TWO MEROMORPHIC MAPPINGS HAVING THE SAME INVERSE IMAGES OF SOME MOVING HYPERPLANES WITH TRUNCATED MULTIPLICITY

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ABSTRACT. Let \( f \) and \( g \) be two meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \) and let \( a_1, \ldots, a_{2n+2} \) be \( 2n+2 \) moving hyperplanes which are slow with respect to \( f \) and \( g \). In this paper, we will show that if \( f \) and \( g \) have the same inverse images for all \( a_i \) (\( 1 \leq i \leq 2n+2 \)) with multiplicities counted to level \( l_i \) such that \( \sum_{i=1}^{2n+2} \frac{1}{l_i} \leq \frac{2}{\binom{2n+2}{n+1}} \), then the map \( f \times g \) into \( \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \) must be algebraically degenerated over the field \( \mathbb{R} \). Our result extends and improves the previous results in this topic.

1. Introduction

Let \( \mathcal{M} \) be the field of meromorphic functions on \( \mathbb{C}^m \) and \( \mathcal{R} \) be a subfield of \( \mathcal{M} \) which contains \( \mathbb{C} \). Let \( V \) be a projective subvariety of \( \mathbb{P}^N(\mathbb{C}) \) and \( (x_0 : \cdots : x_N) \) be a homogeneous coordinates of \( \mathbb{P}^N(\mathbb{C}) \). For a homogeneous polynomial \( Q \) in \( \mathcal{R}[x_0, x_1, \ldots, x_N] \) of degree \( d \) given by

\[
Q = \sum_{I=(i_0, \ldots, i_N)} a_I x_0^{i_0} \cdots x_N^{i_N}, a_I \in \mathcal{R},
\]

we say that \( Q \) does not identically vanish on \( V \) if there exists a point \( z_0 \in \mathbb{C}^m \) such that the polynomial \( Q(z_0) = \sum_{I=(i_0, \ldots, i_N)} a_I(z_0) x_0^{i_0} \cdots x_N^{i_N} \) does not identically vanish on \( V \).

DEFINITION 1.1. A meromorphic mapping \( f \) from \( \mathbb{C}^m \) into \( V \) is said to be algebraically degenerate over \( \mathcal{R} \) if the image of \( f \) is contained in a proper algebraic subvariety of \( V \) over \( \mathcal{R} \), i.e., there exists a homogeneous polynomial \( Q \in \mathcal{R}[x_0, x_1, \ldots, x_N] \) not identically vanishing on \( V \) such that

\[
Q(f_0, f_1, \ldots, f_N) \equiv 0
\]

for a presentation \((f_0 : f_1 : \cdots : f_N)\) of \( f \) (when we consider \( f \) as a mapping into \( \mathbb{P}^N(\mathbb{C}) \)).

We see that the above definition does not depend on the choice of the presentation of \( f \). If \( \mathcal{R} = \mathbb{C} \), we just say that \( f \) is algebraically degenerate.

In 1999, Fujimoto [2] showed that if two meromorphic mappings \( f \) and \( g \) of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \) have the same inverse images of \( 2n+2 \) hyperplanes in general position with multiplicities truncated by a level \( l_0 \), then \( f \times g \) into \( \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \) is algebraically degenerate. His result is stated as follows.
Theorem 1.2 (see [2, Theorem 1.5]). Let $H_1, \ldots, H_{2n+2}$ be hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Then there exists an integer $l_0$ such that, for any two meromorphic mappings $f$ and $g$ of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$, if $\min(v_{f,H_i}, l_0) = \min(v_{g,H_i}, l_0)$ $(1 \leq i \leq 2n+2)$ then the mapping $f \times g$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate.

Here, $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is considered as a projective subvariety of $\mathbb{P}^{(n+1)^2-1}(\mathbb{C})$ by Segre embedding, $f \times g$ is a mapping from $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ defined by

$$(f \times g)(z) = (f(z), g(z)) \in \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$$

for all $z$ outside the union of the indeterminacy loci of $f$ and $g$, $v_{(f,H_i)}$ is the pullback of divisor $H_i$ by $f$.

In [4], S. D. Quang and L. N. Quynh extended the above result of Fujimoto to the case of moving targets, also the number $l_0$ is explicitly estimated by $l_0 > 3n^3q(q - 2)$, where $q = \binom{2n+2}{n+1}$. In another direction, recently the author [6] have successfully generalized the above result of H. Fujimoto to the case of meromorphic mappings on Kähler manifold having the same inverse images of 2n+2 hyperplanes. Motivated by the technique in [6], in this note we will extend and improve the result in [4] by considering the case where the truncated levels $l_0$ are different for each moving hyperplanes, moreover this number is also reduced. To state our result, we first recall the following from [4].

Let $f$ be a meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$ and let $a$ be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})^*$. We call $a$ is a moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$. Such $a$ is said to be slow with respect to $f$ if $\| T(r,a) = o(T(r,f))$ as $r \rightarrow \infty$ (see Section 2 for the notations). Similarly, a meromorphic function $\varphi$ on $\mathbb{C}^n$ is said to be “small” with respect to $f$ if $\| T(r,\varphi) = o(T(r,f))$ as $r \rightarrow \infty$. Suppose that $f$ and $a$ have reduced representations $(f_0 : \cdots : f_n)$ and $(a_0 : \cdots : a_n)$ respectively. By changing the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$ if necessary, throughout this paper, for each such $a$, we always assume that $a_0 \neq 0$ and set $\tilde{a} = \left(\frac{a_0}{a_0}, \frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right)$. We also define

$$(f, a) = \sum_{i=0}^{n} a_i f_i \text{ and } (f, \tilde{a}) = \sum_{j=0}^{n} \tilde{a}_j f_j.$$  

Hence the divisor of zeros $v_{(f,a)}$ of the function $(f,a)$ does not depend on the choice of these representations. Also the function $(f, \tilde{a})$ does not depend on the choice of the reduced representation of $a$.

Definition 1.3. Let $a_1, \ldots, a_q$ $(q \geq n+1)$ be $q$ moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$ with reduced representations $a_i = (a_{i1} : \cdots : a_{in})$ $(1 \leq i \leq q)$. We say that $a_1, \ldots, a_q$ are located in general position if $\det(a_{ij}) \neq 0$ for any $1 \leq i_0 < i_1 < \cdots < i_n \leq q$.

Throughout this paper, we will denote by $R_{(a)}(\mathbb{M}_{\mathbb{C}})$ the smallest subfield of $\mathbb{M}$ containing $\mathbb{C}$ and all $a_{jk}/a_{ji}$ with $a_{ji} \neq 0$. Our main theorem is stated as follows.

Main Theorem. Let $f$ and $g$ be two meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$. Let $a_1, \ldots, a_{2n+2}$ be $(2n+2)$ meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position, which are slow with respect to $f$ and $g$. Let $l_i$ $(1 \leq i \leq 2n+2)$ be positive integers or infinite. Assume that $\min(v_{(f,a_i)}, l_i) =$
where \( q = \binom{2n+2}{n+2} \). Then the map \( f \times g \) into \( \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \) is algebraically degenerate over \( R_{(a_i)}^{2n+2} \).

Our result will valid with \( l_1 = \cdots = l_{2n+2} = l_0 > 3n^2(n+1)q(q-2) \), which is better than the result of H. Fujimoto and S. D. Quang-L. N. Quynh.

**Remark 1.4.** 1) In [4], the author and L. N. Quynh have obtained the estimate \( l_0 \geq 3n^3(n+1)q(q-2) \). However, in the proof of [4, Theorem 1.1], the inequalities:

\[
\frac{n_1 q}{2} \sum_{i=1}^{2n+2} (N_{h_i}^{[1]}(r) + N_{1/h_i}^{[1]}(r)) + o(T(r,g)) \leq \frac{\nu q}{2} \sum_{i=1}^{2n+2} N_{(g,a_i),>l_0}^{[1]}(r) + o(T(r,g))
\]

(in lines 12-14, page 1562) are not correct and must be corrected by

\[
\frac{n_1 q}{2} \sum_{i=1}^{2n+2} (N_{h_i}^{[1]}(r) + N_{1/h_i}^{[1]}(r)) + o(T(r,g)) \leq \frac{n_1 q}{2} \sum_{i=1}^{2n+2} N_{(g,a_i),>l_0}^{[1]}(r) + o(T(r,g))
\]

Then the number \( (l_0 + 1) \) in the rest part of the proof of [4, Theorem 1.1] should be replaced by \( l_0 \).

Hence the corrected estimated for \( l_0 \) in [4, Theorem 1.1] should be \( l_0 > 3n^3(n+1)q(q-2) \). This gap will be fixed in this paper.

2) With the same reason, the inequality (in line 12, page 16 of the proof of [6, Theorem 1.1])

\[
\frac{q-2}{2} \sum_{i \in J} \sum_{i=1}^{2n+2} (v_{h_i}^{[1]} + v_{1/h_i}^{[1]}) \leq \frac{q(q-2)}{2(l_0+1)} \sum_{i=1}^{2n+2} v_{(g,H_i)}^{[1]}
\]

should be corrected by

\[
\frac{q-2}{2} \sum_{i \in J} \sum_{i=1}^{2n+2} (v_{h_i}^{[1]} + v_{1/h_i}^{[1]}) \leq \frac{q(q-2)}{2l_0} \sum_{i=1}^{2n+2} v_{(g,H_i)}^{[1]}
\]

Then the number \( (l_0 + 1) \) in [6, Theorem 1.1] should be replaced by \( l_0 \). The correction of [6, Theorem 1.1] is stated as follows.

**Theorem 1.5** (cf. [6, Theorem 1.1]). Let \( M \) be an \( m \)-dimensional complete Kähler manifold whose universal covering is biholomorphic to a ball \( \mathbb{B}^m(R_0) \) \( (0 < R_0 \leq +\infty) \) of \( \mathbb{C}^m \). Let \( f \) and \( g \) be two linearly nondegenerate meromorphic mappings of \( M \) into \( \mathbb{P}^n(\mathbb{C}) \), and let \( H_1, \ldots, H_{2n+2} \) be \( 2n+2 \) hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) in general position. Let \( q = \binom{2n+2}{n+1} \) and \( p \) be a positive number such that \( (q-
2) \((q - 1)\rho < 2\). Assume that \(f\) and \(g\) satisfy the condition \((C_\rho)\) and \(\min\{v_{(f,H_1),l_0}\} = \min\{v_{(x,H_1),l_0}\}\) for all \(1 \leq i \leq 2n + 2\). If

\[
\frac{3n^2(q + 1)q(q - 2)}{l_0} + \rho \frac{(q - 2)(q - 1)}{2} < 1,
\]

then the mapping \(f \times g\) into \(\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})\) is algebraically degenerate over \(\mathbb{C}\).

Here, we say that \(f\) satisfies the condition \((C_\rho)\) if there exists a nonzero bounded continuous real-valued function \(h\) on \(M\) such that

\[
\rho \Omega_f + dd^c \log h^2 \geq \text{Ric}\omega,
\]

where \(\Omega_f\) denotes the pull-back of the Fubini-Study metric form on \(\mathbb{P}^n(\mathbb{C})\) by \(f\), \(\omega\) is the Kähler form of \(M\).

2. Basic notions and auxiliary results from Nevanlinna theory

We recall some basic notions and auxiliary results from Nevanlinna theory from \([4, 5]\).

(a) We set \(|z| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}\) for \(z = (z_1, \ldots, z_m) \in \mathbb{C}^m\) and define

\[
\begin{align*}
B(r) &:= \{z \in \mathbb{C}^m; |z| < r\}, \quad S(r) := \{z \in \mathbb{C}^m; |z| = r\} \quad (0 < r < \infty), \\
\end{align*}
\]

Define

\[
v_{m-1}(z) := (dd^c|z|^2)^{m-1} \quad \text{and} \quad \sigma_m(z) := d^c \log |z|^2 \wedge (dd^c|z|^2)^{m-1}\text{ on } \mathbb{C}^m \setminus \{0\}.
\]

Let \(v\) be a divisor on \(\mathbb{C}^m\) which is given by a formal sum \(v = \sum v_\mu X_\mu\), where \(\{X_\mu\}\) is a locally finite family of irreducible analytic hypersurfaces in \(\mathbb{C}^m\) and \(v_\mu \in \mathbb{Z}\). We identify the divisor \(v\) with the function \(v(z)\) from \(\mathbb{C}^m\) into \(\mathbb{Z}\) defined as follows:

\[
v(z) = \sum_{X_\mu \ni z} v_\mu.
\]

For positive integers \(k, M\) or \(M = \infty\), we set \(v^{[M]}(z) = \min \{M, v(z)\}\) and define:

\[
\begin{align*}
v^{[M]}_{> k}(z) &= \begin{cases} v^{[M]}(z) & \text{if } v(z) > k, \\ 0 & \text{if } v(z) \leq k, \end{cases} \quad n(t) = \begin{cases} \int_{|v| \cap B(t)} v(z) v_{m-1} & \text{if } m \geq 2, \\ \sum_{|z| \leq t} v(z) & \text{if } m = 1. \end{cases}
\end{align*}
\]

Similarly, we define \(n^{[M]}(t)\) and \(n^{[M]}_{> k}(t)\).

The counting function of \(v\) is defined as follows

\[
N(r, v) = \int_{1}^{r} \frac{n(t)}{t^{2m-1}} dt \quad (1 < r < \infty).
\]

Similarly, we define \(N(r, v^{[M]}), N(r, v^{[M]}_{> k})\) and denote them by \(N^{[M]}(r, v), N^{[M]}_{> k}(r, v)\), respectively. For brevity, we will omit the character \([M]\) if \(M = \infty\).
For a nonzero meromorphic function $\varphi$ on $\mathbb{C}^m$, denote by $v_\varphi^0$ its divisor of zeros and set

$$N_\varphi(r) = N(r, v_\varphi^0), \quad N^{[M]}_\varphi(r) = N^{[M]}(r, v_\varphi^0), \quad N^{[M]}_{\varphi, > k}(r) = N^{[M]}_{\varphi, > k}(r, v_\varphi^0).$$

(b) Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with a reduced representation $f = (f_0 : \cdots : f_n)$. The characteristic function of $f$ is defined by

$$T(r, f) = \int_{S(r)} \log \|f\| \sigma_m - \int_{S(1)} \log \|f\| \sigma_m,$$

where $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$. We note that this definition does not depend on the choice of the reduced representation of $f$.

Let $a$ be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ with a reduced representation $a = (a_0 : \cdots : a_n)$. If $(f, a) \neq 0$, then we define

$$m_{f,a}(r) = \int_{S(r)} \frac{\|f\| \cdot |a|}{|(f,a)|} \sigma_m - \int_{S(1)} \frac{\|f\| \cdot |a|}{|(f,a)|} \sigma_m,$$

where $|a| = (|a_0|^2 + \cdots + |a_n|^2)^{1/2}$.

The first main theorem for moving hyperplanes (see [3]) states

$$T(r, f) + T(r, a) = m_{f,a}(r) + N_{(f,a)}(r).$$

Let $\varphi$ be a nonzero meromorphic function on $\mathbb{C}^m$, which is occasionally regarded as a meromorphic map into $\mathbb{P}^1(\mathbb{C})$. The proximity function of $\varphi$ is defined by

$$m(r, \varphi) := \int_{S(r)} \log \max (|\varphi|, 1) \sigma_m.$$

(c) The following play essential roles in Nevanlinna theory.

**Theorem 2.1** (see [7, Theorem 2.1]). \emph{Let $f = (f_0 : \cdots : f_n)$ be a reduced representation of a meromorphic mapping $f$ of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Assume that $f_{n+1}$ is a holomorphic function with $f_0 + \cdots + f_n + f_{n+1} = 0$. If $\sum_{i \in I} f_i \neq 0$ for all subsets $I \subseteq \{0, \ldots, n+1\}$, then}

$$\|T(r, f)\| \leq \sum_{i=0}^{n+1} N^{[i]}_{f_i}(r) + o(T(r, f)).$$

**Theorem 2.2** (see [3] and [2, Theorem 5.5]). \emph{Let $f$ be a nonzero meromorphic function on $\mathbb{C}^m$. Then}

$$\left\| m\left(r, \frac{\varphi^\alpha(f)}{f}\right) \right\| = O(\log^+ T(r, f)) \quad (\alpha \in \mathbb{Z}^m_+).$$

Here, by the notation “$\| P \|$, we mean the assertion $P$ holds for all $r \in [0, \infty)$ excluding a Borel subset $E$ of the interval $[0, \infty)$ with $\int_E dr < \infty$.

**Theorem 2.3** (see [3, Theorem 5.2.29]). \emph{Let $f$ be a meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_n)$. Suppose that $f_k \neq 0$. Then}

$$T(r, \frac{f_j}{f_k}) \leq T(r, f) \leq \sum_{j=0}^n T(r, \frac{f_j}{f_k}) + O(1).$$
3. Proof of Main Theorem

In order to prove Main Theorem, we need the following algebraic propositions.

Let $H_1, \ldots, H_{2n+1}$ be $(2n+1)$ hyperplanes of $\mathbb{P}(\mathbb{C})$ in general position given by

$$H_i : x_0\omega_0 + x_1\omega_1 + \cdots + x_n\omega_n = 0 \quad (1 \leq i \leq 2n+1).$$

We consider the rational map $\Phi : \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^{2n}(\mathbb{C})$ as follows:

For $v = (v_0 : v_1 : \cdots : v_n), \ w = (w_0 : w_1 : \cdots : w_n) \in \mathbb{P}^n(\mathbb{C})$, we define the value $\Phi(v, w) = (u_1 : \cdots : u_{2n+1}) \in \mathbb{P}^{2n}(\mathbb{C})$ by

$$u_i = \frac{x_0 v_0 + x_1 v_1 + \cdots + x_n v_n}{x_0 w_0 + x_1 w_1 + \cdots + x_n w_n}.$$

**Proposition 3.1** (see [2, Proposition 5.9]). The map $\Phi$ is a birational map of $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ to $\mathbb{P}^{2n}(\mathbb{C})$.

Now let $b_1, \ldots, b_{2n+1}$ be $(2n+1)$ moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with reduced representations

$$b_i = (b_{i0} : b_{i1} : \cdots : b_{in}) \quad (1 \leq i \leq 2n+1).$$

Let $f$ and $g$ be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representations

$$f = (f_0 : \cdots : f_n) \quad \text{and} \quad g = (g_0 : \cdots : g_n).$$

Define $h_i = (f, b_i)/(g, b_i)$ $(1 \leq i \leq 2n+1)$ and $h_I = \prod_{i \in I} h_i$ for each subset $I$ of $\{1, \ldots, 2n+1\}$. Set

$$\mathcal{I} = \{(i_1, \ldots, i_n) ; 1 \leq i_1 < \cdots < i_n \leq 2n+1\}.$$ We have the following proposition.

**Proposition 3.2** (see [4, Proposition 3.2]). If there exist functions $A_I \in \mathcal{R}^{2n+1}_{b_i} (I \in \mathcal{I})$, not all zero, such that

$$\sum_{I \in \mathcal{I}} A_I h_I \equiv 0,$$

then the map $f \times g$ into $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^n(\mathbb{C})$ is algebraically degenerate over $\mathcal{R}^{2n+1}_{b_i}$.

**Proposition 3.3** (see [4, Proposition 3.3]). Let $f$ be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $b_1, \ldots, b_{n+1}$ be moving hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position with reduced representations

$$f = (f_0 : \cdots : f_n), \ b_i = (b_{i0} : \cdots : b_{in}) \quad (1 \leq i \leq n+1).$$

Then for each regular point $z_0$ of the analytic subset $\bigcup_{i=1}^{n+1}\{z ; (f, b_i)(z) = 0\}$, which does not belong to the indeterminacy set of $f$, we have

$$\min_{1 \leq i \leq n+1} v^0_{(f, b_i)}(z_0) \leq v^0_{\det \Phi}(z_0),$$

where $\Phi$ is the matrix $(b_{ij})$.

**Lemma 3.4.** Let $f$ and $g$ be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $a_1, \ldots, a_q$ be $q \geq 2n+1$ meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position, which are slow with respect to $f$ and $g$. Assume that $\min(v_{(f, a_i)}, 1) = \min(v_{(g, a_i)}, 1) \quad (1 \leq i \leq q)$. Then $\|T(r, f) = O(T(r, g))$ and $\|T(r, g) = O(T(r, f))$. 


Proof. Assume that $f, g, a_i$ have reduced representations

$$f = (f_0 : \cdots : f_n), \quad g = (g_0 : \cdots : g_n), \quad a_i = (a_{i_0} : \cdots : a_{i_m}).$$

Then by the second main theorem for moving hyperplanes \cite[Corollary 1.2]{5}, we have

$$\| \frac{q}{n(n+2)} T(r,f) \| \leq \sum_{i=1}^{q} N_{(f,a_i)}^{[1]}(r) + o(T(r,f))$$

$$= \sum_{i=1}^{q} N_{(g,a_i)}^{[1]}(r) + o(T(r,f))$$

$$\leq q(T(r,g)) + o(T(r,f)).$$

Then we have $\| T(r,f) \| = O(T(r,g)).$ Similarly, we also have $\| T(r,g) \| = O(T(r,f)).$ \hfill \Box

PROOF OF MAIN THEOREM. Assume that $f, g, a_i$ have reduced representations

$$f = (f_0 : \cdots : f_n), \quad g = (g_0 : \cdots : g_n), \quad a_i = (a_{i_0} : \cdots : a_{i_m}).$$

We suppose contrarily that the map $f \times g$ is algebraically non-degenerate over $\mathcal{R}_{\{a_i\}_{i=1}^{2n+2}}$. In particular, we suppose that $f$ and $g$ are linear non-degenerate over $\mathcal{R}_{\{a_i\}_{i=1}^{2n+2}}$.

Define $h_i = (f, a_i)/(g, a_i)$ $(1 \leq i \leq 2n+2).$ Then $h_i/h_j$ does not depend on the choice of representations of $f$ and $g$. Since $\sum_{k=0}^{n} a_{ik}f_k - h_i \cdot \sum_{k=0}^{n} a_{ik}g_k = 0$ $(1 \leq i \leq 2n+2),$ we have

$$\Phi := \det (\tilde{a}_{0i}, \cdots, \tilde{a}_{im}, \tilde{a}_{0i}h_i, \cdots, \tilde{a}_{im}h_i; 1 \leq i \leq 2n+2) \equiv 0.$$

For each subset $I \subset \{1,2,\ldots,2n+2\}$, put $h_I = \prod_{i \in I} h_i, \quad \tilde{h}_I = \prod_{i \in I} \frac{h_i}{h_1}.$ Denote by $\mathcal{I}$ the set

$$\mathcal{I} = \{(i_1, \ldots, i_{n+1}) : 1 \leq i_1 < \cdots < i_{n+1} \leq 2n+2\}.$$

For each $I = (i_1, \ldots, i_{n+1}) \in \mathcal{I}$, define

$$A_I = (-1)^{(n+1)(n+2)/2+i_1+\cdots+i_{n+1}} \times \det (\tilde{a}_{i_I}, 1 \leq r \leq n+1, 0 \leq l \leq n)$$

$$\times \det (\tilde{a}_{j_I}, 1 \leq s \leq n+1, 0 \leq l \leq n),$$

where $I^c = (j_1, \ldots, j_{n+1}) \in \mathcal{I}$ such that $I \cup I^c = \{1,2,\ldots,2n+2\}.$ We have

$$\sum_{I \in \mathcal{I}} A_I h_I = 0.$$

Then there is a partition $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_k$ of $\mathcal{I}$ satisfying the following properties:

- $\mathcal{I}_t \cap \mathcal{I}_s = \emptyset$, $1 \leq t < s \leq k.$
- $\sum_{I \in \mathcal{I}_t} A_I h_I = 0$, $1 \leq t \leq k.$
- $\sum_{I \in J} A_I h_I \neq 0$ for any proper subset $J$ of $\mathcal{I}_t$, $1 \leq t \leq k.$

For each $1 \leq t \leq k$, we set $n_t = \frac{1}{2} \mathcal{I}_t - 2$ and assume that $\mathcal{I}_t = \{I_{0t}, \ldots, I_{n_{t}+1}t\}$. We denote by $F_t$ the meromorphic mapping from $\mathbb{C}^m$ into $\mathbb{P}^m(\mathbb{C})$ given by $F_t = (A_{I_{0t}} h_{I_{0t}} : \cdots : A_{I_{n_{t}+1}t} h_{I_{n_{t}+1}t})$ (outside an analytic set).
For each \( 1 \leq i \leq 2n+2 \), we define \( S(i) \) the set of all indices \( j \neq i \) such that there exist \( I, I' \in \mathcal{I} \) satisfying
\[
\frac{h_I}{h_{I'}} = \frac{h_i}{h_j},
\]

CLAIM 3.5. For each \( 1 \leq p \leq 2n+2 \), \( \# S(p) \geq n+1 \),
\[
T(r, \frac{h_I}{h_p}) \leq \sum_{j=1}^{2n+2} \frac{q(q-2)}{2l_j} T(r, g) + o(T(r, g)), \forall l \in S(p),
\]
and
\[
T(r, \frac{h_I}{h_p}) \leq \sum_{j=1}^{2n+2} \frac{q(q-2)}{I_j} T(r, g) + o(T(r, g)), \forall l \not\in S(p),
\]
where \( q = \binom{2n+2}{n+1} \).

Without loss of generality, we prove the claim for \( p = n+2 \). Indeed, suppose contrarily that \( \# S(n+2) \leq n \), we may assume that \( 1, \ldots, n+1 \not\in S(n+2) \). Put \( I_0 = (1, \ldots, n+1) \) and suppose that \( I_0 \in \mathcal{I} \). Since \( f \times g \) is supposed to be algebraically non-degenerate over \( \mathcal{R}_{(a_i)_{i=1}^{2n+2}} \),
\[
\sum_{s=0}^{n+1} A_{I_i}(z) \frac{W^{I_i}(z)}{V^{I_i}(z)} \equiv 0, \forall z \in \mathbb{C}^m,
\]
where \( W^I(z) = \prod_{i \in I} z_i \) and \( V^I(z) = \prod_{i \in I} 1 \). We take a point \( z_0 \) which is not zero neither pole of any \( A_I \) nor pole of any \( a_{ij} \), not in the indeterminacy loci of all \( a_i \) and such that \( I_0 \not\in \mathcal{I} \). Hence, we have
\[
\sum_{I_i \subset \{1, \ldots, n+2\}}^{n+1} A_{I_i}(z_0) \frac{W^{I_i}(z_0)}{V^{I_i}(z_0)} \equiv 0, \forall v.
\]
This yields that there exists \( 1 \leq s' \leq n+1 \) such that \( I_{s'} \subset \{1, \ldots, n+2\}, I_{s'} \neq I_0 \). Therefore \( n+2 \in I' \) and
\[
\frac{h_{I_0}}{h_I} = \frac{h_{I_{s'}}}{h_{n+2}},
\]
for some \( j \in \{1, \ldots, n+1\} \). This contradicts the supposition that \( 1, \ldots, n+1 \not\in S(n+2) \). Hence, we must have \( \# S(n+2) \geq n+1 \).

Now for \( l \in S(n+2) \), we may assume that there exist two elements \( I, I' \in \mathcal{I} \) satisfying
\[
\frac{h_I}{h_{I'}} = \frac{h_l}{h_{n+2}},
\]
We suppose that \( F_1 \) has a reduced representation \( F_1 = (uA_{I_0}a_{I_0}; \ldots; uA_{I_{n+1}}a_{I_{n+1}}) \), where \( u \) is a meromorphic function. Thus, by Theorem 2.1 and Theorem 2.3, we have
\[
T(r, \frac{h_I}{h_{n+2}}) \leq T(r, F_1) \leq \sum_{i=0}^{n+1} I_{uA_{I_i}a_{I_i}}(r) + o(T(r, g)).
\]
On the other hand, consider a point \( z \), which is neither a zero nor a pole of any \( A_{l_{1i}} \)'s. Then there is a neighborhood \( U \) of \( z \), which contains neither zeros nor poles of any \( A_{l_{1i}} \)'s. Hence, in this neighborhood \( F_1 \) will have a presentation

\[
\left( \frac{A_{h_{11}}(\Pi_{j \in h_{11}}(f, a_j))(\Pi_{j \in I_{h_{11}}}^j(g, a_j))}{\prod_{j=1}^{2n+2} v_j} : \cdots : \frac{A_{l_{1n}}(\Pi_{j \in l_{1n}}(f, a_j))(\Pi_{j \in I_{l_{1n}}}^j(g, a_j))}{\prod_{j=1}^{2n+2} v_j} \right),
\]

where \( v_j (1 \leq j \leq 2n+2) \) is a holomorphic function on \( U \) such that \( v_j = \min\{v(f, a_j), v(g, a_j)\} \). This implies that

\[
\sum_{i=0}^{n_1+1} v_{\lambda_{h_{11}}}^{[n_{11}]}(z) \leq n_1 \sum_{i=0}^{n_1+1} \left( \sum_{j \in h_{11}} v_{f, a_j}^{[1]}(z) + \sum_{j \in l_{1n}} v_{g, a_j}^{[1]}(z) \right)
\]

\[
\leq (q-2) \sum_{l \not \in I} \left( \sum_{j \in l} v_{f, a_j}^{[1]}(z) + \sum_{j \in l} v_{g, a_j}^{[1]}(z) \right)
\]

\[
\leq (q-2) \sum_{l \not \in I} \sum_{j \in l} \min\{1, |v_{f, a_j}^{[1]}(z) - v_{g, a_j}^{[1]}(z)|\}
\]

\[
\leq (q-2) \sum_{l \not \in I} \sum_{j \in l} v_{g, a_j}^{[1]}(z)
\]

\[
= \frac{q(q-2)}{2} \sum_{j \in l} v_{g, a_j}^{[1]}(z).
\]

Hence, it implies that

\[
T(r, \frac{h_{1}}{h_{n+2}}) \leq \frac{q(q-2)}{2} \sum_{j=1}^{2n+2} N_{g, a_j}^{[1]}(r) + o(T(r, g))
\]

\[
\leq \frac{q(q-2)}{2} \sum_{j=1}^{2n+2} \frac{1}{l_j} N_{g, a_j}^{[1]}(r) + o(T(r, g))
\]

\[
\leq \sum_{j=1}^{2n+2} \frac{q(q-2)}{2l_j} T(r, g) + o(T(r, g)).
\]

For \( l \not \in S(n+2) \), since \( l \not \in S(l) \geq n+1 \), there exists \( l' \in S(l) \cap S(n+2) \). Hence, by the above proof, we have

\[
T(r, \frac{h_{1}}{h_{n+2}}) \leq T(r, \frac{h_{1}}{h_{l'}}) + T(r, \frac{h_{l'}}{h_{n+2}}) \leq \sum_{j=1}^{2n+2} \frac{q(q-2)}{l_j} T(r, g) + o(T(r, g)).
\]

The claim is proved.

We now continue the proof of Main Theorem. We see that there exist functions \( b_{ij} \in \mathcal{R}_{[a_i]}^{2n+2} \) \( (n + 2 \leq i \leq 2n+2, 1 \leq j \leq n+1) \) such that

\[
\tilde{a}_i = \sum_{j=1}^{n+1} b_{ij} \tilde{a}_j.
\]
By the identity (3.1), we have
\[ \det (\bar{a}_0, \ldots, \bar{a}_n, \bar{a}_0 h_i, \ldots, \bar{a}_n h_i; 1 \leq i \leq 2n+2) \equiv 0. \]

It easily implies that
\[ \det (\bar{a}_0 h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j, \ldots, \bar{a}_n h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j; n+2 \leq i \leq 2n+2) \equiv 0. \]

Therefore, the matrix
\[ \Psi = (\bar{a}_0 h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j, \ldots, \bar{a}_n h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j; n+2 \leq i \leq 2n+2) \]

has the rank at most \( n \).

Suppose that rank \( \Psi < n \). Then, the determinant of the square submatrix
\[ (\bar{a}_i h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j, \ldots, \bar{a}_i h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j; n+2 \leq i \leq 2n+1) \]

vanishes identically. It follows that \( f \times g \) is algebraically degenerate over \( \mathbb{R} \{ a \}_{i=1}^{2n+2} \) by Proposition 3.2. This is a contradiction. Hence rank \( \Psi = n \).

Without loss of generality, we may assume that the determinant of the square submatrix
\[ (\bar{a}_i h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j, \ldots, \bar{a}_n h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j; n+2 \leq i \leq 2n+1) \]

of \( \Psi \) does not vanish identically. On the other hand, we have
\[ (\bar{a}_0 h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j) g_0 + \cdots + (\bar{a}_n h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j) g_n = 0 \ (n+2 \leq i \leq 2n+1). \]

Thus
\[ (\bar{a}_0 h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j) \frac{h_0}{h_1} + \cdots + (\bar{a}_i h_i - \sum_{j=1}^{n+1} b_{ij} \bar{a}_j h_j) \frac{h_i}{h_1} g_{n-1} \frac{g_n}{h_1} = 0 \]
\[ = -\bar{a}_0 h_i \frac{h_i}{h_1} + \sum_{j=1}^{n+1} b_{ij} \bar{a}_j \frac{h_j}{h_1} (n+2 \leq i \leq 2n+1). \]

We regard the above identities as a system of \( n \) equations in unknown variables \( g_0/h_1, \ldots, g_{n-1}/h_1 \) and solve these to obtain that \( g_i/h_i \ (0 \leq i \leq n-1) \) has the form
\[ g_i/h_i = P_i/P_n, \]
where \( P_i \ (0 \leq i \leq n) \) are homogeneous polynomials in \( h_j/h_1 \ (1 \leq j \leq 2n+1) \) of degree \( n \) with coefficients in \( \mathbb{R} \{ a \}_{i=1}^{2n+2} \). Then, there are holomorphic functions \( \phi, \varphi \) such that \( g_j = \phi(\Psi) P_j \ (0 \leq j \leq n) \) and
\[ V_\phi = \min \{ V_{P_i}^0; 0 \leq i \leq n \}, \ V_\varphi = \max \{ V_{P_i}^\infty; 0 \leq i \leq n \} \leq n \sum_{j=2}^{2n+1} V_{h_j/h_1}^\infty. \]
By the definition of the characteristic function, we have

\[
T(r, g) = \lim_{S(r)} \int \log \sqrt{|g_0|^2 + \cdots + |g_n|^2} \sigma_m
\]

\[
= N_\varphi(r) - N_\varphi(r) + \lim_{S(r)} \int \log \sqrt{|P_0|^2 + \cdots + |P_n|^2} \sigma_m
\]

\[
\leq n \sum_{j=2}^{2n+1} N_{h_j/h_j}(r) + n \sum_{j=2}^{2n+1} m \left( r, \frac{h_j}{h_1} \right) + o(T(r, g))
\]

\[
= n \sum_{j=2}^{2n+1} T(r, \frac{h_j}{h_1}) + o(T(r, g))
\]

\[
\leq n \left( \sum_{2 \leq j \leq 2n+1} \sum_{s=1}^{2n+2} \frac{q(q-2)}{2I_s} T(r, g) + 2 \sum_{2 \leq j \leq 2n+1} \sum_{s=1}^{2n+2} \frac{q(q-2)}{2I_s} T(r, g) \right) + o(T(r, g)).
\]

\[
\leq 3n^2 \sum_{s=1}^{2n+2} \frac{q(q-2)}{2I_s} T(r, g) + o(T(r, g)),
\]

where the last inequality comes from the fact that there are at most \( n \) indices \( j \notin S(1) \). Letting \( r \to +\infty \), we get

\[
1 \leq 3n^2 \sum_{s=1}^{2n+2} \frac{q(q-2)}{2I_s}, \quad \text{i.e.,} \quad \sum_{s=1}^{2n+2} \frac{1}{I_s} \leq \frac{2}{3n^2(q(q-2))}.
\]

This is a contradiction. Thus the supposition is untrue.

Hence, \( f \times g \) is algebraically degenerate over \( \mathcal{R}_{\{a_i\}} \). The theorem is proved. \( \square \)

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References


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