UNIQUENESS AND ULAM-HYERS-MITTAG-LEFFLER STABILITY RESULTS FOR THE DELAYED FRACTIONAL MULTI-TERMS DIFFERENTIAL EQUATION INVOLVING THE Φ-CAPUTO FRACTIONAL DERIVATIVE

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ABSTRACT. The principal aim of the present paper is to establish the uniqueness and Ulam-Hyers Mittag-Leffler (UHML) stability of solutions for a new class of multi-terms fractional time-delay differential equations in the context of the Φ-Caputo fractional derivative. To achieve this purpose, the generalized Laplace transform method alongside facet with properties of the Mittag-Leffler functions (M-LFs), are utilized to give a new representation formula of the solutions for the aforementioned problem. Besides that, the uniqueness of the solutions of the considered problem is also proved by applying the well-known Banach contraction principle coupled with the Φ-fractional Bielecki-type norm. While the Φ-fractional Gronwall type inequality and the Picard operator (PO) technique combined with abstract Gronwall lemma are used to prove the UHML stability results for the proposed problem. Lastly, an example is offered to assure the validity of the obtained theoretical results.

1. Introduction

In recent years, the subject of fractional differential equations (FDEs) has aroused considerable attention among scientists. The mentioned branch has many applications in diverse disciplines of sciences and engineering. For more details, see [7, 11, 21, 31]. In 2017, a novel fractional derivative with respect to another function Φ was formulated by Almeida. This new operator is called the Φ-Caputo fractional derivative. In a subsequent year, the concerned derivative was generalized by Sousa and was named the Φ-Hilfer fractional derivative. These novel operators can be regarded as a unification of some well-known fractional operators in the literature. For a detailed discussion on the basic theory of Φ-Caputo and Φ-Hilfer fractional derivatives, we refer to the recent papers [2, 26] and the references therein. Furthermore, many researchers have been paying increased attention to the study of conformable fractional derivatives. This is owing to the fact that it is crucial in a variety of practical applications. For more details on conformable fractional calculus we refer the readers to [1, 5, 9, 18, 35] and the references therein.

Currently, many mathematicians have addressed the existence and uniqueness of solutions as well as different types of Ulam’s stabilities of nonlinear FDEs involving various categories of fractional derivatives with the help of fixed point theory. For details, we refer the reader to [3, 4, 10, 12, 13, 14, 15, 16, 17, 20, 25, 27, 29, 30, 32, 33, 34].

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To the best of our knowledge, the uniqueness and the UHML stability of solutions for the delayed fractional multi-terms differential equation involving the $\Phi$–Caputo fractional derivative is not yet investigated. So getting motivation from the results proved in [19, 34], in this paper, we mainly focus on the uniqueness and the UHML stability of solutions for the following $\Phi$–Caputo fractional multi-terms differential equation ($\Phi$–Caputo FMTDE) with a finite delay of the form

$$
\left\{
\begin{array}{ll}
\mathcal{D}_{m+}^{\mu,\Phi} \beta(\ell) + \rho \mathcal{D}_{m+}^{\kappa,\Phi} \beta(\ell) = Q(\beta(\ell), \beta(f(\ell))), & \ell \in \Omega := [m, n], \\
\beta(\ell) = \alpha(\ell), & \ell \in [m - \sigma, m],
\end{array}
\right.
$$

where $\mathcal{D}_{m+}^{\mu,\Phi}$ and $\mathcal{D}_{m+}^{\kappa,\Phi}$ denote the $\Phi$–Caputo fractional derivatives, with the orders $\mu$ and $\kappa$ respectively such that $0 < \kappa < \mu \leq 1$, $\rho, \sigma > 0$, $Q \in C(\Omega \times \mathbb{R}^2, \mathbb{R})$, $f \in C(\Omega, [m - \sigma, n])$, $f(\ell) \leq \ell$, $\alpha \in C([m - \sigma, m], \mathbb{R})$ and $m, n \in \mathbb{R}^+$, such that $m < n$.

The outline of the paper is as follows. In Section 2 we introduce some basic concepts needed throughout this paper. Section 3 is devoted to establishing the main results in which the uniqueness of the solutions for the problem (1) can be obtained under the famous Banach’s Banach fixed point theorem along with the $\Phi$-fractional Bielecki-type norm. Then, we present the UHML stability result of the problem (1) in Section 4. A specific example is given in Section 5. Finally, the conclusion is presented in Section 6.

### 2. Basic concepts

In this section, we introduce some necessary definitions and preliminary facts which will be used throughout this paper.

Let us consider on the space $\mathcal{X} := C(\Omega, \mathbb{R})$ the $\Phi$-fractional Bielecki-type norm $\| \cdot \|_{\phi, \beta, \mu}$ given by previous studies [25, 27] and defined by

$$
\| \beta \|_{\phi, \beta, \mu} := \sup_{\ell \in [m, n]} \frac{|\beta(\ell)|}{M_{\mu}(\beta(\Phi(\ell) - \Phi(m))^\mu)}, \quad \beta > 0.
$$

Then, $(\mathcal{X}, \| \cdot \|_{\phi, \beta, \mu})$ is a Banach space. In addition, let us denote by $\mathcal{Y} := C([m - \sigma, n], \mathbb{R})$ the Banach space of all continuous functions $\beta$ from $[m - \sigma, n]$ into $\mathbb{R}$ equipped with the norm

$$
\| \beta \|_{\phi, \beta, \mu} := \sup_{\ell \in [m - \sigma, n]} \frac{|\beta(\ell)|}{M_{\mu}(\beta(\Phi(\ell) - \Phi(m))^\mu)}, \quad \beta > 0.
$$

It’s clear that $\| \beta \|_{\phi, \beta, \mu} \leq \| \beta \|_{\phi, \beta, \mu}$.

Now, we recall the definition of the Mittag–Leffler functions (M-LFs).

**Definition 2.1** ([6]). For $p, q > 0$ and $\vartheta \in \mathbb{R}$, the M-LFs of one and two parameters are given by

$$
M_{p}(\vartheta) = \sum_{k=0}^{\infty} \frac{\vartheta^k}{\Gamma(pk + 1)}, \quad M_{p,q}(\vartheta) = \sum_{k=0}^{\infty} \frac{\vartheta^k}{\Gamma(pk + q)}.
$$

Clearly, $M_{p,1}(\vartheta) = M_{p}(\vartheta)$.

**Lemma 2.2** ([6, 32]). Let $p \in (0, 1), q > p$ be arbitrary and $\vartheta \in \mathbb{R}$. The functions $M_{p}, M_{p,p}$ and $M_{p,q}$ are nonnegative and have the following properties:
\( (1) \ M_p(\vartheta) \leq 1, M_{p,q}(\vartheta) \leq \frac{1}{\Gamma(q)}, \ \text{for any } \vartheta < 0, \)

\( (2) \ M_{p,q}(\vartheta) = \vartheta M_{p,p+q}(\vartheta) + \frac{1}{\Gamma(q)}, \ p, q > 0, \vartheta \in \mathbb{R}. \)

Let \( \Phi : \Omega \rightarrow \mathbb{R} \) be an increasing differentiable function such that \( \Phi'(\ell) \neq 0, \) for all \( \ell \in \Omega. \)

**Definition 2.3** ([2, 11]). The R-L fractional integral of order \( \mu > 0 \) for an integrable function \( z : \Omega \rightarrow \mathbb{R} \) with respect to \( \Phi \) is described by

\[
\frac{\eta^\mu}{m^+} \Phi(\ell) = \int_\ell^\eta \frac{\Phi(\eta) (\Phi(\eta) - \Phi(\ell))^{\mu - 1}}{\Gamma(\mu)} d\eta,
\]

where \( \Gamma(\mu) = \int_0^\infty t^{\mu - 1} e^{-t}dt, \mu > 0 \) is the Gamma function.

**Definition 2.4** ([2]). Let \( \Phi, z \in C^\infty(\Omega, \mathbb{R}). \) The Caputo fractional derivative of \( z \) of order \( n - 1 < \mu < n \) with respect to \( \Phi \) is defined by

\[
c^D^\mu \Phi \Phi(\ell) = \frac{\eta^{n - \mu} - \Phi [n] \Phi(\ell)}{m^+} \Phi(\ell),
\]

where \( n = [\mu] + 1 \) for \( n \notin \mathbb{N}, n = \mu \) for \( \mu \in \mathbb{N}, \) and \( \Phi [n] \Phi(\ell) = \left( \frac{d}{\Phi(\ell)} \right)^n \Phi(\ell). \)

Some basic properties of the \( \Phi \)-fractional operators are listed in the following Lemma.

**Lemma 2.5** ([2, 3]). Let \( \mu, \kappa, \beta > 0, \) and \( z \in C(\Omega, \mathbb{R}). \) Then for each \( \ell \in \Omega,

\[
(1) \ c^D^\mu \Phi \Phi(\ell) = \frac{\eta^{n - \mu} - \Phi [n] \Phi(\ell)}{m^+} \Phi(\ell),
\]

\[
(2) \ c^D^\mu \Phi \Phi(\ell) = \frac{\eta^{n - \mu} - \Phi [n] \Phi(\ell)}{m^+} \Phi(\ell), \quad 0 < \mu \leq 1,
\]

\[
(3) \ c^D^\mu \Phi (\Phi(\ell) - \Phi(m))^{\kappa - 1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa + \mu)} (\Phi(\ell) - \Phi(m))^{\kappa + \mu - 1},
\]

\[
(4) \ c^D^\mu \Phi (\Phi(\ell) - \Phi(m))^{\kappa - 1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa + \mu)} (\Phi(\ell) - \Phi(m))^{\kappa + \mu - 1},
\]

\[
(5) \ c^D^\mu \Phi \left( M_\mu (\beta (\Phi(\ell) - \Phi(m)))^\mu \right) + \frac{1}{\beta} (M_\mu (\beta (\Phi(\ell) - \Phi(m))^\mu - 1)\),
\]

in particular if we take \( \beta = 1 \) we can get the following estimate

\[
(6) \ c^D^\mu \Phi \left( M_\mu (\Phi(\ell) - \Phi(m))^\mu \right) \leq M_\mu (\Phi(\ell) - \Phi(m))^\mu.
\]

**Definition 2.6** ([8]). A function \( z : [m, \infty) \rightarrow \mathbb{R} \) is said to be of \( \Phi(\ell) \)-exponential order if there exist non-negative constants \( c_1, c_2, n \) such that

\[
|z(\ell)| \leq c_1 e^{c_2 (\Phi(\ell) - \Phi(m))}, \quad \ell \geq n.
\]

**Definition 2.7** ([8]). Let \( z, \Phi : [m, \infty) \rightarrow \mathbb{R} \) be real valued functions such that \( \Phi(\ell) \) is continuous and \( \Phi'(\ell) > 0 \) on \( [m, \infty). \) The generalized Laplace transform of \( z \) is denoted by

\[
\mathcal{L}_\Phi \left\{ z(\ell) \right\} = \int_m^\infty e^{-\lambda (\Phi(\ell) - \Phi(m))} z(\ell) d\ell, \quad \text{for all } \lambda > 0,
\]

provided that the integral in (4) exists.

**Definition 2.8** ([8]). Let \( z_1 \) and \( z_2 \) be two functions which are piecewise continuous at each interval \( [m, n] \) and of exponential order. We define the generalized convolution of \( z_1 \) and \( z_2 \) by

\[
(z_1 * z_2)(\ell) = \int_m^\ell \Phi'(\eta) z_1(\eta) z_2(\Phi^{-1}(\Phi(\ell) + \Phi(m) - \Phi(\eta))) d\eta.
\]
Lemma 2.9 ([8]). Let 3_1 and 3_2 be two functions which are piecewise continuous at each interval [m, n] and of exponential order. Then
\[ L\Phi\{3_1 * \Phi 3_2\} = L\Phi\{3_1\}L\Phi\{3_2\}. \]

In the following lemma, we present the generalized Laplace transforms of some elementary functions as well as the generalized Laplace transforms of the generalized fractional integrals and derivatives.

Lemma 2.10 ([8]). The following properties are satisfied:
1. \( L\Phi\{1\} = \frac{1}{\lambda}, \lambda > 0, \)
2. \( L\Phi\{(\Phi(\ell) - \Phi(m))^\ell\} = \Gamma(\ell) \frac{\lambda^{\ell-1}}{\lambda^{\ell}}, \quad r, \lambda > 0, \)
3. \( L\Phi\{\mathbb{M}_p(\pm \rho(\Phi(\ell) - \Phi(m)))^\rho\} = \frac{\lambda^{\rho-1}}{\lambda^{\rho} + \rho}, \quad p > 0 \) and \( |\frac{\rho}{\lambda^{\rho}}| < 1, \)
4. \( L\Phi\{(\Phi(\ell) - \Phi(m))^{\ell-1} \mathbb{M}_p(\pm \rho(\Phi(\ell) - \Phi(m)))^\rho\} = \frac{\lambda^{\rho-1}}{\lambda^{\rho} + \rho}, \quad p > 0 \) and \( |\frac{\rho}{\lambda^{\rho}}| < 1, \)
5. \( L\Phi\{\mu_{\ell} \Phi(\ell)\} = \frac{\lambda^{\rho}}{\lambda^{\rho}}, \mu, \lambda > 0, \)
6. \( L\Phi\{c \Phi(\ell)\} = \frac{\lambda^{\rho}}{\lambda^{\rho}}, 0 < \mu \leq 1 \) and \( \lambda > 0, \)

The following lemma is a generalization of Gronwall’s inequality.

Lemma 2.11 ([28]). Let \( \Omega \) be the domain of the nonnegative integrable functions \( c_1, c_2. \) Also, \( c_3 \) be a continuous, nonnegative and nondecreasing function defined on \( \Omega \) and \( \Phi \in C^1(\Omega, \mathbb{R}) \) be an increasing function with the restriction that \( \Phi'(\ell) \neq 0, \forall \ell \in \Omega. \) If
\[ c_1(\ell) \leq c_2(\ell) + c_3(\ell) \int_{m}^{\ell} \Phi'(\eta)(\Phi(\ell) - \Phi(\eta))^{\mu-1} c_1(\eta)d\eta, \ell \in \Omega. \]

Then
\[ c_1(\ell) \leq c_2(\ell) + \int_{m}^{\ell} \sum_{n=0}^{\infty} \frac{(c_3(\ell) \Gamma(\mu))^n}{\Gamma(n\mu)} \Phi'(\eta)(\Phi(\ell) - \Phi(\eta))^{n\mu-1} c_2(\eta)d\eta, \ell \in \Omega. \]

Corollary 2.12 ([28]). Under the conditions of Lemma 2.11, let \( c_2 \) be a nondecreasing function on \( \Omega. \) Then we get that
\[ c_1(\ell) \leq c_2(\ell) \mathbb{M}_\mu \left( \Gamma(\mu) c_1(\ell)(\Phi(\ell) - \Phi(m))^{\mu} \right), \ell \in \Omega. \]

Definition 2.13 ([22]). Let \((E, d)\) be a metric space. An operator \( S : E \rightarrow E \) is a Picard operator (PO) if there exists \( b^* \in E \) such that
1. \( F_S = \{b^*\} \) where \( F_S = \{b \in E : S(b) = b\} \) is the fixed point set of \( S; \)
2. the sequence \( \{S^n(b_0)\}_{n \in \mathbb{N}} \) converges to \( b^* \) for all \( b_0 \in E. \)

Lemma 2.14 ([23] abstract Gronwall lemma). Let \((E, d, \leq)\) be an ordered metric space and \( S : E \rightarrow E \) be an increasing PO. Then, for \( b \in E, b \leq Sb \) implies \( b \leq b^*. \)

3. Uniqueness result for the problem (1)

Before going to our main results, we state the following special linear cases of the problem (1).
Lemma 3.1. For a given $h \in C(\Omega, \mathbb{R}), 0 < \kappa < \mu \leq 1$ and $\rho > 0$, the linear $\Phi$–Caputo FMTDE

\begin{align}
(6) \quad \left\{ \begin{align*}
\mathbb{D}_m^\mu \Phi,^\kappa \Phi \zeta (\ell) &+ \rho \mathbb{D}_m^\kappa \Phi \zeta (\ell) = h(\ell), \quad \ell \in [m, n], \\
\zeta (\ell) & = \mathcal{A}(\ell), \quad \ell \in [m-\sigma, m]
\end{align*} \right.
\end{align}

has a unique solution given explicitly as

\begin{align}
\zeta (\ell) = \left\{ \begin{align*}
\mathcal{A}(m) + \int_m^\ell \mathbb{W}^\mu_\Phi (\ell, \eta) \mathbb{M}_{\mathcal{U} - \kappa, \mu} ( - \rho (\Phi (\ell) - \Phi (\eta))^{\mu - \kappa} ) h(\eta) d\eta, & \quad \ell \in [m, n], \\
\mathcal{A}(\ell), & \quad \ell \in [m-\sigma, m],
\end{align*} \right.
\end{align}

\text{where}

\begin{align}
\mathbb{W}^\mu_\Phi (\ell, \eta) = \Phi' (\eta) (\Phi (\ell) - \Phi (\eta))^{\mu - 1}.
\end{align}

Proof. Applying the generalized Laplace transform to both sides of the first equation of (6) and using Lemma 2.10, we obtain

\begin{align}
\lambda^\mu \mathbb{L}_\Phi \{ \zeta (\ell) \} - \lambda^{\mu - 1} \mathcal{A}(m) + \rho \lambda^{\kappa \mathbb{L}_\Phi} \{ \zeta (\ell) \} - \rho \lambda^{\kappa - 1} \mathcal{A}(m) = \mathbb{L}_\Phi \{ h(\ell) \}.
\end{align}

So,

\begin{align}
\mathbb{L}_\Phi \{ \zeta (\ell) \} = \rho \lambda^{\mu - 1} \mathcal{A}(m) + \frac{\lambda^{\mu - 1} \mathcal{A}(m)}{\rho \lambda^{\mu - 1}} + \frac{\lambda^{\kappa - 1} \mathcal{A}(m)}{\rho \lambda^{\mu - 1}} \mathbb{L}_\Phi \{ h(\ell) \} \\
= \rho \mathbb{L}_\Phi \{ (\Phi (\ell) - \Phi (m))^{\mu - \kappa} M_{\mu - \kappa, \mu - \kappa + 1} ( - \rho (\Phi (\ell) - \Phi (m))^{\mu - \kappa} ) \} \mathcal{A}(m) \\
+ \mathbb{L}_\Phi \{ (\Phi (\ell))^{\mu - 1} M_{\mu - \kappa, \mu} ( - \rho (\Phi (\ell) - \Phi (m))^{\mu - \kappa} ) \} \mathcal{A}(m) \\
+ \int_m^\ell \Phi' (\eta) (\Phi (\ell) - \Phi (\eta))^{\mu - 1} M_{\mu - \kappa, \mu} ( - \rho (\Phi (\ell) - \Phi (\eta))^{\mu - \kappa} ) h(\eta) d\eta, & \quad \ell \in [m, n].
\end{align}

Taking the inverse generalized Laplace transform to both sides of the last expression, we get

\begin{align}
\zeta (\ell) & = (\Phi (\ell))^{\mu - 1} M_{\mu - \kappa, \mu} ( - \rho (\Phi (\ell) - \Phi (m))^{\mu - \kappa} ) + \rho (\Phi (\ell) - \Phi (m))^{\mu - \kappa} M_{\mu - \kappa, \mu - \kappa + 1} ( - \rho (\Phi (\ell) - \Phi (m))^{\mu - \kappa} ) \} \mathcal{A}(m) \\
& + \int_m^\ell \Phi' (\eta) (\Phi (\ell) - \Phi (\eta))^{\mu - 1} M_{\mu - \kappa, \mu} ( - \rho (\Phi (\ell) - \Phi (\eta))^{\mu - \kappa} ) h(\eta) d\eta, & \quad \ell \in [m, n].
\end{align}

This ends the proof of Lemma 3.1. \hfill \Box

As a result of Lemma 3.1, the problem (1) can be converted to an integral equation which takes the following form

\begin{align}
(7) \quad \zeta (\ell) = \left\{ \begin{align*}
\mathcal{A}(m) + \int_m^\ell \mathbb{W}^\mu_\Phi (\ell, \eta) M_{\mathcal{U} - \kappa, \mu} ( - \rho (\Phi (\ell) - \Phi (\eta))^{\mu - \kappa} ) \\
\mathbb{Q} (\eta, \zeta (\eta), \zeta (f(\eta))) d\eta, & \quad \ell \in [m, n], \\
\mathcal{A}(\ell), & \quad \ell \in [m-\sigma, m].
\end{align*} \right.
\end{align}

We are now in position to present and prove our main results.

Theorem 3.2. Assume that the following statements are valid:

\( (H1) \) The function $f : \Omega \rightarrow [m-\sigma, n]$ is continuous function with $f(\ell) \leq \ell$. \\
\( (H2) \) The function $Q : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and there exist $L_Q > 0$ such that

\begin{align}
|Q(\ell, b_2, a_2) - Q(\ell, b_1, a_1)| \leq L_Q \left( |b_2 - b_1| + |a_2 - a_1| \right), & \quad \ell \in \Omega, a_1, a_2, b_1, b_2 \in \mathbb{R}.
\end{align}

Then the problem (1) possesses a unique solution which belong to the space $\mathcal{Y} \cap \mathcal{X}$.
**Proof.** Transform the integral representation (7) of the problem (1) into a fixed point problem as follows:

$$\tilde{j} = \mathbb{P}_3, \quad \tilde{j} \in \mathcal{V},$$

where \( \mathbb{P} : \mathcal{V} \rightarrow \mathcal{V} \) is defined by

$$\mathbb{P}_3(\ell) = \begin{cases} \alpha(m) + \int_m^\ell \mathcal{W}_\Phi^H(\ell, \eta) M_{\mu-\kappa, \mu} \left( -\rho(\Phi(\ell) - \Phi(\eta))^{\mu-\kappa} \right) \times Q(\eta, \tilde{j}(\eta), \tilde{j}(\eta)) \, d\eta, \\
\alpha(\ell), \quad \ell \in [m, n], \\
\alpha(\ell), \quad \ell \in [m - \sigma, m]. \end{cases} \tag{8}$$

Clearly, the operator \( \mathbb{P} \) is well-defined. Moreover, the existence of a fixed point for the operator \( \mathbb{P} \) will ensure the existence of the solution of the problem (1). Our aim is to check that \( \mathbb{P} \) is a contraction operator with respect to the \( \Phi \)-fractional Bielecki-type norm. Note that by definition of operator \( \mathbb{P} \), for any \( \tilde{j}_1, \tilde{j}_2 \in \mathcal{V} \) we have

$$|\mathbb{P}\tilde{j}_2(\ell) - \mathbb{P}\tilde{j}_1(\ell)| = 0, \quad \text{for all } \ell \in [m - \sigma, m].$$

On the other hand, keeping in mind the definition of the operator \( \mathbb{P} \) on \([m, n]\) together with assumptions (H1), (H2) and Lemmas 2.2, 2.5 we can get

$$|\mathbb{P}\tilde{j}_2(\ell) - \mathbb{P}\tilde{j}_1(\ell)| \leq \frac{2\mathcal{L}_Q}{m} \|\tilde{j}_2 - \tilde{j}_1\|_{\mathcal{V}, \mathfrak{A}_\mathfrak{B}, \mu} \int_m^\ell \mathcal{W}_\Phi^H(\ell, \eta) M_{\mu} \left( \beta(\Phi(\ell) - \Phi(m))^{\mu} \right) d\eta \\
\leq \frac{2\mathcal{L}_Q}{\beta} \left[ M_{\mu} \left( \beta(\Phi(\ell) - \Phi(m))^{\mu} \right) - 1 \right] \|\tilde{j}_2 - \tilde{j}_1\|_{\mathcal{V}, \mathfrak{A}_\mathfrak{B}, \mu}.$$

Hence, the above inequality yields

$$\|\mathbb{P}\tilde{j}_2 - \mathbb{P}\tilde{j}_1\|_{\mathcal{V}, \mathfrak{A}_\mathfrak{B}, \mu} \leq \frac{2\mathcal{L}_Q}{\beta} \|\tilde{j}_2 - \tilde{j}_1\|_{\mathcal{V}, \mathfrak{A}_\mathfrak{B}, \mu}.$$

Thus

$$\|\mathbb{P}\tilde{j}_2 - \mathbb{P}\tilde{j}_1\|_{\mathcal{V}, \mathfrak{A}_\mathfrak{B}, \mu} \leq \frac{2\mathcal{L}_Q}{\beta} \|\tilde{j}_2 - \tilde{j}_1\|_{\mathcal{V}, \mathfrak{A}_\mathfrak{B}, \mu}.$$

Let us choose \( \beta > 0 \) such that \( \frac{2\mathcal{L}_Q}{\beta} < 1 \). It is easy to see that the operator \( \mathbb{P} \) is a contraction with respect to Bielecki’s norm \( \| \cdot \|_{\mathcal{V}, \mathfrak{A}_\mathfrak{B}, \mu} \). Now, by applying the Banach’s fixed point theorem, we can find that \( \mathbb{P} \) has a unique fixed point, and thus the problem (1) has a unique solution in the space \( \mathcal{V} \cap \mathcal{X} \). This completes the proof. \( \square \)

**Remark 3.3.** It is worth noting that the results obtained in this paper are generalizations and partial continuation of some results obtained in [4, 15, 19, 20, 33, 34]. For example in our analysis we don’t assume that \( \frac{2\mathcal{L}_Q}{\Gamma(\mu+1)} |\Phi(n) - \Phi(m)|^\mu < 1 \) in Theorem 3.2, while it is required in Theorem 3.4 in the article of Wang and Zhang [34]. Moreover, problem (1) unifies several classes of fractional differential equations because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives by a proper choice of the function \( \Phi \).
4. Ulam-Hyers-Mittag-Leffler stability results for the problem (1)

Motivated by [24, 34], we introduce the Ulam–Hyers–Mittag-Leffler stability of solutions to our problem (1).

Let \( \varepsilon, \rho > 0 \) and \( \zeta : \Omega \to \mathbb{R}^+ \), be a continuous function. We focus on the following inequality:

\[
(9) \quad |cD_{m^+}^{m\Phi}(\varepsilon) + \rho cD_{m^+}^{m\Phi}(\varepsilon) - Q(\varepsilon, 3(\varepsilon), 3(f(\varepsilon)))| \leq \varepsilon M_{\mu}(\varepsilon, \Phi(\varepsilon) - \Phi(m))^\mu, \quad \ell \in \Omega.
\]

**Definition 4.1** ([34]). Equation (1) is \textit{UHML} stable, with respect to \( \varepsilon M_{\mu}(\varepsilon, \Phi(\varepsilon) - \Phi(m))^\mu \) if there exists a real number \( cM_{\mu} > 0 \) such that, for each \( \varepsilon > 0 \) and for each solution \( \tilde{z} \in Y \) of the inequality (9), there is a unique solution solution \( \tilde{z} \in Y \) of Eq. (1) with

\[
\begin{align*}
&\left| \tilde{z}(\varepsilon) - 3(\varepsilon) \right| = 0, \quad \ell \in [m - \sigma, m], \\
&\left| \tilde{z}(\varepsilon) - 3(\varepsilon) \right| \leq cM_{\mu} \varepsilon M_{\mu}(\varepsilon, \Phi(\varepsilon) - \Phi(m))^\mu, \quad \ell \in [m, n].
\end{align*}
\]

**Remark 4.2** ([34]). A function \( \tilde{z} \in \mathcal{Y} \) is a solution of inequality (9) if and only if there exists a function \( \Theta \in \mathcal{X} \) (which depends on solution \( \tilde{z} \)) such that

(i) \( |\Theta(\varepsilon)| \leq \varepsilon M_{\mu}(\varepsilon, \Phi(\varepsilon) - \Phi(m))^\mu \), \( \ell \in \Omega \).

(ii) \( cD_{m^+}^{m\Phi}(\tilde{z}(\varepsilon)) + \rho cD_{m^+}^{m\Phi}(\varepsilon) = Q(\varepsilon, \tilde{z}(\varepsilon), \tilde{z}(f(\varepsilon))) + \Theta(\varepsilon), \quad \ell \in \Omega \).

**Lemma 4.3.** Let \( \tilde{z} \in \mathcal{Y} \) be a solution of of inequality (9), then \( \tilde{z} \) satisfies the following integral inequality

\[
\left| \tilde{z}(\varepsilon) - 3(\varepsilon) - \int_{m}^{\varepsilon} W_{\Phi}(\varepsilon, \eta) M_{\mu - \kappa, \mu}(-p(\Phi(\varepsilon) - \Phi(\eta))^\mu - \kappa) Q(\eta, \tilde{z}(\eta), \tilde{z}(f(\eta))) d\eta \right| \\
\leq \varepsilon M_{\mu}(\varepsilon, \Phi(\varepsilon) - \Phi(m))^\mu.
\]

**Proof.** In fact, by the second part of Remark 4.2, we have

\[
(10) \quad cD_{m^+}^{m\Phi}(\tilde{z}(\varepsilon)) + \rho cD_{m^+}^{m\Phi}(\varepsilon) = Q(\varepsilon, \tilde{z}(\varepsilon), \tilde{z}(f(\varepsilon))) + \Theta(\varepsilon), \quad \ell \in \Omega.
\]

Thanks to Lemma 3.1, the integral representation of (10) is expressed as

\[
(11) \quad \tilde{z}(\varepsilon) = 3(\varepsilon) + \int_{m}^{\varepsilon} W_{\Phi}(\varepsilon, \eta) M_{\mu - \kappa, \mu}(-p(\Phi(\varepsilon) - \Phi(\eta))^\mu - \kappa) \\
\times \left\{ Q(\eta, \tilde{z}(\eta), \tilde{z}(f(\eta))) + \Theta(\eta) \right\} d\eta.
\]

It follows from (11), together with the first part of Remark 4.2, and Lemma 2.2 that

\[
\left| \tilde{z}(\varepsilon) - 3(\varepsilon) - \int_{m}^{\varepsilon} W_{\Phi}(\varepsilon, \eta) M_{\mu - \kappa, \mu}(-p(\Phi(\varepsilon) - \Phi(\eta))^\mu - \kappa) Q(\eta, \tilde{z}(\eta), \tilde{z}(f(\eta))) d\eta \right| \\
\leq \varepsilon M_{\mu}(\varepsilon, \Phi(\varepsilon) - \Phi(m))^\mu.
\]

Using the sixth part of Lemma 2.5, we can get

\[
\left| \tilde{z}(\varepsilon) - 3(\varepsilon) - \int_{m}^{\varepsilon} W_{\Phi}(\varepsilon, \eta) M_{\mu - \kappa, \mu}(-p(\Phi(\varepsilon) - \Phi(\eta))^\mu - \kappa) Q(\eta, \tilde{z}(\eta), \tilde{z}(f(\eta))) d\eta \right| \\
\leq \varepsilon M_{\mu}(\varepsilon, \Phi(\varepsilon) - \Phi(m))^\mu.
\]

\( \square \)
Now, we discuss the UHML stability of solutions for the problem (1).

**Theorem 4.4.** Under the assumptions of Theorem 3.2, the problem (1) is UHML stable.

**Proof.** Let \( \varepsilon > 0 \) and let \( \tilde{z} \in \mathcal{V} \cap \mathcal{K}^\ast \) be a function which satisfies the inequality (9), and denote the unique solution of equation (1) by \( \tilde{z} \in \mathcal{K} \), that is,

\[
\begin{cases}
\tilde{z}\left(\ell,\nu;\Phi,\eta\right) - \tilde{z}\left(\ell,\nu;\Phi,\eta\right) - \int_{\mu}^{\ell} \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) \mathcal{W}_{\mathcal{V}}^{\mu-\kappa,\mu}\left(-\rho(\Phi(\ell) - \Phi(\eta))^{\mu-\kappa}\right) \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) d\eta = \mathcal{Q}(\ell,\nu;\Phi,\eta), & \ell \in [\nu,\sigma], \\
\tilde{z}\left(\ell,\nu;\Phi,\eta\right) = \tilde{z}\left(\ell,\nu;\Phi,\eta\right), & \ell \in [\sigma,\nu].
\end{cases}
\]

By Theorem 3.2, we have

\[
\tilde{z}\left(\ell,\nu;\Phi,\eta\right) = \left\{ \begin{array}{ll}
\tilde{z}(m) + \int_{\mu}^{\ell} \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) \mathcal{W}_{\mathcal{V}}^{\mu-\kappa,\mu}\left(-\rho(\Phi(\ell) - \Phi(\eta))^{\mu-\kappa}\right) \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) d\eta & , \ell \in [\nu,\sigma], \\
\tilde{z}(\ell) & , \ell \in [\sigma,\nu].
\end{array} \right.
\]

Note that, when \( \ell \in [\sigma,\nu] \), we have

\[|\tilde{z}(\ell) - \tilde{z}(\ell)| = 0.\]

On the other side, for each \( \ell \in [\nu,\sigma] \) we obtain

\[
|\tilde{z}(\ell) - \tilde{z}(\ell)| \leq |\tilde{z}(\ell) - \tilde{z}(m)| + \int_{\mu}^{\ell} \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) \mathcal{W}_{\mathcal{V}}^{\mu-\kappa,\mu}\left(-\rho(\Phi(\ell) - \Phi(\eta))^{\mu-\kappa}\right) \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) d\eta.
\]

Using (H2) and Lemma 4.3, we can arrive at

(12)

\[
|\tilde{z}(\ell) - \tilde{z}(\ell)| \leq \varepsilon \mathcal{M}_{\mathcal{V}}\left((\Phi(\ell) - \Phi(\mu))^{\mu}\right) + \mathcal{Q} \int_{\mu}^{\ell} \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) \mathcal{W}_{\mathcal{V}}^{\mu-\kappa,\mu}\left(-\rho(\Phi(\ell) - \Phi(\eta))^{\mu-\kappa}\right) \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) d\eta.
\]

Now, for each \( b \in C([\nu,\sigma],\mathbb{R}_{+}) \), we define an operator \( S : C([m-\sigma,n],\mathbb{R}_{+}) \to C([m-\sigma,n],\mathbb{R}_{+}) \) by

(13)

\[
Sb(\ell) = \left\{ \begin{array}{ll}
\varepsilon \mathcal{M}_{\mathcal{V}}\left((\Phi(\ell) - \Phi(\mu))^{\mu}\right) + \mathcal{Q} \int_{\mu}^{\ell} \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) \mathcal{W}_{\mathcal{V}}^{\mu-\kappa,\mu}\left(-\rho(\Phi(\ell) - \Phi(\eta))^{\mu-\kappa}\right) \mathcal{W}_{\mathcal{V}}^{\mu}\left(\ell,\nu;\Phi,\eta\right) d\eta & , \ell \in [\nu,\sigma], \\
0 & , \ell \in [m-\sigma,m].
\end{array} \right.
\]

We prove that \( S \) is a Picard operator. Let \( b_1, b_2 \in C([m-\sigma,n],\mathbb{R}_{+}) \). Then,

\[|Sb_2(\ell) - Sb_1(\ell)| = 0, \ell \in [m-\sigma,m].\]

Now, for any \( \ell \in [m,n] \), it follows from (H1) and (H2) that

\[
|Sb_2(\ell) - Sb_1(\ell)| \leq \frac{2\mathcal{Q}}{b} \left[ \mathcal{M}_{\mathcal{V}}\left((\Phi(\ell) - \Phi(\mu))^{\mu}\right) - 1 \right] b_2 - b_1 \|_{\mathcal{V},\mathcal{K},\mu}.
\]

which leads to

\[
\|Sb_2 - Sb_1\|_{\mathcal{V},\mathcal{K},\mu} \leq \frac{2\mathcal{Q}}{b} \|b_2 - b_1\|_{\mathcal{V},\mathcal{K},\mu}.
\]
Choosing $\beta > 0$ such that $\frac{2L_Q}{\beta} < 1$, we have that $S$ is a contraction with respect to Bielecki’s norm $\| \cdot \|_{\mathcal{Y}, \mathcal{B}, \mu}$. According to Banach fixed point theorem, we deduce that $S$ is a Picard operator and $F_S = \{b^*\}$. Thus

\begin{equation}
(14) \quad b^*(\ell) = \epsilon M_\mu \left((\Phi(\ell) - \Phi(m))^\mu\right) + L_Q \int^\ell_m \frac{\mathcal{W}_\mu(\ell, \eta)}{\Gamma(\mu)} \left((b^*(\eta) + b^*(\eta))^\mu\right) d\eta, \, \ell \in [m, n].
\end{equation}

Next, we show that $b^*$ is increasing. For this end, let $\ell_1, \ell_2 \in [m - \sigma, n]$. If $\ell_1, \ell_2 \in [m - \sigma, m]$ with $\ell_1 < \ell_2$, then $b^*(\ell_2) - b^*(\ell_1) = 0$, and if $\ell_1, \ell_2 \in [m, n]$ provided that $\ell_1 < \ell_2$. Denote $\nu = \min_{\eta \in [m, n]} (b^*(\eta) + b^*(\eta))$. Then,

\[
b^*(\ell_2) - b^*(\ell_1) = \epsilon \left(M_\mu \left((\Phi(\ell_2) - \Phi(m))^\mu\right) - M_\mu \left((\Phi(\ell_1) - \Phi(m))^\mu\right)\right)
+ \frac{L_Q}{\Gamma(\mu)} \int^{\ell_2}_{\ell_1} \mathcal{W}_\mu(\ell, \eta) \left(b^*(\eta) + b^*(\eta)^\mu\right) d\eta
+ \frac{L_Q}{\Gamma(\mu + 1)} \left((\Phi(\ell_2) - \Phi(m))^\mu - (\Phi(\ell_1) - \Phi(m))^\mu\right)
\geq \epsilon \left(M_\mu \left((\Phi(\ell_2) - \Phi(m))^\mu\right) - M_\mu \left((\Phi(\ell_1) - \Phi(m))^\mu\right)\right)
+ \frac{L_Q}{\Gamma(\mu + 1)} \left((\Phi(\ell_2) - \Phi(m))^\mu - (\Phi(\ell_1) - \Phi(m))^\mu\right)
\geq \epsilon \left(M_\mu \left((\Phi(\ell_2) - \Phi(m))^\mu\right) - M_\mu \left((\Phi(\ell_1) - \Phi(m))^\mu\right)\right)
+ \frac{L_Q}{\Gamma(\mu + 1)} \left((\Phi(\ell_2) - \Phi(m))^\mu - (\Phi(\ell_1) - \Phi(m))^\mu\right)
> 0.
\]

This means that $b^*$ is increasing. Keeping in mind (H1) we arrive to $b^*(f(\ell)) \leq b^*(\ell), \, \ell \in [m, n]$.

Therefore, Eq. (14) reduces to

\[
b^*(\ell) \leq \epsilon M_\mu \left((\Phi(\ell) - \Phi(m))^\mu\right) + \frac{2L_Q}{\Gamma(\mu)} \int^\ell_m \mathcal{W}_\mu(\ell, \eta) b^*(\eta) d\eta, \, \ell \in [m, n].
\]

Applying Corollary 2.12 (the $\Phi$-fractional Gronwall’s inequality Eq. (5)), to above inequality with $c_1(\ell) = b^*(\ell), \, c_2(\ell) = \epsilon M_\mu \left((\Phi(\ell) - \Phi(m))^\mu\right)$ and $c_3(\ell) = \frac{2L_Q}{\Gamma(\mu)}$. Since $c_2(\ell)$ is nondecreasing function on $\Omega$, we conclude that

\[
b^*(\ell) \leq \epsilon M_\mu \left(2L_Q \left(\Phi(\ell) - \Phi(m))^\mu\right)\right)
\leq \epsilon M_\mu \left(2L_Q \left(\Phi(n) - \Phi(m))^\mu\right)\right)
= c_{M_\mu} \epsilon M_\mu \left(\Phi(\ell) - \Phi(m))^\mu\right), \quad \ell \in \Omega,
\]

where $c_{M_\mu} = M_\mu \left(2L_Q \left(\Phi(n) - \Phi(m))^\mu\right)\right)$.

In particular, if $b = [\bar{b} - \bar{b}]$, from (12), $b(\ell) \leq Sb(\ell)$ and applying the abstract Gronwall lemma (Lemma 2.14) we obtain $b(\ell) \leq b^*(\ell)$, where $S$ is an increasing Picard operator. Combining this fact with (15), it yields that

\[
\left|\bar{b}(\ell) - \bar{b}(\ell)\right| = 0, \quad \ell \in [m - \sigma, m],
\left|\bar{b}(\ell) - \bar{b}(\ell)\right| \leq c_{M_\mu} \epsilon M_\mu \left(\Phi(\ell) - \Phi(m))^\mu\right), \quad \ell \in [m, n].
\]

Thus, the problem (1) is UHML stable. \qed
5. An Example

In this fragment, we present an example where we apply both of Theorems 3.2 and 4.4 to some particular cases.

Example 5.1. Let us consider problem (1) with specific data:

\[ \mu = 0.5, \ k = 0.45, \ \rho = 1, \ f(\ell) = \ell - \sigma. \]

In order to illustrate Theorems 3.2 and 4.4, we take

\[ Q(\ell, \psi(\ell), \psi(f(\ell))) = \frac{\sin \ell}{2} \left( \psi(\ell) + \sqrt{1 + \psi^2(\ell)} \right) + \sin(\ell - \sigma), \]

in (1). Obviously, the hypotheses (H1) and (H2) hold with \( L_Q = 1 \). It follows from Theorem 3.2 that the problem (1) with the data (16) and (17) has a unique solution in \( C([m - \sigma, n], \mathbb{R}) \cap C([m, n], \mathbb{R}) \). Also, by Theorem 4.4 the corresponding problem is UHML stable. Moreover, the integral representation of the aforementioned problem is given by

\[ \psi(\ell) = \begin{cases} \alpha(m) + \int_0^m \sqrt{\Phi} \psi^{0.5}(\ell, \eta)\mathcal{M}_{0.05,0.5}(-\Phi(\ell) - \Phi(\eta))^{0.05} \times \left( \frac{\sin \eta}{2} \left( \Psi(\eta) + \sqrt{1 + \Psi^2(\eta)} \right) + \sin(\eta - \sigma) \right) d\eta, & \ell \in [m, n], \\ \alpha(\ell), & \ell \in [m - \sigma, m]. \end{cases} \]

Remark 5.2. It is worth noting that in the previous example \( \beta \) can be determined according to the assumptions of Theorem 3.2. for example, we can choose \( \beta = L_Q + 1 \).

6. Conclusion

In this paper, we studied some basic problems such as uniqueness and Ulam-Hyers Mittag-Leffler (UHML) stability of solutions for a new class of multi-terms fractional time-delay differential equations involving \( \Phi \)-Caputo fractional derivative. The results are obtained by using the Banach contraction principle coupled with the \( \Phi \)-fractional Bielecki-type norm, the \( \Phi \)-fractional Gronwall type inequality, and the Picard operator (PO) technique combined with abstract Gronwall lemma. An example is also given to illustrate the effectiveness of our main results.

References

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