BOUNDING THE ORDER OF NILPOTENT SUBGROUPS OF SOLVABLE LINEAR GROUPS

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Abstract. In this paper, we prove that a nilpotent subgroup $H$ of a finite solvable group $G$ has order at most $|V|^{\beta}/2$ if $V$ is a faithful and completely reducible $G$-module where $\beta = \ln(32)/\ln(9)$. We also find related bounds for nilpotent subgroups of odd order in a solvable linear group. We then further generalize these results to certain chief factors of an arbitrary linear group.

1. Introduction

The order of a finite group is perhaps one of the most fundamental quantities in group theory one can study. Accordingly, the concept of bounding the order of a finite group is a very natural one and has long been a subject of vigorous research. For example, Wolf obtained results [12] in bounding the order of nilpotent linear groups and solvable linear groups by the size of the vector spaces. Pálfy [10, Theorem 1] also derived the same upper bound for the order of solvable linear groups. For solvable linear groups of odd order, Pálfy obtained some more delicate bounds in [9].

In this paper, we will mainly focus on studying nilpotent linear groups, which was motivated by the following result of Wolf [12, Theorem 1.6]. Let $G$ be a finite nilpotent linear group that acts faithfully and completely reducibly on $V$. Then $|G| \leq |V|^{\beta}/2$ where $\beta = \log(32)/\log(9) \approx 1.5774$.

In light of this result as well as some recent developments in [2, 14, 15], it is natural to ask whether one can extend this to a similar result for the order of a nilpotent subgroup $H$ of a completely reducible solvable linear group $G$ (note that $H$ need not be completely reducible on $V$). In this paper, we show that $|H| \leq |V|^{\beta}/2$ if $H$ is a nilpotent subgroup of $G$. We will also discuss the case when $H$ is of odd order.

We then further generalize these results to certain chief factors of an arbitrary linear group. For example, we use the main result of this paper as well as some recent developments in [11] to give a new proof to a result in [5].

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2. Preliminary Results

In this section, we fix some notation and prove some preliminary results.

Notation:
(1) We define a semi-linear group to be

\[ \Gamma(q^n) = \{ x \mapsto ax^\sigma \mid a \in \text{GF}(q^n)^*, \sigma \in \text{Cal}(\text{GF}(q^n)/\text{GF}(q)) \} \]

(2) We use \( H \wr S \) to denote the wreath product of \( H \) with \( S \) where \( S \) is a permutation group.

(3) We use \( N \rtimes K \) to denote the semidirect product of \( N \) with \( K \) where the action of \( K \) on \( N \) is given.

Now, we state some lemmas which will be used later.

Lemma 2.1. [8, Corollary 1.10] Suppose \( G \neq 1 \) is solvable and every normal abelian subgroup of \( G \) is cyclic. Assume that \( F = F(G) \) and let \( Z \) be the socle of the cyclic group \( Z(F) \). Let \( A = C_G(Z) \). Then \( G \) has normal subgroups \( E, U \) and \( T \) such that

1. \( F = ET, Z = E \cap T \) and \( T = C_F(E) \).
2. A Sylow \( q \)-subgroup of \( E \) is cyclic of order \( q \) or extra-special of exponent \( q \) or 4.
3. \( U \leq T, |T : U| \leq 2, U \) cyclic and \( U = C_T(U) \).
4. \( E/Z \cong F/T \) is a completely reducible \( G/F \)-module and faithful \( A/F \)-module.

Lemma 2.2. [8, Corollary 2.3] Suppose that \( V \) is a faithful irreducible \( G \)-module for a solvable group \( G \). We use the notation in Lemma 2.1. If \( V \) is quasi-primitive and \( F = T \), then \( G \leq \Gamma(V) \).

Lemma 2.3. We use the notation in Lemma 2.1. Suppose that \( V \) is a faithful irreducible \( G \)-module and \( W \) is a faithful irreducible \( U \)-module. Then \( \dim(V) = t \cdot e \cdot \dim(W) \) for integers \( t \) and \( e \) with \( e^2 = |F/T| = |E/Z| \).

Proof. This follows from [8, Corollary 2.6].

Lemma 2.4. [8, Corollary 2.12] Suppose that \( G \) is a solvable, irreducible quasi-primitive subgroup of \( \text{GL}(p,q) \) where \( p \) is a prime and \( q \) is a prime power. Then one of the following occurs:

1. \( G \leq \Gamma(q^n) \); or
(2) there exists an extra-special group $D \leq G$ which has order $p^3$, $T \leq \text{Z}(\text{GL}(p,q))$, $T \cap D = \text{Z}(D)$ and $D/\text{Z}(D)$ is a faithful irreducible $G/F(G)$-module of order $p^2$. Also $q \neq 2$ and $G/F(G) \leq \text{Sp}(2,p)$.

**Lemma 2.5.** [8, Corollary 2.15] Suppose that $G$ is a solvable irreducible subgroup of $\text{GL}(2n,2)$ where $n$ is a prime. Then one of the following occurs:

1. $G \leq \Gamma(2^n) \wr \mathbb{Z}_2$, or $G \leq S_3 \wr S$ where $Z_n \leq S \leq Z_n \cdot Z_{n-1} \leq S_n$.
2. $G \leq \Gamma(2^{2n})$; or
3. $n = 3$, $F(G)$ is extra-special of order $3^3$, $F(G)/\text{Z}(F(G))$ is a faithful irreducible $G/F(G)$-module and $|G/F(G)|$ is even, dividing 48.

If $G$ is quasi-primitive, conclusion (2) or (3) must occur.

**Lemma 2.6.** Let $G$ be an odd-order solvable subgroup of the symmetric group $S_m$. Then

$$|G| \leq (\sqrt{3})^{m-1}.$$

**Proof.** This is a well-know result. See for example [4, Theorem 1′] or [1, Theorem 1].

**Lemma 2.7.** Let $G$ be a finite solvable subgroup of $\Gamma(V)$ and let $V \neq 0$ be a finite, faithful and completely reducible $G$-module with $\text{char}(V) = q > 0$. Let $H$ be an odd order nilpotent subgroup of $G$. Then $|H| \leq |V| - 1$.

**Proof.** By [14, Theorem 3.1], we know that $H$ has at least one regular orbit on $V$. Thus $|H| \leq |V| - 1$.

**Theorem 2.8.** Let $G$ be a finite solvable group and let $V \neq 0$ be a finite, faithful and completely reducible $G$-module with $\text{char}(V)$ is odd. Let $H$ be an odd order nilpotent subgroup of $G$. Then $|H| \leq \frac{|V| - 1}{2}$.

**Proof.** By [14, Theorem 4.1], $H$ has at least two regular orbits on $V$ and the result follows.

3. Main Results

Let $H$ be a nilpotent linear group, Wolf showed that $|H| \leq |V|^{3/2}$ in [12, Theorem 1.6]. Now we consider the case where $H$ a nilpotent subgroup of a solvable linear group $G$. We will generalize [12, Theorem 1.6] in the following. We also provide a result for nilpotent subgroups of odd order.
Theorem 3.1. Let $G$ be a finite solvable group and let $V \neq 0$ be a finite, faithful and completely reducible $G$-module with $\text{char}(V) = q$. Let $H$ be a nilpotent subgroup of $G$.

(a) Then $|H| \leq |V|^\beta/2$;

(b) If $H$ is of odd order, then $|H| \leq |V|^l/\sqrt{3}$ where $l = \ln(7\sqrt{3})/\ln(8)$.

Proof. We work by induction on $|H||V|$.

Step 1. We may assume that $V$ is irreducible.

Proof. If $V$ is not irreducible, then $V = V_1 \oplus \cdots \oplus V_m$ for irreducible $G$-modules $V_i$ and $m \geq 2$. Let $C_i = C_G(V_i)$. We know that $C_i H/C_i \cong H/(H \cap C_i)$ is a nilpotent subgroup of $G/C_i$ and $V_i$ is a finite, faithful and completely reducible $G/C_i$-module. By induction, we have

$$|H/(H \cap C_i)| = |C_i H/C_i| \leq |V_i|^{\beta/2}$$

for $i = 1, \ldots, m$. Since $\bigcap_i (H \cap C_i) = 1$, then

$$H = H/\bigcap_i (H \cap C_i) \leq H/(H \cap C_1) \times \cdots \times H/(H \cap C_m).$$

We get that

$$|H| \leq \prod_i |H/(H \cap C_i)| \leq \prod_i (|V_i|^{\beta/2}) \leq |V|^{\beta/2m} \leq |V|^{\beta/2}. $$

Similarly for part (b), we have $|H| \leq |V|^l/\sqrt{3}$.

Thus we can assume that $V$ is an irreducible $G$-module.

Step 2. We may assume that $V$ is quasi-primitive.

Proof. If not, we choose $N \lhd G$ maximal such that $V_N$ is not homogeneous $N$-module. Write $V_N = W_1 \oplus \cdots \oplus W_m$ for the homogeneous components $W_i$ of $V_N$. By [8, Proposition 0.2], we have that $G/N$ faithfully and primitively permutes $W_1, \ldots, W_m$. Let $M/N$ be a chief factor of $G$. Since $G$ is solvable, $M/N$ is a minimal normal elementary abelian subgroup of $G/N$. Then $|M/N| = m$ is a prime power and $C_{G/N}(M/N) = M/N$ (see [8, Pages 39-40]).

Thus $M/N$ is a faithful irreducible $G/M$-module. Using the inductive hypothesis and the
argument in Step 1, we have

\[
|H \cap N| \leq \begin{cases} 
|V|^\beta /2^m, & \text{if } H \text{ is of odd order.} \\
|V|^l/(\sqrt{3})^m, & \text{if } H \text{ is of odd order.}
\end{cases}
\tag{1.1}
\]

Since \(H/H \cap M \cong HM/M\) is a nilpotent subgroup of \(G/M\), by induction we get

\[
|H/(H \cap M)| \leq \begin{cases} 
m^{\beta/2}, & \text{if } H \text{ is of odd order.} \\
m^l/\sqrt{3}, & \text{if } H \text{ is of odd order.}
\end{cases}
\]

At the same time,

\[
|(H \cap M)/(H \cap N)| = |(H \cap M)N/N| \leq |M/N| \leq m.
\]

Thus

\[
|H| \leq \begin{cases} 
|V|^\beta /2 \cdot m^{\beta+1}/2^m, & \text{if } H \text{ is of odd order.} \\
|V|^l/\sqrt{3} \cdot m^{l+1}/(\sqrt{3})^m, & \text{if } H \text{ is of odd order.}
\end{cases}
\]

Note that \(5/2 < \beta + 1 < 13/5\). If \(m^{\beta+1} \leq m^{13/5} \leq 2^m\), then \(|H| \leq |V|^\beta /2\). If \(m^{13/5} > 2^m\), i.e. \(m^{13} > 2^{5m}\), we can calculate that \(2 \leq m \leq 7\). On the other hand, by inequality (1.1), we only need to prove \(|H/(H \cap N)| = |HN/N| \leq 2^{m-1}\) to get the desired result.

If \(m\) is a prime, we know that \(G/M \leq GL(1,m) \cong Z_{m-1}\) and \(G/N \leq Z_m \rtimes Z_{m-1}\).

When \(m = 2\) or \(7\), since \(HN/N\) is a nilpotent subgroup of \(G/N\), we must have that \(|HN/N| \leq |G/N| \leq m \cdot (m-1) \leq 2^{m-1}\).

When \(m = 3\), \(Z_3 \rtimes Z_2\) is not a nilpotent group, we have \(|HN/N| \leq 3 \leq 2^2\).

When \(m = 5\), \(Z_5 \rtimes Z_4\) is not a nilpotent group, thus \(|HN/N| \leq 10 \leq 2^4\).

When \(m = 4\), it is clear that \(G/N \leq S_4\). Note that a nilpotent subgroup of \(S_4\) has order at most \(8 = 2^3\).

For part (b), by inequality (1.1) we just need to show that \(|HN/N| \leq (\sqrt{3})^{m-1}\). By Lemma \[2.6\] and the fact that \(G/N\) faithfully and primitively permutes \(W_1, \ldots, W_m\), we can get

\[
|HN/N| \leq (\sqrt{3})^{m-1}.
\]

This step is complete.

Step 3. Let \(|V| = q^n\). We may assume that \(n \geq 2\) and \(q^n \geq 16\).
Proof. If \( n = 1 \), i.e. \(|V| = q\). Then

\[ |H| \leq |G| \leq |GL(V)| \leq q - 1 \leq q^{3/2}/2 \leq |V|^{\beta}/2. \]

When \( H \) is of odd order, we can assume that \( q > 2 \). Then

\[ |H| \leq (q - 1)/2 \leq (q - 1)/\sqrt{3} \leq |V|^{\ell}/\sqrt{3}. \]

Now we assume that \( n \neq 1 \) and \( q^n < 16 \). Then \(|V| = 2^2, 2^3 \) or \( 3^2 \).

If \(|V| = 2^2\), then \(|GL(V)| = 6\) and a nilpotent subgroup of \( GL(V) \) has order at most 3. Thus \(|H| \leq 3 \leq |V|^{\ell}/\sqrt{3} \leq |V|^{\beta}/2\).

If \(|V| = 2^3\), then \(|GL(V)| = 168 = 2^3 \cdot 3 \cdot 7\) and a nilpotent subgroup of \( GL(V) \) has order at most 8. Thus \(|H| \leq 8 \leq |V|^{\beta}/2\). At the same time, we have \(|H| \leq 7 = |V|^{\ell}/\sqrt{3}\) when \( H \) is of odd order.

If \(|V| = 3^2\), then \(|GL(V)| = 48 = 2^4 \cdot 3\) and a nilpotent subgroup of \( GL(V) \) has order at most 16. Thus \(|H| \leq 16 \leq |V|^{\beta}/2\). Obviously we have \(|H| \leq 3 \leq |V|^{\ell}/\sqrt{3}\) when \( H \) is of odd order.

This step is proven.

Step 4. We may assume that

(i) \( G \not\leq \Gamma(q^n) \);
(ii) \( n \geq 3 \);
(iii) if \( q = 2 \), then \( n \geq 8 \).

Proof. (i) Suppose that \( G \leq \Gamma(q^n) \). Then \(|G| \leq n(q^n - 1)\). We can assume that \( n(q^n - 1) > q^{n\beta}/2 \). This can only happen when \( q = 2 \) and \( 1 \leq n \leq 6 \). What’s more, the cases \( q = 2 \) and \( 1 \leq n \leq 3 \) have been resolved in Step 3.

If \( q = 2 \) and \( n = 4 \), since a nilpotent subgroup of \( \Gamma(2^4) \) has order at most 15, we get that \(|H| \leq 15 \leq |V|^{\beta}/2\).

If \( q = 2 \) and \( n = 5 \), since a nilpotent subgroup of \( \Gamma(2^5) \) has order at most 31, we have \(|H| \leq 31 \leq |V|^{\beta}/2\).

If \( q = 2 \) and \( n = 6 \), note that a nilpotent subgroup of \( \Gamma(2^6) \) has order at most 63, so \(|H| \leq 63 \leq |V|^{\beta}/2\).
For part (b), by Theorem 2.8 we only have to worry about \( q = 2 \). If \( q = 2 \), by Lemma 2.7 we have \(|H| \leq q^n - 1\). Since \( 2^n - 1 \leq 2^{n^2}/\sqrt{3} \) is always true for any positive integer \( n \), \(|H| \leq |V| - 1 \leq |V|^1/\sqrt{3} \).

Thus we may assume that \( G \nleq \Gamma(q^n) \).

(ii) The cases \( n = 1 \) have been resolved in Step 3. We consider \( n = 2 \) or 3. Since \( V \) is quasi-primitive and \( G \nleq \Gamma(q^n) \), it follows from Lemma 2.4 that \( |T| \mid q - 1, |F/T| = n^2, F/T \) is a faithful irreducible \( G/F \)-module and \( |G/F| \leq \text{Sp}(2, n) \).

If \( n = 2 \), \(|HF/F| \leq 3 \) since a nilpotent subgroup of \( G/F \) has order \( \leq 3 \). What’s more, the cases \( n = 2 \) and \( q = 2 \) or 3 have been resolved in Step 3. When \( n = 2 \) and \( q \geq 5 \), then

\[ |H| \leq |T||F/T||HF/F| \leq (q - 1) \cdot 2^2 \cdot 3 \leq 12(q - 1) \leq |V|^3/2. \]

If \( n = 3 \), then \(|HF/F| \leq 8 \), and we don’t have to think about \( q = 2, 3 \). When \( n = 3 \) and \( q \geq 5 \), then

\[ |H| \leq |T||F/T||HF/F| \leq (q - 1) \cdot 3^2 \cdot 8 \leq 72(q - 1) \leq |V|^3/2. \]

Similarly for part (b), by Step 3, we just have to consider \( n = 2, q \geq 5 \) and \( n = 3, q \geq 3 \). Now \( |V| \) is odd. By Theorem 2.8 we know that \(|H| \leq (|V| - 1)/2 \leq |V|^1/\sqrt{3}\).

Thus we may assume that \( n > 3 \) when \( G \nleq \Gamma(q^n) \).

(iii) Now we assume that \( q = 2, 4 \leq n \leq 7 \) and \( G \nleq \Gamma(q^n) \). By [8, Corollary 2.13], \( n \) is not a prime. Since \( V \) is quasi-primitive, Lemma 2.5 implies that \( n = 6 \) and \(|G/F| \leq \text{GL}(2, 3) \). A nilpotent subgroup of \( \text{GL}(2, 3) \) has order at most 16. If \(|HF/F| = |H/(H \cap F)| = 16 \), then \(|(F \cap H)/(Z(F) \cap H)| \neq 9 \) because \((Z_3 \times Z_3) \times Z_8) \times Z_2 \) is not a nilpotent group. Thus

\[ |H| = |H/(H \cap F)||H \cap F| \leq 8 \cdot 3^3 \leq |V|^3/2. \]

Similarly for part (b), we just have to consider the case \( q = 2 \) and \( n = 6 \). In this case, by Lemma 2.5 we have \(|G| = |G/F||F| \mid 48 \cdot 3^2, \) so \(|H| \leq 3^3 \leq |V|^1/\sqrt{3} \).

This completes Step 4.

Step 5. Conclusion.

Proof. Since \( V \) is quasi-primitive, by Lemma 2.1 there exist normal subgroups \( F, Z, U, T, \)...
$E$ and $A$ with $Z = \text{socle}(U) = Z(E)$, $F \leq A = C_G(Z)$ and $E/Z \cong F/T$ is a completely reducible and faithful $A/F$-modules of order $e^2$ for an integer $e$. Since $V$ is quasi-primitive, $V_{EU}$ is a direct sum of $t \geq 1$ isomorphic faithful irreducible $EU$-modules. By Lemma 2.3 it follows that $V_U \cong te \cdot W$ where $W$ is a faithful irreducible $U$-module. Note that $|U| \mid |W| - 1$.

Note that $H$ is a nilpotent subgroup of $G$, there exist normal subgroups $H \cap F, H \cap A, H \cap U, H \cap T$ and $|F \cap H|/|T \cap H| \leq e^2, |(T \cap H) : (U \cap H)| \leq |T : U| \leq 2$. Since $Z$ is cyclic, we see that

$$|H/(H \cap A)| \leq |G/A| \leq |\text{Aut}(Z)| \leq |Z| \leq |U|.$$

When $|T : U| = 1$, obviously

$$|H/(H \cap A)||T \cap H| \leq |U|^2.$$

When $|T : U| = 2$, then

$$|H/(H \cap A)| \leq |G/A| \leq |\text{Aut}(Z)| \leq |Z|/2 \leq |U|/2$$

and $|H/(H \cap A)||T \cap H| \leq |U|^2$.

If $e = 1$, then $F = T$. By Lemma 2.2, $G \leq \Gamma(V)$. The cases have been resolved in Step 4.

If $e = 2$, we have $|F/T| = 2^2$ and $2 \mid |F$, $O_2(G) \neq 1$ and $q \neq 2$. At the same time, we also note that $(A \cap H)F/F \leq A/F \leq \text{GL}(2, 2)$ and $(A \cap H)/(F \cap H) \cong (A \cap H)F/F$ is a nilpotent group. It is obvious that a nilpotent subgroup of $\text{GL}(2, 2)$ has order at most 3. If $|(A \cap H)F/F| = 3$, then $|(F \cap H)/(T \cap H)| \neq 4$ because $(Z_2 \times Z_2) \rtimes Z_3 \cong A_4$ is not a nilpotent group. So

$$|H| = |H/(A \cap H)||(A \cap H)/(F \cap H)||(F \cap H)/(T \cap H)| \leq |U|^2 \cdot 3 \cdot 2 \leq 6|V| \leq |V|^\beta/2.$$

If $|(A \cap H)F/F| \leq 2$, then

$$|H| = |H/(A \cap H)||H \cap T||(A \cap H)/(F \cap H)||(F \cap H)/(T \cap H)| \leq |U|^2 \cdot 2 \cdot 2^2 \leq 8|U|^2 \leq 8|V|.$$

We can assume that $8|V| > |V|^\beta/2$, which only happens if $q = 3$ and $n = 4$. In this case, $|V| = |W|^{2t} = 3^4$ for integer $t \geq 1$, then $|W| = 3^2$ or 3, $\dim(W) \leq 2$ and $|W| - 1 \leq 8$. By Theorem 2.2, we have $|G/C_G(U)| \mid \dim(W)$ and $C_G(U)/F \leq \text{GL}(2, 2)$. By Lemma 2.3, we get

$$|H| \leq |G| \leq \dim(W) \cdot |C_G(U)/F| \cdot e^2 \cdot (|W| - 1) \leq 2 \cdot 6 \cdot 4 \cdot 8 = 384 \leq |V|^\beta/2.$$

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For part (b), note that $q \neq 2$ and $H$ is an odd order nilpotent subgroup of $G$. By Theorem 2.8 we can get $|H| \leq (|V| - 1)/2 \leq |V|^t/\sqrt{3}$.

If $e = 3$, we have $|F/T| = 3^2$ and $3 \mid |F|$, then $O_3(G) \neq 1$ and $q \neq 3$. Observe that $|U|^3 \leq |W|^{3t} = |V|$. By Step 4 (ii)(iii), it follows that $|V| \geq 2^8$. At the same time, we get $(A \cap H)F/F \leq A/F \leq \text{GL}(2,3)$ and a nilpotent subgroup of $\text{GL}(2,3)$ has order at most 16. If $|(A \cap H)F/F| = 16$, then $|(F \cap H)/(T \cap H)| \neq 9$ because $((Z_3 \times Z_3) \times Z_8) \times Z_2$ is not a nilpotent group. Thus

$$|H| = |H/(A \cap H)||H \cap T||(A \cap H)/(F \cap H)||(F \cap H)/(T \cap H)| \leq |U|^2 \cdot 8 \cdot 3^2 \leq 72|V|^{2/3} \leq |V|^{\beta/2}.$$ 

For part (b), by Theorem 2.8 we only have to consider $q = 2$. We can get

$$|G| = |G/A||T||A/F||F/T| \mid |U|^2 \cdot 48 \cdot 3^2 = 2^9 \cdot 2^4 \cdot 3^3 \cdot |U|^2.$$ 

Since $H$ is an odd order nilpotent subgroup of $G$ and $|U|^3 \leq |W|^{3t} = |V|$, we have $|H| \leq 3^3 \cdot |U|^2 \leq 27|V|^{2/3}$. We may assume that $27|V|^{2/3} > |V|^t/\sqrt{3}$, i.e. $27 > |V|^{(t-2/3)}/\sqrt{3}$. Thus $q = 2$ and $8 \leq n \leq 10$. On the other hand, since $|V| = |W|^{3t}$ where $t$ and $|W|$ is a positive integer, we get $q = 2$ and $n = 8, 10$ which is impossible.

When $q = 2$ and $n = 9$, since $|V| = 2^9 = |W|^{3t}$ with $t$ and $|W|$ is a positive integer, we get $|W| = 2^3$ or 2, $\dim(W) \leq 3$ and $|W| - 1 \leq 7$. By [13, lemma 2.3], we get that

$$|G| \leq \dim(W) \cdot |C_G(U)/F| \cdot e^2 \cdot (|W| - 1) \leq 3 \cdot 6 \cdot 4 \cdot 7 = 504.$$ 

Clearly

$$|H| \leq |G| \leq 504 \leq |V|^t/\sqrt{3}.$$ 

Thus we can assume that $e \geq 4$. Since $F/T$ is a completely reducible and faithful $A/F$-module and $|F/T| = e^2$, using the inductive hypothesis we get

$$|(A \cap H)/(F \cap H)| = |(A \cap H)F/F| \leq \begin{cases} e^{2\beta}/2 \\ e^{2l}/\sqrt{3}, \text{ if } H \text{ is of odd order.} \end{cases}$$

Since $|H| = |H/(A \cap H)||H \cap T||(A \cap H)/(F \cap H)||(F \cap H)/(T \cap H)|$ and $|U| \leq |W|$, we have

$$|H| \leq \begin{cases} e^{2+2\beta} \cdot |W|^2/2 \\ e^{2+2l} \cdot |W|^2/\sqrt{3}, \text{ if } H \text{ is of odd order.} \end{cases}$$
By $|V| = |W|^t e$, we may assume that

$$
\begin{aligned}
&\begin{cases}
e^{2+2\beta} \cdot |W|^2 > |W|^{t-e\beta} \\
e^{2+2t} \cdot |W|^2 > |W|^{t-e},
\end{cases}
\end{aligned}
$$

(1.2)

To prove (a), since $|U| = |W| - 1$, we know that $|W| \geq 3$. We can also get inequality $e^{10/3} > |W|^{e-4/3}$ holds by $\beta > 3/2$ and inequality (1.2). Thus we just have to compute inequality $e^{10/3} > 3^{e-4/3}$. In the case, we figure out that $e \leq 7$.

If $e = 5$, then $5 \mid |U|$ and $|W| \geq 11$. If $e = 6$, then $6 \mid |U|$, we know that $|W| \geq 7$. If $e = 7$, then $7 \mid |U|$, we have $|W| \geq 8$. The above three cases are contradictory to inequality (1.2).

Thus $e = 4$. When $e \geq 4$, then $2 \mid |U|$ and the inequality $e^{2+2\beta} \cdot |W|^2 > |W|^{e\beta}$ implies that $|W| = 3$ or $5$ and $t = 1$.

(1) $|W| = 3$ and $t = 1$, then $|V| = 3^4$.

In this case, $T = U = Z$ cyclic, $|U| = 2$ and $|F| = 2^5$. Since $A = C_G(Z) = G$, $F/T$ is a completely reducible and faithful $G/F$-module. By Lemma 2.1 $F/T$ is irreducible or the direct sum of two irreducible $G/F$-modules of order $2^2$. By Lemma 2.5 $G/F \leq \Gamma(2^4)$ or $G/F \leq S_3 \l S_2$.

If $G/F \leq \Gamma(2^4)$, a nilpotent subgroup of $\Gamma(2^4)$ has order at most 15. So

$$
|H| = |H/(H \cap F)||F \cap H| \leq 15 \cdot 2^5 \leq |V|^\beta/2.
$$

If $G/F \leq S_3 \l S_2$, a nilpotent subgroup of $G/F \leq S_3 \l S_2$ has order at most 9. So

$$
|H| = |H/(H \cap F)||F \cap H| \leq 9 \cdot 2^5 \leq |V|^\beta/2.
$$

(2) $|W| = 5$ and $t = 1$, then $|V| = 5^4$.

In this case, $|U| = 4$ or 2.

If $|U| = 2$, this is the same thing as (1), obviously $|H| \leq 3^4 \beta / 2 \leq 5^4 \beta / 2 \leq |V|^\beta / 2$.

If $|U| = 4$, $|Z| = 2$ and $F \mid 2^7$. Since $A = C_G(Z) = G$, $F/T$ is a completely reducible and faithful $G/F$-module. By Lemma 2.1 $F/T$ is irreducible or the direct sum of two irreducible $G/F$-modules of order $2^2$. By Lemma 2.5, $G/F \leq \Gamma(2^4)$ or $G/F \leq S_3 \l S_2$.

If $G/F \leq \Gamma(2^4)$, a nilpotent subgroup of $\Gamma(2^4)$ has order at most 15. So

$$
|H| = |H/(H \cap F)||F \cap H| \leq 15 \cdot 2^7 \leq |V|^\beta / 2.
$$
If $G/F \leq S_3 \wr S_2$, a nilpotent subgroup of $G/F \leq S_3 \wr S_2$ has order at most 9. So
\[ |H| = |H/(H \cap F)||F \cap H| \leq 9 \cdot 2^7 \leq |V|^9/2. \]

To prove (b), by $|U| \mid |W| - 1$ and the inequality (1.2), we can assume that $|W| \geq 3$ and $e^{2+2l} > 3^{d-2}$. In this case, we figure out that $e \leq 9$.

If $e = 5, 6, 7$, this is the same thing as (a), then a contradiction is obtained from (1.2).

If $e = 9$, then $3 \mid |U|$ and $|W| \geq 4$. A contradiction is obtained from (1.2).

Thus $e = 4$ or 8. In both cases, $2 \mid |U|$. Since $U \mid |W| - 1$ and $|V| = |W|^{1e}$, we get that $|V|$ is odd. By Theorem 2.8 we can get $|H| \leq (|V| - 1)/2 \leq |V|^1/\sqrt{3}$.

\[ \square \]

4. Applications

As applications of our results, we first provide a new proof for [5, Theorem 14.3]. Our proof is quite different from the original one since we first reduce the problem to solvable linear groups. The idea is based on the main result of this paper and some recent developments in [11].

Given a chief series
\[ \Delta : 1 = G_0 < G_1 < \cdots < G_n = G \]
of $G$. Let $\mu(G)$ be the number of nonabelian chief factors in $\Delta$. Let $\text{Ord}_\Delta(G)$ denote the product of orders of all central chief factors $G_j/G_{j-1}$ (that is, $[G, G_j] \leq G_{j-1}$) in $\Delta$. Let $\text{Ord}_{\Delta,\text{odd}}(G)$ to denote the odd part of the previous term.

We first quote two results from [11].

Lemma 4.1. Let $G$ be a solvable group. Then $G$ has a nilpotent subgroup $L$ such that $|L| \geq \text{Ord}_{\Delta}(G)$.

Proof. This is [11, Lemma 2.4]. \[ \square \]

Proposition 4.2. Let $G$ be a finite group with $O_p(G) = 1$ for some prime $p$. Then $G$ has a solvable subgroup $H$ with $O_p(H) = 1$ such that $\text{Ord}_\Delta(H) \geq 2^\mu(G) \cdot \text{Ord}_\Delta(G)$.

Proof. This follows from [11] Proposition 2.5]. \[ \square \]

The following strengthens [5, Theorem 14.3]. Note that our proof is quite different.

Theorem 4.3. Let $G$ be a finite group acting faithfully and completely reducibly on a finite vector space $V$. Then
\[ 2^\mu(G) \cdot \text{Ord}_\Delta(G) \leq |V|^{\beta}/2. \]
Proof. By induction, we may assume the action is irreducible and \( V \) is of characteristic \( p \). Then the result follows by Proposition 4.2, Lemma 4.1, and Theorem 3.1 (a). □

We next consider the odd order portion of \( \text{Ord}_N(G) \) and generalize Theorem 3.1 (b) to arbitrary finite groups in a same fashion.

Lemma 4.4. Let \( S \) be a nonabelian simple group and \( r \) be a given prime. Let \( S \trianglelefteq G \leq \text{Aut}(S) \) be such that \( G/S \) is of odd order. Then there exists an abelian \( r' \)-subgroup \( A \) of \( S \) of odd order and \( |A| \geq 2|G/S| \).

Proof. Suppose that \( |G/S| = 1 \). Note that by Burnside’s \( p^aq^b \) theorem the largest prime divisor of the order of \( S \) is at least 5 and \( |S| \) has at last three prime factors. Let \( p = \max\{\pi(S)\} \geq 5 \) where \( \pi(S) \) is the set of all distinct prime factors of \( |S| \). If \( r = p \), then there must be a prime factor \( p_1 \in \pi(S) \) and \( p_1 \geq 3 \). Thus \( S \) has an abelian \( r' \)-group \( A \) of order \( p_1 \) such that \( |A| = p_1 \geq 3 > 2 = 2|G/S| \). If \( r \neq p \), then \( S \) has an abelian subgroup \( A \) of order \( p \) such that \( |A| = p \geq 5 > 2 = 2|G/S| \). This implies

Claim (*) If \( |G/S| = 1 \), then the result follows.

Write \( |G/S| = k \).

We now go through the classification of the finite simple groups. Suppose that \( S \) is isomorphic to a sporadic simple group or the Tits group or an alternating group \( A_n \). As is well-known, we have \( |\text{Out}(S)| \mid 4 \). Then \( k = 1 \) and Claim (*) implies the required result.

In what follows, we assume that \( S \) is a simple group of Lie type over a field of characteristic \( p \). If \( S \) is of type \( ^2A_l, ^2D_l \) or \( ^2E_6 \), then the ground field is assumed to be \( \text{GF}(q^2) \) and let \( q^2 = p^f \); if \( S \) has type \( ^3D_4 \), then the ground field is \( \text{GF}(q^3) \) and let \( q^3 = p^f \); if \( S \) is one of the other types, then the ground field is \( \text{GF}(q) \) and let \( q = p^f \). Observe that

\( |\text{Out}(S)| = dfg, \)

where the numbers \( d, f, g \) are tabulated in [3, Table 5]. We always write

\( q = p^a. \)

Case 1. \( S \cong A_l(q), q = p^a = p^f. \)

Then \( |S| = d^{-1}q^{n(n+1)/2}\prod_{i=1}^{n}(q^{i+1} - 1) \) where \( d = (l + 1, q - 1) \).

Suppose that \( l = 1 \). Then \( d \mid 2 \) and \( g = 1 \). Thus \( |\text{Out}(S)| \mid 2a \) and \( k \mid a \). By Claim (*), we may assume that \( a \geq 3 \). Assume there exists a Zsigmondy prime \( L_1 \) for \( p^{2a} - 1. \)
Then $L_1 \mid p^{2a} - 1$ and thus $L_1 \mid p^a + 1$ and $L_1 \geq 2a \geq 2k$. If $L_1 \neq r$, then we can find a cyclic subgroup $A_1$ of $S$ such that $|A_1| = L_1 \geq 2k$. If $L_1 = r$, by [7] Lemma 3.2, with a few exceptions, there exists a prime $L_2$ for $p^a - 1$ such that $L_2 \geq 2a \geq 2k$, or $L_2^2 \mid p^a - 1$. This implies that $L_2^2 \geq 2a \geq 2k$. It is clear that $L_1 \neq L_2$. Thus we find an abelian subgroup $A_1$ of $S$ such that $|A_2| = L_2 \geq 2k$.

Suppose that $l \geq 2$. Then $d \mid (l + 1)$ and $g = 2$. Thus $|\text{Out}(S)| \mid 2(l + 1)a$.

If $l$ is odd, then $k \mid \frac{(n+1)a}{2}$. Assume there exists a Zsigmondy prime $L_1$ for $p^{(l+1)a} - 1$. Then $L_1 \mid p^{(l+1)a} - 1$ and $L_1 \geq (l + 1)a \geq 2k$. If $L_1 \neq r$, we find a subgroup $A_1$ of $S$ such that $|A_1| = L_1 \geq 2k$. If $L_1 = r$, by [7] Lemma 3.2, with a few exceptions, there exists a prime $L_2$ for $p^{la} - 1$ such that $L_2 \geq 2al \geq (l + 1)a \geq 2k$, or $L_2^2 \mid p^{la} - 1$. Clearly $L_2^2 \geq 2k$ and $L_1 \neq L_2$. Thus we find an abelian subgroup $A_2$ of $S$ such that $|A_2| = L_2$ or $L_2^2$ and $|A_2| \geq 2k$.

If $l$ is even, then $k \mid (n + 1)a$. By [7] Lemma 3.1, with a few exceptions, there exists a large Zsigmondy prime $L_1$ for $p^{(l+1)a} - 1$ such that $L_1 \geq 2(l + 1)a \geq 2k$, or $L_1^2 \mid p^{(l+1)a} - 1$ and $L_1^2 \geq 2(l + 1)a \geq 2k$. If $L_1 \neq r$, then we find a subgroup $A_1$ of $S$ such that $|A_1| = L_1$ or $L_1^2$ and $|A_1| \geq 2k$. If $L_1 = r$, by [7] Lemma 3.3, with a few exceptions, there exists a prime $L_2$ for $p^{la} - 1$ such that $L_2 \geq 3al \geq 2(l + 1)a \geq 2k$, or $L_2^2 \mid p^{la} - 1$. This implies that $L_2^2 \geq 2k$ and $L_1 \neq L_2$. Thus we find an abelian subgroup $A_2$ of $S$ such that $|A_2| = L_2$ or $L_2^2$ and $|A_2| \geq 2k$.

By Claim (*), we can assume $k > 1$. All exceptions to $A_4(q)$ are listed in Table 1 except $k = 1$.

Case 2. $S \cong 2^2A_1(q), l \geq 2, q^2 = p^f = p^{2a}$. Note that $S \not\cong 2^2A_2(2)$.

Then $|S| = d^{-1}q^{an+1+1} \prod_{i=1}^{n}(q^{i+1} - (-1)^{i+1}), d = (l + 1, q + 1), g = 2$.

If $l$ is odd, then $k \mid \frac{(n+1)a}{2}$. Assume there exists a Zsigmondy prime $L_1$ for $p^{(l+1)a} - 1$. Then $L_1 \mid p^{(l+1)a} - 1$ and $L_1 \geq (l + 1)a \geq 2k$. If $L_1 \neq r$, we find a subgroup $A_1$ of $S$ such that $|A_1| = L_1 \geq 2k$. If $L_1 = r$, Assume there exists a prime $L_2$ for $p^{2a} - 1$. Then $L_2 \mid p^{2a} + 1$ and $L_2 \geq 2a \geq (l + 1)a \geq 2k$. Obviously $L_1 \neq L_2$. Thus we find an abelian subgroup $A_2$ of $S$ such that $|A_2| = L_2 \geq 2k$.

If $l$ is even, then $k \mid (n + 1)a$. Assume there exists a Zsigmondy prime $L_1$ for $p^{2(l+1)a} - 1$. Then $L_2 \mid p^{(l+1)a} + 1$ and $L_2 \geq 2(l + 1)a \geq 2k$. If $L_1 \neq r$, then we find a subgroup $A_1$ of $S$ such that $|A_1| = L_1 \geq 2k$. If $L_1 = r$, by [7] Lemma 3.3, with a few exceptions, there exists a prime $L_2$ for $p^{la} - 1$ such that $L_2 \geq 3al \geq 2(l + 1)a \geq 2k$, or $L_2^2 \mid p^{la} - 1$. This implies that $L_2^2 \geq 2k$ and $L_1 \neq L_2$. Thus we find an abelian subgroup $A_2$ of $S$ such that $|A_2| = L_2$ or $L_2^2$. 

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Table 1. Exceptional cases for $A_l(q)$

| $p$ | $l$ | $a$ | $d$ | $|\text{Out}(S)|$ | $k$ | $|A_1|$ | $|A_2|$ |
|-----|-----|-----|-----|-----------------|-----|--------|--------|
| 2   | 1   | 3   | 1   | 3               | $\leq 3$ | 7      | 9      |
| 2   | 1   | 6   | 1   | 6               | $\leq 3$ | 7      | 9      |
| 2   | 1   | 12  | 1   | 12              | $\leq 3$ | 13     | 7      |
| 2   | 2   | 2   | 3   | 12              | $\leq 3$ | 7      | 9      |
| 2   | 2   | 4   | 3   | 24              | $\leq 3$ | 13     | 17     |
| 2   | 2   | 6   | 3   | 36              | $\leq 9$ | 19     | 7$^2$  |
| 2   | 8   | 2   | 3   | 12              | $\leq 3$ | 19     | 7      |
| 2   | 2   | 3   | 1   | 6               | $\leq 3$ | 73     | 7      |
| 2   | 4   | 3   | 1   | 6               | $\leq 3$ | 31     | 7      |
| 2   | 2   | 10  | 3   | 60              | $\leq 15$ | 151   | 31     |
| 2   | 4   | 5   | 1   | 10              | $\leq 5$ | 31     | 11     |
| 3   | 2   | 3   | 1   | 6               | $\leq 3$ | 13     | 7      |
| 5   | 2   | 2   | 3   | 12              | $\leq 3$ | 31     | 7      |
| 5   | 2   | 3   | 3   | 18              | $\leq 9$ | 31     | 19     |

By Claim (*), we can assume $k > 1$. All exceptions to $^2A_l(q)$ are listed in Table 2 except $k = 1$.

Table 2. Exceptional cases for $^2A_l(q)$

| $p$ | $l$ | $a$ | $d$ | $|\text{Out}(S)|$ | $k$ | $|A_1|$ | $|A_2|$ |
|-----|-----|-----|-----|-----------------|-----|--------|--------|
| 2   | 5   | 1   | 3   | 6               | $\leq 3$ | 7      | 31     |
| 2   | 2   | 3   | 3   | 18              | $\leq 9$ | 19     | 3$^3$  |
| 2   | 4   | 2   | 5   | 20              | $\leq 5$ | 41     | 17     |
| 2   | 8   | 1   | 3   | 6               | $\leq 3$ | 19     | 17     |
| 2   | 2   | 6   | 1   | 12              | $\leq 3$ | 37     | 13     |
| 2   | 4   | 3   | 1   | 6               | $\leq 3$ | 11     | 13     |
| 2   | 2   | 10  | 1   | 20              | $\leq 5$ | 61     | 41     |
| 2   | 4   | 5   | 1   | 1               | $\leq 5$ | 31     | 11     |
| 2   | 20  | 1   | 3   | 6               | $\leq 3$ | 41     | 31     |
| 3   | 2   | 3   | 1   | 6               | $\leq 3$ | 19     | 37     |
| 5   | 2   | 3   | 3   | 18              | $\leq 9$ | 31     | 7$^2$  |
Case 3. \( S \) is one of the other types.

If \( k \leq a \), we may assume that \( a \geq 3 \). By Claim (*), we can assume \( k > 1 \).

We now provide a table (Table 3) for the simple groups of Lie types other than \( A_n(q) \) and \( ^2A(q^2) \). The fourth and the fifth columns in the table are the terms where we find two large prime divisors.

If \( S \cong B_2(8) \), then \( k \mid 3 \). Since \( |S| = 8^4(8^2 - 1)(8^4 - 1) \), we may choose \( L_1 = 13 \) and \( L_2 = 7 \).

If \( S \cong G_2(8) \), then \( k \mid 3 \). Since \( |S| = 8^6(8^6 - 1)(8^2 - 1) \), we may choose \( L_1 = 19 \) and \( L_2 = 7 \).

If \( S \cong ^3D_4(2) \), then \( k \mid 3 \). Since \( |S| = 2^{12}(2^8 + 2^4 + 1)(2^6 - 1)(2^2 - 1) \), we may choose \( L_1 = 13 \) and \( L_2 = 7 \).

This completes the proof.

Table 3. Other Lie Type Groups

| type          | \( f \) | \( k \) | \( |L_1| \) | \( |L_2| \) | exceptional cases |
|---------------|--------|--------|----------|----------|------------------|
| \( B_l(q) \), \( l \geq 2 \) | \( a \) | \( \leq a \) | \( p^{a(l-1)} - 1 \) | \( p^{a(l-2)} - 1 \) | \((p = 2, l = 2, a = 3)\) |
| \( C_l(q) \), \( l \geq 3 \) | \( a \) | \( \leq a \) | \( p^{a(l-1)} - 1 \) | \( p^{a(l-2)} - 1 \) | |
| \( D_l(q) \), \( l \geq 4 \) | \( a \) | \( \leq a \) | \( p^{a(l-2)} - 1 \) | \( p^{a(l-4)} - 1 \) | |
| \( ^2D_l(q) \), \( l \geq 4 \) | \( 2a \) | \( \leq a \) | \( p^{a(l-2)} - 1 \) | \( p^{a(l-2)} - 1 \) | |
| \( E_6(q) \) | \( 3a \) | \( \leq 3a \) | \( p^{12a} - 1 \) | \( p^{8a} - 1 \) | |
| \( E_7(q) \) | \( a \) | \( \leq a \) | \( p^{18a} - 1 \) | \( p^{14a} - 1 \) | |
| \( E_8(q) \) | \( a \) | \( \leq a \) | \( p^{30a} - 1 \) | \( p^{24a} - 1 \) | |
| \( F_4(q) \) | \( a \) | \( \leq a \) | \( p^{12a} - 1 \) | \( p^{8a} - 1 \) | |
| \( G_2(q) \) | \( a \) | \( \leq a \) | \( p^{6a} - 1 \) | \( p^{2a} - 1 \) | \((p = 2, a = 3)\) |
| \( ^2E_6(q) \) | \( 2a \) | \( \leq 3a \) | \( p^{12a} - 1 \) | \( p^{8a} - 1 \) | |
| \( ^3D_4(q) \) | \( 3a \) | \( \leq 3a \) | \( p^{12a} - 1 \) | \( p^{6a} - 1 \) | \((p = 2, a = 1)\) |
| \( ^2B_2(q) \), \( q = 2^{2m+1} \) | \( 2m + 1 \) | \( \leq 2m + 1 \) | \( 2^{4(2m+1)} - 1 \) | \( 2^{2m+1} - 1 \) | |
| \( ^2F_4(q) \), \( q = 2^{2m+1} \) | \( 2m + 1 \) | \( \leq 2m + 1 \) | \( 2^{4(2m+1)} - 1 \) | \( 2^{2m+1} - 1 \) | |
| \( ^2G_2(q) \), \( q = 3^{2m+1} \) | \( 2m + 1 \) | \( \leq 2m + 1 \) | \( 3^{3(2m+1)} - 1 \) | \( 3^{2m+1} - 1 \) | |

\[ \Box \]

**Lemma 4.5.** Let \( G \) be a group and let \( N \leq G \) such that \( G/N \) is of odd order. Let \( r \) be a given prime and \( N = S_1 \times \cdots \times S_t \) be a direct product of isomorphic nonabelian simple groups
$S_i$. Assume that $S_i$ are all normal in $G$ and that $C_G(N) = 1$. Then there exists an abelian $r'$-subgroup $A$ of $G$ of odd order such that $|A| \geq 2^t|G/N|$.

Proof. Clearly $G/S_iC_G(S_i)$ is an odd order subgroup of $\text{Out}(S_i)$. Note that $1 = C_G(N) = \bigcap_{i=1}^t C_G(S_i)$ and $C_G(S_i)S_i = C_G(S_i) \times S_i$. Repeatedly using the Dedekind’s Modular Law, we have

\[
\bigcap_{i=1}^t S_iC_G(S_i) = (C_G(S_1) \cap C_G(S_2)S_2 \cap \cdots \cap C_G(S_t)S_t)S_1 = \cdots = (C_G(S_1) \cap C_G(S_2) \cap \cdots \cap C_G(S_t))S_t \cdots S_2S_1 = C_G(N)N = N.
\]

Observe that

\[
G/N \leq G/S_iC_G(S_i) \times \cdots \times G/S_tC_G(S_t).
\]

By Lemma 4.4, there exists an abelian $r'$-subgroup $A_i$ of $S_i$ of odd order such that $|A_i| \geq 2|G/S_iC_G(S_i)|$. Let $A = A_1 \times \cdots \times A_t$. Since $|G/N| \leq \prod_{i=1}^t |G/S_iC_G(S_i)|$, we get that

\[
|A| = \prod_{i=1}^t |A_i| \geq \prod_{i=1}^t 2|G/S_iC_G(S_i)| \geq 2^t|G/N|,
\]

and we are done. \hfill $\square$

**Lemma 4.6.** Let $N \trianglelefteq G$, $H/N \leq G/N$ and $\text{Ord}_{\text{N}_{\text{odd}}}(H/N) \geq m \cdot \text{Ord}_{\text{N}_{\text{odd}}}(G/N)$ where $m$ is a positive real number. Then $\text{Ord}_{\text{N}_{\text{odd}}}(H) \geq m \cdot \text{Ord}_{\text{N}_{\text{odd}}}(G)$.

**Proof.** Let $A$ and $B$ be the product of orders of central chief factors above $N$ and below $N$, respectively, of a given chief series of $G$; let $A'$ and $B'$ be the product of orders of central chief factors above $N$ and below $N$, respectively, of a given chief series of $H$.

Let $a$ and $b$ be the odd part of $A$ and $B$, respectively; let $a'$ and $b'$ be the odd part of $A'$ and $B'$, respectively.

Apparentely, $\text{Ord}_{\text{N}_{\text{odd}}}(G) = ab$ and $\text{Ord}_{\text{N}_{\text{odd}}}(H) = a'b'$. Note that a central chief factor below $N$ of $G$ is a direct product of some central chief factors below $N$ of $H$, it shows that $b' \geq b$. Since $a' = \text{Ord}_{\text{N}_{\text{odd}}}(H/N) \geq m \cdot \text{Ord}_{\text{N}_{\text{odd}}}(G/N) = mb$, we get that

\[
\text{Ord}_{\text{N}_{\text{odd}}}(H) = a'b' \geq ma \cdot b = m \cdot \text{Ord}_{\text{N}_{\text{odd}}}(G).
\]

$\square$

**Lemma 4.7.** Let $G$ be a solvable group. Then $G$ has a nilpotent subgroup $L$ of odd order such that $|L| \geq \text{Ord}_{\text{N}_{\text{odd}}}(G)$. 

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Proof. We work by induction on $|G|$.

We may assume that $G$ is not nilpotent. Let $G^N$ be the smallest normal subgroup of $G$ such that $G/G^N$ is nilpotent. Let $G^N/D$ be a chief factor of $G$. Since $G/D$ is not nilpotent, we have $\Phi(G/D) \cap (G^N/D) = 1$. Then there exists a maximal subgroup $X/D$ of $G/D$ such that $G/D = G^N/D \rtimes X/D$ and $X/D \cong G/G^N$. Note that $G^N/D$ is not a central chief factor of $G$, we get that

$$\text{Ord}_N(X/D) = |G : G^N| = \text{Ord}_N(G/D).$$

It is apparent that

$$\text{Ord}_{N,\text{odd}}(X/D) = \text{Ord}_{N,\text{odd}}(G/D)$$

and so that $\text{Ord}_{N,\text{odd}}(X) \geq \text{Ord}_{N,\text{odd}}(G)$ (see Lemma 4.6). Applying the inductive hypothesis in $X$, we get the required result. □

Proposition 4.8. Let $G$ be a finite group with $O_p(G) = 1$ for some prime $p$. Then $G$ has a solvable subgroup $H$ with $O_p(H) = 1$ such that $\text{Ord}_{N,\text{odd}}(H) \geq 2^{\mu(G)} \cdot \text{Ord}_{N,\text{odd}}(G)$.

Proof. We may assume $G$ is nonsolvable, and we work by induction on $|G|$. Let $E$ be a minimal normal subgroup of $G$ and write $D/E = O_p(G/E)$. We distinguish two cases.

(1) Assume that $G/D$ is nonsolvable.

By induction, $G/D$ has a solvable subgroup $B/D$ with $O_p(B/D) = 1$ such that $\text{Ord}_{N,\text{odd}}(B/D) \geq 2^{\mu(G/D)} \cdot \text{Ord}_{N,\text{odd}}(G/D)$. By Lemma 4.6 we have

$$\text{Ord}_{N,\text{odd}}(B) \geq 2^{\mu(G/D)} \cdot \text{Ord}_{N,\text{odd}}(G).$$

Note that $O_p(B) \leq O_p(D) \leq O_p(G) = 1$.

If $B$ is solvable, then $\mu(G) = \mu(G/D)$ and $B$ meets the requirements.

If $B$ is nonsolvable, by induction, $B$ has a solvable subgroup $H$ with $O_p(H) = 1$ such that $\text{Ord}_{N,\text{odd}}(H) \geq 2^{\mu(B)} \cdot \text{Ord}_{N,\text{odd}}(B)$. Then

$$\text{Ord}_{N,\text{odd}}(H) \geq 2^{\mu(B)} \cdot \text{Ord}_{N,\text{odd}}(B) \geq 2^{\mu(G)} \cdot \text{Ord}_{N,\text{odd}}(G),$$

so $H$ meets the requirements.

(2) Assume that $G/D$ and $G/E$ are solvable for all minimal normal subgroups $E$ of $G$. 


Since $G$ is nonsolvable, $G$ has a unique minimal normal subgroup $E$ and $E$ is nonsolvable. In particular, $C_G(E) = 1$ and $\mu(G) = 1$. By Lemma 4.7, $G/E$ has a nilpotent subgroup $U/E$ of odd order such that

$$|U/E| \geq \text{Ord}_{N_{\text{odd}}}(G/E) = \text{Ord}_{N_{\text{odd}}}(G).$$

Observe that $O_p(U) \leq C_G(E) = 1$, $\mu(U) \geq \mu(G) = 1$, and that

$$\text{Ord}_{N_{\text{odd}}}(U) = \text{Ord}_{N_{\text{odd}}}(U/E) \geq \text{Ord}_{N_{\text{odd}}}(G).$$

Suppose that $U < G$. Applying the inductive hypothesis in $U$, we get the required result.

Suppose that $U = G$. Write $E = S_1 \times \cdots \times S_t$, where $S_1, \ldots, S_t$ are isomorphic nonabelian simple groups. Let us investigate $D = \bigcap_{i=1}^t N_G(S_i)$. Since $S_i \trianglelefteq D$ for $i = 1, \ldots, t$, by Lemma 4.5 there exists an abelian $p'$-subgroup $H$ of $D$ of odd order such that $|H| \geq 2^t|D/E|$. Observe that $G$ acts on $\{S_1, \ldots, S_t\}$ with kernel $D$, it shows that $G/D$ is a nilpotent permutation group of degree $t$. As is well-known, we have that $|G/D| \leq 2^{t-1}$. Now

$$\text{Ord}_{N_{\text{odd}}}(H) = |H| \geq 2|G/E| \geq 2^{\mu(G)} \cdot \text{Ord}_{N_{\text{odd}}}(G),$$

and $H$ meets the requirements.

\[ \square \]

**Theorem 4.9.** Let $G$ be a finite group acting faithfully and completely reducibly on a finite vector space $V$. Then $2^{\mu(G)} \cdot \text{Ord}_{N_{\text{odd}}}(G) \leq |V|^l/\sqrt{3}$.

**Proof.** By induction on $|G||V|$, we may assume the action is irreducible and $V$ is of characteristic $p$. Then the result follows by Proposition 4.8, Lemma 4.7 and Theorem 3.1(b). \[ \square \]

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**References**


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