FUNCTION SPACES OVER PRODUCTS WITH ORDINALS

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Abstract. We are concerned with the question of when a homeomorphism between $C_p(X \times \tau)$ and $C_p(Y \times \tau)$ implies the existence of a homeomorphism between $C_p(X \times \tau')$ and $C_p(Y \times \tau')$, where $\tau$ and $\tau'$ are spaces of ordinals with some additional desired properties.

1. Introduction

We will be concerned with the following general problem.

Problem. Let $C_p(X \times Z)$ be homeomorphic to $C_p(Y \times Z)$. Suppose that $Z'$ is a subspace or a superspace of $Z$. What additional conditions on $X, Y, Z, Z'$ guarantee that $C_p(X \times Z')$ is homeomorphic to $C_p(Y \times Z')$? That $X$ is homeomorphic to $Y$?

Note that for any $X$ and $Y$ we can find $Z$ such that $X \times Z$ is homeomorphic to $Y \times Z$ (in particular, $Z = X^\omega \times Y^\omega$ works). Therefore, it is natural to require that candidates for $Z$ in the above problems have rigid structures. There are many interesting results around this problem. Since most of $Z$’s in this study will be ordered, we would like to mention O. Okunev’s result (see, for example, [1, the argument of Theorem 0.6.2]) that if $C_p(X \times \mathbb{R})$ and $C_p(Y \times \mathbb{R})$ are homeomorphic, then so are $C_p(X)^\omega$ and $C_p(Y)^\omega$. In this work, we will primarily target cases when $Z$ is a space of ordinals. Our major restriction on the topology of a space will be a network size. We will show that under certain restrictions on a network size of $X$ and $Y$, a homeomorphism between $C_p(X \times \tau)$ and $C_p(Y \times \tau)$, where $\tau$ is regular and uncountable, implies the existence of a homeomorphism between $C_p(X \times \lambda)$ and $C_p(Y \times \lambda)$ for some $\lambda < \tau$. We also show in Theorem 2.10 that for compact $X$ and $Y$, if $C_p(X \times \omega_1)$ and $C_p(Y \times \omega_1)$ are homeomorphic, then so are $C_p(X \times (\omega_1+1))$ and $C_p(Y \times (\omega_1+1))$. These statements lead to some natural generalizations as well as questions for possible further exploration.

Recall that a network of $X$ is a family of sets with the same properties as a basis except that elements of the family need not be open. By $A(\tau)$ we denote the
Alexandroff single-point compactification of a discrete space of size $\tau$ with $\infty$ being its only non-isolated point. All spaces under consideration are Tychonov. In notation and terminology of general topological nature we follow [3]. To distinguish ordered pairs from open intervals, we will denote the former by $(a, b)$ and the latter by $\langle a, b \rangle$. We also refer the reader to [1] for general facts and notations involving function spaces endowed with the topology of point-wise convergence.

2. Study

In our arguments we will use the following folklore facts for which we outline proofs for convenience.

**Proposition 2.1.** (Folklore)

(1) Let an ordinal $\tau$ have cofinality greater than the density of an infinite space $X$ and let $f \in C_p(X \times \tau) \cup C_p(X \times (\tau + 1))$. Then there exists $\lambda < \tau$ such that $f|_{\{x\} \times [\lambda, \tau)}$ is constant for every $x \in X$.

(2) Let $X$ be compact, $\tau$ an uncountable regular ordinal, and $f \in C_p(X \times \tau)$. Then, there exists $\lambda < \tau$ such that $f$ is constant on $\{x\} \times [\lambda, \tau)$ for every $x \in X$.

**Proof.** To see why (1) is true let $D = \{d_\alpha : \alpha < d(X)\}$ be dense in $X$, where $d(X)$ is the density of $X$. For each $\alpha < d(X)$, fix $\tau_\alpha < \tau$ such that $f$ is constant on $\{d_\alpha\} \times [\tau_\alpha, \tau)$. Such a $\tau_\alpha$ exists due to countable compactness of $\tau$. Since $d(X) < cf(\tau)$, we conclude that $\lambda = \sup\{\tau_\alpha : \alpha < d(X)\} < \tau$. By continuity of $f$ and density of $D$, $\lambda$ is as desired.

For (2), assume the contrary and for each $\alpha < \tau$, pick $x_\alpha \in X$ and distinct $\beta_\alpha, \gamma_\alpha > \alpha$ such that $f(x_\alpha, \beta_\alpha) \neq f(x_\alpha, \gamma_\alpha)$. For cardinality reason, we may assume that there exist a natural number $N$ and a rational number $q$ such that $f(x_\alpha, \beta_\alpha)$ and $f(x_\alpha, \gamma_\alpha)$ are on the left and right sides of $q$, respectively, and are at distance at least $1/N$ from $q$. By compactness, there exists $(x, \tau)$ a complete accumulation point for $\{\langle x_\alpha, \beta_\alpha \rangle : \alpha < \tau\}$ in $X \times (\tau + 1)$. By the product topology of $X \times (\tau + 1)$, $(x, \tau)$ is a complete accumulation point for $\{\langle x_\alpha, \gamma_\alpha \rangle : \alpha < \tau\}$ in $X \times (\tau + 1)$ too. Since $f$ must be continuously extendable to $\tilde{f}$ over $X \times (\tau + 1)$, we arrive at the fact that $\tilde{f}(x, \tau)$ must be strictly to the left and strictly to the right of $q$. \qed

**Theorem 2.2.** Let $X$ and $Y$ have networks of cardinality less than $\tau$, where $\tau$ is regular and uncountable. Let $C_p(X \times \tau)$ be homeomorphic to $C_p(Y \times \tau)$. Then $C_p(X \times \lambda)$ is homeomorphic to $C_p(Y \times \lambda)$ for some non-zero $\lambda < \tau$. 

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Proof. Fix a homeomorphism $H : C_p(X \times \tau) \rightarrow C_p(Y \times \tau)$. Next for $\alpha < \tau$, define $F(X, \alpha)$ as follows: $f \in F(X, \alpha)$ if and only if $f$ is constant on $\{x\} \times [\alpha, \tau)$ for every $x \in X$. Similarly, we define $F(Y, \alpha)$.

Claim 1. $F(X, \mu)$ is homeomorphic to $C_p(X \times (\mu + 1))$.

To prove the claim, for each $f \in C_p(X \times (\mu + 1))$, let $f_{\mu} \in F(X, \mu)$ be the function such that $f(x, \alpha) = f_{\mu}(x, \alpha)$ for each $(x, \alpha) \in X \times (\mu + 1)$. Clearly, the map $M : C_p(X \times (\mu + 1)) \rightarrow F(X, \mu)$ defined by $M(f) = f_{\mu}$ is a homeomorphism.

Claim 2. $F(X, \alpha)$ is closed in $C_p(X \times \tau)$ for any $\alpha < \tau$.

To prove the claim fix $f \in C_p(X \times \tau) \setminus F(X, \alpha)$. By the description of $F(X, \alpha)$, there exist $x \in X$ and $\lambda \in (\alpha, \tau)$ such that $f(x, \lambda) \neq f(x, \alpha)$. Put $\epsilon = |f(x, \lambda) - f(x, \alpha)|$. Then the following open set contains $f$ and misses $F(X, \alpha)$:

$$\{g : g(x, \lambda) \in (f(x, \lambda) - \epsilon/3, f(x, \lambda) + \epsilon/3))\} \cap \{g : g(x, \alpha) \in (f(x, \alpha) - \epsilon/3, f(x, \alpha) + \epsilon/3))\}$$

The claim is proved.

Claim 3. Let $\alpha < \tau$. Then $H(F(X, \alpha)) \subset F(Y, \alpha')$ for some $\alpha' < \tau$.

To prove the claim, recall that $X$ has netweight less than $\tau$ and $\tau$ is regular. Therefore, $X \times (\alpha + 1)$ has netweight less than $\tau$. Therefore, $C_p(X \times (\alpha + 1))$ has density less than $\tau$. By Claim 1, $F(X, \alpha)$ has density less than $\tau$ too. By Proposition 2.1, Part (1), any subset of $C_p(Y \times \tau)$ of density less than $\tau$ is a subset of $F(Y, \alpha')$ for some $\alpha' < \tau$.

By Claim 3, we can create a sequence $\alpha_1 < \alpha_2 < ...$ with the following two properties:

P1: $H(F(X, \alpha_n)) \subset F(Y, \alpha_{n+1})$ if $n$ is odd.
P2: $H^{-1}(F(Y, \alpha_n)) \subset F(X, \alpha_{n+1})$ if $n$ is even.

Put $\mu = \sup\{\alpha_n\}_n$. By Claim 1, it remains to show that $H(F(X, \mu)) = F(Y, \mu)$. Since $\{\alpha_n\}_n$ is an increasing sequence we conclude that that $H(\bigcup_n F(\alpha_n, X)) = \bigcup_n F(\alpha_n, Y)$. Since $H$ is a homeomorphism, $H(\bigcup_n F(\alpha_n, X)) = \bigcup_n F(\alpha_n, Y)$. It remains to show that $\bigcup_n F(\alpha_n, X) = F(\mu, X)$ and $\bigcup_n F(\alpha_n, Y) = F(\mu, Y)$. We will prove the former equality. Since $F(\alpha_n, X)$ is a subset of $F(\mu, X)$ and the latter is closed by Claim 2, we conclude that $\bigcup_n F(\alpha_n, X) \subset F(\mu, X)$. To show the reverse inclusion, fix $f \in F(\mu, X)$ and a standard open neighborhood $U$ of $f$. We may assume that $U = U_1 \cap U_2$, where $U_1 = U_1(p_1, ..., p_n, B_1, ..., B_n)$ and $U_2 = U_2(q_1, ..., q_k, O_1, ..., O_k)$ for some $p_1, ..., p_n \in X \times \mu$ and $q_1, ..., q_k \in (X \times \tau) \setminus (X \times \mu)$. To show that $U$ meets $\bigcup_n F(\alpha_n, X)$ let $N$ be such that $p_1, ..., p_n \in X \times \alpha_N$. Define $g$ by letting $g(x, \beta) = f(x, \mu)$ for every $\beta > \alpha_N$ and $g(x, \beta) = f(x, \beta)$ for every $\beta \leq \alpha_N$. Clearly, $g \in U \cap F(\alpha_n, X)$. 

\[ \square \]
The following is a by-product of the argument of Theorem 2.2.

**Theorem 2.3.** Let $X$ and $Y$ have networks of cardinality less than $\tau$, where $\tau$ is regular and uncountable. Let $C_p(X \times (\tau + 1))$ be homeomorphic to $C_p(Y \times (\tau + 1))$. Then $C_p(X \times \lambda)$ is homeomorphic to $C_p(Y \times \lambda)$ for some non-zero $\lambda < \tau$.

One may wonder if a homeomorphism between $C_p(X \times \kappa)$ and $C_p(Y \times \kappa)$ for some ordinal $\kappa$ implies the existence of a homeomorphism between $C_p(X \times \lambda)$ and $C_p(Y \times \lambda)$ for some $0 < \lambda < \kappa$. It is, however, not the case. Note that the function spaces over any discrete space $X$ is homeomorphic to $\mathbb{R}^{|X|}$. Therefore, $C_p(2 \times \omega)$ and $C_p(3 \times \omega)$ are homeomorphic but $C_p(2 \times n)$ and $C_p(3 \times n)$ are not homeomorphic for any positive integer $n$.

For future reference let us isolate the case of $\tau = \omega_1$ of Theorem 2.2 into a corollary.

**Corollary 2.4.** Let $X$ and $Y$ have countable networks. If $C_p(X \times \omega_1)$ is homeomorphic to $C_p(Y \times \omega_1)$, then $C_p(X \times \lambda)$ is homeomorphic to $C_p(Y \times \lambda)$ for some non-zero countable ordinal $\lambda$.

The above corollary prompts the following question.

**Question 2.5.** Let $C_p(X \times (\omega + 1))$ and $C_p(Y \times (\omega + 1))$ be homeomorphic. Is it true that $C_p(X \times \mathbb{Q})$ and $C_p(Y \times \mathbb{Q})$ are homeomorphic? What additional conditions on $X$ and $Y$ guarantee an affirmative answer?

An argument similar to that in Theorem 2.2 leads to our next statement. Since the argument of Theorem 2.2 needs only notation changes for our next result, we will provide the strategy without repeating the proof details of Theorem 2.2.

**Theorem 2.6.** Let $X$ and $Y$ have countable networks and let $C_p(X \times A(\tau))$ be homeomorphic to $C_p(Y \times A(\tau))$ for some infinite cardinal $\tau$. Then $C_p(X \times A(\omega))$ is homeomorphic to $C_p(Y \times A(\omega))$.

**Proof.** We may assume that $\tau$ is uncountable. Fix a homeomorphism $\phi : C_p(X \times A(\tau)) \to C_p(Y \times A(\tau))$. Given $A \subset \tau$, define $F(A, X) \subset C_p(X \times A(\tau))$ as follows:

**Definition of $F(A, X)$:** $f \in C_p(X \times A(\tau))$ is in $F(A, X)$ if and only if $f(x, \beta) = f(x, \infty)$ for every $\beta \notin A$ and every $x \in X$.

**Claim 1.** For every $f \in C_p(X \times A(\tau))$ there exists a countable $A \subset \tau$ such that $f \in F(A, X)$.
To prove the claim, assume the contrary. Then there exists a subset \( \{ \langle x_\alpha, \lambda_\alpha \rangle : \alpha < \omega_1 \} \subset X \times A(\tau) \) such that \( f(x_\alpha, \lambda_\alpha) \neq f(x_\alpha, \infty) \) for each \( \alpha < \omega_1 \), and \( \lambda_\alpha \neq \lambda_\beta \) for distinct \( \alpha, \beta < \omega_1 \). Without loss of generality we may assume that there exist \( p, q \in \mathbb{Q} \) such that \( f(x_\alpha, \lambda_\alpha) < p < q < f(x_\alpha, \infty) \) for all \( \alpha < \omega_1 \). Since \( X \) is Lindelöf, there exists \( \langle x, \infty \rangle \) a complete accumulation point of \( \{ \langle x_\alpha, \infty \rangle : \alpha < \omega_1 \} \), and therefore, \( f(x, \infty) \geq q \). By the product topology, \( \langle x, \infty \rangle \) is also a complete accumulation point for \( \{ \langle x_\alpha, \lambda \rangle : \alpha < \omega_1 \} \), and therefore, \( f(x, \infty) \leq p \). We have arrived at a contradiction with \( p < q \), which proves the claim.

The next two claims (as Claim 1) are proved using the same arguments as the corresponding claims of Theorem 2.2.

**Claim 2.** Let \( K \) be an infinite subset of \( \tau \) of cardinality \( \kappa \). Then \( F(X, K) \) is homeomorphic to \( C_p(X \times A(\kappa)) \).

**Claim 3.** \( F(X, A) \) is closed in \( C_p(X \times A(\tau)) \) for any \( A \subset \tau \).

By Claim 2, it remains to find a countable \( A \subset \tau \) such that \( \phi(F(X, A)) = F(Y, A) \). Inductively, we will define \( A_n \) that will be building blocks for our desired \( A \).

**Step 0:** \( A_0 = \omega \).

**Assumption:** \( A_k \) is defined for all \( k = 0, \ldots, n - 1 \) and \( A_m \subset A_k \) if \( m < k \).

**Step n:** Our definition of \( A_n \) will depend on whether \( n \) is odd or even.

Assume that \( n \) is odd. Since \( X \) has a countable network, by Claim 2, \( F(A_{n-1}, X) \) is separable. By Claims 1 and 3, there exists a countable \( A_n \subset \tau \) that contains \( A_{n-1} \) such that \( \phi(F(A_{n-1}, X)) \subset F(A_n, Y) \).

Assume that \( n \) is even. Similarly, we choose a countable \( A_n \subset \tau \) such that \( A_n \) contains \( A_{n-1} \) and \( \phi^{-1}(F(A_{n-1}, Y)) \subset F(A_n, X) \).

Our induction construction is complete. Put \( A = \bigcup_n A_n \). By Claim 2, it remains to show that \( \phi(F(A, X)) = F(A, Y) \). The argument is identical to that in Theorem 2.2. \( \square \)

A more general statement is as follows.

**Theorem 2.7.** Let \( \tau \) be an infinite cardinal. Let \( X \) and \( Y \) have networks of cardinality \( \tau < \kappa \). If \( C_p(X \times A(\kappa)) \) is homeomorphic to \( C_p(Y \times A(\kappa)) \), then \( C_p(X \times A(\tau)) \) is homeomorphic to \( C_p(Y \times A(\tau)) \).

It would be interesting to consider scenarios in a direction somewhat opposite to Theorems 2.2 - 2.7, namely, in a direction aligned with the following question.
**Question 2.8.** Assume that $C_p(X \times \alpha)$ is homeomorphic to $C_p(Y \times \alpha)$ for every $\alpha < \omega_1$. Is it true that $C_p(X \times \omega_1)$ is homeomorphic to $C_p(Y \times \omega_1)$? What if $\omega_1$ is replaced by an arbitrary fixed ordinal?

Note that there are many examples of pairs $X$ and $Y$ that satisfy the assumption of Question 2.8. For example, it was shown by van Mill [5] that any two non-locally compact subspaces of rational numbers have homeomorphic $C_p$’s. This result was later generalized to any infinite non-discrete subspaces of rationals by Dobrowolski, Marciszewski, and Mogilski [2]. In connection with these nice facts and the statement of Theorem 2.4, the following special case of Question 2.8 might be of interest:

**Question 2.9.** Let $X$ and $Y$ be infinite non-discrete subspaces of rationals. Is it true that $C_p(X \times \omega_1)$ is homeomorphic to $C_p(Y \times \omega_1)$? What if $\omega_1$ is replaced by $\omega_1 + 1$ or any other larger ordinal?

Next is a positive result in the scenario ”from smaller to larger” ordinal.

**Theorem 2.10.** Let $X$ and $Y$ be compact spaces. If $C_p(X \times \omega_1)$ is homeomorphic to $C_p(Y \times \omega_1)$, then $C_p(X \times (\omega_1 + 1))$ is homeomorphic to $C_p(Y \times (\omega_1 + 1))$.

*Proof.* Let $H : C_p(X \times \omega_1) \to C_p(Y \times \omega_1)$ be a homeomorphism. Let $H^* : C_p(X \times (\omega_1 + 1)) \to C_p(Y \times (\omega_1 + 1))$ be defined by letting $H^*(\tilde{f}) = \tilde{g}$, where $\tilde{f}$ and $\tilde{g}$ are the continuous extensions of $f \in C_p(X \times \omega_1)$ and $g \in C_p(Y \times \omega_1)$ and $H(f) = g$. Clearly, $H^*$ is a bijection. Note that the inverse of $H^*$ has the analogous definition based on continuity of $H^{-1}$. Therefore, to prove that $H^*$ is a homeomorphism, it suffices to show that $H^*$ is continuous. To prove this, let us agree to refer to elements and subsets of $C_p(X \times (\omega_1 + 1))$ and $C_p(Y \times (\omega_1 + 1))$ by $\tilde{f}$ and $\tilde{A}$, where $f$ and $A$ are their counterparts in $C_p(X \times \omega_1)$ or $C_p(Y \times \omega_1)$.

Fix $\tilde{P} \subset C_p(X \times (\omega_1 + 1))$ and $\tilde{p}$ in the closure of $\tilde{P}$. We need to show that $H^*(\tilde{p})$ is in the closure of $H^*(\tilde{P})$. Since $C_p(X \times (\omega_1 + 1))$ has countable tightness, we may assume that $P$ is countable. By Part (2) of Proposition 2.1 and smallness of $P$ there exists $\lambda < \omega_1$ such that $\tilde{f}$ is constant on $\{x\} \times [\lambda, \omega_1]$ for each $x \in X$ and $\tilde{f} \in \tilde{P}$ in the closure of $P$. By continuity of $H$, $s = H(\tilde{p})$ is in the closure of $\tilde{S}$. It remains to show that $\tilde{s}$ is in the closure of $S$.

Let $W$ be an open basic neighborhood of $\tilde{s}$. We need to show that $W$ meets $\tilde{S}$. Since $s$ is in the closure of $S$, it suffices to show that $W = \{\tilde{f} : \tilde{f}(c, \omega_1) \in (\tilde{s}(c, \omega_1) - \epsilon, \tilde{s}(c, \omega_1) + \epsilon)\} \cap S$ by an infinite subset, where $c \in Y$ and $\epsilon > 0$ are arbitrary and fixed. By Part (2) of Proposition 2.1, we can fix a $\mu < \omega_1$ such that $f$ is constant on $\{y\} \times [\mu, \omega_1]$ for each $f \in S \cup \{s\}$ and $y \in Y$. Since
\[ s(c, \mu) \in (\tilde{s}(c, \omega_1) - \epsilon, \tilde{s}(c, \omega_1) + \epsilon), \] we conclude that 
\[ V = \{ f \in C_p(Y \times \omega_1) : 
(f(c, \mu) \in (\tilde{s}(c, \omega_1) - \epsilon, \tilde{s}(c, \omega_1) + \epsilon) \} \]

is a neighborhood of \( s \). Since \( s \) is in the closure of \( S \), the intersection of \( V \) and \( S \) is infinite. Pick any \( f \in V \cap S \). Since
\[ f(c, \omega_1) \in (\tilde{s}(c, \omega_1) - \epsilon, \tilde{s}(c, \omega_1) + \epsilon), \]
we conclude that \( \tilde{f}(c, \omega_1) \in (\tilde{s}(c, \omega_1) - \epsilon, \tilde{s}(c, \omega_1) + \epsilon) \). Therefore, \( \tilde{V} \cap \tilde{S} \subset \tilde{W} \), which completes the proof. \( \square \)

A few natural questions arise from the above result.

**Question 2.11.** Let \( X \) and \( Y \) be compact spaces. Suppose that \( C_p(X \times (\omega_1+1)) \) is homeomorphic to \( C_p(Y \times (\omega_1+1)) \). Is it true that \( C_p(X \times \omega_1) \) is homeomorphic to \( C_p(Y \times \omega_1) \) ?

**Question 2.12.** Are there spaces \( X \) and \( Y \) (not necessarily compact) such that \( C_p(X \times (\omega_1+1)) \) is homeomorphic to \( C_p(Y \times (\omega_1+1)) \) but \( C_p(X \times \omega_1) \) is not homeomorphic to \( C_p(Y \times \omega_1) \)? Or, such that \( C_p(X \times \omega_1) \) is homeomorphic to \( C_p(Y \times \omega_1) \) but \( C_p(X \times (\omega_1+1)) \) is not homeomorphic to \( C_p(Y \times (\omega_1+1)) \)?

The proof of Theorem 2.10 works for any ordinal \( \tau \) of uncountable cofinality without changes.

**Theorem 2.13.** Let \( X \) and \( Y \) be compact spaces and let an ordinal \( \tau \) have uncountable cofinality. If \( C_p(X \times \tau) \) is homeomorphic to \( C_p(Y \times \tau) \), then \( C_p(X \times (\tau + 1)) \) is homeomorphic to \( C_p(Y \times (\tau + 1)) \).

We would like to finish with two general questions related to our study that are also particular cases of the general problem that motivated this work.

**Question 2.14.** Assume that \( C_p(X \times \tau) \) and \( C_p(Y \times \tau) \) are homeomorphic for some ordinal \( \tau > 1 \). What additional conditions on \( X, Y, \tau \) guarantee that \( C_p(X) \) and \( C_p(Y) \) homeomorphic? That \( X \) and \( Y \) are homeomorphic?

**Question 2.15.** Is there an example of \( X \) and \( Y \) with homeomorphic \( C_p \)'s such that \( C_p(X \times \tau) \) and \( C_p(Y \times \tau) \) are not homeomorphic for some \( \tau \)?
In connection with this question, a notable pair to test is the set of reals and a closed segment. It is due to Gulko and Khmyleva [4] that $C_p(\mathbb{R})$ and $C_p([0,1])$ are homeomorphic. It might be interesting to know if there is an ordinal $\lambda$ such that $C_p(\mathbb{R} \times \lambda)$ and $C_p([0,1] \times \lambda)$ are not homeomorphic. For an overview of non-homeomorphic spaces with homeomorphic $C_p$’s, which may address Question 2.15, we refer the reader to [6].

**REFERENCES**


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