TRANSCENDENTAL ENTIRE SOLUTIONS OF SEVERAL FERMAT TYPE PDES AND PDDES WITH TWO COMPLEX VARIABLES

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ABSTRACT. This article is concerned with the describe of transcendental entire solutions of several Fermat type functional equations in $C^2$ concerning partial differential-difference equations and partial differential equations. By utilizing the Nevanlinna theory of meromorphic functions in several complex variables, we establish some results about the existence and the forms of entire solutions for such equations, which are some improvements and generalizations of the previous theorems. Moreover, some examples are given explain that our results are precise to some extent.

1. Introduction and Main Results

As is known to all, Nevanlinna’s value distribution theory occupying the central position for more than one century is an important tool in studying the properties of meromorphic functions in the field of complex analysis. A large number of articles show that Nevanlinna theory has been connected with other areas of mathematics (topology, differential geometry, measure theory, potential theory and others), and also widely been applied to the differential equations and partial differential equations. With the development of the difference analogues of Nevanlinna theory (see [4, 7, 8, 9]), many scholars paid considerable attention on the properties of meromorphic solutions on complex difference equations (see [3]). In recent, there was an increasing interest in studying the properties on the solutions of complex differential-difference equations by making use of Nevanlinna theory. An equation is called differential-difference equation, if the equation includes derivatives, shifts or differences of $f$, which can be called DDE for short (see [22]).

The study of complex DDEs can be traced back to Naftalevich’s research [24, 25]. He investigated the meromorphic solutions on complex DDEs with one complex variable, by employing the operator theory and iteration method. However, in the past two decades, there were a number of articles focusing on DDEs by using the difference analogues of Nevanlinna theory (see [18, 19, 20, 21, 27, 28]). In particular, Liu et al. [20] investigated the existence of entire solutions with finite order of the
Fermat type DDEs

\[(1.1) \quad f'(z)^2 + f(z + c)^2 = 1, \]

and obtained

**Theorem A** (see [20]). *The transcendental entire solutions with finite order of equation (1.1) must satisfy* \(f(z) = \sin(z \pm Bi)\), where \(B\) is a constant and \(c = 2k\pi\) or \(c = (2k + 1)\pi\), \(k\) is an integer, and \(B\) is a constant.

In 2019, Liu and Gao [21] further studied the entire solutions of second order DDE with single complex variable and obtained

**Theorem B** (see [21, Theorem 2.1]). *Suppose that* \(f\) *is a transcendental entire solution with finite order of the complex DDE*

\[f''(z)^2 + f(z + c)^2 = Q(z),\]

*then* \(Q(z) = c_1c_2\) *is a constant, and* \(f(z)\) *satisfies*

\[f(z) = \frac{c_1e^{a(z+b)} + c_2e^{-a(z-b)}}{2a^2},\]

*where* \(a, b \in \mathbb{C}\), *and* \(a^4 = 1, c = \frac{\log(-ia^2) + 2k\pi i}{a}, k \in \mathbb{Z}\).

Now, let us recall some results about complex functional equations in several complex variables as follows. An equation is called partial differential equation, if the equation includes partial differential of \(f\), which can be called PDE for short. In lots of previous articles [10, 12, 15, 23, 29] about complex partial differential equations with several complex variables, D. Khavinson [12] pointed out that any entire solution of the partial differential equation

\[(1.2) \quad \left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = 1\]

in \(\mathbb{C}^2\) is necessarily linear. These partial differential equations in real variable case occur in the study of characteristic surfaces and wave propagation theory, and it is the two dimensional eiconal equation, one of the main equations of geometric optics (see [5, 6]). For complex PDE, E. G. Saleeby [29] in 1999 studied the entire solution of Fermat type partial differential equation (1.2) and obtain

**Theorem C** (see [29, Theorem 1]). *If* \(f\) *is an entire solution of equation (1.2) in* \(\mathbb{C}^2\), *then* \(f = c_1z_1 + c_2z_2 + c\), *where* \(c_1, c_2, c \in \mathbb{C}\) *and* \(c_1^2 + c_2^2 = 1\).

Later, B. Q. Li and his co-authors discussed some variations of the partial differential equation (1.2) such as \(\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = f^n\), \(\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = p\), \(\left(\frac{\partial f}{\partial z_1}\right)^2 + \left(\frac{\partial f}{\partial z_2}\right)^2 = e^g\), etc, where \(n\) is a positive integer and \(p, g\) are polynomials in \(\mathbb{C}^2\), and obtained a lot of interesting and important results, readers can refer to [14, 15, 16, 17].

Very recently, Xu and Cao [2, 34, 35], Xu, Liu and Li [33] investigated the existence of the solutions for some Fermat-type partial differential-difference equations with several complex variables by making use of the difference logarithmic derivative lemma of several complex variables (see [1, 2, 13]) and obtained the following theorem.
Theorem D (see [34, Theorem 1.2]). Let \( c = (c_1, c_2) \in \mathbb{C}^2 \). Then any transcendental entire solutions with finite order of the partial differential-difference equation

\[
(f(z_1, z_2) + \frac{\partial f}{\partial z_1})^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1
\]

has the form of \( f(z_1, z_2) = \sin(Az_1 + B) \), where \( A \) is a constant on \( \mathbb{C} \) satisfying \( Ae^{iAc_1} = 1 \), and \( B \) is a constant on \( \mathbb{C} \); in the special case whenever \( c_1 = 0 \), we have \( f(z_1, z_2) = \sin(z_1 + B) \).

Remark 1.1. An equation is called partial differential-difference equation, if the equation includes partial derivatives, shifts or differences of \( f \), which can be called PDDE for short.

This article is concerned with the description of transcendental entire solutions for the above partial differential-difference of Fermat-type. The main tools are used in this paper are the Nevanlinna theory and difference Nevanlinna theory with several complex variables. This article is organized as follows. In Section 2, our main results about the existence and the forms of entire solutions of several PDEs and PDDEs and some indispensable lemmas are introduced, and some examples are given show that the conclusions on the forms of transcendental entire solutions are precise. The detailed proofs of Theorems 2.1-2.3 will be contained in Sections 3, and the detailed proofs of Theorems 2.4-2.5 will be contained in Section 4.

2. Results

Here and in the following, for convenience, let \( z + w = (z_1 + w_1, z_2 + w_2) \) for any \( z = (z_1, z_2), w = (w_1, w_2) \). Inspired by the ideas of Theorems B, C and D, we mainly investigate the properties of entire solutions of some variations of the PDE (1.2) and PDDE (1.3) in \( \mathbb{C}^2 \) as follows

\[
(1) \quad \left( f(z) + \frac{\partial f}{\partial z_1} \right)^2 + \left( f(z) + \frac{\partial f}{\partial z_2} \right)^2 = 1,
\]

\[
(2) \quad \left( f(z) + \frac{\partial f}{\partial z_1} \right)^2 + \left( f(z) + \frac{\partial^2 f}{\partial z_1^2} \right)^2 = 1,
\]

\[
(3) \quad \left( f(z) + \frac{\partial f}{\partial z_1} \right)^2 + \left( f(z) + \frac{\partial^2 f}{\partial z_1 \partial z_2} \right)^2 = 1,
\]

\[
(4) \quad \left( f(z + c) + \frac{\partial f}{\partial z_1} \right)^2 + \left( f(z + c) + \frac{\partial f}{\partial z_2} \right)^2 = 1,
\]

\[
(5) \quad \left( f(z + c) + \frac{\partial f}{\partial z_1} \right)^2 + \left( f(z + c) + \frac{\partial^2 f}{\partial z_1^2} \right)^2 = 1,
\]

where \( c = (c_1, c_2) \in \mathbb{C}^2 \), and obtain

**Theorem 2.1.** Let \( f(z_1, z_2) \) be a transcendental entire solution with finite order of the partial differential equation (2.1). Then \( f(z_1, z_2) \) is of the form

\[
f(z_1, z_2) = \pm \sqrt{2} \sqrt{2} e^{-(z_1 + z_2)},
\]

or

\[
f(z_1, z_2) = \frac{1}{2} \sin(z_2 - z_1 + \eta_1) + \frac{1}{2} \cos(z_2 - z_1 + \eta_1) + \eta_2 e^{-(z_1 + z_2)},
\]
where \( \eta, \eta_1, \eta_2 \in \mathbb{C} \).

**Theorem 2.2.** The partial differential equation \((2.2)\) does not admit any transcendental entire solution with finite order.

**Theorem 2.3.** Let \( f(z_1, z_2) \) be a transcendental entire solution with finite order of the partial differential equation \((2.3)\). Then \( f(z_1, z_2) \) is of the form

\[
f(z_1, z_2) = \pm \frac{\sqrt{2}}{2} + \eta e^{z_2 - z_1},
\]

where \( \eta \in \mathbb{C} \).

**Theorem 2.4.** Let \( c = (c_1, c_2)(\neq (0, 0)) \in \mathbb{C}^2 \) and \( s_0 = c_1 + c_2 \), and let \( f(z_1, z_2) \) be a transcendental entire solution with finite order of the partial differential equation \((2.4)\). Then \( f(z_1, z_2) \) is of the following forms

(i)

\[
f(z_1, z_2) = \phi(z_1 + z_2),
\]

where \( \phi(s) \) is a transcendental entire function with finite order in \( s := z_1 + z_2 \) satisfying

\[
\phi(s + s_0) + \phi'(s) = \pm \frac{\sqrt{2}}{2}.
\]

(ii)

\[
f(z_1, z_2) = \frac{1 + i}{2(\alpha_1 - \alpha_2)} e^{L(z)+B} - \frac{1 - i}{2(\alpha_1 - \alpha_2)} e^{-L(z)-B} + \varphi(z_1 + z_2),
\]

where \( L(z) = \alpha_1 z_1 + \alpha_2 z_2 \), \( \alpha_1, \alpha_2, B \in \mathbb{C} \) and \( \varphi(s) \) satisfies

\[
\alpha_1^2 + \alpha_2^2 = -2, \quad e^{L(c)} = -\frac{i\alpha_1 + \alpha_2}{1 + i} = -\frac{1 - i}{i\alpha_1 - \alpha_2}, \quad \varphi'(s) + \varphi(s + s_0) = 0.
\]

The following examples show that the conclusions on the forms of transcendental entire solutions with finite order are precise.

**Example 2.1.** Let \( f(z_1, z_2) = \pm \frac{\sqrt{2}}{2} + e^{z_1 + z_2} \). Then \( f(z_1, z_2) \) is a transcendental entire solution with finite order of equation \((2.4)\) with \((c_1, c_2) = (\pi, 2\pi i)\).

**Example 2.2.** Let \( \alpha_1 = \sqrt{3}i \), \( \alpha_2 = 1 \), and

\[
f(z_1, z_2) = \frac{1 + i}{2(\sqrt{3}i - 1)} e^{L(z)+B} - \frac{1 - i}{2(\sqrt{3}i - 1)} e^{-L(z)+B} + e^{z_1 + z_2}.
\]

Then \( f(z_1, z_2) \) is a transcendental entire solution with finite order of equation \((2.4)\) with \((c_1, c_2) = (\beta - \frac{\pi}{\sqrt{3}i - 1}, -\frac{\sqrt{3} \pi + \beta}{\sqrt{3}i - 1})\), where \( \beta = \log((\sqrt{3} - 1)(1 - i)) - \log 2 \) and \( B \in \mathbb{C} \).

**Theorem 2.5.** Let \( c = (c_1, c_2)(\neq (0, 0)) \in \mathbb{C}^2 \), and let \( f(z_1, z_2) \) be a transcendental entire solution with finite order of the partial differential equation \((2.5)\). Then \( f(z_1, z_2) \) is of the form

\[
f(z_1, z_2) = \pm \frac{\sqrt{2}}{2} + \eta e^{z_1 + \frac{2k\pi + \pi - \alpha_1}{2}z_2},
\]

where \( k \in \mathbb{Z}_+ \) and \( \eta \in \mathbb{C} \); or

\[
f(z_1, z_2) = -\frac{1 + i}{2\alpha_1(\alpha_1 - 1)} e^{L(z)+B} - \frac{1 - i}{2\alpha_1(\alpha_1 - 1)} e^{-L(z)+B} + \eta e^{z_1 + \frac{2k\pi + \pi - \alpha_1}{2}z_2},
\]

where \( L(z) = \alpha_1 z_1 + \alpha_2 z_2 \) and \( \eta \in \mathbb{C} \), \( \alpha_1, \alpha_2, c_1, c_2 \) satisfy

\[
(\alpha_1, \alpha_2)^2 = 2, \quad e^{2L(c)} = -i.
\]
The following examples show that the conclusions on the forms of transcendental entire solutions with finite order are precise.

**Example 2.3.** Let \( f(z_1, z_2) = \pm \sqrt{\frac{1}{2}} + e^{z_1 - z_2}. \) Then \( f(z_1, z_2) \) is a transcendental entire solution with finite order of equation (2.5) with \((c_1, c_2) = (-2\pi i, \pi i)\).

**Example 2.4.** Let \( \alpha_1 = \sqrt{\frac{4\sqrt{2} - 1 - i}{2}}, \alpha_2 = 1, \) \( L(z) = \sqrt{\frac{4\sqrt{2} - 1 - i}{2}}z_1 + z_2 \) and
\[
f(z_1, z_2) = -\frac{1}{\gamma_1}e^{L(z)} - \frac{1}{\gamma_2}e^{-(L(z))} + e^{z_1 + \gamma_3 z_2},
\]
where \( \gamma_1 = \sqrt{2 - \sqrt{4\sqrt{2} - 1}} - (\sqrt{2} - 1)i, \gamma_2 = \sqrt{2} - 1 + i(\sqrt{2} - \sqrt{4\sqrt{2} - 1}) \) and
\( \gamma_3 = \frac{1 + i}{2\sqrt{4\sqrt{2} - 1 + i}}. \) Then \( f(z_1, z_2) \) is a transcendental entire solution with finite order of equation (2.5) with \((c_1, c_2) = (\pi, \frac{2\pi\sqrt{4\sqrt{2} - 1 + i}}{4})\) and \( B = 0. \)

The following lemmas play the key roles in proving our results.

**Lemma 2.1.** ([30, 31]). For an entire function \( F \) on \( \mathbb{C}^n, F(0) \neq 0 \) and put \( \rho(n_F) = \rho < \infty. \) Then there exist a canonical function \( f_F \) and a function \( g_F \in \mathbb{C}^n \) such that \( F(z) = f_F(z)e^{g_F(z)}. \) For the special case \( n = 1, f_F \) is the canonical product of Weierstrass.

**Remark 2.1.** Here, denote \( \rho(n_F) \) to be the order of the counting function of zeros of \( F. \)

**Lemma 2.2.** ([26]). If \( g \) and \( h \) are entire functions on the complex plane \( \mathbb{C} \) and \( g(h) \) is an entire function of finite order, then there are only two possible cases: either

(a) the internal function \( h \) is a polynomial and the external function \( g \) is of finite order; or else

(b) the internal function \( h \) is not a polynomial but a function of finite order, and the external function \( g \) is of zero order.

**Lemma 2.3.** ([11, Lemma 3.1]). Let \( f_j \neq 0, j = 1, 2, 3, \) be meromorphic functions on \( \mathbb{C}^m \) such that \( f_1 \) is not constant, and \( f_1 + f_2 + f_3 = 1, \) and such that
\[
\sum_{j=1}^{3} \left\{ N_2(r, \frac{1}{f_j}) + 2N(r, f_j) \right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),
\]
for all \( r \) outside possibly a set with finite logarithmic measure, where \( \lambda < 1 \) is a positive number. Then either \( f_2 = 1 \) or \( f_3 = 1. \)

**Remark 2.2.** Here, \( N_2(r, \frac{1}{f}) \) is the counting function of the zeros of \( f \) in \(|z| \leq r, \) where the simple zero is counted once, and the multiple zero is counted twice.

3. Proofs of Theorems 2.1-2.3

3.1. The Proof of Theorem 2.1. Suppose that \( f(z) \) is a transcendental entire solution with finite order of equation (2.1). Two cases will be discussed below.

**Case 1.** Suppose that \( f(z) + \frac{\partial f}{\partial z_1} \) is a constant. Set
\[
f(z) + \frac{\partial f}{\partial z_1} = K_1, \quad K_1 \in \mathbb{C}.
\]
In view of equation (2.1), it follows that \( f(z) + \frac{\partial f}{\partial z_2} \) is a constant, let

\[
(3.2) \quad f(z) + \frac{\partial f}{\partial z_2} = K_2, \quad K_2 \in \mathbb{C}.
\]

Thus, it leads to \( K_1^2 + K_2^2 = 1 \). In view of (3.1) and (3.2), it yields that

\[
(3.3) \quad \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2} = K_1 - K_2.
\]

The characteristic equations of (3.3) are

\[
\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = -1, \quad \frac{df}{dt} = K_1 - K_2.
\]

Using the initial conditions: \( z_1 = 0, \ z_2 = s \), and \( f = f(0, s) := \phi(s) \) with a parameter \( s \). Thus, we obtain the following parametric representation for the solutions of the characteristic equations: \( z_1 = t, \ z_2 = -t + s \),

\[
f(t, s) = \int_0^t (K_1 - K_2) dt + \phi(s) = (K_1 - K_2)t + \phi(s),
\]

where \( \phi(s) \) is a transcendental entire function with finite order in \( s \). Then, by combining with \( t = z_1 \) and \( s = z_2 + z_1 \), the solution of equation (3.3) is of the form

\[
(3.4) \quad f(z_1, z_2) = (K_1 - K_2)z_1 + \phi(z_1 + z_2).
\]

On the other hand, differentiating both two sides of the equations (3.1), (3.2) for the variables \( z_2, z_1 \), respectively, and combining with the fact \( \frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial^2 f}{\partial z_2 \partial z_1} \), it follows that

\[
\frac{\partial f}{\partial z_1} = \frac{\partial f}{\partial z_2},
\]

which implies that \( K_1 = K_2 = \pm \frac{\sqrt{2}}{2} \). Thus, it follows \( f(z_1, z_2) = \phi(z_1 + z_2) \).

Substituting this into (3.2) or (3.3), we obtain

\[
\phi'(z_1 + z_2) + \phi(z_1 + z_2) = \pm \frac{\sqrt{2}}{2}.
\]

This means that \( f(z_1, z_2) = \phi(z_1 + z_2) = \pm \frac{\sqrt{2}}{2} + ye^{-(z_1 + z_2)} \).

**Case 2.** Suppose that \( f(z) + \frac{\partial f}{\partial z_1} \) is not a constant. In view of the fact that the entire solutions of the functional equation \( f^2 + g^2 = 1 \) are \( f = \cos a(z), g = \sin a(z) \), where \( a(z) \) is an entire function, it follows that there exists an entire function \( h(z) \) such that

\[
(3.5) \quad f(z) + \frac{\partial f}{\partial z_1} = \sin h(z), \quad f(z) + \frac{\partial f}{\partial z_2} = \cos h(z).
\]

This leads to

\[
(3.6) \quad \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2} = \sin h(z) - \cos h(z).
\]

On the other hand, from (3.5), we have

\[
(3.7) \quad \frac{\partial f}{\partial z_2} + \frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial h}{\partial z_2} \cos h(z), \quad \frac{\partial f}{\partial z_1} + \frac{\partial^2 f}{\partial z_2 \partial z_1} = -\frac{\partial h}{\partial z_1} \sin h(z).
\]
By combining with \( \frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial^2 f}{\partial z_2 \partial z_1} \), we deduce from (3.7) that
\[
(3.8) \quad \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2} = \frac{\partial h}{\partial z_1} \sin h(z) - \frac{\partial h}{\partial z_2} \cos h(z).
\]
Thus, it follows from (3.6) and (3.8) that
\[
(3.9) \quad \left( 1 + \frac{\partial h}{\partial z_1} \right) \sin h(z) = \left( 1 - \frac{\partial h}{\partial z_2} \right) \cos h(z).
\]
If \( 1 + \frac{\partial h}{\partial z_1} \neq 0 \) and \( 1 - \frac{\partial h}{\partial z_2} \neq 0 \), then it follows from (3.9) that
\[
(3.10) \quad \tan h(z) = \frac{1 - \frac{\partial h}{\partial z_2}}{1 + \frac{\partial h}{\partial z_1}}.
\]

Since \( f(z) \) is a finite order transcendental entire solution of equation (2.1), by Lemma 2.1, 2.2 and (3.5), we conclude that \( h(z) \) is a nonconstant polynomial in \( \mathbb{C}^2 \). Thus, a contradiction can be obtained from (3.10) using Nevanlinna theory. In fact, if \( T(r, F) \) denotes the Nevanlinna characteristic function of a meromorphic function \( F \) in \( \mathbb{C}^2 \), then by (3.10) we deduce that \( T(r, \tan h) = O\{T(r, h) + \log r\} \), outside possibly a set of finite Lebesgue measure, using the results (see e.g. [32, p.99], [31]) that \( T(r, F_{z_2}) = O\{T(r, F)\} \) for any meromorphic function \( F \) outside a set of finite Lebesgue measure and that \( T(r, P) = O\{\log r\} \) for any polynomial \( P \). But, \( \lim_{r \to \infty} \frac{T(r, \tan h)}{T(r, h) + \log r} = +\infty \) when \( h \) is a nonconstant polynomial. Therefore, \( h \) must be constant, a contradiction.

If \( 1 + \frac{\partial h}{\partial z_1} = 0 \) or \( 1 - \frac{\partial h}{\partial z_2} = 0 \), then it yields that \( \cos h(z) = 0 \) or \( \sin h(z) = 0 \), which imply that \( h(z) \) is a constant, a contradiction.

If \( 1 + \frac{\partial h}{\partial z_1} = 0 \) and \( 1 - \frac{\partial h}{\partial z_2} = 0 \), then it follows that \( h(z) = z_2 - z_1 + \eta_1 \), where \( \eta_1 \in \mathbb{C} \). Substituting this into (3.6), we have
\[
(3.11) \quad \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2} = \sin(z_2 - z_1 + \eta_1) - \cos(z_2 - z_1 + \eta_1).
\]
The characteristic equations of (3.11) are
\[
\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = -1, \quad \frac{df}{dt} = \sin(z_2 - z_1 + \eta_1) - \cos(z_2 - z_1 + \eta_1).
\]
Using the initial conditions: \( z_1 = 0, z_2 = s \), and \( f = f(0, s) := \phi_0(s) \) with a parameter \( s \). Thus, we obtain the following parametric representation for the solutions of the characteristic equations: \( z_1 = t, z_2 = -t + s \),
\[
f(t, s) = \int_0^t \left[ \sin(-2t + s + \eta_1) - \cos(-2t + s + \eta_1) \right] dt + \phi_0(s)
\]
\[
= \frac{1}{2} \cos(-2t + s + \eta_1) + \frac{1}{2} \sin(-2t + s + \eta_1) + \phi_1(s),
\]
where \( \phi_1(s) = \phi_0(s) - \frac{1}{2} \cos(s + \eta_1) - \frac{1}{2} \sin(s + \eta_1) \) is a transcendental entire function with finite order in \( s \). By combining with \( t = z_1, s = z_2 + z_1 \), we have
\[
(3.12) \quad f(z_1, z_2) = \frac{1}{2} \cos(z_2 - z_1 + \eta_1) + \frac{1}{2} \sin(z_2 - z_1 + \eta_1) + \phi_1(z_1 + z_2).
\]
Substituting (3.12) into the first equation of (3.5), it yields
\[
\phi'_1(z_1 + z_2) + \phi_1(z_1 + z_2) = 0,
\]
which implies \( \phi_1(z_1 + z_2) = \eta_2 e^{-(z_1 + z_2)}, \eta_2 \in \mathbb{C} \).
Therefore, this completes the proof of Theorem 2.1.

3.2. The Proof of Theorem 2.2. Suppose that \( f(z) \) is a transcendental entire solution with finite order of equation (2.5). We will discuss two cases as follows.

Case 1. Suppose that \( f(z) + \frac{\partial f}{\partial z_1} \) is a constant. Similar to the argument as in the proof of Case 1 in Theorem 2.1, we get

\[
(3.13) \quad f(z) + \frac{\partial f}{\partial z_1} = \pm \frac{\sqrt{2}}{2},
\]

and

\[
(3.14) \quad f(z_1, z_2) = e^{z_1 + \varphi_1(z_2)} + \varphi_2(z_2),
\]

where \( \varphi_1(z_2), \varphi_2(z_2) \) are two functions in \( z_2 \). Substituting (3.14) into (3.13), we have

\[
2e^{z_1 + \varphi_1(z_2)} + \varphi_2(z_2) = \pm \frac{\sqrt{2}}{2},
\]

which implies that \( \varphi_2(z_2) = \pm \frac{\sqrt{2}}{2} \) and \( 2e^{z_1 + \varphi_1(z_2)} = 0 \), this is impossible.

Case 2. Suppose that \( f(z) + \frac{\partial f}{\partial z_1} \) is not a constant. In view of the fact that the entire solutions of the functional equation \( f^2 + g^2 = 1 \) are \( f = \cos a(z), g = \sin a(z) \), where \( a(z) \) is an entire function, it follows that there exists an entire function \( h(z) \) such that

\[
(3.15) \quad f(z) + \frac{\partial f}{\partial z_1} = \sin h(z), \quad f(z) + \frac{\partial^2 f}{\partial z_1^2} = \cos h(z).
\]

Thus, it yields that

\[
(3.16) \quad \frac{\partial f}{\partial z_1} - \frac{\partial^2 f}{\partial z_1^2} = \sin h(z) - \cos h(z),
\]

\[
(3.17) \quad \frac{\partial f}{\partial z_1} + \frac{\partial^2 f}{\partial z_1^2} = \frac{\partial h}{\partial z_1} \cos h(z).
\]

In view of (3.15) and (3.17), we have

\[
(3.18) \quad f(z) - \frac{\partial f}{\partial z_1} = \cos h(z) - \frac{\partial h}{\partial z_1} \cos h(z).
\]

Thus, it follows from (3.16) and (3.18) that

\[
\frac{\partial f}{\partial z_1} - \frac{\partial^2 f}{\partial z_1^2} = - \frac{\partial h}{\partial z_1} \sin h(z) - \frac{\partial^2 h}{\partial z_1^2} \cos h(z) + \left( \frac{\partial h}{\partial z_1} \right)^2 \sin h(z) = \sin h(z) - \cos h(z),
\]

that is,

\[
(3.19) \quad \left( 1 + \frac{\partial h}{\partial z_1} - \left( \frac{\partial h}{\partial z_1} \right)^2 \right) \sin h(z) = \left( 1 - \frac{\partial^2 h}{\partial z_1^2} \right) \cos h(z).
\]

Since \( f(z) \) is a finite order transcendental entire solution of equation (2.2), by Lemma 2.1, 2.2 and (3.15), we conclude that \( h(z) \) is a nonconstant polynomial in \( \mathbb{C}^2 \).

If \( 1 - \frac{\partial^2 h}{\partial z_1^2} = 0 \), then \( \frac{\partial h}{\partial z_1} = z_1 + \varphi(z_2) \), where \( \varphi(z_2) \) is a polynomial in \( z_2 \). Thus, it follows that

\[
1 + \frac{\partial h}{\partial z_1} - \left( \frac{\partial h}{\partial z_1} \right)^2 = 1 + z_1 + \varphi(z_2) - (z_1 + \varphi(z_2))^2 \neq 0.
\]

In view of (3.19), we conclude that \( h(z) \) is a constant, this is impossible.
If \(1 + \frac{\partial h}{\partial z_1} - \left(\frac{\partial h}{\partial z_1}\right)^2 = 0\), then it follows that \(\frac{\partial^2 h}{\partial z_1^2} (1 - 2 \frac{\partial h}{\partial z_1}) = 0\). Thus, it yields that \(\frac{\partial^2 h}{\partial z_1^2} = 0\) or \(\frac{\partial h}{\partial z_1} = \frac{1}{2}\). Hence, we have \(1 - \frac{\partial^2 h}{\partial z_1^2} \neq 0\). Thus, we also deduce from (3.19) that \(h(z)\) is a constant, this is impossible.

From the above discussion, we can see that \(1 + \frac{\partial h}{\partial z_1} - \left(\frac{\partial h}{\partial z_1}\right)^2 \neq 0\) and \(1 - \frac{\partial^2 h}{\partial z_1^2} \neq 0\), then it follows from (3.19) that

\[
\tan h(z) = \frac{1 - \frac{\partial^2 h}{\partial z_1^2}}{1 + \frac{\partial h}{\partial z_1} - \left(\frac{\partial h}{\partial z_1}\right)^2}.
\]

Thus, similar to the argument as in the proof of Case 2 in Theorem 2.1, a contradiction can be obtained from (3.20) using Nevanlinna theory.

Therefore, this completes the proof of Theorem 2.2.

3.3. The Proof of Theorem 2.3. Suppose that \(f(z)\) is a transcendental entire solution with finite order of equation (2.3). We will discuss two cases as follows.

Case 1. Suppose that \(f(z) + \frac{\partial f}{\partial z_1}\) is a constant. Set

\[
f(z) + \frac{\partial f}{\partial z_1} = K_1, \quad K_1 \in \mathbb{C}.
\]

In view of equation (2.3), it follows that \(f(z) + \frac{\partial^2 f}{\partial z_1 \partial z_2}\) is a constant, let

\[
f(z) + \frac{\partial^2 f}{\partial z_1 \partial z_2} = K_2, \quad K_2 \in \mathbb{C}.
\]

Thus, it leads to \(K_1^2 + K_2^2 = 1\). In view of (3.21) and (3.22), it yields that

\[
\frac{\partial f}{\partial z_1} - \frac{\partial^2 f}{\partial z_1 \partial z_2} = K_1 - K_2,
\]

\[
f(z) - \frac{\partial f}{\partial z_2} = K_2.
\]

Thus, it follows from (3.24) that

\[
\frac{\partial f}{\partial z_1} - \frac{\partial^2 f}{\partial z_2 \partial z_1} = 0,
\]

which means that \(K_1 = K_2 = \pm \frac{\sqrt{2}}{2}\) from \(\frac{\partial^2 f}{\partial z_2 \partial z_1} = \frac{\partial^2 f}{\partial z_1 \partial z_2}\). And by (3.21) and (3.24), we have

\[
\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2} = 0.
\]

By solving equation (3.26), it follows that \(f(z) = \varphi(z_1 - z_2)\), where \(\varphi(s)\) is a transcendental entire function with finite order in \(s := z_1 - z_2\). Substituting \(f(z) = \varphi(z_1 - z_2)\) into (3.21), it yields

\[
\varphi(s) + \varphi'(s) = \pm \frac{\sqrt{2}}{2},
\]

which means that \(\varphi(s) = \pm \frac{\sqrt{2}}{2} + \eta e^{-s}, \eta \in \mathbb{C}\). Thus, we have \(f(z) = \pm \frac{\sqrt{2}}{2} + \eta e^{z_2 - z_1}\). 

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**Case 2.** Suppose that \( f(z) + \frac{\partial f}{\partial z_1} \) is not a constant. Similar to the argument as in Case of Theorem 2.2, we can obtain that there exists an entire function \( h(z) \) such that

\[
(3.27) \quad f(z) + \frac{\partial f}{\partial z_1} = \sin h(z), \quad f(z) + \frac{\partial^2 f}{\partial z_1 \partial z_2} = \cos h(z),
\]
and

\[
(3.28) \quad \left(1 + \frac{\partial h}{\partial z_1} - \frac{\partial h}{\partial z_1} \frac{\partial h}{\partial z_2} \right) \sin h(z) = \left(1 - \frac{\partial^2 h}{\partial z_1 \partial z_2} \right) \cos h(z).
\]

Since \( f(z) \) is a finite order transcendental entire solution of equation (2.3), by Lemmas 2.1, 2.2 and (3.27), we conclude that \( h(z) \) is a nonconstant polynomial in \( \mathbb{C}^2 \).

If \( 1 - \frac{\partial^2 h}{\partial z_1 \partial z_2} = 0 \), then it follows \( h(z) = z_1 z_2 + \varphi_1(z_1) + \varphi_2(z_2) \), where \( \varphi_1(z_1), \varphi_2(z_2) \) are polynomials in \( z_1, z_2 \) in \( \mathbb{C} \), respectively. Substituting this into \( 1 + \frac{\partial h}{\partial z_1} - \frac{\partial h}{\partial z_1} \frac{\partial h}{\partial z_2} \), we have

\[
1 + \frac{\partial h}{\partial z_1} - \frac{\partial h}{\partial z_1} \frac{\partial h}{\partial z_2} = z_1 z_2 + z_2 (\varphi_2'(z_2) - 1) + (z_1 - 1) \varphi_1'(z_1) + \varphi_1'(z_1) \varphi_2'(z_2) - 1 \neq 0.
\]

Combining this with (3.28), we have that \( \sin h(z) = 0 \), that is, \( h(z) \) is a constant, a contradiction.

If \( 1 - \frac{\partial^2 h}{\partial z_1 \partial z_2} \neq 0 \) and \( 1 + \frac{\partial h}{\partial z_1} - \frac{\partial h}{\partial z_1} \frac{\partial h}{\partial z_2} \neq 0 \), then it follows from (3.28) that

\[
(3.29) \quad \tan h(z) = \frac{1 - \frac{\partial^2 h}{\partial z_1 \partial z_2}}{1 + \frac{\partial h}{\partial z_1} - \frac{\partial h}{\partial z_1} \frac{\partial h}{\partial z_2}}.
\]

Thus, similar to the argument as in the proof of Case 2 in Theorem 2.1, a contradiction can be obtained from (3.29) using Nevanlinna theory.

Therefore, this completes the proof of Theorem 2.3.

4. PROOFS OF THEOREMS 2.4-2.5

4.1. **The Proof of Theorem 2.4.** Suppose that \( f(z) \) is a transcendental entire solution with finite order of equation (2.4). Two cases will be discussed below.

**Case 1.** Suppose that \( f(z + c) + \frac{\partial f}{\partial z_1} \) is a constant. Set

\[
(4.1) \quad f(z + c) + \frac{\partial f}{\partial z_1} = K_1, \quad K_1 \in \mathbb{C}.
\]

In view of equation (2.4), it follows that \( f(z + c) + \frac{\partial f}{\partial z_2} \) is a constant, let

\[
(4.2) \quad f(z + c) + \frac{\partial f}{\partial z_2} = K_2, \quad K_2 \in \mathbb{C}.
\]

Thus, it leads to \( K_1^2 + K_2^2 = 1 \). In view of (4.1) and (4.2), it yields that

\[
(4.3) \quad \frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2} = K_1 - K_2.
\]

The characteristic equations of (4.3) are

\[
\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = -1, \quad \frac{df}{dt} = K_1 - K_2.
\]
Using the initial conditions: \( z_1 = 0, z_2 = s \), and \( f = f(0, s) := \phi(s) \) with a parameter \( s \). Thus, we obtain the following parametric representation for the solutions of the characteristic equations: \( z_1 = t, z_2 = -t + s \),

\[
f(t, s) = \int_0^t (K_1 - K_2) dt + \phi(s) = (K_1 - K_2)t + \phi(s),
\]

where \( \phi(s) \) is a transcendental entire function with finite order in \( s \). Then, by combining with \( t = z_1 \) and \( s = z_2 + z_1 \), the solution of equation (4.3) is of the form

\[
(4.4) \quad f(z_1, z_2) = (K_1 - K_2)z_1 + \phi(z_1 + z_2).
\]

Substituting (4.4) into (4.2) or (4.3), we obtain that

\[
(4.5) \quad (K_1 - K_2)z_1 + \phi(s + s_0) + (K_1 - K_2) + \phi'(s) = K_1.
\]

On the other hand, differentiating both two sides of the equations (4.1), (4.2) for the variables \( z_2, z_1 \), respectively, and combining with the fact \( \frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial^2 f}{\partial z_2 \partial z_1} \), it follows that

\[
\frac{\partial f(z + c)}{\partial z_1} = \frac{\partial f(z + c)}{\partial z_2},
\]

which implies that

\[
(4.6) \quad \frac{\partial f(z)}{\partial z_1} = \frac{\partial f(z)}{\partial z_2}, \quad K_1 = K_2 = \pm \frac{\sqrt{2}}{2}.
\]

In view of (4.4)-(4.6) and (2.4), we have

\[
f(z_1, z_2) = \phi(z_1 + z_2),
\]

where \( \phi(s) \) is a transcendental entire function in \( s \) with finite order satisfying \( \phi'(s) + \phi(s + s_0) = \pm \frac{\sqrt{2}}{2} \).

**Case 2.** Suppose that \( f(z + c) + \frac{\partial f}{\partial z_1} \) is not a constant. The fact that the entire solutions of equation \( f^2 + g^2 = 1 \) are \( f = \cos a(z), g = \sin a(z) \) can deduce that \( f(z + c) + \frac{\partial f}{\partial z_1} \) is transcendental, where \( a(z) \) is an entire function. Thus, we rewrite (2.4) as the following form

\[
(4.7) \quad \left[ f(z + c) + \frac{\partial f}{\partial z_1} + i \left( f(z + c) + \frac{\partial f}{\partial z_2} \right) \right] \left[ f(z + c) + \frac{\partial f}{\partial z_1} - i \left( f(z + c) + \frac{\partial f}{\partial z_2} \right) \right] = 1,
\]

which implies that both \( f(z + c) + \frac{\partial f}{\partial z_1} + i \left( f(z + c) + \frac{\partial f}{\partial z_2} \right) \) and \( f(z + c) + \frac{\partial f}{\partial z_1} - i \left( f(z + c) + \frac{\partial f}{\partial z_2} \right) \) have no poles and zeros. Thus, by Lemmas 2.1 and 2.2, there thus exists a polynomial \( p(z) \) such that

\[
f(z + c) + \frac{\partial f}{\partial z_1} + i \left( f(z + c) + \frac{\partial f}{\partial z_2} \right) = e^{p(z)},
\]

\[
f(z + c) + \frac{\partial f}{\partial z_1} - i \left( f(z + c) + \frac{\partial f}{\partial z_2} \right) = e^{-p(z)}.
\]
these lead to
\begin{equation}
(4.8)
   f(z + c) + \frac{\partial f(z)}{\partial z_1} = \frac{e^{p(z)} + e^{-p(z)}}{2},
\end{equation}
\begin{equation}
(4.9)
   f(z + c) + \frac{\partial f(z)}{\partial z_2} = \frac{e^{p(z)} - e^{-p(z)}}{2i}.
\end{equation}

Hence, it yields
\begin{equation}
(4.10)
   \frac{\partial f(z)}{\partial z_1} - \frac{\partial f(z)}{\partial z_2} = \frac{1 + i}{2} e^{p(z)} + \frac{1 - i}{2} e^{-p(z)}.
\end{equation}

Differentiating on $z_2, z_1$ for both two sides of equations (4.8), (4.9), respectively, and combining with the fact $\frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial^2 f}{\partial z_2 \partial z_1}$, we conclude that
\begin{equation}
(4.11)
   \frac{\partial f(z + c)}{\partial z_1} - \frac{\partial f(z + c)}{\partial z_2} = -\frac{1}{2} (i \frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2}) e^{p(z)} + \frac{1}{2} (1 - i) e^{-p(z)}.
\end{equation}

Thus, it follows from (4.10) and (4.11) that
\begin{equation}
(4.12)
   \frac{1}{1 - i} \left( i \left( \frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} \right) e^{p(z) + p(z+c)} - 1 \left( i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2} \right) e^{p(z+c) - p(z)} - \frac{1 + i}{1 - i} e^{2p(z+c)} \right) = 0.
\end{equation}

If $i \frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} = 0$, then $i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2} \neq 0$. Otherwise, $-\frac{1 + i}{1 - i} e^{2p(z+c)} = 1$, this means that $p(z)$ is a constant, which implies $f(z + c) + \frac{\partial f(z)}{\partial z_1}$ is a constant. This is a contradiction with the assumption at the begin of Case 2. Then (4.12) becomes
\begin{equation}
(4.13)
   e^{2p(z+c)} = -\frac{1}{1 + i} (i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2}) e^{p(z+c) - p(z)} - \frac{1 - i}{1 + i}.
\end{equation}

Combining with the fact that $p(z)$ is a polynomial, it is easy to get that $N(r, \frac{1}{e^{2p(z+c)}}) = 0$, $N(r, e^{2p(z+c)}) = 0$ and $N(r, \frac{1}{1 + i} (i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2}) e^{p(z+c) - p(z)}) = o(T(r, e^{2p}))$. Then by the Nevanlinna second main theorem in several complex variables, and in view of (4.13), we conclude
\begin{align*}
T(r, e^{2p(z+c)}) & \leq N(r, \frac{1}{e^{2p(z+c)}}) + N(r, \frac{1}{1 + i} (i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2}) e^{p(z+c) - p(z)}) + \frac{1}{1 + i} (i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2}) e^{p(z+c)} + S(r, e^{2p(z+c)}) \\
& \leq N(r, \frac{1}{1 + i} (i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2}) e^{p(z+c) - p(z)}) + S(r, e^{2p(z+c)}) \\
& = o(T(r, e^{2p(z+c)})),
\end{align*}
which leads to a contradiction. Hence, it yields $i \frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} \neq 0$. Similarly, we have $i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2} \neq 0$. By Lemma 2.3, and combining with the fact that $e^{p(z+c) + p(z)}$ is a nonconstant, it yields that
\begin{equation}
(4.14)
   \frac{1}{1 - i} \left( i \frac{\partial p}{\partial z_1} - \frac{\partial p}{\partial z_2} \right) e^{p(z+c) - p(z)} = 1.
\end{equation}

Thus, in view of (4.12), it follows that
\begin{equation}
(4.15)
   \frac{1}{1 + i} \left( i \frac{\partial p}{\partial z_1} + \frac{\partial p}{\partial z_2} \right) e^{p(z) - p(z+c)} = 1.
\end{equation}

Since $p(z)$ is a polynomial, then Eq. (4.14) (or (4.15)) implies $p(z + c) - p(z) = \zeta$, where $\zeta$ is a constant in $\mathbb{C}$. Thus, it follows that $p(z) = L(z) + H(z) + B$, where
\( L(z) = \alpha_1 z_1 + \alpha_2 z_2, \) \( H(z) := H(s_1), \) \( H(s_1) \) is a polynomial in \( s_1 = c_2 z_1 - c_1 z_2. \) Here, we will prove that \( H(z) \equiv 0. \)

In view of (4.14) and (4.15), it yields that

\[
\begin{align*}
(4.16) & \quad -\frac{1}{1-i} [i\alpha_1 - \alpha_2 + H'(c_1 + ic_2)] = A^{-1}, \\
(4.17) & \quad -\frac{1}{1+i} [i\alpha_1 + \alpha_2 - H'(c_1 - ic_2)] = A,
\end{align*}
\]

where \( A = e^{L(c)}. \) This implies that both \( H'(c_1 + ic_2) \) and \( H'(c_1 - ic_2) \) are constants. By combining with the fact \( (c_1, c_2) \neq (0, 0), \) it follows that \( H' \) is a constant, that is, \( \deg_s H \leq 1. \) Thus, the form of \( L(z) + H(z) + B \) is still the linear form of \( \alpha_1 z_1 + \alpha_2 z_2 + B, \) which means that \( H(z) \equiv 0. \) Hence, it follows that \( p(z) = L(z) + B = \alpha_1 z_1 + \alpha_2 z_2 + B. \) Thus, we can deduce from (4.16) and (4.17) that

\[
-\frac{1}{1-i} (i\alpha_1 - \alpha_2) e^{L(c)} = 1, \quad -\frac{1}{1+i} (i\alpha_1 + \alpha_2) e^{-L(c)} = 1,
\]

this leads to

\[
(4.18) \quad \alpha_1^2 + \alpha_2^2 = -2, \quad e^{2L(c)} = \frac{(i\alpha_1 + \alpha_2)^2}{2i}.
\]

On the other hand, it follows from (4.10) that

\[
\begin{align*}
(4.19) & \quad \frac{\partial f(z)}{\partial z_1} - \frac{\partial f(z)}{\partial z_2} = \frac{1+i}{2} e^{\alpha_1 z_1 + \alpha_2 z_2 + B} + \frac{1-i}{2} e^{-(\alpha_1 z_1 + \alpha_2 z_2 + B)}.
\end{align*}
\]

The characteristic equations of (4.19) are

\[
\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = -1, \quad \frac{df}{dt} = \frac{1+i}{2} e^{\alpha_1 z_1 + \alpha_2 z_2 + B} + \frac{1-i}{2} e^{-(\alpha_1 z_1 + \alpha_2 z_2 + B)}.
\]

Using the initial conditions: \( z_1 = 0, z_2 = s, \) and \( f = f(0,s) := \varphi_0(s) \) with a parameter \( s. \) Thus, we obtain the following parametric representation for the solutions of the characteristic equations: \( z_1 = t, \) \( z_2 = t + s, \)

\[
\begin{align*}
f(t,s) & = \int_0^t \left( \frac{1+i}{2} e^{(\alpha_1 - \alpha_2) t + \alpha_2 s + B} + \frac{1-i}{2} e^{-[(\alpha_1 - \alpha_2) t + \alpha_2 s + B]} \right) dt + \varphi_0(s) \\
& = \frac{1+i}{2(\alpha_1 - \alpha_2)} e^{(\alpha_1 - \alpha_2) t + \alpha_2 s + B} - \frac{1-i}{2(\alpha_1 - \alpha_2)} e^{-[(\alpha_1 - \alpha_2) t + \alpha_2 s + B]} + \varphi(s),
\end{align*}
\]

where \( \varphi(s) \) is an entire function with finite order in \( s \) such that

\[
\varphi(s) = \varphi_0(s) - \frac{1+i}{2(\alpha_1 - \alpha_2)} e^{\alpha_2 s + B} + \frac{1-i}{2(\alpha_1 - \alpha_2)} e^{-(\alpha_2 s + B)}.
\]

Thus, it yields that

\[
(4.20) \quad f(z_1,z_2) = \frac{1+i}{2(\alpha_1 - \alpha_2)} e^{\alpha_1 z_1 + \alpha_2 z_2 + B} - \frac{1-i}{2(\alpha_1 - \alpha_2)} e^{-(\alpha_1 z_1 + \alpha_2 z_2 + B)} + \varphi(s).
\]

Substituting (4.20) into (4.8), and combining with (4.18), we can deduce that \( \varphi(s) \) satisfies

\[
(4.21) \quad \varphi'(s) + \varphi(s + s_0) = 0.
\]

Therefore, from Case 1 and Case 2, this completes the proof of Theorem 2.4.
4.2. **The Proof of Theorem 2.5.** Suppose that \( f(z) \) is a transcendental entire solution with finite order of equation (2.5). Two cases will be discussed below.

**Case 1.** Suppose that \( f(z + c) + \frac{\partial f}{\partial z_1} \) is a constant. Set

\[
(4.22) \quad f(z + c) + \frac{\partial f}{\partial z_1} = K_1, \quad K_1 \in \mathbb{C}.
\]

In view of equation (2.5), it follows that \( f(z + c) + \frac{\partial^2 f}{\partial z_1^2} \) is a constant, let

\[
(4.23) \quad f(z + c) + \frac{\partial^2 f}{\partial z_1^2} = K_2, \quad K_2 \in \mathbb{C}.
\]

Thus, it leads to \( K_1^2 + K_2^2 = 1 \). In view of (4.22) and (4.23), it yields that

\[
\frac{\partial f(z + c)}{\partial z_1} - f(z + c) = K_2,
\]

which leads to

\[
(4.24) \quad \frac{\partial^2 f(z + c)}{\partial z_1^2} - \frac{\partial f(z + c)}{\partial z_1} = 0.
\]

On the other hand, in view of (4.23) and (4.22), we have

\[
(4.25) \quad \frac{\partial^2 f(z)}{\partial z_1^2} - \frac{\partial f(z)}{\partial z_1} = K_2 - K_1.
\]

Eqs. (4.24) and (4.25) imply that \( K_2 - K_1 = 0 \), that is, \( K_2 = K_1 \). By combining with \( K_1^2 + K_2^2 = 1 \), it follows that \( K_1 = \pm \frac{1}{\sqrt{2}} \).

By solving equation (4.25), we obtain that

\[
(4.26) \quad f(z_1, z_2) = e^{z_1 + \varphi_1(z_2)} + \varphi_2(z_2),
\]

where \( \varphi_1(z_2), \varphi_2(z_2) \) are two functions in \( z_2 \).

Substituting (4.26) into (4.22) and (4.23), it yields that

\[
(4.27) \quad e^{(z_1 + c_1) + \varphi_1(z_2 + c_2)} + \varphi_2(z_2 + c_2) + e^{z_1 + \varphi_1(z_2)} = K_1.
\]

Thus, it follows from (4.27) that

\[
e^{\varphi_1(z_2 + c_2) - \varphi_1(z_2)} = -e^{-c_1}, \quad \varphi_2(z_2) = K_1,
\]

which implies that

\[
\varphi_1(z_2) = A_1 z_2 + B, \quad A_1 = -c_1 + 2k\pi i \pm \pi i.
\]

Thus, we conclude that

\[
f(z_1, z_2) = \pm \frac{\sqrt{2}}{2} + \eta_1 e^{z_1 + A_1 z_2},
\]

where \( \eta_1 = e^B \).

Therefore, this proves the conclusion of Theorem 2.5 (i).

**Case 2.** Suppose that \( f(z + c) + \frac{\partial f}{\partial z_1} \) is not a constant. Similar to the argument as in the proof of Case 2 in Theorem 2.4, there exists a nonconstant polynomial
7. In view of (4.28) and (4.29), we have

\begin{align*}
(4.30) \quad & f(z + c) - \frac{\partial f(z + c)}{\partial z_1} = -i + \frac{i \partial p}{\partial z_1} e^p + \frac{i \partial p}{\partial z_1} e^{-p}, \\
(4.31) \quad & \frac{\partial f(z)}{\partial z_1} - \frac{\partial^2 f(z)}{\partial z_1^2} = \frac{1 + i}{2} e^p + \frac{1}{2} - i \frac{e^{-p}}{2}.
\end{align*}

This leads to

\begin{align*}
(4.32) \quad & (1 + i) e^{p(z + c)} + (1 - i) e^{-p(z + c)} \\
& = - [Q_1(z) + Q_2(z)] e^p(z) + [Q_1(z) - Q_2(z)] e^{-p(z)},
\end{align*}

where

\begin{align*}
Q_1(z) = \frac{\partial^2 p}{\partial z_1^2}, \quad & Q_2(z) = i \frac{\partial p}{\partial z_1} + \left( \frac{\partial p}{\partial z_1} \right)^2.
\end{align*}

Next, we will prove that $Q_1(z) - Q_2(z)$ and $Q_1(z) + Q_2(z)$ can not be equal to 0.

Obviously, $Q_1(z) - Q_2(z) = 0$ and $Q_1(z) + Q_2(z) = 0$ can not hold at the same time. Otherwise, we have $(1 + i) e^{2p(z + c)} = 1 - i$, this is a contradiction. If $Q_1(z) + Q_2(z) = 0$ and $Q_1(z) - Q_2(z) \neq 0$, then it follows from (4.32) that

\begin{align*}
(4.33) \quad & (1 + i) e^{2p(z + c)} + (1 - i) = -2Q_2(z) e^{p(z + c)} - p(z).
\end{align*}

Since $p(z)$ is a nonconstant polynomial, then $e^{p(z + c)}$ is not a constant. the Nevanlinna second main theorem in several complex variables, and in view of (4.33), we conclude

\begin{align*}
T(r, e^{2p(z + c)}) \\
\leq N(r, \frac{1}{e^{2p(z + c)}}) + N(r, \frac{1}{e^{2p(z + c)} - \frac{i}{1 + i}}) + N(r, e^{2p(z + c)}) + S(r, e^{2p(z + c)}) \\
\leq N(r, \frac{1}{-2Q_2(z) e^{p(z + c)} - p(z)}) + S(r, e^{2p(z + c)}) \\
= o(T(r, e^{2p(z + c)})).
\end{align*}

This is impossible. If $Q_1(z) + Q_2(z) \neq 0$ and $Q_1(z) - Q_2(z) = 0$, then equation (4.32) leads to that $Q_1(z) = Q_2(z)$ and

\begin{align*}
(4.34) \quad & (1 + i) e^{2p(z + c)} + (1 - i) = -2Q_2(z) e^{p(z + c)} - p(z).
\end{align*}

Similarly, we can obtain a contradiction. Hence, we conclude that $Q_1(z) - Q_2(z) \neq 0$ and $Q_1(z) + Q_2(z) \neq 0$. Thus, it follows from (4.32) that

\begin{align*}
(4.35) \quad & - \frac{Q_1(z) + Q_2(z)}{1 - i} e^{p(z + c)} + \frac{Q_1(z) - Q_2(z)}{1 - i} e^{p(z + c)} - \frac{1 + i}{1 - i} e^{p(z + c)} = 1.
\end{align*}
Since \( p(z + c) + p(z) \) is not a constant, and by Lemma 2.3, it follows from (4.35) that
\[
\frac{Q_1(z) - Q_2(z)}{1 - i} e^{p(z+c) - p(z)} \equiv 1,
\]
and
\[
\frac{-Q_1(z) + Q_2(z)}{1 - i} e^{p(z) - p(z+c)} \equiv 1.
\]

Since \( p(z) \) is a polynomial, then Eq. (4.36) (or (4.36)) implies \( p(z + c) - p(z) = \zeta \), where \( \zeta \) is a constant in \( \mathbb{C} \). Similar to the argument as in the proof of Case 2 in Theorem 2.4, it follows that \( p(z) = L(z) + B \), where \( L(z) = \alpha_1 z_1 + \alpha_2 z_2 \), \( \alpha_1, \alpha_2, B \in \mathbb{C} \). Then, we have \( Q_1(z) = 0 \) and \( Q_2(z) = i\alpha_1 + \alpha_2^2 \). Equations (4.36) and (4.37) lead to
\[
-i\alpha_1 + \alpha_2^2 \frac{e^{L(c)}}{1 - i} = 1, \quad \frac{-i\alpha_1 + \alpha_2^2}{1 + i} e^{-L(c)} = 1,
\]
which lead to
\[
\alpha_2^2 (i + \alpha_1)^2 = 2, \quad e^{2L(c)} = e^{2(\alpha_1c_1 + \alpha_2c_2)} = -i.
\]

By combining with (4.30) and \( p(z) = L(z) + B = \alpha_1 z_1 + \alpha_2 z_2 + B \), we obtain
\[
f(z + c) = \frac{\alpha_1 + i}{2\alpha_1(\alpha_1 - 1)} e^{L(z) + B} + \frac{\alpha_1 + i}{2\alpha_1(\alpha_1 - 1)} e^{-L(z) - B} + \varphi(z_2) e^{z_1},
\]
where \( \varphi(z_2) \) is an entire function in \( z_2 \).

Substituting (4.40) into (4.28), and combining with (4.38), we obtain that
\[
\varphi(z_2 + c_2) = -\varphi(z_2) e^{-c_1},
\]
which implies that
\[
\varphi(z_2) = e^{A_2 z_2 + B_0}, \quad A_2 = \frac{-c_1 + 2k\pi i \pm \pi i}{c_2}.
\]

Thus, in view of (4.38), (4.40) and (4.41), we obtain
\[
f(z) = -\frac{1 + i}{2\alpha_1(\alpha_1 - 1)} e^{L(z) + B} - \frac{1 - i}{2\alpha_1(\alpha_1 - 1)} e^{-L(z) - B} + \eta_2 e^{z_1 + A_2 z_2},
\]
where \( \eta_2 = e^{B - c_1 + A_2 c_2} \). This completes the proof of Theorem 2.5 (ii).

Therefore, we prove the conclusions of Theorem 2.5.

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**Data Availability**

No data were used to support this study.
Competition of interests

The authors declare that none of the authors have any competing interests in the manuscript.

Author's contributions

Conceptualization, H. Y. Xu; writing-original draft preparation, H.Y. Xu; writing-review and editing, H. Y. Xu, J. Tu and H. Wang; funding acquisition, H. Y. Xu and J. Tu.

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