OSCILLATORY BEHAVIOR OF NABLA DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. Consider the first-order nabla dynamic equations
\[ x^\nabla (t) - \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \in [t_0, \infty)_T, \]
where \( p_i(t) (i = 1, 2, \ldots, m) \in C_{ld}([t_0, \infty)_T, \mathbb{R}^+), \tau_i(t) (i = 1, 2, \ldots, m) \in C_{ld}([t_0, \infty)_T, T) \) and \( \tau_i(t) \geq t \). Under the assumption that the \( \tau_i(t) (i = 1, 2, \ldots, m) \) are not necessarily monotone, we present new sufficient conditions for the oscillation of first-order nabla dynamic equations on time scales. We can say that this paper is the first in the literature in obtaining the oscillations of the solutions of the above equation. An example illustrating the results is also given.

Keywords: Nabla dynamic equations, time scale, non-monotone, oscillatory solutions.

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1. Introduction

Consider the first-order nabla dynamic equations
\[(1.1) \quad x^\nabla (t) - p(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_T \]
and
\[(1.2) \quad x^\nabla (t) - \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \in [t_0, \infty)_T, \]
where \( T \) is a time scale unbounded above with \( t_0 \in T \), \( p_i(t) (i = 1, 2, \ldots, m) \) are ld-continuous and nonnegative, the advanced function \( \tau_i : T \rightarrow T \) are not necessarily monotone and satisfies
\[(1.3) \quad \tau_i(t) \geq t \quad \text{for all } t \in T, \]
and \( \sup T = \infty \).

In this paper, our aim is to obtain some oscillation criteria for first order nabla dynamic equations on time scales, which contains well-known criteria for advanced differential equations and advanced difference equations as special cases. The problem of establishing sufficient conditions for the oscillation of all solutions to the dynamic equations (1.1) and (1.2) when \( T = \mathbb{R} \) or \( T = \mathbb{Z} \) have been the subject of many investigations, see for example [1-16] and the references cited therein. First, we give a short review on the time scales calculus extracted from [1] and [2]. If \( T \) has a right-scattered minimum \( m \), define \( T_\kappa := T - \{ m \} \); otherwise, set \( T_\kappa = T \).

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The backwards graininess \( \nu : \mathbb{T}_\kappa \to \mathbb{R}^+_0 \) is defined by \( \nu(t) = t - \rho(t) \). If \( \mathbb{T} = \mathbb{R} \), then \( x^{\nabla}(t) = x'(t) \) (the usual derivative), while if \( \mathbb{T} = \mathbb{Z} \), then \( x^{\nabla}(t) = \nabla x(t) = x(t) - x(t - 1) \) (the usual backwards difference).

A function \( p \in C_{ld}(\mathbb{T}, \mathbb{C}) \) is called \( \nu \)-regressive if \( 1 - p \nu \neq 0 \) on \( \mathbb{T}_\kappa \), and \( p \in C_{ld}(\mathbb{T}, \mathbb{C}) \) is called positively regressive if \( 1 - p \nu > 0 \) on \( \mathbb{T}_\kappa \). The set of regressive functions are denoted by \( \mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}; \mathbb{R}) \).

We know from [1, Theorem 8.48] that: assume \( a, b \in \mathbb{T} \) and \( f : \mathbb{T} \to \mathbb{R} \) is \( ld \)-continuous. If \( \mathbb{T} = \mathbb{R} \), then
\[
\int_a^b f(t) \nabla t = \int_a^b f(t) dt,
\]
where the integral on the right is the Riemann integral from calculus. If \( \mathbb{T} = \mathbb{Z} \), then
\[
\int_a^b f(t) \nabla t = \sum_{t=a+1}^b f(t).
\]

A function \( x : \mathbb{T} \to \mathbb{R} \) is called a solution of the equation (1.1), if \( x(t) \) is nabla differentiable for \( t \in \mathbb{T}_\kappa \) and satisfies equation (1.1) for \( t \in \mathbb{T}_\kappa \). We say that a solution \( x \) of equation (1.1) has a generalized zero at \( t \) if \( x(t) = 0 \) or if \( \nu(t) > 0 \) and \( x(\rho(t))x(t) < 0 \). Let \( \sup \mathbb{T} = \infty \) and then a nontrivial solution \( x \) of equation (1.1) is called oscillatory on \( [t, \infty) \) if it has arbitrarily large generalized zeros in \( [t, \infty) \).

Next, let us recall some known oscillation results on this subject. For \( \mathbb{T} = \mathbb{R} \) and \( \mathbb{T} = \mathbb{Z} \), equation (1.1) reduce to
\[
(1.4) \quad x'(t) - p(t)x(\tau(t)) = 0, \quad t \in \mathbb{R}
\]
and
\[
(1.5) \quad \nabla x(t) - p(t)x(\tau(t)) = 0, \quad t \in \mathbb{N},
\]
respectively.

For Eq. (1.4), in 1983, Fukagai and Kusano [9] proved that if \( \tau(t) \) is nondecreasing and
\[
(1.6) \quad \liminf_{t \to \infty} \int_t^{\tau(t)} p(s)ds > \frac{1}{e},
\]
then all solutions of (1.4) oscillate.

In 1987, Ladde et al. [13, Theorem 2.4.1] proved that if \( \tau(t) \) is not necessarily monotone and (1.6) holds, then all solutions of (1.4) oscillate. They also proved that if \( \tau(t) \) is nondecreasing and
\[
(1.7) \quad \limsup_{t \to \infty} \int_t^{\tau(t)} p(s)ds > 1,
\]
then all solutions of (1.4) oscillate.
For Eq. (1.5), in 2012 Chatzarakis and Stavroulakis [6] proved that if \( \{ \tau(t) \} \) is nondecreasing and
\[
\limsup_{t \to \infty} \sum_{j=t}^{\tau(t)} p(j) > 1,
\]
then all solutions of (1.5) oscillate.

In 2016, Öcalan and Özkan [15] proved that if \( \{ \tau(t) \} \) is not necessarily monotone and
\[
\limsup_{t \to \infty} \sum_{j=t}^{h(t)} p(j) > 1,
\]
where \( h(t) = \min_{t \leq s} \{ \tau(s) \} \), then all solutions of (1.5) oscillate.

Very recently, Ş. Öcalan et al. [14] proved that if \( \{ \tau(t) \} \) is not necessarily monotone and
\[
\liminf_{t \to \infty} \sum_{j=t+1}^{h(t)} p(j) = \liminf_{t \to \infty} \sum_{j=t+1}^{\tau(t)} p(j) > \frac{1}{e},
\]
where \( h(t) = \min_{t \leq s} \{ \tau(s) \} \), then all solutions of (1.5) oscillate.

In 2016, Öcalan and Özkan [15] tried to obtain some criteria for the oscillatory solution of the equation (1.1) when \( \tau(t) \) is not necessarily monotone. Unfortunately, the authors have made a mistake in the proof of Theorem 2.4 in [15] caused by induction. So, the proof of Theorem 2.4 in [15] is invalid. Therefore, one of the aim of this paper is to obtain sufficient condition, involving \( \liminf \), for the equation (1.1) to be oscillatory.

2. SOME AUXILIARY LEMMAS AND MAIN RESULTS

Set
\[
h(t) := \inf_{t \leq s} \tau(s), \quad t \in \mathbb{T}.
\]
Clearly, \( h(t) \) is nondecreasing and \( \tau(t) \geq h(t) \) for all \( t \geq t_0 \). The following lemmas are needed to proof of the main result.

**Lemma 2.1.** [15] Assume that (2.1) holds and \( \alpha > 0 \). Then we have
\[
\alpha = \liminf_{t \to \infty} \int_{\rho(b)}^{u} f(s) g(h(s)) \nabla s \geq g(h(b)) \int_{\rho(b)}^{u} f(s) \nabla s.
\]

**Lemma 2.2.** [15] Assume that \( f : \mathbb{T} \to \mathbb{R} \) is \( \text{ld-continuous} \), \( g : \mathbb{T} \to \mathbb{R} \) is nondecreasing and \( h : \mathbb{T} \to \mathbb{T} \) is nondecreasing. If \( b < u \), then
\[
\int_{\rho(b)}^{u} f(s) g(h(s)) \nabla s \geq g(h(b)) \int_{\rho(b)}^{u} f(s) \nabla s.
\]

**Lemma 2.3.** For nonnegative \( p \) with \( p \in \mathcal{R}_\rho^+ \), we have the following inequalities
\[
\hat{e}_p(t,s) \geq \exp \left\{ \int_{s}^{t} p(u) \nabla u \right\} \quad \text{for all } t \geq s.
\]
Proof. We use the representation \[ 2 \]
\[
\exp(t; s) = \exp\left\{ \int_s^t \xi_{\nu(u)}(p(u)) \nabla u \right\}.
\]
Here, we have for any \( p \) with \( p \in \mathbb{R}^+ \)
if \( \nu(u) = 0 \), and if \( \nu(u) > 0 \)
\[
\xi_{\nu(u)}(p(u)) = \frac{-\log(1 - \nu(u)p(u))}{\nu(u)} = \frac{-\log(1 - \nu(u)p(u))}{\nu(u)}
\]
\[
= p(u) + \frac{f(-\nu(u)p(u))}{\nu(u)} \geq p(u),
\]
where \( f : (-1, \infty) \to \mathbb{R} \) is defined by \( f(x) = x - \log(1 + x) \) and hence satisfies \( f(x) \geq 0 \) for all \( x > -1 \). So, the proof is complete. \( \square \)

Lemma 2.4. Suppose \( p \in \mathbb{R}^+ \) and \( s \in \mathbb{T} \). If \( x(t) \) is positive solution the following inequalities
\[
x^{\nabla}(t) - p(t)x(t) \geq 0 \quad \text{for all } t \geq s,
\]
then
\[
x(t) \geq \hat{e}_p(t, s)x(s) \quad \text{for all } t \geq s.
\]
Proof. We put \( f := x^{\nabla} - px \) and use \[2, \text{Theorem 3.42}\] to solve
\[
x^{\nabla} = p(t)x + f(t), \quad x(s) \quad \text{given}.
\]
Thus \( t \geq s \),
\[
x(t) = \hat{e}_p(t, s)x(s) + \int_s^t \hat{e}_p(t, \rho(u))f(u)\nabla u.
\]
The integrand is nonnegative as \( p \in \mathbb{R}^+ \) and \( f \geq 0 \), so our claim follows. \( \square \)

In this section, we present a new sufficient condition for the oscillation of all solutions of \( (1.1) \), under the assumption that the argument \( \tau(t) \) is not necessarily monotone. The following result was given in [15].

Theorem 2.5. [15] Assume that \( (1.3) \) holds. If \( \tau(t) \) is not necessarily monotone and
\[
(2.2) \quad \lim_{t \to \infty} \sup_{t \to \infty} \int_{\rho(t)}^{h(t)} p(s) \nabla s > 1,
\]
where \( h(t) \) is defined by \( (2.1) \), then all solutions of \( (1.1) \) oscillate.

Theorem 2.6. Assume that \( (1.3) \) holds. If \( \tau(t) \) is not necessarily monotone and
\[
(2.3) \quad \lim_{t \to \infty} \int_{t}^{\tau(t)} p(s) \nabla s = \lim_{t \to \infty} \int_{t}^{h(t)} p(s) \nabla s > 1,
\]
then all solutions of \( (1.1) \) oscillate.
Proof. Assume, for the sake of contradiction, that there exists a positive nonoscil-
latory solution \( x(t) \) of (1.1). Since \(-x(t)\) is also a solution of (1.1), we can confine
our discussion only to the case where the solution \( x(t) \) is eventually positive. Then
there exists \( t_1 > t_0 \) such that \( x(t), \ x(\tau(t)) > 0, \) for all \( t \geq t_1. \) Thus, from (1.1) we have
\[
x^\nabla(t) = p(t)x(\tau(t)) \geq 0, \quad \text{for all } t \geq t_1,
\]
which means that \( x(t) \) is an eventually nondecreasing function. In view of this and
taking into account that \( \tau(t) \geq h(t) \geq t, \) (1.1) gives
\[
x^\nabla(t) - p(t)x(h(t)) \geq 0, \quad t \geq t_1
\]
and
\[
x^\nabla(t) - p(t)x(t) \geq 0, \quad t \geq t_1
\]
and so we have Lemma 2.4. On the other hand, using by Lemma 2.1 and from (2.3)
it follows that there exists a constant \( c > 0 \) such that
\[
\int_{t}^{h(t)} p(s)\nabla s \geq c > \frac{1}{c}, \quad t \geq t_2 > t_1.
\]
So, by Lemma 2.4, we obtain
\[
x(h(t)) \geq \hat{\epsilon}_p(h(t), t)x(t) \quad \text{for all } h(t) \geq t.
\]
Thus, by Lemma 2.3, (2.6) and (2.7), we get
\[
x(h(t)) \geq \hat{\epsilon}_p(h(t), t)x(t) \geq \exp \left\{ \int_{t}^{h(t)} p(u)\nabla u \right\} x(t)
\]
and
\[
x(h(t)) \geq e^c x(t) \geq (ec) x(t),
\]
where \( ec > 1. \) Thus, from (2.4) and (2.8) we have
\[
x^\nabla(t) - p(t)(ec)x(t) \geq 0, \quad t \geq t_2.
\]
Let \( p_1(t) := (ec)p(t). \) So, we obtain
\[
x^\nabla(t) - p_1(t)x(t) \geq 0, \quad t \geq t_2.
\]
Using by Lemma 2.4, we get
\[
x(h(t)) \geq \hat{\epsilon}_{p_1}(h(t), t)x(t) \quad \text{for all } h(t) \geq t.
\]
Thus, by Lemma 2.3, (2.6) and (2.10), we get
\[
x(h(t)) \geq \hat{\epsilon}_{p_1}(h(t), t)x(t) \geq \exp \left\{ \int_{t}^{h(t)} p_1(u)\nabla u \right\} x(t)
\]
\[
= \exp \left\{ ec \int_{t}^{h(t)} p(u)\nabla u \right\} x(t) \geq \exp \{ ec^2 \} x(t)
\]
\[
\geq (ec)^2 x(t).
\]
Repeating the above procedure, it follows by induction that for any positive integer $k$
\begin{equation}
\frac{x(h(t))}{x(t)} \geq (ec)^k \quad \text{for sufficiently large } t.
\end{equation}

On the other hand, from (2.6), there exists $t^* \in (t, h(t)], t^* \in \mathbb{T}$, such that
\begin{equation}
\int_{\rho(t)}^{t^*} p(s)\nabla s \geq \frac{c}{2} \quad \text{and} \quad \int_{\rho(t^*)}^{h(t)} p(s)\nabla s \geq \frac{c}{2}.
\end{equation}

Integrating (2.4) from $\rho(t)$ to $t^*$, and using the fact that the function $x(t)$ and $h(t)$ are nondecreasing, we obtain
\[x(t^*) - x(\rho(t)) - \int_{\rho(t)}^{t^*} p(s)x(h(s))\nabla s \geq 0.\]

Now, using Lemma 2.2 and (2.12), we have
\[x(t^*) - x(h(t)) \int_{\rho(t)}^{t^*} p(s)\nabla s \geq 0\]
or
\begin{equation}
x(t^*) - x(h(t)) \frac{c}{2} \geq 0.
\end{equation}

Integrating (2.4) from $\rho(t^*)$ to $h(t)$, and using the same arguments we have
\[x(h(t)) - x(\rho(t^*)) - \int_{\rho(t^*)}^{h(t)} p(s)x(h(s))\nabla s \geq 0,
\]
or
\[x(h(t)) - x(h(t^*)) \int_{\rho(t^*)}^{h(t)} p(s)\nabla s \geq 0\]
or
\begin{equation}
x(h(t)) - x(h(t^*)) \frac{c}{2} \geq 0.
\end{equation}

Combining the inequalities (2.13) and (2.14), we obtain
\[x(t^*) \geq x(h(t)) \frac{c}{2} \geq x(h(t^*)) \left(\frac{c}{2}\right)^2,
\]
or
\[\frac{x(h(t^*))}{x(t^*)} \leq \left(\frac{2}{c}\right)^2 < +\infty \]
i.e., \( \liminf_{t \to +\infty} \frac{x(h(t))}{x(t)} \) exists. This contradicts to (2.11). The proof is complete. \( \square \)

A slight modification in the proofs of Theorems 2.5 and 2.6 leads to the following result.
Theorem 2.7. Assume that all the conditions of Theorems 2.5 and 2.6 hold. Then

(i) the dynamic inequality
\[ x^{\nabla}(t) - p(t)x(\tau(t)) \geq 0, \quad t \in [t_0, \infty)_T \]

has no eventually positive solutions;

(ii) the dynamic inequality
\[ x^{\nabla}(t) - p(t)x(\tau(t)) \leq 0, \quad t \in [t_0, \infty)_T \]

has no eventually negative solutions.

Example 2.1. Let \( T = 2\mathbb{Z} = \{2k : k \in \mathbb{Z}\} \). For \( t \in T \), we have
\[ \rho(t) = t - 2, \quad \nu(t) = 2 \quad \text{and} \quad x^{\nabla}(t) = \frac{x(t) - x(t - 2)}{2}. \]

Thus, Eq. (1.1) becomes
\[ (2.15) \quad \frac{x(t) - x(t - 2)}{2} - p(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_T. \]

We take \( p(t) = 0.19 \) and \( \tau(t) = t + 2 \). On the other hand, when \( T = h\mathbb{Z} \), from (iii) in Theorem 8.48 [1], we have the following,
\[ \int_a^b f(t)\nabla t = \sum_{k=\frac{a}{h}}^{b} f(kh)h. \]

So, the condition (2.3) becomes
\[ (2.16) \quad \liminf_{t \to \infty} \int_t^{\tau(t)} p(s)\nabla s = \liminf_{t \to \infty} \sum_{j=\frac{t}{2}}^{\frac{t+2}{2}} p(2j)2 \]

Then, by (2.16), we obtain
\[ \liminf_{t \to \infty} \sum_{j=\frac{t}{2}}^{\frac{t+2}{2}} 2p(2j) = \liminf_{t \to \infty} [2 \times p(t + 2)] = 0.38 > \frac{1}{e}. \]

Thus, by Theorem (2.6), every solution of (2.15) is oscillatory.

3. Equations with several arguments

Now, we consider the first-order nabla dynamic equations with several arguments
\[ (3.1) \quad x^{\nabla}(t) - \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \in [t_0, \infty)_T, \]

where \( p_i(t) (i = 1, 2, \cdots, m) \) are ld-continuous and nonnegative, the \( \tau_i : T \to T \) are not necessarily monotone and satisfy
\[ (3.2) \quad \tau_i(t) \geq t \quad \text{for all} \quad t \in T \quad \text{and} \quad \sup T = \infty. \]

In this section, we present some new sufficient conditions for the oscillation of all solutions of (3.1), under the assumption that the argument \( \tau_i(t) (i = 1, 2, \cdots, m) \) are not necessarily monotone. For \( T = \mathbb{R} \) and \( T = \mathbb{Z} \), equation (3.1) reduce to
\[ (3.3) \quad x'(t) - \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \geq t_0 \]
and
\[ \nabla x(t) - \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) = 0, \quad t \in \mathbb{N}_0, \]
respectively.

In 1978, Ladde [12] and in 1982 Ladas and Stavroulakis [11] proved that if \( \tau_i(t) \) \((i = 1, 2, \cdots, m)\) are nondecreasing and
\[ \liminf_{t \to \infty} \int_{t}^{\tau(t)} \sum_{i=1}^{m} p_i(s)ds < \frac{1}{e}, \]
where \( \tau(t) = \min_{1 \leq i \leq m} \{\tau_i(t)\} \), then all solutions of (3.3) oscillate.

In 1987, Ladde et al. [13] proved that if \( \tau_i(t) \) \((i = 1, 2, \cdots, m)\) are nondecreasing and
\[ \limsup_{t \to \infty} \int_{t}^{\tau(t)} \sum_{i=1}^{m} p_i(s)ds > 1, \]
where \( \tau(t) = \min_{1 \leq i \leq m} \{\tau_i(t)\} \), then all solutions of (3.3) oscillate.

In 2016, Braverman et al. [5] proved that if \( \tau_i(t) \) \((i = 1, 2, \cdots, m)\) are not necessarily monotone, and (3.5) or
\[ \limsup_{t \to \infty} \int_{t}^{h(t)} \sum_{i=1}^{m} p_i(s)ds > 1, \]
hold, where \( \tau(t) = \min_{1 \leq i \leq m} \{\tau_i(t)\} \) and \( h_i(t) := \inf_{t \leq s} \tau_i(s) \) and \( h(t) = \min_{1 \leq i \leq m} h_i(t) \), \( t \geq t_0 \in \mathbb{R} \), then all solutions of (3.3) oscillate.

In 2015, Braverman et al. [4] proved that if \( \tau_i(t) \) \((i = 1, 2, \cdots, m)\) are not necessarily monotone, and
\[ \limsup_{t \to \infty} \sum_{j=i+1}^{m} \sum_{i=1}^{h_i(j)} > 1, \]
where \( \tau(t) = \min_{1 \leq i \leq m} \{\tau_i(t)\} \), \( h_i(t) := \inf_{t \leq s} \tau_i(s) \) and \( h(t) = \min_{1 \leq i \leq m} h_i(t) \), \( t \geq t_0 \in \mathbb{N} \), then all solutions of (3.3) oscillate.

Very recently, Ş. Öcalan et al. [14] proved that if \( \{\tau_i(t)\} \) \((i = 1, 2, \cdots, m)\) are not necessarily monotone and
\[ \liminf_{t \to \infty} \int_{t}^{\tau(t)} \sum_{j=i+1}^{m} p_i(j) > \frac{1}{e}, \]
where \( \tau(t) = \min_{1 \leq i \leq m} \{\tau_i(t)\} \), \( t \geq t_0 \in \mathbb{N} \), then all solutions of (3.4) oscillate.

Set
\[ h_i(t) := \inf_{t \leq s} \tau_i(s) \] \text{ and } \[ h(t) = \min_{1 \leq i \leq m} h_i(t), \quad t \in \mathbb{T}. \]
Clearly, \( h_i(t) \) \((i = 1, 2, \cdots, m)\) are nondecreasing and \( \tau_i(t) \geq h_i(t) \geq h(t) \) for all \( t \in \mathbb{T} \).

Now, we have the following result.
Theorem 3.1. Assume that (3.2) holds. If $\tau_i(t)$ ($i = 1, 2, \cdots, m$) are not necessarily monotone and

\begin{equation}
\limsup_{t \to \infty} \int_{\rho(t)}^{h(t)} \sum_{i=1}^{m} p_i(s) \nabla s > 1,
\end{equation}

or

\begin{equation}
\liminf_{t \to \infty} \int_{t}^{\tau(t)} \sum_{i=1}^{m} p_i(s) \nabla s > \frac{1}{e},
\end{equation}

where $\tau(t) = \min_{1 \leq i \leq m} \{\tau_i(t)\}$ and $h(t)$ is defined by (3.7), then all solutions of (3.1) oscillate.

Proof. Assume, for the sake of contradiction, that there exists a positive nonoscillatory solution $x(t)$ of (3.1). Then there exists $t_1 > t_0$ such that $x(t), x(\tau_i(t)) > 0$ for all $t \geq t_1$. Thus, from (3.1) we have

\begin{equation}
x^{\nabla}(t) - x(\tau(t)) \sum_{i=1}^{m} p_i(t) \geq 0.
\end{equation}

Comparing (3.10), we obtain a contradiction to Theorem 2.7. Here, we have used the following equality

\[
\liminf_{t \to \infty} \int_{t}^{\tau(t)} \sum_{i=1}^{m} p_i(s) \nabla s = \liminf_{t \to \infty} \int_{t}^{h(t)} \sum_{i=1}^{m} p_i(s) \nabla s
\]

which is easily obtained as similar to the proof of Lemma 2.1.

A slight modification in the proof of Theorem 3.1 leads to the following result.

Theorem 3.2. Assume that all the conditions of Theorem 3.1 hold. Then

(i) the dynamic inequality

\[
x^{\nabla}(t) - \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) \geq 0, \quad t \in [t_0, \infty)_T
\]

has no eventually positive solutions;

(ii) the dynamic inequality

\[
x^{\nabla}(t) - \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) \leq 0, \quad t \in [t_0, \infty)_T
\]

has no eventually negative solutions.

References


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