A note on the Banach mapping theorem and its related conjecture

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Abstract

We extend, through an elementary way, the results of Banach [1], Fan [3], and Sanders [7], which concern a finite collection \( \{ f_i : A_i \to A_{i+1} \}_{i=1}^{n} \) of mappings with \( A_{n+1} = A_1 \) is decomposable as follows: \( f_i(B_i) = A_{i+1} \backslash B_{i+1} \), where \( B_i \subseteq A_i \) for all \( i \) and \( B_{n+1} = B_1 \). Our theorem determines when such a collection is decomposable. We also show that such a set \( B_1 \) is unique up to an addition of a certain set, which was conjectured by Sanders in [7].

Let \( A_1 \) and \( A_2 \) be two sets and let \( f_1 : A_1 \to A_2 \) and \( f_2 : A_2 \to A_1 \) be two mappings. If both \( f_1 \) and \( f_2 \) are injections, then the Banach mapping theorem [1] asserts that there exist \( B_1 \subseteq A_1 \) and \( B_2 \subseteq A_2 \) such that \( f_1(B_1) = A_2 \backslash B_2 \) and \( f_2(B_2) = A_1 \backslash B_1 \), and the Cantor-Schroeder-Bernstein theorem [5, Theorem 4.5.5] asserts that there exists a bijection from \( A_1 \) to \( A_2 \). It is well known that the former theorem implies the latter one, by defining a desired bijection to be \( f_1 \) if restricted to \( B_1 \), and the inverse of \( f_2 \) otherwise. In fact, the Banach mapping theorem can be generalized by removing the condition of injections; refer to [5, p.102, Exercise 4.21], and our previous work [6] for an elementary proof.

In order to extend the Banach mapping theorem, let \( n \) be a positive integer, \( \{ A_i \}_{i=1}^{n+1} \) be a collection of sets with \( A_{n+1} = A_1 \), and \( \{ f_i : A_i \to A_{i+1} \}_{i=1}^{n} \) be a collection of mappings. We say that \( \{ f_i \}_{i=1}^{n} \) is decomposable if there exists a subset \( B_1 \) of \( A_1 \), possibly empty, such that the recursive definition \( B_{i+1} = A_{i+1} \backslash f_i(B_i) \) for \( 1 \leq i \leq n \) implies \( B_{n+1} = B_1 \); in this case, we also say that \( \{ f_i \}_{i=1}^{n} \) is decomposed by \( B_1 \). Let \( F = f_n \circ \cdots \circ f_1 \) and let \( Q \) be the union of all subsets \( P \) of \( A_1 \) such that \( P \) is invariant for \( F \), i.e., \( F(P) = P \).

The Banach mapping theorem says exactly that if \( n = 2 \) then \( \{ f_i \}_{i=1}^{n} \) is always decomposable. We briefly summarize its generalizations in literature as follows. Assume that each \( f_i \) is injective and each \( A_i \) is infinite. Fan [3] extended this result for all even \( n \). Froda [4], in a review, pointed out that this is not true for odd \( n \). Sanders [7] showed that when \( n \geq 3 \) is odd, in order that \( \{ f_i \}_{i=1}^{n} \) can be decomposable, it is sufficient and necessary that there exists a subset \( Q' \) of \( Q \) such...
that \( F(Q') = Q \setminus Q' \). Callahan and Kneale [2] proved that if \( n = 2 \), the set \( B_1 \) decomposing \( \{f_i\}^{n}_{i=1} \) is unique up to an addition of a set invariant for \( F \). Sanders conjectured in [7] that a similar uniqueness result holds for all \( n \).

In this note, we extend the above results by demonstrating that the assumption of infinite cardinality is not necessary. We also extend Fan’s result by dropping the injection condition. Furthermore, we prove Sanders’ conjecture.

First, we state the extension on decomposability of \( \{f_i\}^{n}_{i=1} \).

**Theorem 1.** The following hold.

1. If \( n \) is even, then \( \{f_i\}^{n}_{i=1} \) is decomposable.

2. If \( n \) is odd and \( Q \) is empty, then \( \{f_i\}^{n}_{i=1} \) is decomposable.

3. If \( n \) is odd and each \( f_i \) is injective, then \( \{f_i\}^{n}_{i=1} \) is decomposable if and only if there exists \( Q' \subseteq Q \) such that \( F(Q') = Q \setminus Q' \).

It is noted in [3] that the relation \( f_i(A_i) \subseteq A_{i+1} \) for \( 1 \leq i \leq n \) implies the following sequence of inclusion relations

\[
A_1 \supseteq f_n(A_n) \supseteq f_n \circ f_{n-1}(A_{n-1}) \supseteq \cdots \supseteq f_n \circ \cdots \circ f_2(A_2) \\
\supseteq F(A_1) \supseteq F \circ f_n(A_n) \supseteq F \circ f_{n-1}(A_{n-1}) \supseteq \cdots .
\]

(1)

Even more, it is mentioned that when \( n \) is even and each \( f_i \) is injective, \( \{f_i\}^{n}_{i=1} \) is decomposed by the set \( B^* \), defined by

\[
B^* = [A_1 \setminus f_n(A_n)] \cup [f_n \circ f_{n-1}(A_{n-1}) \setminus f_n \circ f_{n-1} \circ f_{n-2}(A_{n-2})] \cup \cdots,
\]

a union of infinitely many terms in which the \( k \)-th term is obtained by subtracting the \( 2k \)-th term of (1) from the \( (2k - 1) \)-th term of (1).

Next, we use \( B^* \) to characterize the sets decomposing \( \{f_i\}^{n}_{i=1} \) in the following, which affirms the conjecture of Sanders in [7].

**Theorem 2.** Assume that each \( f_i \) is injective. Let \( B_1 \) be a subset of \( A_1 \). Then the following hold.

1. If \( n \) is even, then \( \{f_i\}^{n}_{i=1} \) is decomposed by \( B_1 \) if and only if \( B_1 = B^* \cup P \), where \( P \subseteq A_1 \) is invariant for \( F \).

2. If \( n \) is odd, then \( \{f_i\}^{n}_{i=1} \) is decomposed by \( B_1 \) if and only if \( B_1 = B^* \cup Q' \), where \( Q' \subseteq Q \) satisfies \( F(Q') = Q \setminus Q' \).

To this end, we use the approach introduced in [6]. Let \( \mathcal{P}(T) \) denote the power set of a set \( T \). For each \( 1 \leq i \leq n \), we define \( \varphi_i : \mathcal{P}(A_i) \rightarrow \mathcal{P}(A_{i+1}) \) by for \( S \in \mathcal{P}(A_i) \),

\[
\varphi_i(S) = A_{i+1} \setminus f_i(S).
\]

Let \( \Phi = \varphi_n \circ \cdots \circ \varphi_1 \). Then for any \( B_1 \in \mathcal{P}(A_1) \), we have that \( \{f_i\}^{n}_{i=1} \) is decomposed by \( B_1 \) if and only if \( B_1 \) is a fixed point for \( \Phi \), i.e., \( \Phi(B_1) = B_1 \). It is clear that each \( \varphi_i \) is order-reversing with respect to set inclusion, that is, if \( S \subseteq T \) then \( \varphi_i(S) \supseteq \varphi_i(T) \). Thus \( \Phi \) is order-preserving for even \( n \) and order-reversing for odd
n. Let \( \Omega = \{ S \subseteq A_1 : \Phi(S) \subseteq S \} \), and \( \Omega' = \{ S \subseteq A_1 : \Phi \circ \Phi(S) \subseteq S \} \). Then \( \Omega \) and \( \Omega' \) are non-empty since \( A_1 \in \Omega \cap \Omega' \). Let

\[
B^e = \bigcap_{S \in \Omega} S \quad \text{and} \quad B^o = \bigcap_{S \in \Omega'} S.
\]

Thus if \( \Phi(B^e) = B^e \), then \( B^e \) is the minimal set decomposing \( \{ f_i \}_{i=1}^n \). Moreover, \( B^e \) and \( B^o \) will play the same role as \( B^* \) in items 1 and 2, respectively, of Theorem 2.

Now we prove our results.

**Proof of Theorem 1.** For item 1, it is sufficient to show that \( B^e \) is a fixed point for \( \Phi \). By the definition of \( \Omega \) and \( B^e \), we get \( \Phi(B^e) = \Phi(\bigcap_{S \in \Omega} S) \subseteq \bigcap_{S \in \Omega} \Phi(S) \subseteq \bigcap_{S \in \Omega} S = B^e \). On the other hand, since \( n \) is even, \( \Phi \) is order preserving; in particular, we have \( \Phi(\Phi(B^e)) \subseteq \Phi(B^e) \). By the definition of \( \Omega \), we obtain \( \Phi(B^e) \in \Omega \). The definition of \( B^o \) implies \( B^e \subseteq \Phi(B^e) \). Therefore, \( \Phi(B^e) = B^e \).

For item 2, we use the same argument as above to get \( \Phi \circ \Phi(B^o) = B^o \). Thus, \( B^o \) belongs to \( \Omega' \), and so does \( \Phi(B^o) \). By the definition of \( B^o \), we have \( B^o \subseteq \Phi(B^o) \). It remains to show that \( \Phi(B^o) \subseteq B^o \). To do so, we need an observation. Let \( S \) and \( T \) be subsets of \( A_1 \). By the definition of \( \varphi_i \), we get \( \varphi_1(S) \setminus \varphi_1(T) = f_1(T) \setminus f_1(S) \subseteq f_1(T \setminus S) \) and hence \( \varphi_2 \circ \varphi_1(S) \setminus \varphi_2 \circ \varphi_1(T) \subseteq f_2(\varphi_1(T) \setminus \varphi_1(S)) \subseteq f_2 \circ f_1(S \setminus T) \). Thus, we get recursively that for \( 1 \leq i \leq n \),

\[
\varphi_1 \circ \cdots \circ \varphi_i(S) \setminus \varphi_1 \circ \cdots \circ \varphi_i(T) \subseteq \begin{cases} f_1 \circ \cdots \circ f_i(S \setminus T), & \text{if } i \text{ is even,} \\ f_1 \circ \cdots \circ f_i(T \setminus S), & \text{if } i \text{ is odd.} \end{cases}
\tag{2}
\]

Since \( n \) is odd, (2) implies \( \Phi(B^o) \setminus B^o = \Phi(B^o) \setminus (\Phi(B^o) \setminus B^o) \subseteq F(\Phi(B^o) \setminus B^o) \). It follows that \( \{ F^k(\Phi(B^o) \setminus B^o) \}_{k=0} \) is a nested sequence of subsets of \( A_1 \) such that \( F^k(\Phi(B^o) \setminus B^o) \subseteq F^{k+1}(\Phi(B^o) \setminus B^o) \), where \( F^0 \) denotes the identity mapping, and \( F^k \) denotes the composition of \( F \) with itself for \( k \geq 1 \) times. Let \( P = \bigcup_{k=0}^\infty F^k(\Phi(B^o) \setminus B^o) \). Obviously, \( F(P) = P \). Since \( Q \) is empty, \( P \) is empty and so is \( \Phi(B^o) \setminus B^o \). Therefore, \( \Phi(B^o) \subseteq B^o \). The proof of item 2 is complete.

Since item 3 immediately follows from item 2 of Theorem 2, we shall prove the latter below.

Before proving Theorem 2, we derive some preliminary observations in an elementary way. Assume that each \( f_i \)'s is injective. Then equality holds in (2). Moreover, \( \varphi_1(S \cup T) = A_2 \setminus [f_1(S) \cup f_1(T)] = \varphi_1(S) \setminus f_1(T) \), and hence \( \varphi_2 \circ \varphi_1(S \cup T) = A_3 \setminus [f_2(\varphi_1(S)) \setminus f_2 \circ f_1(T)] = \varphi_2 \circ \varphi_1(S) \setminus f_2 \circ f_1(T) \). Thus, we obtain recursively that for \( 1 \leq i \leq n \),

\[
\varphi_1 \circ \cdots \circ \varphi_i(S \cup T) = \begin{cases} \varphi_1 \circ \cdots \circ \varphi_i(S) \cup f_i \circ \cdots \circ f_1(T), & \text{if } i \text{ is even,} \\ \varphi_1 \circ \cdots \circ \varphi_i(S) \setminus f_i \circ \cdots \circ f_1(T), & \text{if } i \text{ is odd.} \end{cases}
\tag{3}
\]

Next, we prove that the set \( Q \) is itself invariant for \( F \) and in fact, \( Q = \bigcap_{k=0}^\infty F^k(A_1) \). Let \( P \subseteq A_1 \) be invariant for \( F \). Then \( P = F^k(P) \subseteq F^k(A_1) \) for all \( k \geq 0 \) and so \( P = \bigcap_{k=0}^\infty F^k(P) \subseteq \bigcap_{k=0}^\infty F^k(A_1) \). By the definition of \( Q \), it remains to show that \( \bigcap_{k=0}^\infty F^k(A_1) \) is invariant for \( F \). Since \( \bigcap_{k=0}^\infty F^k(A_1) \subseteq F^m(A_1) \) for all \( m \geq 0 \), we get \( F(\bigcap_{k=0}^\infty F^k(A_1)) \subseteq F^{m+1}(A_1) \) for all \( m \geq 0 \). Thus \( F(\bigcap_{k=0}^\infty F^k(A_1)) \subseteq \bigcap_{k=0}^\infty F^k(A_1) \).
On the other hand, let \( x \in \bigcap_{k=0}^{\infty} F^k(A_1) \). Then there exists a sequence \( \{x_k\}_{k=0}^{\infty} \) in \( A_1 \) such that \( F^k(x_k) = x \) for all \( k \geq 0 \). Since each \( f_i \) is injective, \( F^k \) is injective for all \( k \geq 0 \). Thus \( F(x_k) = x_{k-1} \) for all \( k \geq 1 \) and so \( F^k(x_{k+1}) = x_1 \) for all \( k \geq 0 \). Thus \( x_1 \in \bigcap_{k=0}^{\infty} F^k(A_1) \) and \( F(x_1) = x \). This shows \( \bigcap_{k=0}^{\infty} F^k(A_1) \subseteq F(\bigcap_{k=0}^{\infty} F^k(A_1)) \).

Furthermore, we are able to obtain \( B^c \cap Q = \emptyset \) for even \( n \), and \( B^o \cap Q = \emptyset \) for odd \( n \). In order to do so, let

\[
\begin{align*}
\hat{A}_1 &= A_1 \setminus Q \text{ and } \hat{A}_{n+1} = \hat{A}_1, \\
\hat{A}_i &= A_i \setminus f_{i-1} \circ \cdots \circ f_1(Q) \text{ for all } 2 \leq i \leq n, \\
\hat{f}_i &= f_i|_{\hat{A}_i} \text{ the restriction of } f_i \text{ to } \hat{A}_i, \text{ for all } 1 \leq i \leq n, \\
\hat{\varphi}_i(U) &= \hat{A}_{i+1} \setminus \hat{f}_i(U), \text{ for all } U \subseteq \hat{A}_i \text{ and } 1 \leq i \leq n, \\
\hat{\Phi} &= \hat{\varphi}_n \circ \cdots \circ \hat{\varphi}_1.
\end{align*}
\]

Then there is a relation between \( \hat{\varphi}_i \)'s and \( \varphi_i \)'s as follows. Let \( V \subseteq \hat{A}_1 \). Since \( f_1(Q) \cap \hat{A}_2 = \emptyset \), we have that \( \varphi_1(V) = A_2 \setminus f_1(V) = [A_2 \setminus f_1(Q)] \cup f_1(Q) = \hat{\varphi}_1(V) \cup f_1(Q) \), and hence \( \varphi_2 \circ \varphi_1(V) = A_3 \setminus f_2(\varphi_1(V)) = [A_3 \setminus f_2 \circ f_1(Q)] \cup f_2(\varphi_1(V)) = \hat{\varphi}_2 \circ \hat{\varphi}_1(V) \). Thus, we get recursively that for \( 1 \leq i \leq n \),

\[
\varphi_i \circ \cdots \circ \varphi_1(V) = \begin{cases} \\
\hat{\varphi}_i \circ \cdots \circ \hat{\varphi}_1(V), & \text{if } i \text{ is even,} \\
\hat{\varphi}_i \circ \cdots \circ \hat{\varphi}_1(V) \cup f_i \circ \cdots \circ f_1(Q), & \text{if } i \text{ is odd.}
\end{cases}
\]

(4)

If \( n \) is even, by applying the same argument as in the proof of item 1 of Theorem 1 to \( \{\hat{f}_i\}_{i=1}^{n} \), there exists the minimal subset \( \tilde{B}_1 \) of \( \hat{A}_1 \) such that \( \hat{\Phi}(\tilde{B}_1) = \tilde{B}_1 \). By (4) with \( V = \tilde{B}_1 \), we get \( \hat{\Phi}(\tilde{B}_1) = \tilde{B}_1 \). By the minimality of \( B^o \), we have \( B^c \subseteq \tilde{B}_1 \subseteq A_1 \setminus Q \) and hence \( B^c \cap Q = \emptyset \).

Similarly, if \( n \) is odd, then \( B^o \) is disjoint from the union of all invariant sets for \( F \circ F \), and hence \( B^o \cap Q = \emptyset \), since \( Q \) is also invariant for \( F \circ F \). Moreover, in the proof of item 2 of Theorem 1, we have shown \( B^o \subseteq \Phi(B^o) \) without requiring \( Q = \emptyset \). Since \( B^o \subseteq \hat{A}_1 \) and \( n \) is odd, the equality in (2) implies \( \Phi(B^o) \setminus B^o = \Phi(B^o) \setminus \Phi(B^o) = F(\Phi(B^o) \setminus B^o) \) and hence \( B^o \subseteq \Phi(B^o) \subseteq B^c \cup Q \). By (4) and the invariance of \( Q \), we get \( \Phi(B^o) = \Phi(B^o) \cup Q \). Hence \( B^o \subseteq \Phi(B^o) \cup Q \subseteq B^c \cup Q \). Since \( Q \) is disjoint from \( B^o \) and \( \Phi(B^o) \), we obtain \( \hat{\Phi}(B^o) = B^o \).

Now, we are ready to prove theorem 2.

**Proof of Theorem 2.** First, we prove the counterparts of items 1 and 2 with \( B^* \) replaced by \( B^c \) and \( B^o \) respectively; for convenience, we name them items 1’ and 2’, respectively. For the “if” part of item 1’, let \( P \subseteq A_1 \) be invariant for \( F \) and let \( B_1 = B^c \cup P \). In the proof of item 1 of Theorem 1, we have shown \( \Phi(B^c) = B^c \).

Thus, by (3), since \( n \) is even, \( \Phi(B_1) = \Phi(B^c) \cup F(P) = B^c \cup P = B_1 \). This shows that \( \{f_i\}_{i=1}^{n} \) is decomposed by \( B_1 \). For the “only if” part of item 1’, let \( \{f_i\}_{i=1}^{n} \) be decomposed by \( B_1 \). Since \( n \) is even, by the equality in (2), we get \( B_1 \setminus B^c = \Phi(B_1) \setminus \Phi(B^c) = F(B_1 \setminus B^c) \). Thus \( B_1 = B^c \cup P \) for some invariant set \( P \) for \( F \).
For the “if” part of item 2’, let $Q' \subseteq Q$ satisfy $F(Q') = Q \setminus Q'$ and let $B_1 = B^o \cup Q'$.

Then

$$
\Phi(B_1) = \Phi(B^o) \setminus F(Q'), \text{ since (3) and } n \text{ is odd,}
$$

$$
= \Phi(B^o) \cup Q \setminus F(Q'), \text{ since (4), } B^o \subseteq A_1, \text{ and } \Phi(B^o) = B^o,
$$

$$
= B^o \cup Q
$$

$$
= B_1.
$$

This shows that $\{ f_i \}_{i=1}^n$ is decomposed by $B_1$. For the “only if” part of item 2’, let $\{ f_i \}_{i=1}^n$ be decomposed by $B_1$. Then $\Phi \circ \Phi(B_1) = B_1$, and so $B^o \subseteq B_1$ by the minimality of $B^o$. Let $Q' = B_1 \setminus B^o$. Then

$$
Q' = \Phi(B^o \cup Q') \setminus B^o, \text{ since } B_1 = \Phi(B_1),
$$

$$
= \Phi(B^o) \setminus F(Q') \setminus B^o, \text{ since (3) and } n \text{ is odd,}
$$

$$
= B^o \cup Q \setminus F(Q') \setminus B^o, \text{ since (4), } \Phi(B^o) = B^o \text{ and } F(Q) = Q,
$$

$$
= Q \setminus F(Q'), \text{ since } B^o \cap Q = \emptyset.
$$

This implies $F(Q') \subseteq F(Q) = Q$ and hence $F(Q') = Q \setminus Q'$.

Finally, we prove items 1 and 2. First, consider the case when $n$ is even. By the definitions of $\varphi_i$’s and of $B^*$, we have $\varphi_{i+1} \circ \varphi_i(S) = A_{i+2} \setminus f_{i+1}(A_{i+1} \setminus f_i(S)) = [A_{i+2} \setminus f_{i+1}(A_{i+1})] \cup f_{i+1} \circ f_i(S)$ for all $S \subseteq A_i$ and hence $\Phi(B^*) = B^*$. Moreover, one has $B^* \cap Q = \emptyset$. Indeed, if $x \in B^*$, then since $n$ is even, the definition of $B^*$ implies $x \in F^m(A_1 \setminus f_n \circ \cdots \circ f_2(A_2))$ for some $m \geq 0$. Since $f_n \circ \cdots \circ f_2(A_2) \supseteq F(A_1)$, we get $x \notin F^{m+1}(A_1)$ and hence $x \notin \bigcap_{k=0}^{\infty} F^k(A_1) = Q$. Now, the “only if” part of item 1’, together with the facts $B^* \cap Q = \emptyset$ and $\Phi(B^*) = B^*$ just verified, implies $B^* = B^o$. Therefore, item 1 follows from item 1’. Next, consider the case when $n$ is odd. Similar to the above argument, the definition of $B^*$ implies $\Phi \circ \Phi(B^*) = B^*$ and $B^* \cap R = \emptyset$, where $R$ is the union of all invariant subsets of $A_1$ for $F^2$.

Moreover, since $2n$ is even, one can restate item 1’ as follows: $\Phi \circ \Phi(B_1) = B_1$ if and only if $B_1 = B^o \cup P$, where $P \subseteq A_1$ is invariant for $F^2$. This shows $B^* = B^o$. Therefore, item 2 follows from item 2’.

We have finished the proof of Theorem 2. □

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**References**


