MULTIPLE-TERM REFINEMENTS OF
ALZER-FONSECA-KOVAČEC’S INEQUALITIES

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Abstract. In this paper, by the weighted arithmetic-geometric mean inequality we present multiple-term refinements of one of the most important extensions to Young’s inequalities due to Alzer-Fonseca-Kovačec [Linear Multilinear Algebra 63(3) (2015), 622–635]. As applications, we give some related inequalities for operators and matrices.

1. Introduction

The weighted arithmetic-geometric mean (AM-GM) inequality states as follows:

**Theorem 1.1.** Let \( N \) be a positive integer. For \( k = 1, 2, \ldots, N \), let \( x_k > 0 \), and let \( \mu_k \geq 0 \) satisfy \( \sum_{k=1}^{N} \mu_k = 1 \). Then, we have

\[
\prod_{k=1}^{N} x_k^{\mu_k} \leq \sum_{k=1}^{N} \mu_k x_k.
\] (1.1)

The special case of the weighted AM-GM inequality (\( N = 2 \)) is the well-known Young’s inequality, for positive real numbers \( a, b \) and \( 0 \leq \mu \leq 1 \), we have

\[
a^{1-\mu} b^\mu \leq (1 - \mu)a + \mu b,
\] (1.2)

with equality if and only if \( a = b \). This inequality, though very simple, has attracted researchers working in operator theory due to its applications in this field.

Refining this inequality has taken the attention of many researchers in the field, where adding a positive term to the left side is possible.

One of the first refinement of Young’s inequality is the squared version presented in [11] as follows

\[
(a^{1-\mu} b^\mu)^2 + r_0^2(a - b)^2 \leq ((1 - \mu)a + \mu b)^2,
\] (1.3)

where \( r_0 = \min\{\mu, 1 - \mu\} \).

Later, Kittaneh and Manasrah [21], obtained the other interesting refinement of Young’s inequality

\[
a^{1-\mu} b^\mu + r_0(\sqrt{a} - \sqrt{b})^2 \leq (1 - \mu)a + \mu b,
\] (1.4)

where \( r_0 = \min\{\mu, 1 - \mu\} \).

Zhao and Wu [30], obtained the following refinement of inequality (1.4) as follows

(1) If \( 0 < \mu \leq \frac{1}{2} \), then

\[
a^{1-\mu} b^\mu + \mu(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt{ab} - \sqrt{a})^2 \leq (1 - \mu)a + \mu b.
\] (1.5)
Theorem 1.4 that happened to be better than (1.4)

Theorem 1.3

let be refinements of Young’s inequality

Kovačec’s inequalities, then we recapture the inequalities (1.8).

In 2020, Ren refined the Alzer-Fonseca-Kovačec inequalities for the case

Very recently, Ighachane and Akkouchi established the following multiple-term ref

Theorem 1.2 ([19]). Let \( a, b > 0 \) and \( 0 \leq \mu \leq 1 \). Then for all a positive integer \( N \), we have

\[
(1 + \frac{L(2^N \mu)}{2^{2N}} \ln^2(\frac{a}{b}))a\mu b^{1-\mu} + \sum_{t=0}^{N-1} r_1(v) \sum_{k=1}^{2^t} f_{t,k}(a, b) = \left(\sqrt{a^{\frac{1-\lambda}{2}} b^{1-\frac{1-\lambda}{2}}} - \sqrt{b^{\frac{1-\lambda}{2}} a^{1-\frac{1-\lambda}{2}}} \right)^2
\]

where \( \chi_I \) stands for a characteristic function of an interval \( I \), \( r_0 = \min\{\mu, 1 - \mu\} \), \( r_n(\mu) = \min\{2r_{n-1}(\mu), 1 - 2r_{n-1}(\mu)\} \), \( f_{t,k}(a, b) \), \( L(t) \) is periodic with period 1 given by \( L(t) = \frac{t^2}{2} \left(\frac{1}{4} \right)^{2t} \) for \( t \in (0, 1) \).

The inequalities (1.3) and (1.4), happened to be special cases of a more general refinement stating that

Theorem 1.3 ([1]). Let \( a, b > 0 \) and \( m \) be a positive integer, then we have

\[
\left( a^{1-\mu} b^\mu \right)^m + r_0^m \left( a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2 \leq \left( (1 - \mu)a + \mu b \right)^m,
\]

where \( r_0 = \min\{\mu, 1 - \mu\} \).

Here, it is worthy of pointing out that Akkouchi and Ighachane [3] gave another proof to inequality (1.8).

Recently, Choi [7] gave a further generalized refinement of inequalities (1.3) and (1.4) as follows

\[
\left( a^{1-\mu} b^\mu \right)^m + r_0^m \left( (a + b)^m - 2^m(ab)^{\frac{m}{2}} \right) \leq \left( (1 - \mu)a + \mu b \right)^m.
\]

For further reading related to generalized refinement of Young’s inequality, the reader is referred to recent papers [2, 3, 10, 14, 15, 16, 17, 24, 26, 27, 28, 29].

Throughout this paper, we denote \( (1 - \mu)a + \mu b \) and \( a^{1-\mu} b^\mu \), respectively by \( a\nabla_\mu b \) and \( a^{2\mu} b \). In [4], H. Alzer et al. proved an important refinement of Young’s inequality that happened to be better than (1.4)

Theorem 1.4 (Alzer-Fonseca-Kovačec). Let \( a, b > 0 \) and let \( \lambda, \mu \) and \( \tau \) be real numbers with \( \lambda \geq 1 \) and \( 0 \leq \mu < \tau \leq 1 \). Then

\[
\left( \frac{\mu}{\tau} \right)^\lambda \leq \frac{(a\nabla_\mu b)^\lambda - (a^{2\mu} b)^\lambda}{(a\nabla_\tau b)^\lambda - (a^{2\mu} b)^\lambda} \leq \left( \frac{1 - \mu}{1 - \tau} \right)^\lambda.
\]

In fact, if we replace \( \lambda \) by the positive integer \( m \) and \( \tau \) by \( \frac{1}{2} \) in the Alzer-Fonseca-Kovačec’s inequalities, then we recapture the inequalities (1.9).

In 2020, Ren refined the Alzer-Fonseca-Kovačec inequalities for the cases \( \lambda = 1, 2 \) as follows.
**Theorem 1.5** ([25]). Let \( a, b > 0 \) and let \( \mu \) and \( \tau \) be real numbers with \( 0 < \mu, \tau < 1 \). Then we have
\[
\frac{a \nabla_\mu b - a^\tau \mu b}{a \nabla_\tau b - a^\tau \mu b} \leq \frac{\mu(1 - \mu)}{\tau(1 - \tau)} \quad \text{and} \quad \frac{(a \nabla_\mu b)^2 - (a^\tau \mu b)^2}{(a \nabla_\tau b)^2 - (a^\tau \mu b)^2} \leq \frac{\mu(1 - \mu)}{\tau(1 - \tau)},
\]
for \((b - a)(\tau - \mu) \geq 0\), and the inequalities are reversed if \((b - a)(\tau - \mu) \leq 0\).

It is worthy of pointing out that in [20] Ighachane et al. present two refining terms of Theorem 1.4.

The significance of the above results in not only these inequalities themselves. Rather, these inequalities have been used to obtain new bounds for operator means, trace, determinant, singular values and norm inequalities of matrices.

Our main goal in this paper is to present multiple refining terms of Theorem 1.4, with applications to operator means and matrix inequalities.

### 2. Multiple-term refinements of Alzer-Fonseca-Kovačec’s inequalities

We start this section with some basic lemmas which are important in terms of proving the main results.

**Lemma 2.1** ([14]). Let \( m \) be a positive integer and let \( \mu \) be a positive number, such that \( 0 \leq \mu \leq 1 \). Then we have
\[
\sum_{k=1}^{m} \binom{m}{k} \mu^k (1 - \mu)^{m-k} = m \mu,
\]
where \( \binom{m}{k} \) is the binomial coefficient.

**Lemma 2.2.** Let \( a, b > 0 \) and \( m \) and \( N \) be positive integers with \( N \geq 2 \), and \( 0 \leq \tau \leq 1 \). Then
\[
M_N(a, b) = (a^\tau \mu b)^m \sum_{k=2}^{N} 2^{k-2} \left( \frac{a^m}{(a^\tau \mu b)^m} - 1 \right)^2 = \left( 2^{N-1} - 1 \right)(a^\tau \mu b)^m + a^m - 2^{N-1} a^m b \frac{2^{N-1} \mu (1 - \mu) + m \tau}{2^{N-1} \mu (2^{N-1})} \quad (2.3)
\]
and
\[
R_N(a, b) = \sum_{k=2}^{N} 2^{k-2} \left( \sqrt{a^m} - \frac{a^{k-1}}{a^{\tau \mu b} m^a m(2^{k-2}-1)} \right)^2 = \left( 2^{N-1} - 1 \right)a^m + (a^\tau \mu b)^m - 2^{N-1} \sqrt{a^m} \frac{2^{N-1} (a^\tau \mu b)^m a^m (2^{N-2}-1)}{2^{N-1}} \quad (2.4)
\]
Proof. A straightforward calculation show that

\[ M_N(a, b) = (a^\#_\tau b)^m \sum_{k=2}^{N} 2^{k-2} \left( \frac{a^m}{(a^\#_\tau b)^m} - 1 \right)^2 \]

\[ = (a^\#_\tau b)^m \sum_{k=2}^{N} 2^{k-2} \left( \frac{a^m}{(a^\#_\tau b)^m} - 2 \frac{a^m}{(a^\#_\tau b)^m} \right) \]

\[ + 2 \sqrt{\frac{a^m}{(a^\#_\tau b)^m}} - 2^2 \sqrt{\frac{a^m}{(a^\#_\tau b)^m}} \]

\[ + \ldots \]

\[ + 2^{N-3} \sum_{k=2}^{N-3} \frac{a^m}{(a^\#_\tau b)^m} - 2^{N-2} \sum_{k=2}^{N-2} \frac{a^m}{(a^\#_\tau b)^m} \]

\[ + 2^{N-2} \sum_{k=2}^{N-2} \frac{a^m}{(a^\#_\tau b)^m} - 2^{N-1} \sum_{k=2}^{N-1} \frac{a^m}{(a^\#_\tau b)^m} \]

\[ = (2^{N-1} - 1)(a^\#_\tau b)^m + (a^\#_\tau b)^m \left( \frac{a^m}{(a^\#_\tau b)^m} - 2 \frac{a^m}{(a^\#_\tau b)^m} \right) \]

\[ = (2^{N-1} - 1)(a^\#_\tau b)^m + (a^\#_\tau b)^m - 2^{N-1} \left( \frac{a^m}{(a^\#_\tau b)^m} \right) \]

\[ = (2^{N-1} - 1)(a^\#_\tau b)^m + a^m - 2^{N-1} a \frac{2^{N-1}(1+b^m)}{2^{N-1}} \frac{a^m}{b^{2N-1}} \]

\( \square \)

Similarly, equality (2.4) holds.

Lemma 2.3. Let \( \mu \) and \( \tau \) be two positive numbers such that \( 0 \leq \mu \leq 1 \) and \( m \) and \( N \) be positive integers with \( N \geq 2 \).

(1) For \( 0 \leq k \leq m \), we have

\[ \mu^k(1-\mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^m \tau^k(1-\tau)^{m-k} \geq 0. \]

(2) If \( 0 \leq \mu \leq \frac{\tau}{2^{N-1}} \), then

\[ (1-\mu)^m - (1-\tau)^m \left( \frac{\mu}{\tau} \right)^m \geq 0. \]

(3) If \( \frac{(2^{N-1}-1)\tau}{2^{N-1}} \leq \mu \leq \tau \), then

\[ (1-\mu)^m - (1-\tau)^m \left( \frac{\mu}{\tau} \right)^m \geq 0 \]

and

\[ \left( \frac{\mu}{\tau} \right)^m - (2^{N-1}-1) \left( 1 - \frac{\mu}{\tau} \right)^m \geq 0. \]
\textbf{Proof.} \hspace{1em} (1) Under the condition $0 \leq \mu \leq \tau \leq 1$, we have $\frac{\mu(1-\tau)}{\tau(1-\mu)} \leq 1$. So,

$$\mu^k(1-\mu)^{m-k} - \left(\frac{\mu}{\tau}\right)^m \tau^k(1-\tau)^{m-k} = \mu^k(1-\mu)^{m-k} \left(1 - \left(\frac{\mu}{\tau}\right)^m \left(\frac{1-\tau}{1-\mu}\right)^{m-k}\right) \geq 0.$$

(2) Suppose that $0 \leq \mu \leq \frac{\tau}{2N-1}$, set

$$g(\mu) = (1-\mu)^m - (1-\tau)^m \left(\frac{\mu}{\tau}\right)^m - (2^{N-1} - 1) \left(\frac{\mu}{\tau}\right)^m,$$

then we have

$$g'(\mu) = -m\left((1-\mu)^{m-1} + \frac{1}{\tau}(1-\tau)^m \left(\frac{\mu}{\tau}\right)^{m-1} + (2^{N-1} - 1) \frac{1}{\tau} \left(\frac{\mu}{\tau}\right)^{m-1}\right) \leq 0.$$

So $f$ is decreasing, then

$$g(\mu) \geq g\left(\frac{\tau}{2N-1}\right) = \frac{(2^{N-1} - \tau)^m - ((1-\tau)^m + (2^{N-1} - 1))}{2mN-m} \geq 0.$$

(3) Suppose that $\frac{2^{N-1} - \tau}{2N-1} \leq \mu \leq \tau$, we have

$$\begin{align*}
(1-\mu)^m - (1-\tau)^m \left(\frac{\mu}{\tau}\right)^m &- (1-\mu)^m \\
= &\left(1 - \frac{\mu}{\tau}\right) \left((1-\mu)^{m-1} + \ldots + (1-\tau)^{m-1} \left(\frac{\mu}{\tau}\right)^{m-1}\right) - (1-\mu)^m \\
= &\left(1 - \frac{\mu}{\tau}\right) \left((1-\mu)^{m-1} + \ldots + (1-\tau)^{m-1} \left(\frac{\mu}{\tau}\right)^{m-1}\right) - (1-\mu)^m \\
= &\left(1 - \frac{\mu}{\tau}\right) \left((1-\mu)^{m-1} - (1-\mu)^{m-1} + \ldots + (1-\tau)^{m-1} \left(\frac{\mu}{\tau}\right)^{m-1}\right) \\
\geq &0.
\end{align*}$$

Set,

$$h(\mu) = \left(\frac{\mu}{\tau}\right)^m - (2^{N-1} - 1) \left(1 - \frac{\mu}{\tau}\right)^m,$$

then we have

$$h'(\mu) = m\left(\frac{\mu}{\tau}\right)^{m-1} \frac{1}{\tau} + (2^{N-1} - 1) \left(1 - \frac{\mu}{\tau}\right)^{m-1} \frac{1}{\tau} \geq 0.$$

So $h$ is increasing, then

$$h(\mu) \geq h\left(\frac{(2^{N-1} - 1)\tau}{2^{N-1}}\right) = \frac{(2^{N-1} - 1)^m - (2^{N-1} - 1)}{2mN-m} \geq 0.$$

The proof is complete. \hfill \Box

Now we are ready to prove our main results about Alzer-Fonseca-Kovačec’s inequalities. Also, we will present the significance of these results in Remark 2.1 below.

\textbf{Theorem 2.1.} Let $a, b > 0$, $0 \leq \mu \leq \tau \leq 1$ and $m$ and $N$ be positive integers with $N \geq 2$. We have
(1) If $0 \leq \mu \leq \frac{\tau}{2^{N-1}}$, then

$$
\frac{\mu}{\tau}^m (a \nabla_\tau b)^m - (a^\#_\tau b)^m \\
+ \left(\frac{\mu}{\tau}\right)^m \sum_{k=2}^{N} 2^{k-2} \left(\sqrt{a^m} - 2^{k-1} \sqrt{(a^\#_\tau b)^m a^{m(2^{k-2}-1)}} \right)^2 \\
\leq (a \nabla_\mu b)^m - (a^\#_\mu b)^m.
$$

(2.5)

(2) If $\frac{(2^{N-1}-1)\tau}{2^{N-1}} \leq \mu \leq \tau$, then

$$
\frac{\mu}{\tau}^m (a \nabla_\tau b)^m - (a^\#_\tau b)^m \\
+ \left(1 - \frac{\mu}{\tau}\right)^m (a^\#_\tau b)^m \sum_{k=2}^{N} 2^{k-2} \left(\sqrt{a^m} - 2^{k-1} \sqrt{(a^\#_\tau b)^m a^{m(2^{k-2}-1)}} \right)^2 \\
\leq (a \nabla_\mu b)^m - (a^\#_\mu b)^m.
$$

(2.6)

(3) If $\mu \leq \tau \leq \frac{(2^{N-1}-1)\mu+1}{2^{N-1}}$, then

$$
\frac{1-\tau}{1-\mu}^m (a \nabla_\mu b)^m - (a^\#_\mu b)^m \\
+ \left(\frac{1-\tau}{1-\mu}\right)^m (a^\#_\mu b)^m \sum_{k=2}^{N} 2^{k-2} \left(\sqrt{b^m} - 2^{k-1} \sqrt{(a^\#_\mu b)^m b^{m(2^{k-2}-1)}} \right)^2 \\
\leq (a \nabla_\tau b)^m - (a^\#_\tau b)^m.
$$

(2.7)

(4) If $\frac{(2^{N-1}-1)+\mu}{2^{N-1}} \leq \tau \leq 1$, then

$$
\frac{1-\tau}{1-\mu}^m (a \nabla_\mu b)^m - (a^\#_\mu b)^m \\
+ \left(\frac{1-\tau}{1-\mu}\right)^m \sum_{k=2}^{N} 2^{k-2} \left(\sqrt{b^m} - 2^{k-1} \sqrt{(a^\#_\mu b)^m b^{m(2^{k-2}-1)}} \right)^2 \\
\leq (a \nabla_\tau b)^m - (a^\#_\tau b)^m.
$$

(2.8)

Proof. (1) Suppose that $0 \leq \mu \leq \frac{\tau}{2^{N-1}}$. We claim that

$$
(a \nabla_\mu b)^m - \left(\frac{\mu}{\tau}\right)^m (a \nabla_\tau b)^m - (a^\#_\tau b)^m \\
- \left(\frac{\mu}{\tau}\right)^m \sum_{k=2}^{N} 2^{k-2} \left(\sqrt{a^m} - 2^{k-1} \sqrt{(a^\#_\tau b)^m a^{m(2^{k-2}-1)}} \right)^2 \geq (a^\#_\mu b)^m.
$$
By using Lemma 2.2, we have the following identities

\[
\left( a \nabla_{\mu} b \right)^{m} - \left( \frac{\mu}{\tau} \right)^{m} \left( a \nabla_{\tau} b \right)^{m} - (a_{\tau} b)^{m} \\
- \left( \frac{\mu}{\tau} \right)^{m} \sum_{k=2}^{N} 2^{k-2} \left( \sqrt{a^{m}} - 2^{k-1} \sqrt{(a_{\tau} b)^{m} a^{m(2^{k-2} - 1)}} \right)^{2} \\
= \sum_{k=0}^{m} \left( \frac{m}{k} \right) \mu^{k} (1 - \mu)^{m-k} a^{m-k} b^{k} - \left( \frac{\mu}{\tau} \right)^{m} \left( \sum_{k=0}^{m} \left( \frac{m}{k} \right) \tau^{k} (1 - \tau)^{m-k} a^{m-k} b^{k} - (a_{\tau} b)^{m} \right) \\
- (2^{N-1} - 1) \left( \frac{\mu}{\tau} \right)^{m} a^{m} - (a_{\tau} b)^{m} + 2^{N-1} \left( \frac{\mu}{\tau} \right)^{m} \sqrt{a^{m} 2^{N-1} \sqrt{(a_{\tau} b)^{m} a^{m(2^{N-2} - 1)}}} \\
= \sum_{k=1}^{m} \left( \frac{m}{k} \right) \left( \mu^{k} (1 - \mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^{m} \tau^{k} (1 - \tau)^{m-k} \right) a^{m-k} b^{k} \\
+ \left( (1 - \mu)^{m} - (1 - \tau)^{m} \left( \frac{\mu}{\tau} \right)^{m} - (2^{N-1} - 1) \left( \frac{\mu}{\tau} \right)^{m} \right) a^{m} \\
+ 2^{N-1} \left( \frac{\mu}{\tau} \right)^{m} \sqrt{a^{m} 2^{N-1} \sqrt{(a_{\tau} b)^{m} a^{m(2^{N-2} - 1)}}} \\
= \sum_{k=0}^{m+1} \mu_{k} x_{k},
\]

where \( x_{k} \) and \( \mu_{k} \) are given by:

\[
x_{0} := a^{m}, \quad \text{with} \quad \mu_{0} := (1 - \mu)^{m} - (1 - \tau)^{m} \left( \frac{\mu}{\tau} \right)^{m} - (2^{N-1} - 1) \left( \frac{\mu}{\tau} \right)^{m},
\]

and for \( 1 \leq k \leq m, \)

\[
x_{k} := a^{m-k} b^{k}, \quad \text{with} \quad \mu_{k} := \left( \frac{m}{k} \right) \left( \mu^{k} (1 - \mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^{m} \tau^{k} (1 - \tau)^{m-k} \right),
\]

and

\[
x_{m+1} := \sqrt{a^{m} 2^{N-1} \sqrt{(a_{\tau} b)^{m} a^{m(2^{N-2} - 1)}}}, \quad \text{with} \quad \mu_{m+1} := 2^{N-1} \left( \frac{\mu}{\tau} \right)^{m}.
\]

By using Lemma 2.3, we have

(a) \( x_{k} > 0 \) for all \( k \in \{0, 1, \ldots, m+1\}, \)

(b) \( \mu_{k} \geq 0 \) for all \( k \in \{0, 1, \ldots, m+1\}, \) with \( \sum_{k=0}^{m+1} \mu_{k} = 1. \)

Hence by Theorem 1.1, we get

\[
\left( a \nabla_{\mu} b \right)^{m} - \left( \frac{\mu}{\tau} \right)^{m} \left( a \nabla_{\tau} b \right)^{m} - (a_{\tau} b)^{m} \\
- \left( \frac{\mu}{\tau} \right)^{m} \sum_{k=2}^{N} 2^{k-2} \left( \sqrt{a^{m}} - 2^{k-1} \sqrt{(a_{\tau} b)^{m} a^{m(2^{k-2} - 1)}} \right)^{2} \geq \prod_{k=0}^{m+1} x_{k}^{\mu_{k}} = a^{\alpha(m)} b^{\beta(m)},
\]
\[
\alpha(m) = \sum_{k=1}^{m-1} \binom{m}{k} (m-k) \left( \mu^k (1-\mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^m \tau^k (1-\tau)^{m-k} \right)
\]
\[
+ m \left( (1-\mu)^m - (1-\tau)^m \left( \frac{\mu}{\tau} \right)^m - (2^{N-1} - 1) \left( \frac{\mu}{\tau} \right)^m \right)
\]
\[
+ 2^{N-1} \left( \frac{\mu}{\tau} \right)^m \left( \frac{m}{2} + \frac{m(1-\tau) + m(2^{N-2} - 1)}{2^{N-1}} \right)
\]
\[
= \sum_{k=0}^{m-1} \binom{m}{k} (m-k) \left( \mu^k (1-\mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^m \tau^k (1-\tau)^{m-k} \right)
\]
\[
- m (2^{N-1} - 1) \left( \frac{\mu}{\tau} \right)^m + (2^{N-1} m - m\tau) \left( \frac{\mu}{\tau} \right)^m
\]
\[
= \sum_{k=0}^{m-1} \binom{m}{k} (m-k) \left( \mu^k (1-\mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^m \tau^k (1-\tau)^{m-k} \right)
\]
\[
+ m(1-\tau) \left( \frac{\mu}{\tau} \right)^m
\]
\[
= m(1-\mu) - m(1-\tau) \left( \frac{\mu}{\tau} \right)^m + m(1-\tau) \left( \frac{\mu}{\tau} \right)^m = m(1-\mu), \quad \text{(by Lemma 2.1)}
\]

and

\[
\beta(m) = \sum_{k=1}^{m} \binom{m}{k} k \left( \mu^k (1-\mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^m \tau^k (1-\tau)^{m-k} \right) + 2^{N-1} \left( \frac{\mu}{\tau} \right)^m \frac{m\tau}{2^{N-1}}
\]
\[
= \sum_{k=1}^{m} \binom{m}{k} k \left( \mu^k (1-\mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^m \tau^k (1-\tau)^{m-k} \right) + m\tau \left( \frac{\mu}{\tau} \right)^m
\]
\[
= m\mu - m\tau \left( \frac{\mu}{\tau} \right)^m + m\tau \left( \frac{\mu}{\tau} \right)^m = m\mu \quad \text{ (by Lemma 2.1)}
\]

(2) Suppose that \( \frac{(2^{N-1} - 1)\tau}{2^{N-1}} \leq \mu \leq \tau \). We claim that

\[
\left( a_{\nabla} \mu^m \right)^m - \left( \frac{\mu}{\tau} \right)^m \left( (a_{\nabla} \tau^m \mu^m - (a_{\nabla} \tau^m \mu^m) \right)
\]
\[
- \left( 1 - \frac{\mu}{\tau} \right)^m (a_{\nabla} \tau^m)^m \sum_{k=2}^{N} 2^{k-2} \left( \frac{a_{\nabla} \tau^m}{a_{\nabla} \tau^m - 1} \right)^{2} \geq \left( a_{\nabla} \tau^m \mu^m \right)^m.
\]
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By using Lemma 2.2, we have the following identities

\[
\left( a \nabla \mu b \right)^m - \left( \frac{\mu}{\tau} \right)^m \left( (a \nabla \tau b)^m - (a_{\#}^\tau b)^m \right)
- \left( 1 - \frac{\mu}{\tau} \right)^m (a_{\#}^\tau b)^m \sum_{k=2}^{N} 2^{k-2} \left( 2^{k-1} \sqrt{\frac{a^m}{(a_{\#}^\tau b)^m}} - 1 \right)^2
= \sum_{k=0}^{m} \binom{m}{k} \mu^k (1 - \mu)^{m-k} a^{m-k} b^k - \left( \frac{\mu}{\tau} \right)^m \sum_{k=0}^{m} \binom{m}{k} \tau^k (1 - \tau)^{m-k} a^{m-k} b^k - (a_{\#}^\tau b)^m
-(2^{N-1} - 1) \left( 1 - \frac{\mu}{\tau} \right)^m (a_{\#}^\tau b)^m - \left( 1 - \frac{\mu}{\tau} \right)^m a^m
+ 2^{N-1} \left( 1 - \frac{\mu}{\tau} \right)^m a^{\frac{2^{N-1}m(1-\tau) + m\tau}{2^{N-1}}} \frac{m(2^{N-1}-1)}{2^{N-1}}
\]

where \( x_k \) and \( \mu_k \) are given by:

\[
x_0 := a^m, \quad \text{with} \quad \mu_0 := (1 - \mu)^m - (1 - \tau)^m \left( \frac{\mu}{\tau} \right)^m - \left( 1 - \frac{\mu}{\tau} \right)^m,
\]

and for \( 1 \leq k \leq m, \)

\[
x_k := a^{m-k} b^k, \quad \text{with} \quad \mu_k := \binom{m}{k} \mu^k (1 - \mu)^{m-k} - \left( \frac{\mu}{\tau} \right)^m \tau^k (1 - \tau)^{m-k},
\]

\[
x_{m+1} := (a_{\#}^\tau b)^m, \quad \text{with} \quad \mu_{m+1} := \left( \frac{\mu}{\tau} \right)^m - (2^{N-1} - 1) \left( 1 - \frac{\mu}{\tau} \right)^m,
\]

\[
x_{m+2} := a^{\frac{2^{N-1}m(1-\tau) + m\tau}{2^{N-1}}} \frac{m(2^{N-1}-1)}{2^{N-1}}, \quad \text{with} \quad \mu_{m+2} := 2^{N-1} \left( 1 - \frac{\mu}{\tau} \right)^m,
\]

By using Lemma 2.3, we have

(a) \( x_k > 0 \) for all \( k \in \{0, 1, \ldots, m + 2\}, \)

(b) \( \mu_k \geq 0 \) for all \( k \in \{0, 1, \ldots, m + 2\}, \) with \( \sum_{k=0}^{m+2} \mu_k = 1. \)

Hence by Theorem 1.1, we get

\[
\left( a \nabla \mu b \right)^m - \left( \frac{\mu}{\tau} \right)^m \left( (a \nabla \tau b)^m - (a_{\#}^\tau b)^m \right)
- \left( 1 - \frac{\mu}{\tau} \right)^m (a_{\#}^\tau b)^m \sum_{k=2}^{N} 2^{k-2} \left( 2^{k-1} \sqrt{\frac{a^m}{(a_{\#}^\tau b)^m}} - 1 \right)^2 \geq \prod_{k=0}^{m+2} x_k^\mu = a^{\alpha(m)} b^{\beta(m)},
\]

where
\[
\alpha(m) = \sum_{k=1}^{m-1} \binom{m}{k} (m-k) \left( \mu^k (1-\mu)^{m-k} - \left(\frac{\mu}{\tau}\right)^m \tau^k (1-\tau)^{m-k} \right) \\
+ m \left( (1-\mu)^m - (1-\tau)^m \left(\frac{\mu}{\tau}\right)^m \right) \\
- m(1-\tau) \left( \left(\frac{\mu}{\tau}\right)^m - (2^{N-1} - 1)(1-\mu)^m \right) \\
+ 2^{N-1} \left( (1-\mu)^m \cdot \frac{2^{N-1}m(1-\tau) + m\tau}{2^{N-1}} \right) \\
= \sum_{k=0}^{m-1} \binom{m}{k} (m-k) \left( \mu^k (1-\mu)^{m-k} - \left(\frac{\mu}{\tau}\right)^m \tau^k (1-\tau)^{m-k} \right) + m(1-\tau) \left(\frac{\mu}{\tau}\right)^m \\
= m(1-\mu) - m(1-\tau) \left(\frac{\mu}{\tau}\right)^m + m(1-\tau) \left(\frac{\mu}{\tau}\right)^m = m(1-\mu) \quad \text{(by Lemma 2.1)},
\]

and
\[
\beta(m) = \sum_{k=1}^{m} \binom{m}{k} k \left( \mu^k (1-\mu)^{m-k} - \left(\frac{\mu}{\tau}\right)^m \tau^k (1-\tau)^{m-k} \right) \\
+ m\tau \left( \left(\frac{\mu}{\tau}\right)^m - (2^{N-1} - 1)(1-\mu)^m \right) \\
+ 2^{N-1} \left( (1-\mu)^m \cdot \frac{2^{N-1}m(1-\tau) + m\tau}{2^{N-1}} \right) \\
= \sum_{k=1}^{m} \binom{m}{k} k \left( \mu^k (1-\mu)^{m-k} - \left(\frac{\mu}{\tau}\right)^m \tau^k (1-\tau)^{m-k} \right) + m\tau \left(\frac{\mu}{\tau}\right)^m \\
= m\mu - m\tau \left(\frac{\mu}{\tau}\right)^m + m\tau \left(\frac{\mu}{\tau}\right)^m = m\mu \quad \text{(by Lemma 2.1)}.
\]

(3) If \(0 \leq \mu \leq \tau \leq 1\), then we have \(0 \leq 1-\tau \leq 1 - \mu \leq 1\). Suppose that \(\mu \leq \tau \leq (2^{N-1}-1)\mu + 1\), then \(2^{N-1}(1-\mu) \leq 1 - \tau \leq 1 - \mu\). So by changing \(a, b\), \(\mu\), and \(\tau\) by \(b-a, 1-\tau\) and \(1-\mu\), respectively in inequality (2.6), the desired inequality (2.7) is obtained.

(4) Suppose that \(\frac{(2^{N-1}-1)+\mu}{2^{N-1}} \leq \tau \leq 1\), then \(0 \leq 1 - \tau \leq \frac{1-\mu}{2^{N-1}}\). So by changing \(a, b\), \(\mu\), and \(\tau\) by \(b-a, 1-\tau\) and \(1-\mu\), respectively in inequality (2.5), the desired inequality (2.8) is obtained.

\[\square\]

Remark 2.1. Notice that the first inequality in Theorem 2.1 can be written as
\[
\left(\frac{\mu}{\mu}\right)^m [(a\nabla_\mu b)^m - (a_\mu b)^m] \leq (a\nabla_\mu b)^m - (a_\mu b)^m; 0 \leq \mu < \mu \leq 1; m = 1, 2, \ldots \quad (2.9)
\]
while the second inequality in the same theorem can be stated as
\[
[(a\nabla_\mu b)^m - (a_\mu b)^m] \leq \left(\frac{1-\mu}{\mu}\right)^m (a\nabla_\mu b)^m - (a_\mu b)^m; 0 \leq \mu < \mu \leq 1; m = 1, 2, \ldots
\]

Consequently, inequalities (2.5) and (2.6) of Theorem 2.1 presents multiple refining forms for (2.9), while inequalities (2.7) and (2.8) in the same theorem presents multiple
refining terms for (2.10), Consequently Theorem 2.1, give a considerable refinement of Theorem 1.4.

Remark 2.2. If we set \( m = 1 \) in Theorem 2.1, \( \tau = \frac{1}{2} \) in inequalities (2.5) and (2.8) and \( \mu = \frac{1}{2} \) in inequalities (2.6) and (2.7) then we recapture Proposition 2.1 and Proposition 2.2 in [9], respectively.

3. Inequalities for operators

Let \( B(\mathcal{H}) \) denote the \( C^* \)-Algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \). An operator \( A \in B(\mathcal{H}) \) is called positive, denoted by \( A \geq 0 \) if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \). The set of all positive operators is denoted by \( B(\mathcal{H})^+ \). The set of all invertible operators in \( B(\mathcal{H})^+ \) is denoted by \( B(\mathcal{H})^{++} \).

Assume that \( A, B \) are positive operators on a complex Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \). The weighted operator arithmetic mean for the pair \( (A, B) \) is defined by

\[
A \nabla_{\mu} B := (1 - \mu)A + \mu B.
\]

In 1980, Kubo and Ando [13] introduced the weighted operator geometric mean for the pair \( (A, B) \) with \( A \) positive and invertible and \( B \) positive by

\[
A^g_{\mu} B := A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^{\mu} A^{1/2}.
\]

When \( \mu \in [0, 1] \) we have the following fundamental operator means inequalities, or Young’s inequality

\[
A^g_{\mu} B \leq A \nabla_{\mu} B.
\]

In order to obtain operator inequalities from the corresponding scalar inequalities presented in section 2, the following lemma is essential.

Lemma 3.1 ([22, p. 3]). Let \( T \in B(\mathcal{H}) \) be self-adjoint. If \( f \) and \( g \) are both continuous real valued functions with \( f(t) \geq g(t) \) for \( t \in \text{Sp}(T) \) (where the sign \( \text{Sp}(T) \) denotes the spectrum of \( T \)), then \( f(T) \geq g(T) \).

In the next theorem, we show an extension of Theorem 2.1 for operators.

Theorem 3.1. Let \( A, B \in B(\mathcal{H})^{++} \) and \( m \) and \( N \) be positive integers with \( N \geq 2 \), and \( 0 \leq \mu \leq \tau \leq 1 \). Then

(1) If \( 0 \leq \mu \leq \frac{\mu}{\tau} \), then

\[
\left( \frac{\mu}{\tau} \right)^m \left( A^m_{\mu} (A \nabla_{\tau} B) - A^m_{\mu \tau} B \right) + \left( \frac{\mu}{\tau} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( A + A^k_{\mu \tau} B - 2A^k_{\mu \tau} B \right)
\]

\[
\leq A^m_{\mu m} (A \nabla_{\mu} B) - A^m_{\mu m} B. \tag{3.1}
\]

(2) If \( \frac{\tau}{2^{N-1}} \leq \mu \leq \tau \), then

\[
\left( \frac{\mu}{\tau} \right)^m \left( A^m_{\mu} (A \nabla_{\tau} B) - A^m_{\mu \tau} B \right) + \left( 1 - \frac{\mu}{\tau} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( A^k_{\mu \tau} (2^{k-2}) - 2A^k_{\mu \tau} (2^{k-2} - 1) \right)
\]

\[
\leq A^m_{\mu m} (A \nabla_{\mu} B) - A^m_{\mu m} B. \tag{3.2}
\]
(3) If \( \mu \leq \tau \leq \frac{2^{N-1}-1}{2^{N-1}} \mu + 1 \), then
\[
\left( \frac{1}{1-\mu} \right)^m (A_{\leq m}(A \nabla_{\mu} B) - A_{\leq m\mu} B) \\
+ \left( \frac{1}{1-\mu} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( A_{\leq m}(2k-2,\ldots,1) \right) B + A_{\leq m\mu} B - 2A_{\leq m(2k-2,\ldots,1)\mu} B \leq A_{\leq m}(A \nabla_{\tau} B) - A_{\leq m\tau} B. \tag{3.3}
\]

(4) If \( \frac{2^{N-1}-1}{2^{N-1}} + \mu \leq \tau \leq 1 \), then
\[
\left( \frac{1}{1-\mu} \right)^m (A_{\leq m}(A \nabla_{\mu} B) - A_{\leq m\mu} B) \\
+ \left( \frac{1}{1-\mu} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( A_{\leq m} B + A_{\leq m}(2k-2,\ldots,1) \right) B - 2A_{\leq m(2k-2,\ldots,1)\mu} B \leq A_{\leq m}(A \nabla_{\tau} B) - A_{\leq m\tau} B. \tag{3.4}
\]

**Proof.** We prove the first inequality. The other inequalities are shown similarly. Let \( a = 1 \) in inequality (2.5), then we get
\[
\left( \frac{\mu}{\tau} \right)^m \left( ((1-\tau) + \tau b)^m - b^{m\tau} \right) \leq \left( \frac{1}{1-\mu} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( 1 + b^{\frac{m\tau}{2^{N-1}}} - 2b^{\frac{m\tau}{2^{N-1}}} \right)
\]
\[
\leq ((1-\mu)I + \mu C)^m - C^{m\mu} \tag{3.5}
\]

The operator \( C = A_{\leq \tau} BA_{\geq \tau} \) has a positive spectrum, then by Lemma 3.1 and inequality (3.5) we get
\[
\left( \frac{\mu}{\tau} \right)^m \left( ((1-\tau)I + \tau C)^m - C^{m\tau} \right) \leq \left( \frac{1}{1-\mu} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( I + C^{\frac{m\tau}{2^{N-1}}} - 2C^{\frac{m\tau}{2^{N-1}}} \right)
\]
\[
\leq ((1-\mu)I + \mu C)^m - C^{m\mu} \tag{3.6}
\]

Finally, multiplying inequality (3.6) by \( A_{\leq \tau} \) on the left and right hand sides, we can get
\[
\left( \frac{\mu}{\tau} \right)^m (A_{\leq m}(A \nabla_{\tau} B) - A_{\leq m\tau} B) \\
+ \left( \frac{\mu}{\tau} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( A_{\leq m} B + A_{\leq m}(2k-2,\ldots,1) \right) B - 2A_{\leq m(2k-2,\ldots,1)\mu} B \leq A_{\leq m}(A \nabla_{\mu} B) - A_{\leq m\mu} B.
\]

\[\square\]

4. **Inequalities for matrices**

In this section, we will give some new refined Young type inequalities for determinants, norms and traces of positive definite matrices.

The singular values of a matrix \( A \in \mathbb{M}_n(\mathbb{C}) \) are the eigenvalues of the positive semi-definite matrix \( |A| = (A^*A)^{1/2} \), denoted by \( s_i(A) \) for \( i = 1, 2, 3, \ldots, n \). A norm \( |||.|.||| \) on \( \mathbb{M}_n(\mathbb{C}) \) is called unitarily invariant if \( |||UAV||| = |||A||| \) for all \( A \in \mathbb{M}_n(\mathbb{C}) \) and all unitary matrices \( U, V \in \mathbb{M}_n(\mathbb{C}) \). The trace norm is given by \( |||A|||_1 = tr|A| = \)
\[ \sum_{k=1}^{n} s_k(A), \text{ where } tr \text{ is the usual trace. A matrix Young’s inequality due to Ando} \ [5] \text{ asserts that} \]
\[ s_j(A^{1-\mu}B^\mu) \leq s_j((1-\mu)A + \mu B), \]
\[ \text{the above singular value inequality entails the following unitarily invariant norm inequality} \]
\[ |||A^{1-\mu}B^\mu||| \leq |||(1-\mu)A + \mu B|||. \]

4.1. **Refinements of Young’s type inequality for determinants.** A determinant version of Young’s inequalities is also known \[8, p. 467\]: For positive semi-definite matrices \( A, B \) and \( 0 \leq \mu \leq 1 \),
\[ \det(A^{1-\mu}B^\mu) \leq \det((1-\mu)A + \mu B). \] (4.1)

To prove the first main result of this subsection, we need to recall the following lemma that is necessary to obtain our main results, (see, e.g., \[8, p. 482,\]) is the Minkowski inequality for determinants.

**Lemma 4.1.** Let \( A, B \in M_n(\mathbb{C}) \) be positive definite matrices. Then we have
\[ \det(A + B)^{\frac{1}{n}} \geq \det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}}. \] (4.2)

In the next theorem, we show a multiple-term refinements of Alzer-Fonseca-Kovačec’s inequalities for determinants of positive definite matrices.

**Theorem 4.1.** Let \( A, B \in M_n(\mathbb{C}) \) be positive definite matrices and let \( m \) and \( N \) be positive integers with \( N \geq 2 \), and \( 0 \leq \mu \leq \tau \leq 1 \). Then

1. If \( 0 \leq \mu \leq \frac{\tau}{2N-1} \), then
\[ \left( \frac{\mu}{\tau} \right)^{nm} \left( \left( \det(A) \nabla_{\tau} \det(B) \right)^m - (\det(A)^{\frac{1}{n}} \det(B)^{\frac{1}{n}})^m \right) \]
\[ + \left( \frac{\mu}{\tau} \right)^{nm} \sum_{k=2}^{N} 2^{k-2} \left( \sqrt{\det(A)^m} - \frac{1}{2^{k-1}} \sqrt{\det(A)^{\frac{1}{n}} \det(B)^{\frac{1}{n}}} \right)^2 \]
\[ \leq \left( \det((1-\mu)A + \mu B) \right)^m - \left( \det(A^{1-\mu}B^\mu) \right)^m. \] (4.3)

2. If \( \frac{(2N-1-1)}{2N-1} \mu \leq \tau \), then
\[ \left( \frac{\mu}{\tau} \right)^{nm} \left( \left( \det(A) \nabla_{\tau} \det(B) \right)^m - (\det(A)^{\frac{1}{n}} \det(B)^{\frac{1}{n}})^m \right) \]
\[ + (1 - \frac{\mu}{\tau})^{nm} \left( \det(A)^{\frac{1}{n}} \det(B)^{\frac{1}{n}} \right)^m \sum_{k=2}^{N} 2^{k-2} \left( \frac{1}{2^{k-1}} \sqrt{\frac{\det(A)^m}{\det(A)^{\frac{1}{n}} \det(B)^{\frac{1}{n}}}} - 1 \right)^2 \]
\[ \leq \left( \det((1-\mu)A + \mu B) \right)^m - \left( \det(A^{1-\mu}B^\mu) \right)^m. \] (4.4)
(3) If \( \mu \leq \tau \leq \frac{(2^{N-1}-1)\mu+1}{2^{N-1}} \), then
\[
\left( \frac{1 - \tau}{1 - \mu} \right)^{nm} \left( (\det(A)\nabla_\mu \det(B))^m - (\det(A)\nabla_\mu \det(B))^m \right) \\
+ \left( 1 - \frac{1 - \tau}{1 - \mu} \right)^{nm} \left( \det(A)\nabla_\mu \det(B))^m \sum_{k=2}^{N} 2^{k-2} \left( \sqrt[k]{\frac{\det(B)^m}{(\det(A)\nabla_\mu \det(B))^m}} - 1 \right) \right)^2 \\
\leq \left( \det((1 - \tau)A + \tau B) \right)^m - \left( \det(A^{1-\tau}B^m) \right)^m. 
\] (4.5)

(4) If \( \frac{(2^{N-1}-1)+\mu}{2^{N-1}} \leq \tau \leq 1 \), then
\[
\left( \frac{1 - \tau}{1 - \mu} \right)^{nm} \left( (\det(A)\nabla_\mu \det(B))^m - (\det(A)\nabla_\mu \det(B))^m \right) \\
+ \left( 1 - \frac{1 - \tau}{1 - \mu} \right)^{nm} \sum_{k=2}^{N} 2^{k-2} \left( \sqrt[k]{\frac{\det(B)^m}{(\det(A)\nabla_\mu \det(B))^m}} - 1 \right) \right)^2 \\
\leq \left( \det((1 - \tau)A + \tau B) \right)^m - \left( \det(A^{1-\tau}B^m) \right)^m. 
\] (4.6)

Proof. We prove the first inequality. The other inequalities are shown similarly.
Suppose that \( 0 \leq \mu \leq \frac{\tau}{2^{N-1}} \). By using inequality (2.5) of Theorem 2.1, we have
\[
det \left( (1 - \mu)A + \mu B \right)^m = \left[ det \left( (1 - \mu)A + \mu B \right)^{\frac{1}{\tau}} \right]^{nm} \\
\geq \left[ det((1 - \mu)A)^{\frac{1}{\tau}} + det(\mu B)^{\frac{1}{\tau}} \right]^{nm} \quad (by \ Lemma 4.1) \\
= \left[ (1 - \mu) det(A)^{\frac{1}{\tau}} + \mu det(B)^{\frac{1}{\tau}} \right]^{nm} \\
\geq \left( \frac{\mu}{\tau} \right)^{nm} \left( (\det(A)\nabla_\tau \det(B))^m - (\det(A)\nabla_\tau \det(B))^m \right) \\
+ \left( \frac{\mu}{\tau} \right)^{nm} \sum_{k=2}^{N} 2^{k-2} \left( \sqrt[k]{\frac{\det(B)^m}{(\det(A)\nabla_\tau \det(B))^m}} - 1 \right) \right)^2 \\
+ \left( \det(A^{1-\mu}B^m) \right)^m \\
= \left( \frac{\mu}{\tau} \right)^{nm} \left( (\det(A)\nabla_\tau \det(B))^m - (\det(A)\nabla_\tau \det(B))^m \right) \\
+ \left( \frac{\mu}{\tau} \right)^{nm} \sum_{k=2}^{N} 2^{k-2} \left( \sqrt[k]{\frac{\det(B)^m}{(\det(A)\nabla_\tau \det(B))^m}} - 1 \right) \right)^2 \\
+ \left( \det(A^{1-\mu}B^m) \right)^m. 
\]

\[\Box\]

4.2. Refinements of Young's type inequality for norms. The main result to be proved in this subsection, concerns the norms of positive semi-definite matrices.
In order to achieve our further results, furthermore, we need the following lemma.

Lemma 4.2 ([12]). Let \( A, B \in M_n(\mathbb{C}) \) be positive semi-definite matrices. Then we have
\[
\|A^{1-\mu}XB^\mu\| \leq \|AX\|\|1-\mu\|\|XB\|^\mu. 
\] (4.7)
In particular,

$$tr|A^{1-\mu}B^\mu| \leq (trA)^{1-\mu}(trB)^\mu.$$  \hfill (4.8)

In the next theorem, we present multiple-term refinements of Alzer-Fonseca-Kovačec's inequalities for norms of positive definite matrices.

**Theorem 4.2.** Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive definite matrices and $m$ and $N$ be positive integers with $N \geq 2$, and $0 \leq \mu, \tau \leq 1$. Then

1. If $0 \leq \mu \leq \frac{\tau}{2N-1}$, then

$$\left(\frac{\mu}{\tau}\right)^m \left(||AX|||\nabla_\tau||XB|||^m - (||AX|||\nabla_\tau||XB|||^m)^m\right)$$

$$+ \left(\frac{\mu}{\tau}\right)^m \sum_{k=2}^N 2^{k-2} \left(\sqrt{||AX||m} - 2^{k-1} \sqrt{\left(||AX|||\nabla_\tau||XB|||^m\right)m||AX||^m(2^{k-1}-1)}\right)^2$$

$$\leq \left((1 - \mu)||AX||^{m} + \mu||XB||^{m}\right)^m - \left(||A^{1-\mu}XB^\mu||^m\right)^m.$$  \hfill (4.9)

2. If $\frac{(2N-1-1)^\tau}{2N-1} \leq \mu \leq \tau$, then

$$\left(\frac{\mu}{\tau}\right)^m \left(||AX|||\nabla_\tau||XB|||^m - (||AX|||\nabla_\tau||XB|||^m)^m\right)$$

$$+ \left(1 - \frac{\mu}{\tau}\right)^m (||AX|||\nabla_\tau||XB|||^m)^m \sum_{k=2}^N 2^{k-2} \left(2^{k-1} \sqrt{||AX||^m||\nabla_\tau||XB||^m} - 1\right)^2$$

$$\leq \left((1 - \mu)||AX||^{m} + \mu||XB||^{m}\right)^m - \left(||A^{1-\mu}XB^\mu||^m\right)^m.$$  \hfill (4.10)

3. If $\mu \leq \tau \leq \frac{(2N-1-1)^\mu+1}{2N-1}$, then

$$\left(\frac{1 - \tau}{1 - \mu}\right)^m \left(||AX|||\nabla_\mu||XB|||^m - (||AX|||\nabla_\mu||XB|||^m)^m\right)$$

$$+ \left(1 - \frac{1 - \tau}{1 - \mu}\right)^m (||AX|||\nabla_\mu||XB|||^m)^m \sum_{k=2}^N 2^{k-2} \left(2^{k-1} \sqrt{||XB||^m||\nabla_\mu||XB||^m} - 1\right)^2$$

$$\leq \left((1 - \tau)||AX||^{m} + \tau||XB||^{m}\right)^m - \left(||A^{1-\tau}XB^\tau||^m\right)^m.$$  \hfill (4.11)

4. If $\frac{(2N-1-1)+\mu}{2N-1} \leq \tau \leq 1$, then

$$\left(\frac{1 - \tau}{1 - \mu}\right)^m \left(||AX|||\nabla_\mu||XB|||^m - (||AX|||\nabla_\mu||XB|||^m)^m\right)$$

$$+ \left(1 - \frac{1 - \tau}{1 - \mu}\right)^m \sum_{k=2}^N 2^{k-2} \left(\sqrt{||XB||^m} - 2^{k-1} \sqrt{\left(||AX|||\nabla_\mu||XB|||^m\right)m||XB||^m(2^{k-2}-1)}\right)^2$$

$$\leq \left((1 - \tau)||AX||^{m} + \tau||XB||^{m}\right)^m - \left(||A^{1-\tau}XB^\tau||^m\right)^m.$$  \hfill (4.12)

*Proof.* We prove the first inequality. The other inequalities are shown similarly.
By using inequality (2.5) of Theorem 2.1 and Lemma 4.2 we have
\[
\left( \| A^{1-\mu}XB^\mu \| \right)^m + \left( \frac{\mu}{\tau} \right)^m \left( \| \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m - \left( \| \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m
\]
\[
\left( \frac{\mu}{\tau} \right)^m \sum_{k=2}^N 2^{k-2} \left( \sqrt{\| A \|} - 2^{k-1} \sqrt{\| A \|} \right)^m \left( \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m \leq \left( \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m
\]
\[
+ \left( \frac{\mu}{\tau} \right)^m \left( \| \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m \leq \left( \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m
\]
\[
+ \left( \frac{\mu}{\tau} \right)^m \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m \leq \left( \| A \| \| \nabla_{\tau} \| \| XB \| \| \right)^m
\]
\[
\leq \left( \| A \| \| XB \| \| \right)^m.
\]

4.3. Refinements of Young’s type inequality for traces. In the next result, we show a multiple-term refinements of Alzer-Fonseca-Kovačec’s inequalities for traces of positive definite matrices.

**Theorem 4.3.** Let $A, B \in M_n(\mathbb{C})$ be positive definite matrices and $m$ and $N$ be positive integers with $N \geq 2$, and $0 \leq \mu \leq \tau \leq 1$. Then

1. If $0 \leq \mu \leq \frac{\tau}{2^{N-1}}$, then
   \[
   \left( \frac{\mu}{\tau} \right)^m \left( (\tr(A)\nabla_{\tau}\tr(B))^m - (\tr(A)\nabla_{\tau}\tr(B))^m \right)
   + \left( \frac{\mu}{\tau} \right)^m \sum_{k=2}^N 2^{k-2} \left( \sqrt{\tr(A)} - 2^{k-1} \sqrt{\tr(A)} \right)^m \tr(A)^m \tr(B)^m \leq \left( \tr((1-\mu)A + \mu B) \right)^m - \left( \tr(A^{1-\mu}B^\mu) \right)^m.
   \] (4.13)

2. If $\frac{2^{N-1}-1}{2^{N-1}} \leq \mu \leq \tau$, then
   \[
   \left( \frac{\mu}{\tau} \right)^m \left( (\tr(A)\nabla_{\tau}\tr(B))^m - (\tr(A)\nabla_{\tau}\tr(B))^m \right)
   + \left( 1 - \frac{\mu}{\tau} \right)^m \left( \tr(A)^m \right)^\mu \sum_{k=2}^N 2^{k-2} \left( \sqrt{\tr(A)} - 2^{k-1} \sqrt{\tr(A)} \right)^m \tr(A)^m \tr(B)^m \leq \left( \tr((1-\mu)A + \mu B) \right)^m - \left( \tr(A^{1-\mu}B^\mu) \right)^m.
   \] (4.14)

3. If $\mu \leq \tau \leq \frac{2^{N-1}-1}{2^{N-1}}$, then
   \[
   \left( \frac{1-\tau}{1-\mu} \right)^m \left( (\tr(A)\nabla_{\mu}\tr(B))^m - (\tr(A)\nabla_{\mu}\tr(B))^m \right)
   + \left( 1 - \frac{1-\tau}{1-\mu} \right)^m \left( \tr(A)^m \right)^\mu \sum_{k=2}^N 2^{k-2} \left( \sqrt{\tr(A)} - 2^{k-1} \sqrt{\tr(A)} \right)^m \tr(A)^m \tr(B)^m \leq \left( \tr((1-\tau)A + \tau B) \right)^m - \left( \tr(A^{1-\tau}B^\tau) \right)^m.
   \] (4.15)
Proof. We prove the first inequality. The other inequalities are shown similarly.

By using inequality (2.5) of Theorem 2.1 and Lemma 4.2 we have

\[
\left(\text{tr}(A^{1-\mu}B^\mu)\right)^m + \left(\frac{\mu}{\tau}\right)^m \left(\text{tr}(A)\nabla_\tau \text{tr}(B))^m - (tr(A)^\mu \text{tr}(B))^m\right)
\]

\[+
\left(\frac{\mu}{\tau}\right)^m \sum_{k=2}^N 2^{k-2}\left(\sqrt{\text{tr}(A)^m} - \sqrt[2k-1]{\text{tr}(A)^\mu \text{tr}(B))^m tr(A)^{m(2k-2-1)}}\right)^2
\]

\[\leq \left(\text{tr}((1 - \tau)A + \tau B)\right)^m - \left(\text{tr}(A^{1-\tau}B^\tau)\right)^m. \tag{4.16}
\]

5. Concluding remarks

The paper starts with and an introduction in which we make some recalls concern (scalar) Young’s inequality and its refinements obtained by several authors.

The purpose of this work is to give multiple-term refinements of Alzer-Fonseca-Kovačec’s inequalities and provide several applications.

In section 2, we establish in Theorem 2.1 multiple-term refinements Alzer-Fonseca-Kovačec’s inequalities. This theorem will generalize the results (see Proposition 2.1 and Proposition 2.2), obtained by S. Furuichi in [9].

In section 3, As a consequence of Theorem 2.1, we deduce (see Theorem 3.1) a new refinement of Young’s type inequalities for operators.

In section 4, A first application of Theorem 2.1 is to give (see Theorem 4.1) a multiple-term refinements of Alzer-Fonseca-Kovačec’s inequalities for the determinants of positive definite matrices.

A second application of Theorem 2.1 is to give (see Theorem 4.2) a multiple-term refinements of Alzer-Fonseca-Kovačec’s inequalities for the norms of positive definite matrices.

In the last application of Theorem 2.1, we provide a new refinement of Alzer-Fonseca-Kovačec’s inequalities (see Theorem 4.3), for the traces of positive definite matrices.

We hope that our work will provide more other applications.

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References


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