\alpha} m^{-1-\varrho-\alpha} 2^{s(|\alpha|+1-\alpha)} \min\{2^{-l}, 2^{k+1}\} \|f\|_{L^2}. \]

Applying the above estimates, \( \sum_{j=1}^{D_n} |Y_{m,j}(\xi')|^2 \sim m^{2\varrho} \) and Littlewood-Paley theory, we get
\[ \|H_{s,j} f\|_{L^2} \leq C \int_{\mathbb{R}^n} \sum_{j=1}^{D_n} \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} 2^{\alpha s} m^{-2-2\varrho-2\alpha} 2^{s(|\alpha|+1-\alpha)} \min\{2^{-l}, 2^{k+1}\} \|f\|_{L^2} \right| \Delta_{s+k}(f) \right) d\xi \]
\[ \leq C m^{-2-2\alpha} 2^{2s(|\alpha|+1-\alpha)} \min\{2^{-l}, 2^{k+1}\} \|f\|_{L^2} \leq C m^{-2-2\alpha} 2^{2s(|\alpha|+1-\alpha)} \min\{2^{-l}, 2^{k+1}\} \|f\|_{L^2}. \]

Therefore, we get
\[ \left( \sum_{j=1}^{D_n} (V_p(T_{m,j}^3 f))^2 \right)^{1/2} \|_{L^2} \leq C m^{-1-\alpha} \sum_{s \leq \beta} \sum_{k \in \mathbb{Z}} 2^{s(|\alpha|+1-\alpha)} \min\{2^{-l}, 2^{k+1}\} \|f\|_{L^2} \]
\[ \leq C m^{-1-\alpha} \|f\|_{L^2}. \]

Therefore we finish the proof of (2.1).
2.2. Proof of (2.2). For \( t \in [1, 2] \) and \( k \in \mathbb{Z} \), we first define \( T_{a,k,t} \) by
\[
T_{a,k,t}f(x) = \int_{2^{|t-k|} \leq 2^{|t|}} \frac{\Omega(x,y)}{|y|^{n+a}} f(x-y) dy.
\]
By the Minkowski inequality and \( \sum_{k \in \mathbb{Z}} \Delta^2 f = f \), we have
\[
(2.15) \quad S_2(T_a f)(x) = \left( \sum_{k \in \mathbb{Z}} \|T_{a,k,t}f(x)\|_{\mathbb{R}^n}^2 \right)^{1/2} \leq \sum_{k \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \|T_{a,k,t} \Delta^2 f(x)\|_{\mathbb{R}^n}^2 \right)^{1/2} =: \sum_{k \in \mathbb{Z}} S_2(T_a f)(x).
\]
Thus, it suffices to give the estimate of \( ||S_2(T_a f)||_{L^2} \) for \( l \in \mathbb{Z} \). Since \( ||a||_{V_2} \leq C ||a||_{X}^{1/2} \cdot ||a||_{X}^{1/2} \), where \( X = L^2([1, 2], \frac{1}{t}) \) (see [25]), then
\[
[S_2(T_a f)]^2 \leq C \sum_{k \in \mathbb{Z}} \|T_{a,k,t} \Delta^2 f(x)\|_{L^2([1, 2])} \cdot \| \frac{d}{dt}(T_{a,k,t} \Delta^2 f(x))\|_{L^2([1, 2])}.
\]
By the Cauchy-Schwarz inequality, we have
\[
(2.16) \quad ||S_2(T_a f)||_{L^2}^2 \leq \left( \sum_{k \in \mathbb{Z}} \|T_{a,k,t} \Delta^2 f\|_{L^2([1, 2])} \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \left( \frac{d}{dt}(T_{a,k,t} \Delta^2 f)\right)_{L^2([1, 2])} \right)^{1/2} =: ||I_{1,2}(T_a f)||_{L^2}^2.
\]
We now estimate \( ||I_{1,2}(T_a f)||_{L^2}^2 \). By (2.3), the Hölder inequality twice and (2.6) by taking \( 0 < \theta < 1 \) and close 1 sufficiently, we get
\[
||I_{1,2}(T_a f)||_{L^2}^2 \leq \left( \sum_{k \in \mathbb{Z}} \int_1^2 \int \left| \sum_{m=|\alpha|+1}^{D_n} \sum_{j=1}^\infty a_{m,j}(\cdot) \sigma_{a,k,t,m,j} \ast \Delta^2 f \right|^2 \frac{dt}{t} \right)^{1/2} \leq C ||\Omega||_{L^2(\mathbb{R}^n) \times L^2(\mathbb{S}^{n-1})} \int_1^2 \left( \sum_{k \in \mathbb{Z}} \sum_{m=|\alpha|+1}^{D_n} \sum_{j=1}^\infty \left| \sigma_{a,k,t,m,j} \ast \Delta^2 f \right|^2 \right)^{1/2} \frac{dt}{t} = C ||\Omega||_{L^2(\mathbb{R}^n) \times L^2(\mathbb{S}^{n-1})} \int,
\]
where \( \sigma_{a,k,t,m,j}(x) = \frac{Y_m(x)}{|m|^{2\alpha}} X_{2^{|t-k|} \leq 2^{|t|}} \) for \( t \in [1, 2] \). By Lemma 2.1, we get for \( 0 < \beta < 1 \) and \( q = \frac{n-2}{2} \),
\[
|\sigma_{a,k,t,m,j}(\xi)| \leq C |\xi|^{n} m^{-1-\beta/2-q} \min(2^k \xi^{1-\beta}, 2^k \xi^{1-q}) Y_m(\xi'),
\]
uniformly in \( t \in [1, 2] \). Therefore, by the Plancherel theorem, \( \sum_{j=1}^\infty |Y_{m,j}(\xi')|^2 \sim m^{q} \), Littlewood-Paley theory and taking \( 0 < \theta < 1 \) and close 1 sufficiently such that \( 1 - \theta (1 + 2\alpha) + 2\alpha < 1 \) and \( 0 < \beta < 1 - \theta (1 + 2\alpha) + 2\alpha \), we get
\[
I = \sum_{m=|\alpha|+1}^{D_n} \sum_{j=1}^\infty \int \int \left| Y_{m,j}(\xi') \right|^2 \left| \sigma_{a,k,t,m,j}(\xi) \right|^2 |\Delta^{2} f(\xi)|^2 \frac{d\xi}{\xi} \frac{dt}{t}
\]
where

Lemma 2.2. Then by the Minkowski inequality, we have

By (1.3), we have the following elementary fact, that is, for any Schwartz function

Then we get

Case 1. We need the following lemma

Next, we consider \( \|I_{2,l,f}\|_{L^2} \). By the Minkowski inequality, we get

By (1.3), we have the following elementary fact, that is, for any Schwartz function \( h \),

We will estimate \( \|I_{2,l,f}\|_{L^2} \) into three different cases. 1. \( 0 < \alpha < 1 \); 2. \( \alpha > 1 \) and is not an integer; 3. \( \alpha \) is a nonnegative integer.

Case 1. \( 0 < \alpha < 1 \). We need the following lemma

Lemma 2.2. ([11]) Suppose that \( 0 < \alpha < 1, 1 < r < \infty, D^r B \in L^r \). Then

where \( M \) is Hardy-Littlewood Maximal operator, \( b = D^r B \).

Then by Lemma 2.2 and \( t \in [1, 2] \), we have

Then by the Minkowski inequality, we have

(2.19) \[
\left\| \left( \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{a,k} \Delta_{i-k} f(t)^2 \right| \right)^{1/2} \right\|_{L^2} \leq C \int_{S^{n-1}} |\Omega(x, y)| (MD^r h(x - 2^j y') + MD^r h(x)) d\sigma(y').
\]
Then by (2.18),
\begin{equation}
\left\| \left( \sum_{k \in \mathbb{Z}} |M(f_k)|^2 \right) \right\|_{L^2}^{1/2} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right) \right\|_{L^2}^{1/2}
\end{equation}
(see [2]), and Littlewood-Paley theory, we have
\begin{equation}
\|I_{2,j}f\|_{L^2} \leq C \| \left( \sum_{k \in \mathbb{Z}} |M(D^\alpha \Delta_{l-k} f)|^2 \right) \|_{L^2}^{1/2} \left( \int_0^1 \left\| \left( \sum_{k \in \mathbb{Z}} |M(D^\alpha \Delta_{l-k} f)|^2 \right) \right\|_{L^2} \right)^{1/2} dt
\end{equation}
\begin{align*}
&\leq C \|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \left\| \left( \sum_{k \in \mathbb{Z}} |\Delta_{l-k}^2 D^\alpha f|^2 \right) \right\|_{L^2}^{1/2} \\
&\leq C \|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \|f\|_{L^2}^{1/2}.
\end{align*}

**Case 2.** \(\alpha > 1(\alpha)\) is not an integer). By Taylor’s Theorem and (1.3), we have
\begin{align*}
\left| \frac{d}{dt} T_{a,k,h}(x) \right| &= \left| -2^{-j_0} \frac{1}{t^{1+\alpha}} \int_{\mathbb{S}^{n-1}} \Omega(x, y') (h(x - 2^j ty') - h(x)) d\sigma(y') \right| \\
&= \left| -2^{-j_0} \frac{1}{t^{1+\alpha}} \int_{\mathbb{S}^{n-1}} \Omega(x, y') \sum_{|\beta| = |\alpha|} \int_0^1 (1 - s)^{1-|\alpha|} (D^\beta h)(x - 2^{j} s ty') |2^{j} ty'|^\beta d\sigma(y') \right| \\
&= \left| -2^{-j_0} \frac{1}{t^{1+\alpha}} \int_{\mathbb{S}^{n-1}} \Omega(x, y') \sum_{|\beta| = |\alpha|} \int_0^1 (1 - s)^{1-|\alpha|} |(D^\beta h)(x - 2^{j} s ty') - (D^\beta h)(x)| |2^{j} ty'|^\beta d\sigma(y') \right|
\end{align*}
Since \(0 < \alpha - |\alpha| < 1\), applying Lemma 2.2 and \(t \in [1, 2]\), we get
\begin{equation}
\left| \frac{d}{dt} T_{a,k,h}(x) \right| \leq 2^{-j_0} \frac{1}{t^{1+\alpha}} \int_{\mathbb{S}^{n-1}} |\Omega(x, y')| \sum_{|\beta| = |\alpha|} \int_0^1 \left( M(D^{\alpha-|\alpha|} D^\beta h)(x - 2^{j} s ty') + M(D^{\alpha-|\alpha|} D^\beta h)(x) |2^{j} ty'|^\beta d\sigma(y') \right) \\
\leq C \int_{\mathbb{S}^{n-1}} |\Omega(x, y')| \sum_{|\beta| = |\alpha|} \int_0^1 \left( M(D^{\alpha-|\alpha|} D^\beta h)(x - 2^{j} s ty') + M(D^{\alpha-|\alpha|} D^\beta h)(x) \right) d\sigma(y').
\end{equation}

Then by (2.22) we get
\begin{equation}
\left\| \left( \sum_{k \in \mathbb{Z}} \left| \frac{d}{dt} T_{a,k,h} \Delta_{l-k}^2 f \right| \right)^{1/2} \right\|_{L^2} \leq C \|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \sum_{|\beta| = |\alpha|} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \left| M(D^{\alpha-|\alpha|} D^\beta \Delta_{l-k}^2 f)(x) \right|^2 dx.
\end{equation}
Then by the above inequality, (2.18), (2.20) and Littlewood-Paley theory, we have
\begin{equation}
\|I_{2,j}f\|_{L^2} \leq C \|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \sum_{|\beta| = |\alpha|} \left\| \left( \sum_{k \in \mathbb{Z}} |M(D^{\alpha-|\alpha|} D^\beta \Delta_{l-k}^2 f)|^2 \right) \right\|_{L^2}^{1/2} \\
&\leq C \|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \sum_{|\beta| = |\alpha|} \left\| M(D^{\alpha-|\alpha|} D^\beta f) \right\|_{L^2} \\
&\leq C \|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \sum_{|\beta| = |\alpha|} \left\| f \right\|_{L^2}^{1/2} \leq C \|\Omega\|_{L^{\infty}(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \|f\|_{L^2}^{1/2}.
\end{equation}
Case 3. $\alpha$ is a nonnegative integer. Note that $t \in [1, 2]$, by Taylor’s Theorem and (1.3), we have
\[
\frac{d}{dt} T_{\alpha,k,t} h(x) = 2^{-\frac{\beta}{2}} \frac{1}{t^{1+\alpha}} \int_{\mathbb{S}^{n-1}} |\Omega(x, y')(h(x - 2^{j}t' y') - h(x))| ds(y')
\]
\[
\leq \int_{\mathbb{S}^{n-1}} |\Omega(x, y')| \sum_{|\beta|=\alpha} \int_{0}^{1} |D^\beta h(x - 2^{j}t' y')| ds(y').
\]
Therefore, by the above inequality and the Minkowski inequality, we have
\[
(2.24) \quad \left\| \left( \sum_{k \in \mathbb{Z}} |T_{\alpha,k,t} A_{L, f}|^2 \right)^{1/2} \right\|_{L^2}
\]
\[
\leq C \sum_{|\beta|=\alpha} \int_{0}^{1} \int_{\mathbb{S}^{n-1}} |\Omega(x, y')| \left( \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |(D^\beta A_{L, f}(x - 2^{j}t' y'))|^2 dx \right)^{1/2} \right) ds(y') ds
\]
\[
\leq C \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \left\| \left( \sum_{k \in \mathbb{Z}} |D^\beta A_{L, f}|^2 \right)^{1/2} \right\|_{L^2}.
\]
Then, by (2.18) and Littlewood-Paley theory, we have
\[
(2.25) \quad \|I_{2,f}\|_{L^2} \leq C \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \|f\|_{L^2}.
\]
Combining (2.17) with (2.21), (2.23) and (2.25) we get
\[
(2.26) \quad \|S_2(T_{\alpha,t} f)\|_{L^2} \leq C \min \{2^{|(\alpha+1-\alpha)/2|}, 2^{-|\beta/4|}\} \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \|f\|_{L^2}.
\]
Then by (2.15)
\[
\|S_2(T_{\alpha,t} f)\|_{L^2} \leq C \sum_{k \in \mathbb{Z}} \min \{2^{|(\alpha+1-\alpha)/2|}, 2^{-|\beta/4|}\} \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \|f\|_{L^2}
\]
\[
\leq C \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^1(\mathbb{S}^{n-1})} \|f\|_{L^2}.
\]
Therefore, we finish the proof of (2.2).

2.3. The sharpness of Theorem 1.1. We will take the idea from [5] to show that Theorem 1.1 is sharp.

Proposition 2.3. If in Theorem 1.1 we take $q = \frac{2n-1}{n+2\alpha}$, $V_\rho(T_{\alpha,t} f)$ of an $f \in L^2_\rho$ need not be in $L^2$.

Proof. $T_{\alpha,t} f(x) = \int_{\mathbb{R}^n} K_{\alpha}(x,y) f(y) dy$. In general, we shall have
\[
K_{\alpha}(x,y) = \frac{\Omega(x,z')}{|z'|^{n+\alpha}},
\]
where, systematically, $z = x - y$. Thus $K_{\alpha}(x,y)$ depends on the point $x$ and on the direction from $x$ to $y$. Let us take for $f(y) \in \mathcal{F}(\mathbb{R}^n)$ such that $f \geq 0$, $\text{supp} f \subset \{y : |y| \leq 1\}$ and the function equal to 1 for $|y| \leq 1/2$. It belongs to $L^2_{\rho}$. Let $q$ be any positive number. Let us assume that $\Omega(x,z') = 0$ for $|x| \leq 2$. For $|x| > 2$ we define $\Omega(x,z')$ as follows:
1°: It is equal to $|x|^{(n-1)/q}$ for the points $y$ of each ray from $x$ intersecting $\mathbb{S}^{n-1};$
2°: It is equal to $-|x|^{(n-1)/q}$ for the points $y$ of the rays opposite to those in 1°.
3°: It is equal to zero on all other rays from $x$.
Then
\[
\int_{\mathbb{S}^{n-1}} |\Omega(x,z')|^q d\sigma(z') < C,
\]
It is not difficult to see that for the function $f$ just defined,
\[
V_\rho(T_\alpha f)(x) = \left| T_\alpha f(x) \right| = \left| \int_{\mathbb{R}^n} \Omega(x, z') |z|^{-n-\alpha} f(y) \, dy \right|
\]
\[
= \left| \int_{\mathbb{R}^n} \Omega(x, z') |z|^{-n-\alpha} f(y) \, dy \right|
\]
\[
= \frac{C}{|x|^\beta}, \quad \text{as} \quad |x| \to \infty,
\]
where $C$ is a constant and $\beta = n + \alpha - (n-1)/q$. If we want $V_\rho(T_\alpha f)(x)$ to be in $L^2$ we must assume that $\beta > n/2$, or, what is the same thing, $q > \frac{2(n-1)}{n+2\alpha}$. This completes the proof.

3. Spherical Harmonic Polynomials, Proofs of Theorem 1.4 and Theorem 1.5

As done previously, denote by $\mathcal{H}_m$ the space of surface spherical harmonics of degree $m$ on $\mathbb{S}^{n-1}$ with its dimension $D_m$. As it was pointed out in [7], $D_m \leq Cm^{n-2}$, $m \geq 1$. Furthermore, $\{Y_{m,j}\}_{j=1}^{D_m}$ denotes the normalized complete system in $\mathcal{H}_m$. Then $\{Y_{m,j}(x')\}_{j=1}^{D_m}(m = 0, 1, \ldots)$ is a complete orthonormal system in $L^2(\mathbb{S}^{n-1})$ and for any multi-index $\beta$ (see [6])
\[
\sup_{x' \in \mathbb{S}^{n-1}} |(\partial/\partial x')^\beta Y_{m,j}(x')| \leq Cm^{\beta/(n-2)/2}, \quad m = 1, 2, \ldots.
\]

To prove Theorem 1.4 and Theorem 1.5, we will give some useful lemmas for spherical harmonic polynomials.

**Lemma 3.1.** Let $\alpha \geq 0$, $\{Y_{m,j}\}_{j=1}^{D_m}$ denotes the normalized complete system in $\mathcal{H}_m$. Let
\[
\sigma_{a,k,m,j} * f(x) = \int_{2^k e^{|b|} \leq 2^{k+1}} \frac{Y_{m,j}(y')}{|y'|^{\alpha+2}} f(x-y) \, dy
\]
for $m \geq [\alpha] + 1, j = 1, 2, \ldots, D_m$. If $1 < p < \infty$ and $w \in A_p$, there exists a constant $C > 0$ such that
\[
\| \sup_{k \in \mathbb{Z}} \| \sigma_{a,k,m,j} * f \|_{L_p(w)} \| \leq C m^{1/2} \| f \|_{L_p^q(w)}.
\]

**Proof.** Since $m \geq [\alpha] + 1$, we know
\[
\int_{\mathbb{S}^{n-1}} Y_{m,j}(y') y'^\gamma \, d\sigma(y') = 0,
\]
where $\gamma \leq [\alpha]$. Then we get
\[
\sup_{k \in \mathbb{Z}} \| \sigma_{a,k,m,j} * f \| = \sup_{k \in \mathbb{Z}} \left| \int_{2^k} \int_{\mathbb{S}^{n-1}} Y_{m,j}(y')(f(x - ty) - f(x)) \, d\sigma(y') \frac{dt}{t^{1+\alpha}} \right|.
\]
We will estimate $\sup_{k \in \mathbb{Z}} | \sigma_{\epsilon,a,m,j} * f |$ into three different cases. 

**Case 1.** $0 < \alpha < 1$. Applying Lemma 2.2 and $|Y_{m,j}(y')| \leq C m^{-n/2}$, we have
\[
\left| \int_{\mathbb{S}^{n-1}} Y_{m,j}(y')(f(x - ty) - f(x)) \, d\sigma(y') \right|
\]
\[
\leq C m^{\alpha/2} \int_{\mathbb{S}^{n-1}} (M(D^\alpha f)(x - ty') + M(D^\alpha f)(x)) \, d\sigma(y').
\]
From the above estimate, we can get

\[(3.3) \quad \sup_{k \in \mathbb{Z}} |\sigma_{a,k,m,j} \ast f(x)| \leq Cm^{\frac{n-1}{2}} \sup_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_{S_{\alpha-\frac{1}{2}}} (M(D^{\alpha} f)(x - ty) + M(D^{\alpha} f)(x)) d\sigma(y') \frac{dt}{t} \]
\[\approx Cm^{\frac{n}{2}} \left( \sup_{k \in \mathbb{Z}} \int_{2^k < \|y\| \leq 2^{k+1}} \frac{M(D^{\alpha} f)(x - y)}{|y|^n} dy + M(D^{\alpha} f)(x) \right) \]
\[\leq Cm^{\frac{n}{2}} (MM(D^{\alpha} f)(x) + M(D^{\alpha} f)(x)). \]

Then, by the $L^p(w)$, ($1 < p < \infty$, $w \in A_p$) boundedness of the Hardy-Littlewood maximal operator $M$, we obtain

\[(3.4) \quad \| \sup_{k \in \mathbb{Z}} |\sigma_{a,k,m,j} \ast f| \|_{L^p(w)} \leq Cm^{\frac{n}{2}} \| f \|_{L^p(w)}. \]

**Case 2.** $\alpha > 1$ (\alpha is not an integer), by Taylor’s Theorem and (3.2), we get

\[\left| \int_{S_{\alpha-1}} Y_m(y')(f(x - ty') - f(x)) d\sigma(y') \right| \]
\[= \left| \int_{S_{\alpha-1}} \sum_{[\|y\| = \alpha]} \int_{0}^{1} (1 - s)^{\alpha-1}(D^{\alpha} f)(x - sty')(ty'y'Y_m(y')dsd\sigma(y')) \right| \]
\[= \left| \int_{S_{\alpha-1}} \sum_{[\|y\| = \alpha]} \int_{0}^{1} (1 - s)^{\alpha-1}[(D^{\alpha} f)(x - sty') - (D^{\alpha} f)(x)](ty'y'Y_m(y')dsd\sigma(y')) \right|. \]

Since $0 < \alpha - [\alpha] < 1$, applying Lemma 2.2 and $|Y_m(y')| \leq Cm^{\frac{n}{2}}$, we have

\[\left| \int_{S_{\alpha-1}} Y_m(y')(f(x - ty') - f(x)) d\sigma(y') \right| \]
\[\leq Cm^{\frac{n}{2}} \int_{S_{\alpha-1}} \sum_{[\|y\| = \alpha]} \int_{0}^{1} [M(D^{\alpha-[\alpha]} D^{\alpha} f)(x - sty') + M(D^{\alpha-[\alpha]} D^{\alpha} f)(x)] dsd\sigma(y'). \]

From the above estimate, we can get

\[(3.5) \quad \sup_{k \in \mathbb{Z}} |\sigma_{a,k,m,j} \ast f(x)| \]
\[\leq Cm^{\frac{n}{2}} \sup_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_{S_{[\|y\| = \alpha]}} \int_{0}^{1} [M(D^{\alpha-[\alpha]} D^{\alpha} f)(x - sty') + M(D^{\alpha-[\alpha]} D^{\alpha} f)(x)] dsd\sigma(y') \frac{dt}{t} \]
\[\approx Cm^{\frac{n}{2}} \sum_{[\|y\| = \alpha]} \left( \int_{0}^{1} \sup_{k \in \mathbb{Z}} \int_{|t| < \|y\| < 2^{k+1}} \frac{M(D^{\alpha-[\alpha]} D^{\alpha} f)(x - y)}{|y|^n} dyds + M(D^{\alpha-[\alpha]} D^{\alpha} f)(x) \right) \]
\[\leq Cm^{\frac{n}{2}} (MM(D^{\alpha-[\alpha]} D^{\alpha} f)(x) + M(D^{\alpha-[\alpha]} D^{\alpha} f)(x)). \]

Thus, by the $L^p(w)$ boundedness of $M$, we have

\[(3.6) \quad \| \sup_{k \in \mathbb{Z}} |\sigma_{a,k,m,j} \ast f| \|_{L^p(w)} \leq Cm^{\frac{n}{2}} \sum_{[\|y\| = \alpha]} \| D^{\alpha} f \|_{L^p_w(w)} \| f \|_{L^p_w(w)}. \]

**Case 3.** $\alpha$ is a nonnegative integer. By Taylor’s Theorem and (3.2), we have

\[\left| \int_{S_{\alpha-1}} Y_m(y')(f(x - ty') - f(x)) d\sigma(y') \right|. \]
\[
\begin{align*}
&= \left| \int_{\mathbb{R}^n} \sum_{\|b|=\alpha} \int_0^1 (1 - s)^{[\alpha]-1} (D^\beta f)(x - sty')(ty')^\beta Y_m(y') ds d\sigma(y') \right| \\
&\leq Cm^{n-1} \sum_{\|b|=\alpha} \int_0^1 \int_{\mathbb{R}^n} |D^\beta f(x - sty')| d\sigma(y') ds.
\end{align*}
\]

Then, we have
\[
(3.7) \quad \sup_{k \in \mathbb{Z}} |\sigma_{a,k,m,j} * f(x)| \leq Cm^{n-1} \sum_{\|b|=\alpha} \int_0^1 \sup_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} |D^\beta f(x - sty')| d\sigma(y') \frac{dt}{t} ds
\]

\[
= Cm^{n-1} \sum_{\|b|=\alpha} \int_0^1 \sup_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} |D^\beta f(x - y)| \frac{dy}{|y|^\beta} ds
\]

\[
\leq Cm^{n-1} \sum_{\|b|=\alpha} M(D^\beta f)(x).
\]

Thus, by the \(L^p(w)\) boundedness of \(M\), we get
\[
(3.8) \quad \|\sup_{k \in \mathbb{Z}} |\sigma_{a,k,m,j} * f|\|_{L^p(w)} \leq Cm^{n-1}\|f\|_{L^p_w}.
\]

Therefore, we finish the proof of Lemma 3.1. \(\square\)

**Lemma 3.2.** Let \(\alpha \geq 0\). \(\{Y_{m,j}\}_{j=1}^{D_m}\) denotes the normalized complete system in \(\mathcal{H}_m\). Let
\[
T_{a,m,j,f}(x) = p.v. \int_{\mathbb{R}^n} \frac{Y_{m,j}(y')}{|y'|^{\alpha+\beta}} f(x - y) dy,
\]

for \(m \geq [\alpha] + 1\), \(j = 1, 2, \ldots, D_m\). If \(1 < p < \infty\) and \(w \in A_p\), there exists a constant \(C > 0\) such that
\[
\|T_{a,m,j,f}\|_{L^p(w)} \leq Cm^{n-1}\|f\|_{L^p_w}.
\]

**Proof.** Recall that \(\sigma_{a,k,m,j}(x) = Y_{m,j}(x') |x'|^{-\alpha} \chi_{|x'| \leq 2^{k+1}}(x)\) and \(\sum_{k \in \mathbb{Z}} \Delta^2_k f = f\), we have
\[
(3.9) \quad T_{a,m,j,f}(x) = \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sigma_{a,k,m,j} \Delta^2_k f(x) =: \sum_{l \in \mathbb{Z}} T^l_{a,m,j,f}(x)
\]

By Lemma 2.1 and \(|Y_{m,j}(y')| \leq Cm^{\beta-2}\), we get for \(0 < \beta < 1\),
\[
|\sigma_{a,k,m,j}(\xi)| \leq C|\xi|^\alpha \sum_{k \in \mathbb{Z}} |\Delta^2_k f(x)| =: \sum_{k \in \mathbb{Z}} |\Delta^2_k f(x)|
\]

Then
\[
(3.10) \quad \|T^l_{a,m,j,f}\|_{L^2} \leq \left( \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |\xi|^\alpha \sum_{k \in \mathbb{Z}} |\Delta^2_k f(x)| \right)^2 d\xi \right)^{1/2}
\]

\[
\leq Cm^{\beta-2} \sum_{k \in \mathbb{Z}} |\Delta^2_k f(x)| \left( \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |\Delta^2_k f(x)| \right)^2 d\xi \right)^{1/2}
\]

On the other hand, by (3.3), (3.5) and (3.7), we have for \(1 < p < \infty\) and \(w \in A_p\)
\[
(3.11) \quad \|\sup_{k \in \mathbb{Z}} |\sigma_{a,k,m,j} * f_k|\|_{L^p(w)} \leq Cm^{n-1} \sum_{\|b|=\alpha} \|M^2(D^\alpha(D^\beta f_k))\|_{L^p(w)} \quad \alpha \neq [\alpha],
\]

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and
\begin{equation}
\| \sup_{k \in \mathbb{Z}} |\sigma_{\alpha,k,m,j} \ast f_k|_{L^p(w)} \| \leq C m^{\frac{n}{2} - 1} \sum_{|\beta|=\alpha} \| \sup_{k \in \mathbb{Z}} |M(D^\beta f_k)|_{L^p(w)}, \quad \alpha = [\alpha].
\end{equation}

Similar to the proof of Lemma 3.1, we can also get for $1 < p < \infty$ and $w \in A_p$
\begin{equation}
\| \sum_{k \in \mathbb{Z}} |\sigma_{\alpha,k,m,j} \ast f_k|_{L^p(w)} \| \leq C m^{\frac{n}{2} - 1} \sum_{|\beta|=\alpha} \| \sum_{k \in \mathbb{Z}} |M^2(D^{\alpha-\beta}(D^\beta f_k))|_{L^p(w)}, \quad \alpha \neq [\alpha],
\end{equation}
and
\begin{equation}
\| \sum_{k \in \mathbb{Z}} |\sigma_{\alpha,k,m,j} \ast f_k|_{L^p(w)} \| \leq C m^{\frac{n}{2} - 1} \| |M(D^\beta f_k)|_{L^p(w)}, \quad \alpha = [\alpha].
\end{equation}

Then interpolating between (3.11) and (3.13), (3.12) and (3.14), we get
\begin{equation}
\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\alpha,k,m,j} \ast f_k|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C m^{\frac{n}{2} - 1} \sum_{|\beta|=\alpha} \left\| \left( \sum_{k \in \mathbb{Z}} |M(D^{\alpha-\beta}(D^\beta f_k))|^2 \right)^{1/2} \right\|_{L^p(w)}, \quad \alpha \neq [\alpha],
\end{equation}
and
\begin{equation}
\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\alpha,k,m,j} \ast f_k|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C m^{\frac{n}{2} - 1} \sum_{|\beta|=\alpha} \left\| \left( \sum_{k \in \mathbb{Z}} |M(D^\beta f_k)|^2 \right)^{1/2} \right\|_{L^p(w)}, \quad \alpha = [\alpha].
\end{equation}

Here we only consider the case of $\alpha \neq [\alpha]$ because the case of $\alpha = [\alpha]$ can be treated similarly. By (3.15), $L^p(\mathbb{F}^2, w)$, $(1 < p < \infty, w \in A_p)$ of $M$ (see [23]) and weighted Littlewood-Paley theory ([26]), we get
\begin{equation}
\left\| T^{l}_{\alpha,m,j} f \right\|_{L^p(w)} \leq \left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\alpha,k,m,j} \ast \Delta_{l-k} f|^2 \right)^{1/2} \right\|_{L^p(w)}
\end{equation}
\begin{equation}
\leq C m^{-n-1} \left\| \left( \sum_{k \in \mathbb{Z}} |D^{\alpha-\beta}(D^\beta \Delta_{l-k} f)|^2 \right)^{1/2} \right\|_{L^p(w)}
\end{equation}
\begin{equation}
\leq C m^{-n-1} \left\| f \right\|_{L^p(t^\alpha,w)}. \quad \text{Applying Stein and Weiss's interpolation theorem [38] with change of measure between (3.10) and (3.17), we get for some } \delta \in (0, 1),
\end{equation}
\begin{equation}
\left\| T^{l}_{\alpha,m,j} f \right\|_{L^p(w)} \leq C m^{2-1/2-\delta} \left\| f \right\|_{L^p(t^\alpha,w)}.
\end{equation}
Therefore by (3.9),
\begin{equation}
\left\| T^{l}_{\alpha,m,j} f \right\|_{L^p(w)} \leq C \sum_{k \in \mathbb{Z}} m^{2-1/2-\delta} \left\| f \right\|_{L^p(t^\alpha,w)} \leq C m^{2-1} \left\| f \right\|_{L^p(t^\alpha,w)}.
\end{equation}

\textbf{Proof of Theorem 1.4.} By the similar argument of Theorem 1.1, we get actually
\begin{equation}
\Omega(x, y') = \sum_{m=[\alpha]+1}^{\infty} \sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(y').
\end{equation}

By the definition of the strong $\rho$-variation function and (3.18), we have
\begin{equation}
V_\rho(T_{\alpha} f)(x) \leq C \sum_{[\alpha]+1}^{\infty} \sum_{j=1}^{D_{[\alpha]}} |a_{m,j}(x)| V_\rho(T_{\alpha,m,j} f)(x).
\end{equation}
We claim that

\[
\| V_\rho(T_{a,m,j}f) \|_{L^p(w)} \leq C m^{2^{n-1}} \| f \|_{L^p(w)}. \tag{3.19}
\]

Thus by (3.19), \( \| a_{m,j} \|_{L^w} \leq C m^{-2n} \) for \( j = 1, \ldots, D_m \) (see [12]) and \( D_m \leq C m^{n-2} \), we have

\[
\| V_\rho(T_{a,m,j}f) \|_{L^p(w)} \leq C \sum_{m=|a|+1}^\infty \sum_{j=1}^{D_m} \| a_{m,j} \|_{L^w} \| V_\rho(T_{a,m,j}f) \|_{L^p(w)}
\]

\[
\leq C \sum_{m=|a|+1}^\infty m^{-2n}m^{-n-2}m^{2^{n-1}} \| f \|_{L^p(w)}
\]

\[
\leq C \| f \|_{L^p(w)}.
\]

Therefore, we complete the proof of Theorem 1.4.

Let \( T_{a,m,j,2^k} \) be defined as in (2.4). Now we return to prove (3.19). To obtain (3.19), it is sufficient to prove the following two inequalities

\[
\left\| V_\rho(\{ T_{a,m,j,2^k}f \}_{k\in\mathbb{Z}}) \right\|_{L^p(w)} \leq C m^{2^{n-1}} \| f \|_{L^p(w)} \tag{3.20}
\]

and

\[
\| S_2(T_{a,m,j}f) \|_{L^p(w)} \leq C m^{2^{n-1}} \| f \|_{L^p(w)} \tag{3.21}
\]

where \( 1 < p < \infty \) and \( w \in A_p \).

3.1. **Proof of (3.20).** Let \( \sigma_{a,k,m,j} \) be defined as in Section 2. Let \( \phi \) be a Schwartz function such that \( \hat{\phi}(\xi) = 1 \) for \( |\xi| \leq 2 \) and \( \hat{\phi}(\xi) = 0 \) for \( |\xi| > 4 \). We have

\[
T_{a,m,j,2^k}f(x) = \phi_k * T_{a,m,j}f(x) + \sum_{s \geq 0} \left( \delta_0 - \phi_k * \sigma_{a,k+s,m,j} \right) * f(x) - \phi_k * \sum_{s < 0} \sigma_{a,k+s,m,j} * f(x)
\]

\[
= T_{k,m,j,2^k}^1f(x) + T_{k,m,j,2^k}^2f(x) - \delta_0 f(x).
\]

where \( \phi_k(x) = 2^{-kn} \phi(2^{-k}x) \) with \( \hat{\phi}_k(\xi) = \hat{\phi}(2^k \xi) \), \( \delta_0 \) is the Dirac measure at 0. Let \( T_{m,j}^i \) be the family \{\( T_{k,m,j,2^k}^i \)\}_{k\in\mathbb{Z}} \) for \( i = 1, 2, 3 \). Clearly, we need to prove the inequalities below

\[
\| V_\rho(T_{m,j}^i f) \|_{L^p(w)} \leq C m^{2^{n-1}} \| f \|_{L^p(w)}, \quad i = 1, 2, 3.
\tag{3.22}
\]

First we estimate (3.22) for \( i = 1 \). Since

\[
\| V_\rho(\{ \phi_k * f \}_{k\in\mathbb{Z}}) \|_{L^p(w)} \leq C \| f \|_{L^p(w)}
\]

(see [14]), then by Lemma 3.2

\[
\| V_\rho(\{ T_{m,j}^1 f \}_{k\in\mathbb{Z}}) \|_{L^p(w)} \leq C \| T_{a,m,j}f \|_{L^p(w)} \leq C m^{2^{n-1}} \| f \|_{L^p(w)}.
\]

Next, we consider the case \( i = 2 \) of estimate (3.22). By \( \sum_{k\in\mathbb{Z}} \Delta_{l,k} f = f \) and the Minkowski inequality,

\[
V_\rho(T_{m,j}^2 f(x)) \leq C \left( \sum_{k\in\mathbb{Z}} \left| \frac{1}{2} \delta_0 - \phi_k \right|^2 \sigma_{a,k+s,m,j} \right) f(x)^2 \right)^{1/2}
\]

\[
\leq C \sum_{s \geq 0} \left( \sum_{k\in\mathbb{Z}} \left| \frac{1}{2} \delta_0 - \phi_k \right|^2 \sigma_{a,k+s,m,j} \sum_{l\in\mathbb{Z}} \Delta_{l-k}(f(x))^2 \right)^{1/2}
\]

\[
\leq C \sum_{s \geq 0} \sum_{l\in\mathbb{Z}} \left( \sum_{k\in\mathbb{Z}} \left| \frac{1}{2} \delta_0 - \phi_k \right|^2 \sigma_{a,k+s,m,j} \Delta_{l-k}(f(x))^2 \right)^{1/2}
\]
By (2.13), $|Y_{m,j}(x')| \leq Cm^{(n-2)/2}$ and Littlewood-Paley theory, we have
\begin{equation}
\|F_{m,j,i}\|_{L^2} \leq Cm^{-1+\beta/2-\alpha}2^{-\beta/2} \min\{2^{-\beta/2}, 2^{\alpha}\} \|f\|_{L^2}.
\end{equation}

Next, we will consider the $L^p(w)$ boundedness of $F_{m,j,i,f}$ for $1 < p < \infty$ and $w \in A_p$. Here we only consider the case of $\alpha \neq [\alpha]$ because the case of $\alpha = [\alpha]$ can be treated similarly. By (3.15) and $L^p(\ell^2, w)$ boundedness of $M$ (see [23]), we have for $1 < p < \infty$ and $w \in A_p$,
\begin{equation}
\left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\alpha,k+s,m,j} * f_k|^2 \right)^{1/2} \right\|_{L^p(w)} \leq Cm^{\alpha-1} \sum_{[|\alpha|]} \left\| \left( \sum_{k \in \mathbb{Z}} |D^{\alpha-|\alpha|}(\Delta^2 f_k)|^2 \right)^{1/2} \right\|_{L^p(w)}.
\end{equation}

By the above estimate, $L^p(\ell^2, w)$ boundedness of $\{\phi_k * f\}$ (see [23]) and the weighted Littlewood-Paley theory, we can get
\begin{equation}
\|F_{m,j,i,f}\|_{L^p(w)} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} |\sigma_{\alpha,k+s,m,j} * \Delta^2 f_k|^2 \right)^{1/2} \right\|_{L^p(w)}
\leq Cm^{\alpha-1} \sum_{[|\alpha|]} \left\| \left( \sum_{k \in \mathbb{Z}} |D^{\alpha-|\alpha|}(\Delta^2 f_k)|^2 \right)^{1/2} \right\|_{L^p(w)}
\leq Cm^{\alpha-1} \sum_{[|\alpha|]} \|D^{\alpha-|\alpha|}(\Delta^2 f)\|_{L^p(w)} \leq Cm^{\alpha-1}\|f\|_{L^p(w)}.
\end{equation}

Then using Stein and Weiss’s interpolation theorem [38] with change of measure between (3.24) and (3.26), we can get for some $\beta_0, \tau_0 \in (0, 1)$
\begin{equation}
\|F_{m,j,i,f}\|_{L^p(w)} \leq C2^{-\beta_0}2^{-\tau_0}m^{\alpha-1}\|f\|_{L^p(w)}.
\end{equation}

Then,
\begin{equation}
\|V_p(T_{m,j}^{-3}f)\|_{L^p(w)} \leq C \sum_{k \in \mathbb{Z}} \sum_{s \geq 0} 2^{-\beta_0}2^{-\tau_0}m^{\alpha-1}\|f\|_{L^p(w)} \leq Cm^{\alpha-1}\|f\|_{L^p(w)}.
\end{equation}

Finally, we consider the case $i = 3$ of estimate (3.22). By $\sum_{k \in \mathbb{Z}} \Delta^2 f = f$ and the Minkowski inequality,
\begin{equation}
V_p(T_{m,j}^{-3}f(x)) \leq C \left( \sum_{k \in \mathbb{Z}} \left| \sum_{s < 0} \phi_k * \sigma_{\alpha,k+s,m,j} * f(x) \right|^2 \right)^{1/2}
\leq C \sum_{s < 0} \left( \sum_{k \in \mathbb{Z}} \left| \phi_k * \sigma_{\alpha,k+s,m,j} * \sum_{l \in \mathbb{Z}} \Delta^2 f(x) \right|^2 \right)^{1/2}
\leq C \sum_{s < 0} \left( \sum_{k \in \mathbb{Z}} \left| \phi_k * \sigma_{\alpha,k+s,m,j} * \Delta^2 f(x) \right|^2 \right)^{1/2}
=: \sum_{s < 0} I_{m,j,i,f}(x).
\end{equation}

By (2.14), $|Y_{m,j}(x')| \leq Cm^{(n-2)/2}$ and Littlewood-Paley theory, we have
\begin{equation}
\|I_{m,j,i,f}\|_{L^2} \leq Cm^{-1+\alpha/2}[|\alpha|+1]s \min\{2^{-\alpha}, 2^{[(|\alpha|+1)-l]}\} \|f\|_{L^2}.
\end{equation}
Next, we will consider the $L^p(w)$ boundedness of $I_{m,j,l,t} f$. Here we only consider the case of $\alpha \neq [\alpha]$ because the case of $\alpha = [\alpha]$ can be treated similarly. By (3.25), the $L^p(\mathbb{E}, w)$ boundedness of $[\phi_k * f]$ (see [23]) and weighted Littlewood-Paley theory, we can get

\begin{equation}
\|I_{m,j,l,t} f\|_{L^p(w)} \leq Cm^{\frac{3}{2}-1}\|f\|_{L^p(w)}.
\end{equation}

Then using Stein and Weiss’s interpolation theorem [38] with change of measure between (3.29) and (3.30), we can get for some $\gamma, \tau \in (0, 1)$

\begin{equation}
\|I_{m,j,l,t} f\|_{L^p(\mathbb{E})} \leq C \sum_{l \in \mathbb{Z}} \sum_{s < 0} 2^{\tau l - \gamma l} \|m^{\frac{3}{2}-1}\|f\|_{L^p(\mathbb{E})} \leq Cm^{\frac{3}{2}-1}\|f\|_{L^p(\mathbb{E})}.
\end{equation}

Therefore we finish the proof of (3.20).

3.2. Proof of (3.21). For $t \in [1, 2]$, recall that $\sigma_{a,k,l,m,j}(x) = \frac{Y_0(x')}{|x'|^m} \chi_{\{2^{\nu} \leq |x| \leq 2^{\nu+1}\}}(x)$ for $k \in \mathbb{Z}$. Then, by the Minkowski inequality and $\sum_{l \in \mathbb{Z}} \Delta^2_l f = f$, we have

\begin{equation}
S_2(T_{a,m,j} f)(x) = \left( \sum_{k \in \mathbb{Z}} \|V_{2,k}(T_{a,m,j} f)(x)\|^2 \right)^{1/2}.
\end{equation}

On the one hand, for some $0 < \epsilon < 1$, we have

\begin{equation}
\|S_2(T_{a,m,j} f)\|_{L^2} \leq C2^{-\epsilon l}m^{\frac{3}{2}-1}\|f\|_{L^2},
\end{equation}

which can be proved in a similar way as that in the estimate of (2.26).

On the other hand,

\begin{equation}
S_2(T_{a,m,j} f)(x) = \left( \sum_{k \in \mathbb{Z}} \sup_{t_i < \cdots < t_j} \sum_{[t_i, t_{i+1}]}^{j-1} |\sigma_{a,k,l,m,j} * \Delta^2_{l-k} f(x) - v_{a,k,l,m,j} * \Delta^2_{l-k} f(x)|^2 \right)^{1/2}
\end{equation}

\begin{equation}
= \left( \sum_{k \in \mathbb{Z}} \sup_{t_i < \cdots < t_j} \sum_{[t_i, t_{i+1}]}^{j-1} \left| \int_{2^{t_i} \leq |y| \leq 2^{t_{i+1}}} \frac{Y_{m,j}(y') \Delta^2_{l-k} f(x-y)d\eta(y')}{|y|^{\alpha+\frac{3}{2}} d\eta(y')} \right|^2 \right)^{1/2}
\end{equation}

\begin{equation}
= \left( \sum_{k \in \mathbb{Z}} \sup_{t_i < \cdots < t_j} \sum_{[t_i, t_{i+1}]}^{j-1} |T_{a,k,m,l,t_j,t_{i+1}} \Delta^2_{l-k} f| \right)^{1/2}.
\end{equation}

It is easy to verify

\begin{equation}
\sup_{t_i < \cdots < t_j} \sum_{[t_i, t_{i+1}]}^{j-1} |T_{a,k,m,l,t_j,t_{i+1}} f(x)| \leq \sup_{t_i < \cdots < t_j} \sum_{[t_i, t_{i+1}]}^{j-1} \int_{2^{t_i} \leq |y| \leq 2^{t_{i+1}}} \int_{[t_{i+1}]}^{t_{i+1}} \frac{Y_{m,j}(y') f(x-ty')d\sigma(y')}{|y|^{\alpha+\frac{3}{2}}} dt.
\end{equation}
Therefore by (3.32), for (3.39), we can get for some $\epsilon$:

$$\sum_{k \in \mathbb{Z}} |\mathcal{T}_{m,j,k} f_k|^2 \leq C m^{2-1} \sum_{|\beta|=|\alpha|} \left( \sum_{k \in \mathbb{Z}} |M(D^2 |\alpha|) D^\beta f_k|^2 \right)^{1/2} \parallel_{L^p(w)}, \quad \alpha \neq |\alpha|,$$

and

$$\sum_{k \in \mathbb{Z}} \sup_{1 \leq j \leq J} \left( \sum_{|\beta|=|\alpha|} \left( \sum_{k \in \mathbb{Z}} |M(D^2 |\alpha|) D^\beta f_k|^2 \right)^{1/2} \parallel_{L^p(w)}, \quad \alpha \neq |\alpha|.$$
Applying the above estimate and the Minkowski inequality, we get
\[ \leq C m^{2-1} \| f \|_{L^p_w}. \]

Thus, we finish the proof of (3.21). \( \square \)

**Proof of Theorem 1.5.** Let \( I_\gamma \) be the Riesz potential operator of order \( \gamma \), \( 0 < \gamma < n \) (the definition of the Riesz potential operator, we refer the readers to [23]). There exist an integer \( m \) such that \( \gamma m = \alpha \). By Theorem 1.4 and \( (I_\gamma)^m D^\alpha f(x) = f(x) \) to give that
\[
\| V_\rho(T_\alpha(I_\gamma)^m g) \|_{L^p(w)} \leq C \| g \|_{L^p(w)},
\]
where \( g := D^\alpha f \).

For fixed \( t \in \mathbb{R}^n \) and \( r > 0 \), we abbreviate \( B = B(t, r) \). We write
\[ g(y) = g(y) \chi_{2B}(y) + \sum_{k=1}^\infty g(y) \chi_{2^{k+1}B \setminus 2^kB}(y) =: \sum_{k=0}^\infty g_k(y). \]

For \( k = 0 \), by (3.41) with \( w(x) = 1 \), we get
\[
\int_B |V_\rho(T_\alpha(I_\gamma)^m g_0)(x)|^p dx \leq \| V_\rho(T_\alpha(I_\gamma)^m g_0) \|_{L^p}^p \leq C \| g_0 \|_{L^p}^p
\]
\[
= C \int_{2B} |D^\alpha f(y)|^p dy \leq C (2r)^4 \| f \|_{L^2_w}^p.
\]

For \( k > 0 \), note that \( M(\chi_B)(x) \sim 2^{-kn} \) when \( x \in 2^{k+1}B \setminus 2^kB \). Fix any \( \eta \in (\alpha/n, 1) \), we know that \( (M\chi_B)^\eta \in A_1 \subset A_p \) with \( A_1 \) constant depending on \( \lambda \) and \( n \) only. Now for \( k > 0 \), by (3.41), we get for \( 1 < p < \infty \) and \( \lambda \in (0, n) \),
\[
\int_B |V_\rho(T_\alpha(I_\gamma)^m g_k)(x)|^p dx = \int_{\mathbb{R}^n} |V_\rho(T_\alpha(I_\gamma)^m g_k)(x)|^p (\chi_B(x))^p dx
\]
\[
\leq \int_{\mathbb{R}^n} |V_\rho(T_\alpha(I_\gamma)^m g_k)(x)|^p (M\chi_B(x))^p dx
\]
\[
\leq C \int_{\mathbb{R}^n} |g_k(x)|^p (M\chi_B(x))^p dx
\]
\[
\leq C 2^{-kn} \int_{\mathbb{R}^n} |g_k(x)|^p dx
\]
\[
\leq C 2^{-kn(\eta-\lambda/n)2^{-k\lambda}} \int_{2^{k+1}B} |D^\alpha f(x)|^p dx
\]
\[
\leq C 2^{-kn(\eta-\lambda/n)} p^\lambda \| f \|_{L^p_w}^p.
\]
Applying the above estimate and the Minkowski inequality, we get
\[
\left( \frac{1}{r^\lambda} \int_B |V_\rho(T_\alpha f(x)|^p dx \right)^{1/p} = \left( \frac{1}{r^\lambda} \int_B |V_\rho(T_\alpha(I_\gamma)^m g(x)|^p dx \right)^{1/p}
\]
\[
\leq C \sum_{k=0}^\infty 2^{-k \frac{(\eta-\lambda/n)p^\lambda}{p}} \| f \|_{L^p_w}^p
\]
\[
\leq C \| f \|_{L^p_w}^p.
\]
Therefore we finish the proof of Theorem 1.5. \( \square \)
REFERENCES


