Integral representations in the complex plane and iterated boundary value problems

Mohamed Akel, Heinrich Begehr, Alip Mohammed

South Valley University, Qena, Egypt; Math. Institute, FU Berlin, Arnimallee 3, D-14195 Berlin, Germany; Germany;
makel65@yahoo.com; begehrh@zedat.fu-berlin.de; ghalipm@gmail.com

Abstract. Fundamental solutions to differential operators lead to integral operators providing integral representation formulas for solutions to related differential equations. Proper modifications of the fundamental solutions result in integral operators which are related to certain boundary value problems. For complex partial differential operators of arbitrary order in the plane, fundamental solutions are achievable by properly integrating the Cauchy kernel. Particular such complex model differential operators are the poly-analytic and the poly-harmonic operators. A hierarchy of integral operators is available for these model operators leading to poly-analytic Cauchy-Schwarz and to poly-harmonic Green, Neumann, Robin, and hybrid Green integral operators. The theory is supplemented here by constructing a new poly-analytic Pompeiu (Pompeiu-Vekua) integral operator of any order adjusted to (iterated) Neumann boundary conditions.

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1 Complex integral operators.

Any theory of differential equations is based on the fundamental theorem of calculus. In complex analysis as also in the analysis of several real variables this is the Gauss divergence theorem. It leads to integral representation formulas when applied to functions convoluted with proper fundamental kernel functions. In complex analysis
the basic fundamental kernel is the Cauchy kernel $\frac{1}{\pi z}$ and the related representation is the Cauchy-Pompeiu formula linked to the Cauchy-Riemann differential operator $\partial_z = \frac{1}{2}(\partial_x + i\partial_y)$, where $z = x + iy$. A dual formula is available related to the differential operator $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. The Pompeiu integral operator

$$T f(z) = -\frac{1}{\pi} \int_D f(\zeta) \frac{d\xi d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta,$$

for (bounded) domains $D$ of the complex plane $\mathbb{C}$ and $f \in L^p(D; \mathbb{C}), \ 1 < p$, provides a particular (weak) solution to the inhomogeneous Cauchy-Riemann equation $w_z = f$ in $D$. The $T-$operator is extensively studied and applied by I.N. Vekua [1] in the theory of generalized analytic functions. This justifies to call $T$ the Pompeiu-Vekua operator. Related to the weakly singular Cauchy kernel is the singular kernel $\frac{1}{\pi z^2}$ leading to the Ahlfors-Beurling integral operator

$$\Pi f(z) = -\frac{1}{\pi} \int_D f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2},$$

a particular Calderon-Zygmund operator [2] playing an important role in the theory of quasi-conformal mappings and for the Beltrami equation [3]. Pompeiu-Vekua and Ahlfors-Beurling operators are particular cases of a hierarchy of integral operators related to model partial differential operators of arbitrary order, [4], including poly-analytic and poly-harmonic operators. They are just products of powers of the Cauchy-Riemann operator and powers of its complex conjugate, hence products of certain poly-analytic and poly-harmonic operators

$$\partial_z^m \partial_{\bar{z}}^n = (\partial_z \partial_{\bar{z}})^m \partial_z^{n-m}, \ m \leq n.$$

For positive $m$ and $n$ their fundamental solutions are $\frac{1}{\pi} \frac{z^{m-1}\bar{z}^{n-1}}{(m-1)!(n-1)!} \log |z|^2$ or in modified form

$$\frac{1}{\pi} \frac{z^{m-1}\bar{z}^{n-1}}{(m-1)!(n-1)!} \left[ \log |z|^2 - \sum_{\mu=1}^{m-1} \sum_{\nu=1}^{n-1} \frac{1}{\mu \nu} \right].$$

For $m = 0$ and positive $n$, i.e. for the poly-analytic operator $\partial_{\bar{z}}^n$, the fundamental solution is

$$(-1)^n \frac{1}{\pi} \frac{\bar{z}^{n-1}}{n-1)!z^n}.$$

In particular for the poly-harmonic operator ($0 < m = n$), $\partial_z^n \partial_{\bar{z}}^n$, it is $\frac{1}{\pi} \frac{|z|^{2(n-1)}}{(n-1)!^2} \log |z|^2$, and in modified form

$$\frac{1}{\pi} \frac{|z|^{2(n-1)}}{(n-1)!^2} \left[ \log |z|^2 - 2 \sum_{\nu=1}^{n-1} \frac{1}{\nu} \right].
Remark 1. For finding fundamental solutions to higher order differential operators from ones of lower order operators proper integrations are suitable. A fundamental solution to the product of two differential operators $\partial_1, \partial_2$ is found from a fundamental solution to $\partial_1$, say $f_1$, as a primitive with respect to $\partial_2$ of $f_1$, $\partial_2^{-1} f_1$. As $f_1$ satisfies $\partial_1 f_1 = \delta$ with the Dirac $\delta$-operator, then $\partial_1 \partial_2^{-1} f_1 = \delta$.

As the Cauchy kernel is appearing as kernel function in the Pompeiu-Vekua operator also these fundamental solutions lead to integral operators. They together with their partial derivatives form a hierarchy of weakly and strongly singular integral operators also these fundamental solutions lead to integral operators. They satisfy

$$T_{m,n} f(z) = \int_D K_{m,n}(z - \zeta) f(\zeta) d\xi d\eta$$

supplemented by $T_{0,0} f = f$. In particular $T_{0,1} = T, T_{1,0} = T, T_{-1,1} = \Pi, T_{1,-1} = \bar{\Pi}$. They satisfy

$$\partial^k_\zeta \partial^l_\zeta T_{m,n} = T_{m-k,n-l}, k + l \leq m + n, T_{m,-m} T_{k,-k} = T_{m+k,-m-k}.$$ 

Also $T_{m,-m} : L_2(\mathbb{C}; \mathbb{C}) \to L_2(\mathbb{C}; \mathbb{C})$ is a unitary integral operator,

$$\|T_{m,-m} f\|_{L_2(\mathbb{C}; \mathbb{C})} = \|f\|_{L_2(\mathbb{C}; \mathbb{C})}.$$ 

$T_{m,n}$ is weakly singular for $0 < m + n$, and a strongly singular integral operator of Calderon-Zygmund type for $m + n = 0$. For negative indices they can be interpreted as certain differential operators [5, 6]. Recently some were used in [7] to treat a boundary control problem to the Poisson equation in relation to the third (Robin) boundary condition. Potentials for the model operators serve to reduce boundary value problems of general equations the leading part of which is the model operator in question, to singular integral equation, see e.g. [8].

2 Integral representations.

The divergence theorem is the origin of integral representation formulas. In complex analysis there are two such dual forms, see e.g. [9, 10].
Gauss-Ostrogradsky divergence theorem. Complex form. For domains $D \subset \mathbb{C}$ with piecewise smooth boundary $\partial D$ and $w \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$ the relations

$$
\int_D w_{\bar{z}}(z) \, dx \, dy = \frac{1}{2i} \int_{\partial D} w(z) \, dz,
$$
$$
\int_D w_z(z) \, dx \, dy = -\frac{1}{2i} \int_{\partial D} w(z) \, d\bar{z}
$$
are valid.

Immediate consequences are

**Theorem 1.** (Cauchy-Pompeiu representations.) Functions $w \in C^1(D; \mathbb{C}) \cap C(D; \mathbb{C})$ are represented by

$$
w(z) = \frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_D w_{\bar{\zeta}}(\zeta) \frac{d\zeta \, d\eta}{\zeta - z}, \quad z \in D, \tag{1}
$$

and its dual form

$$
w(z) = -\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\bar{\zeta}}{\zeta - z} - \frac{1}{\pi} \int_D w_{\bar{\zeta}}(\zeta) \frac{d\zeta \, d\eta}{\zeta - z}, \quad z \in D. \tag{2}
$$

### 2.1 Poly-analytic integral representations.

Iterating formula (1) applied to $w$ and its $\bar{z}$—derivatives of higher order, poly-analytic Cauchy-Pompeiu representations become available [11].

**Theorem 2.** (Higher order Cauchy-Pompeiu formula.) Functions $w \in C^n(D; \mathbb{C}) \cap C^{n-1}(\overline{D}; \mathbb{C}), \; n \in \mathbb{N}$, are represented by

$$
w(z) = \sum_{\nu=0}^{n-1} \frac{1}{2\pi i} \int_{\partial D} \frac{(-1)^\nu (\zeta - z)^\nu}{\nu!(\zeta - z)} \partial_\zeta^\nu w(\zeta) \, d\zeta,
- \frac{1}{\pi} \int_D \frac{(-1)^{n-1}(\zeta - z)^{n-1}}{(n-1)!(\zeta - z)} \partial_\zeta^{n-1} w(\zeta) \, d\zeta, \quad z \in D. \tag{3}
$$

In order to adjust this representation to proper boundary conditions for achieving a representation of solutions to some boundary value problem for the poly-analytic equation the analytic Cauchy-Schwarz kernel can be generalized to a poly-analytic kernel. For the case of the unit disc $\mathbb{D} = \{|z| < 1\}$ the poly-analytic Cauchy-
Schwarz-Pompeiu representation is for \( w \in C^n(\mathbb{D}; \mathbb{C}) \cap C^{n-1}(\overline{\mathbb{D}}; \mathbb{C}) \), \( n \in \mathbb{N} \), \( z \in \mathbb{D} \),

\[
w(z) = i \sum_{\nu=0}^{n-1} \frac{\text{Im} \partial_{\zeta} \varphi \varphi(0)}{\nu!} (z + \overline{z})^\nu 
+ \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{2\pi i \nu!} \int_{\partial \mathbb{D}} \text{Re} \partial_{\zeta} \varphi \varphi(\zeta + z - \zeta + \overline{\zeta})^\nu \frac{d\zeta}{\zeta} 
- \frac{(-1)^{n-1}}{2\pi (n-1)!} \int_{\mathbb{D}} \left( \frac{\partial_{\zeta} \varphi \varphi(\zeta + z - \zeta + \overline{\zeta})}{\zeta - z} + \frac{\partial_{\zeta} \varphi \varphi(1 + \overline{\zeta})}{\zeta - 1 - \zeta} \right) (\zeta + z - \zeta + \overline{\zeta})^{n-1} d\xi d\eta,
\]

(4)

see [12]. Such a representation holds more general for certain planar domains having a harmonic Green function (see next subsection).

**Definition (Green function).** The function

\[
G_1(z, \zeta) = -\log |\zeta - z|^2 + h_1(z, \zeta)
\]

with a harmonic function \( h_1(\cdot, \zeta) \) in \( D \) for any \( \zeta \in D \) satisfying \( G_1(z, \zeta) = 0 \) for \( z \in \partial D \), \( \zeta \in D \) is called the harmonic Green function for the domain \( D \).

The Green function exists for many domains. It is symmetric in its two variables. From \( G_1(z, \zeta) = 0 \), i.e. \( d_\zeta G_1(z, \zeta) = 0 \) for \( \zeta \) on \( \partial D \), \( z \in D \), the relation

\[
\frac{d\zeta}{\zeta - z} - h_{1\zeta}(z, \zeta) d\zeta = -\left[ \frac{d\zeta}{\zeta - z} - h_{1\zeta}(z, \zeta) d\zeta \right]
\]

(5)

and with \( \partial_{\zeta} \) the outward normal derivative on \( \partial D \), i.e. \( i \partial_{\zeta} d\varsigma = [\partial_\zeta d\zeta - \partial_\zeta d\overline{\zeta}] \), and \( s_\zeta \) the arc length parameter also

\[
-i \partial_{\zeta} G_1(z, \zeta) d\zeta = \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\zeta - z} - h_{1\zeta}(z, \zeta) d\zeta + h_{1\overline{\zeta}}(z, \zeta) d\overline{\zeta}
\]

(6)

follow.

**Lemma 1.** For any \( \gamma \in C(\partial D; \mathbb{R}) \), \( D \) a planar domain with harmonic Green function \( G_1(z, \zeta) = -\log |\zeta - z|^2 + h_1(z, \zeta) \), satisfying \( h_{1\zeta}(z, \zeta) = 0 \) the function

\[
\Gamma(z) = \int_{\partial \mathbb{D}} \gamma(\zeta) \left[ \frac{d\zeta}{\zeta - z} - h_{1\zeta}(z, \zeta) d\zeta \right]
\]

is constant.

As \( \Gamma \) is real-valued and analytic it is constant, [13].

**Definition (Admissible domain for Schwarz problem).** A plane domain with Green function is called an admissible domain for the Schwarz problem if the function

\[
h_{1\zeta}(z, \zeta)
\]

is an analytic function of \( z \in D \) for any \( \zeta \in \partial D \). There are domains which are, but most of them seem not to be admissible [13].
Theorem 3. (Cauchy-Schwarz-Pompeiu representation.) Any $w \in C^n(D; \mathbb{C}) \cap C^{n-1}(\overline{D}; \mathbb{C})$ for an admissible domain $D$, $z_0 \in D$, in the complex plane $\mathbb{C}$ with Green function $G_1(z, \zeta)$ is representable as

$$w(z) = \sum_{\mu=0}^{n-1} \left\{ \frac{i \text{Im} \partial^\mu w(z_0)}{\mu!} (z - z_0 + \overline{z} - \overline{z_0})^\mu ight. + \frac{(-1)^\mu}{2\pi i \mu!} \int_{\partial D} \text{Re} \partial^\mu w(\zeta) (\zeta - z + \overline{\zeta} - \overline{z})^\mu \\
\times \left\{ \frac{\zeta - z_0}{(\zeta - z)(\zeta - z_0)} d\zeta + \left( h_{1\zeta}(z, \zeta) - \frac{1}{\zeta - z_0} \right) d\zeta \right\} (7)
\left. + \frac{(-1)^n}{2\pi(n-1)!} \int_D \partial^\mu w(\zeta) \left\{ \frac{\zeta + z - 2z_0}{(\zeta - z)(\zeta - z_0)} - h_{1\zeta}(z_0, \zeta) \right\} \\
- \partial^\mu w(\zeta) \left[ 2h_{1\zeta}(z, \zeta) - h_{1\zeta}(z_0, \zeta) - \frac{1}{\zeta - z_0} \right] (\zeta - z + \overline{z} - \overline{z})^{n-1} d\xi d\eta. \right.$$

For a proof see [14].

Remark 2. Formula (7) provides a solution to the Schwarz boundary value problem for the poly-analytic operator, [15, 14],

$$\partial^\mu w = f \text{ in } D, \quad \text{Re} \partial^\mu w = \gamma_\mu, \quad \text{on } \partial D, \quad \text{Im} \partial^\mu w(z_0) = c_\mu, \quad 0 \leq \mu \leq n - 1.$$

Related recent papers are [16, 17, 18].

2.2 Harmonic integral representations.

Iterating formula (1) for $w \in C^2(D; \mathbb{C}) \cap C^1(\overline{D}; \mathbb{C})$ and formula (2) applied to the $\overline{\zeta}$-derivative of $w$ results in an integral representation for the Laplace operator with its fundamental solution $\frac{1}{\pi} \log |\zeta|^2$:

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_D w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z}, z \in D,$$

$$w_{\overline{\zeta}}(\zeta) = -\frac{1}{2\pi i} \int_{\partial D} w_{\overline{\zeta}}(\zeta) \frac{d\zeta}{\zeta - \zeta} - \frac{1}{\pi} \int_D w_{\zeta \overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - \zeta}, \zeta \in D,$$

i.e.

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\partial D} w_{\overline{\zeta}}(\zeta) \log |\zeta - z|^2 d\overline{\zeta}$$
$$+ \frac{1}{\pi} \int_D w_{\zeta \overline{\zeta}}(\zeta) \log |\zeta - z|^2 d\xi d\eta, z \in D.$$
But this formula turns out to involve \( w_\sigma \) and not \( w_z \) and thus is asymmetric. Repeating the procedure with (2) for \( w \) and (1) applied to \( w_z \),

\[
w(z) = -\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_D w_\zeta(\zeta) \frac{d\zeta d\eta}{\zeta - z}, \quad z \in D
\]

\[
w_\zeta(\zeta) = \frac{1}{2\pi i} \int_{\partial D} w_\zeta(\tilde{\zeta}) \frac{d\tilde{\zeta}}{\tilde{\zeta} - \zeta} - \frac{1}{\pi} \int_D w_\zeta(\tilde{\zeta}) \frac{d\tilde{\zeta} d\eta}{\tilde{\zeta} - \zeta}, \quad \zeta \in D,
\]

leads to

\[
w(z) = -\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{\partial D} w_\zeta(\zeta) \log |\zeta - z|^2 d\zeta
\]

\[+ \frac{1}{\pi} \int_D w_\zeta(\zeta) \log |\zeta - z|^2 d\xi d\eta, \quad z \in D.
\]

Combining the two representations gives for \( w \in C^2(D; \mathbb{C}) \cap C^1(\overline{D}; \mathbb{C}) \) [10],

\[
w(z) = \frac{1}{4\pi} \int_{\partial D} w(\zeta) \partial_{\zeta} \log |\zeta - z|^2 d\xi d\eta - \frac{1}{4\pi} \int_{\partial D} \partial_{\zeta} w(\zeta) \log |\zeta - z|^2 d\xi d\eta
\]

\[+ \frac{1}{\pi} \int_D w_\zeta(\zeta) \log |\zeta - z|^2 d\xi d\eta, \quad z \in D,
\]

where the outward normal derivative \( \partial_{\zeta} \) and the arc length parameter \( s_\zeta \) on \( \partial D \) are combined via

\[i \partial_{\zeta} ds = \partial_\zeta d\zeta - \partial_\zeta \tilde{d}\zeta.
\]

For adjusting this representation to proper boundary conditions the harmonic Robin function as an interpolation of Green and Neumann functions is introduced [19, 20], see also [21, 22]. Explicit forms of the Robin functions for the unit disc and for the concentric ring are also provided there.

**Definition (Robin function).** For \( \alpha, \beta \in \mathbb{R}, \ 0 < \alpha^2 + \beta^2 \), a real-valued function \( R_{1; \alpha, \beta}(z, \zeta), z, \zeta \in D, z \neq \zeta \), is called Robin function if for any \( \zeta \in D \) it has the properties

- \( R_{1; \alpha, \beta}(\cdot, \zeta) \) is harmonic in \( D \setminus \{ \zeta \} \) and continuously differentiable in \( \overline{D} \setminus \{ \zeta \} \),
- \( h(z, \zeta) = R_{1; \alpha, \beta}(z, \zeta) + \log |z - \zeta|^2 \) is harmonic for \( z \in D \),
- \( \alpha R_{1; \alpha, \beta}(z, \zeta) + \beta \partial_\zeta R_{1; \alpha, \beta}(z, \zeta) = \beta \sigma(s) \) for \( z = z(s) \in \partial D \), where the density function \( \sigma \) is a real-valued, piecewise constant function of \( s \) with finite mass \( \int_{\partial D} \sigma(s) ds \),
- \( \beta \int_{\partial D} \sigma(s) R_{1; \alpha, \beta}(z, \zeta) ds_z = 0 \) (normalization condition).

The Robin function is symmetric in its arguments \( R_{1; \alpha, \beta}(z, \zeta) = R_{1; \alpha, \beta}(\zeta, z) \), and it includes the Green and the Neumann function as particular cases, \( R_{1; \alpha, \beta}(z, \zeta) = G_1(z, \zeta), R_{1; 0, \beta}(z, \zeta) = N_1(z, \zeta) \).
**Theorem 4.** Any function \( w \in C^2(D; \mathbb{C}) \cap C^1(\overline{D}; \mathbb{C}) \) can be represented as

\[
w(z) = \omega_k(z) - \frac{1}{\pi} \int_D \partial_\zeta \partial_\zeta w(\zeta) R_{1; \alpha, \beta}(\zeta, z) d\xi d\eta, \quad k = 1, 2, 3,
\]

\[
\omega_1(z) = -\frac{1}{4\pi} \int_{\partial D} \{ w(\zeta) \partial_\nu R_{1; \alpha, \beta}(\zeta, z) - \partial_\nu w(\zeta) R_{1; \alpha, \beta}(\zeta, z) \} d\zeta,
\]

\[
4\pi \beta \omega_2(z) = \int_{\partial D} \{ \alpha w(\zeta) + \beta \partial_\nu w(\zeta) \} R_{1; \alpha, \beta}(\zeta, z) d\zeta - \beta \int_{\partial D} \sigma w(\zeta) d\zeta
\]

\((\beta \neq 0),\)

\[
4\pi \alpha \omega_3(z) = -\int_{\partial D} \{ \alpha w(\zeta) + \beta \partial_\nu w(\zeta) \} \partial_\nu R_{1; \alpha, \beta}(\zeta, z) d\zeta + \beta \int_{\partial D} \sigma \partial_\nu w(\zeta) d\zeta
\]

\((\alpha \neq 0).\)

This representation provides the key for solving the Robin problem. The resulting statement includes the ones for the Dirichlet and the Neumann problems. In case of the unit disc \( D \) it reads:

**Theorem 5.** For \( f \in L^p(D; \mathbb{C}), 2 < p < \infty, \gamma \in C(\partial D; \mathbb{C}), \) the Robin problem

\[
\partial_z \partial_{\bar{z}} w = f \text{ in } D, \quad \alpha w + \beta \partial_\nu w = \gamma \text{ on } \partial D,
\]

(i) if \( \beta \neq 0 \) is solvable if and only if

\[
\frac{1}{2\pi i} \int_{\partial D} \frac{\partial_\nu \zeta}{\zeta} \frac{d \xi d\eta}{\zeta} - \frac{\alpha}{2\pi i} \int_{\partial D} w(\zeta) \frac{d \zeta}{\zeta} = \frac{2\beta}{\pi} \int_{D} f(\zeta) d\xi d\eta
\]

the solution being then

\[
w(z) = \frac{1}{4\pi i \beta} \int_{\partial D} \gamma(\zeta) R_{1; \alpha, \beta}(\zeta, z) \frac{d \xi d\eta}{\zeta} + \frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d \zeta}{\zeta} - \frac{1}{\pi} \int_{D} f(\zeta) R_{1; \alpha, \beta}(\zeta, z) d\xi d\eta,
\]

(ii) if \( \alpha \neq 0 \) is solvable if and only if

\[
\frac{\beta}{2\pi i} \int_{\partial D} \partial_\nu w(\zeta) \frac{d \zeta}{\zeta} = \frac{2\beta}{\pi} \int_{D} f(\zeta) d\xi d\eta
\]

the solution then being

\[
w(z) = -\frac{1}{4\pi i \alpha} \int_{\partial D} \gamma(\zeta) \partial_\nu R_{1; \alpha, \beta}(\zeta, z) \frac{d \zeta}{\zeta} + \frac{\beta}{4\pi i \alpha} \int_{\partial D} \sigma \partial_\nu w(\zeta) \frac{d \zeta}{\zeta}
\]

\[-\frac{1}{\pi} \int_{D} f(\zeta) R_{1; \alpha, \beta}(\zeta, z) d\xi d\eta.\]
Remark 3. If $\alpha = 0$ then $\gamma = \beta \partial_{\nu} w$ and the solvability condition is the known one for the Neumann problem, see e.g. [10]. In case of $\beta = 0$, i.e. for the Dirichlet problem, there is no solvability condition! The terms
\[
\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta}, \quad \frac{\beta}{4\pi i \alpha} \int_{\partial D} \sigma \partial_{\nu} w(\zeta) \frac{d\zeta}{\zeta}
\]
can be fixed to determine the solution uniquely.

2.3 Poly-harmonic representations. Hybrid poly-harmonic Green functions.

For the poly-harmonic Poisson equation of $n$-th order
\[
(\partial_{\zeta} \partial_{\bar{\zeta}})^n w = f \quad \text{in} \; D
\]  
(8)

many boundary value problems are available, see e.g. [23]. The higher the order the larger is the number of possible problems. $n$ independent conditions may be posed. Besides the classical

poly-harmonic Dirichlet problem
\[
\partial_{\nu}^{\mu} w = \gamma_{\mu}, 0 \leq \mu \leq n - 1, \quad \text{on} \; \partial D,
\]  
(9)

poly-harmonic Neumann problem
\[
\partial_{\nu}^{\mu} w = \gamma_{\mu}, 1 \leq \mu \leq n, \quad \text{on} \; \partial D,
\]  
(10)

poly-harmonic Riquier problem
\[
(\partial_{\zeta} \partial_{\bar{\zeta}})^{\mu} w = \gamma_{\mu}, 0 \leq \mu \leq n - 1, \quad \text{on} \; \partial D,
\]  
(11)

there are all kinds of combinations of $n$ different Robin conditions possible. Also
\[
\partial_{\nu}(\partial_{\zeta} \partial_{\bar{\zeta}})^{\mu} w = \gamma_{\mu}, 0 \leq \mu \leq n - 1, \quad \text{on} \; \partial D,
\]  
(12)

and
\[
(\partial_{\zeta} \partial_{\bar{\zeta}})^{\mu} w = \gamma_{0\mu}, 0 \leq 2\mu \leq n - 1, \\
\partial_{\nu}(\partial_{\zeta} \partial_{\bar{\zeta}})^{\mu} w = \gamma_{1\mu}, 0 \leq 2\mu \leq n - 2, \quad \text{on} \; \partial D,
\]  
(13)

are possible boundary conditions. Decomposing the model equation (8) into the system of Poisson equations
\[
\partial_{\zeta} \partial_{\bar{\zeta}} w_{\mu} = w_{\mu+1} \quad \text{in} \; D, 0 \leq \mu \leq n - 1,
\]  
(14)

with $w_0 = w$ and $w_n = f$, each allows one boundary condition. Choosing them as Robin conditions, possibly with different parameters, leads to a variety of boundary
value problems and as a consequence to hybrid poly-harmonic Green functions. The Riquier problem (11) is of this type, where just Dirichlet conditions are posed. A general theory for all these poly-harmonic boundary value problems is not yet worked out.

However, (9) and (10) are not decomposable in a system of boundary value problems for Poisson equations. For these problems poly-harmonic Green and Neumann functions exist. They are not of hybrid type.

Definition (Poly-harmonic Green-Almansi Function). A real-valued function $G_n(z, \zeta)$, $z, \zeta \in D$, satisfying

- $G_n(\cdot, \zeta)$ is poly-harmonic of order $n$ in $D\setminus\{\zeta\}$,
- $G_n(z, \zeta) + \frac{|\zeta - z|^{2(n-1)}}{(n-1)!} \log |\zeta - z|^2$ is poly-harmonic of order $n$ in $D$ for any $\zeta \in D$,
- $\partial_{\nu_s} G_n(z, \zeta) = 0$ for $z \in \partial D, \zeta \in D, 0 \leq \mu \leq n - 1$,
- $G_n(z, \zeta) = G_n(\zeta, z)$ for $z, \zeta \in D, z \neq \zeta$,

is called poly-harmonic Green-Almansi function, [24], see also [25]. For its explicit form for the unit disc see e.g. [11] and for the upper half plane [26, 27]. Besides for the Dirichlet problem (9) it is useful also for the problem (13) treated in [28] explicitly for the unit disc.

Theorem 6. Any $\omega \in C^{2n}(D; \mathbb{C}) \cap C^{2n-1}(\overline{D}; \mathbb{C})$, $n \in \mathbb{N}$, is representable by

$$
\omega(z) = -\sum_{\mu=0}^{\left[\frac{n}{2}\right]-1} \frac{1}{4\pi} \int_{\partial D} \partial_{\nu_s} \left(\partial_{\zeta} \partial_{\zeta}^{n-\mu-1} G_n(z, \zeta) \partial_{\zeta} \partial_{\zeta}^{\mu} \omega(\zeta)\right) ds_{\zeta}
$$

$$
+ \sum_{\mu=0}^{\left[\frac{n-1}{2}\right]} \frac{1}{4\pi} \int_{\partial D} \left(\partial_{\zeta} \partial_{\zeta}^{n-\mu} G_n(z, \zeta) \partial_{\nu_s} \partial_{\zeta} \partial_{\zeta}^{\mu} \omega(\zeta)\right) ds_{\zeta}
$$

$$
- \frac{1}{\pi} \int_{\overline{D}} G_n(z, \zeta) \left(\partial_{\zeta} \partial_{\zeta}^{n} \omega(\zeta)\right) d\xi d\eta.
$$

Remark 4. Poly-harmonic Green functions for domains in $\mathbb{R}^n$ are exploited in [29], see [30], in particular for balls and half spaces in [31, 32, 33]. For poly-domains and balls in $\mathbb{C}^n$ see [34, 35, 36, 37, 38, 39, 40]. There are some single investigations of mixed type boundary value problems for higher order equations, e.g. [41, 42, 43, 44, 45, 46, 47].
2.3.1 Iterated poly-harmonic Green function.

Rewriting the Riquier boundary value problem (11) for equation (8) as the system (14) with Dirichlet conditions

\[ w_\mu = \gamma_\mu \text{ on } \partial D, 0 \leq \mu \leq n - 1, \]

leads to the representation

\[ w_\mu(z) = -\frac{1}{4\pi} \int_{\partial D} \gamma_\mu(\zeta) \partial_\nu G_1(z, \zeta) d\s_\zeta - \frac{1}{\pi} \int_D w_{\mu+1}(\zeta) G_1(z, \zeta) d\xi d\eta. \]

Composing them gives the solution to the poly-harmonic Riquier problem (8),(11), see [48, 49, 50, 51].

**Definition (Poly-harmonic iterated Green function).** The iterated convolution of the harmonic Green function \( G_1(z, \zeta) \),

\[ G_\mu(z, \zeta) = -\frac{1}{\pi} \int_D G_1(z, \tilde{\zeta}) G_{\mu-1}(\tilde{\zeta}, \zeta) d\tilde{\xi} d\tilde{\eta}, \quad 2 \leq \mu, \]

is called poly-harmonic iterated Green function of order \( \mu \). Its normal derivative at the boundary

\[ g_\mu(z, \zeta) = -\frac{1}{2} \partial_\nu G_\mu(z, \zeta), \quad z \in D, \zeta \in \partial D, \]

is called the polyharmonic Poisson kernel of order \( \mu \).

**Theorem 7.** The solution to the Riquier problem (11) for equation (8) is uniquely given as

\[ w(z) = \sum_{\mu=0}^{n-1} \frac{1}{2\pi} \int_{\partial D} \gamma_\mu(\zeta) g_{\mu+1}(z, \zeta) d\s_\zeta - \frac{1}{\pi} \int_D f(\zeta) G_n(z, \zeta) d\xi d\eta. \]

The \( G_\mu(z, \zeta) \) are poly-harmonic Green functions having different boundary behavior than the Green-Almansi function \( \mathcal{G}_\mu(z, \zeta) \), [52].

**Lemma 2.** The iterated Green function \( G_\mu(\cdot, \zeta), \mu \in \mathbb{N}, \) solves for any \( \zeta \in D \) the Dirichlet problem for the Poisson equation:

\[ \partial_\nu \partial_\mu G_\mu(z, \zeta) = G_{\mu-1}(z, \zeta) \quad \text{in } D, G_\mu(z, \zeta) = 0 \quad \text{on } \partial D. \]

In particular it satisfies for any \( \zeta \in D \) the properties

- \( G_\mu(\cdot, \zeta) \) is a poly-harmonic function of order \( \mu \) in \( D \setminus \{\zeta\} \),
- \( G_\mu(z, \zeta) + \frac{|\zeta - z|^{2(\mu-1)}}{(\mu-1)!^2} \log |\zeta - z|^2 \) is poly-harmonic of order \( \mu \) in \( D \)
- \( (\partial_\nu \partial_\mu)^\nu G_\mu(z, \zeta) = 0, 0 \leq \nu \leq \mu - 1, \quad \text{on } \partial D. \)

Moreover, it is symmetric in its variables:

- \( G_\mu(z, \zeta) = G_\mu(\zeta, z), z, \zeta \in D, z \neq \zeta. \)
2.3.2 Iterated poly-harmonic Neumann function.

Convoluting the harmonic Neumann function in the same way as the Green function in the preceding section leads to an iterated poly-harmonic Neumann function.

**Definition (Poly-harmonic iterated Neumann function).** The iterated convolution of the harmonic Neumann function \( N_1(z, \zeta) \),

\[
N_\mu(z, \zeta) = -\frac{1}{\pi} \int_D N_1(z, \tilde{\zeta}) N_{\mu-1}(\tilde{\zeta}, \zeta) d\tilde{\xi}d\tilde{\eta}, \ 2 \leq \mu,
\]

is called poly-harmonic iterated Neumann function of order \( \mu \).

**Lemma 3.** The iterated Neumann function \( N_\mu(\cdot, \zeta), \mu \in \mathbb{N} \), solves for any \( \zeta \in D \) the Poisson equation:

\[
\partial_z \partial_{\bar{z}} N_\mu(z, \zeta) = N_{\mu-1}(z, \zeta) \text{ in } D.
\]

In particular it satisfies for any \( \zeta \in D \) the properties, see [53],

- \( N_\mu(\cdot, \zeta) \) is a poly-harmonic function of order \( \mu \) in \( D \setminus \{ \zeta \} \),
- \( N_\mu(z, \zeta) + \frac{K}{(\mu-1)!} \log |\zeta - z|^2 \) is poly-harmonic of order \( \mu \) in \( D \),
- \( \partial_{\nu_z} N_\mu(z, \zeta) = \sigma(z) \tau_{\mu-1}(\zeta) \) on \( \partial D \) with \( \sigma \) a continuous, piecewise constant, real function on \( \partial D \) and \( \tau_0(\zeta) \equiv 1, \tau_{\mu-1} = -\frac{1}{\pi} \int_D \tau_{\mu-2}(\tilde{\zeta}) N_1(\tilde{\zeta}, \zeta) d\tilde{\xi}d\tilde{\eta} \) for \( 2 \leq \mu \),
- \( \int_{\partial D} \sigma(z) N_\mu(z, \zeta) ds_z = 0 \).

Moreover, it is symmetric in its variables:

- \( N_\mu(z, \zeta) = N_\mu(\zeta, z), \ z, \zeta \in D, \ z \neq \zeta \).

**Remark 5.** In the particular case of the unit disc \( D \) the boundary behavior of \( N_\mu(\cdot, \zeta) \) is shown, [53], to be for \( z \in \partial D, \zeta \in D \)

\[
\partial_{\nu_z} N_\mu(z, \zeta) = -\frac{2}{(\mu-1)!} (|\zeta|^2 - 1)^{\mu-1} + \sum_{\lambda=1}^{\mu-2} \frac{\mu-\lambda^2}{(\mu-1)! (\mu-1-\lambda)! (2\lambda-\mu+1)!} \partial_{\nu_z} N_{\lambda+1}(z, \zeta).
\]

2.3.3 Iterated poly-harmonic Robin function.

Because of the parameters convolution of Robin functions offer a variety of possibilities. However, in this subsection the parameters \( \alpha, \beta \) are kept fixed.

**Definition (Poly-harmonic iterated Robin function for fixed parameters).** The iterated convolution of the harmonic Robin function \( R_{1;\alpha,\beta}(z, \zeta) \),

\[
R_{\mu;\alpha,\beta}(z, \zeta) = -\frac{1}{\pi} \int_D R_{1;\alpha,\beta}(z, \tilde{\zeta}) R_{\mu-1;\alpha,\beta}(\tilde{\zeta}, \zeta) d\tilde{\xi}d\tilde{\eta}, \ 2 \leq \mu,
\]

is called poly-harmonic iterated Robin function for parameters \( \alpha, \beta \) of order \( \mu \), see [54, 55].
Lemma 4. The iterated Robin function $R_{\mu;\alpha,\beta}(\cdot,\zeta), \mu \in \mathbb{N}$, solves for any $\zeta \in D$ the Poisson equation:

$$\partial_z \partial_{\bar{z}}R_{\mu;\alpha,\beta}(z, \zeta) = R_{\mu-1;\alpha,\beta}(z, \zeta) \text{ in } D.$$ 

In particular it satisfies for any $\zeta \in D$ the properties

- $R_{\mu;\alpha,\beta}(\cdot,\zeta)$ is a poly-harmonic function of order $\mu$ in $D \setminus \{\zeta\}$,
- $R_{\mu;\alpha,\beta}(z, \zeta) + \frac{(z-\zeta)^{2(\mu-1)}}{(\mu-1)!^2} \log |z - \zeta|^2$ is poly-harmonic of order $\mu$ in $D$,
- $(\alpha + \beta \partial_\nu)R_{\mu;\alpha,\beta}(z, \zeta) = \beta \sigma(z) \tau_{\mu-1}(\zeta)$ on $\partial D$ with $\sigma$ a continuous, piecewise constant, real function on $\partial D$ and $\tau_\mu(\zeta) \equiv 1, \tau_{\mu-1} = -\frac{1}{\pi} \int_D \tau_{\mu-2}(\tilde{\zeta})R_{1;\alpha,\beta}(\tilde{\zeta}, \zeta) d\tilde{\xi} d\eta$ for $2 \leq \mu$,
- $\beta \int_{\partial D} \sigma(z)R_{\mu;\alpha,\beta}(z, \zeta) ds_z = 0$.

Moreover, it is symmetric in its variables:

- $R_{\mu;\alpha,\beta}(z, \zeta) = R_{\mu;\alpha,\beta}(\zeta, z), \ z, \zeta \in D, \ z \neq \zeta$.

This iterated Robin function is proper for the problem

$$(\partial_z \partial_{\bar{z}})^n w = f \text{ in } D,$$

$$(\alpha + \beta \partial_\nu)(\partial_z \partial_{\bar{z}})^\mu w = \gamma_\mu, 0 \leq \mu \leq n - 1, \text{ on } \partial D.$$ 

2.3.4 Hybrid poly-harmonic Green function.

Definition (Class of poly-harmonic Green functions of order $n$). Let $PHG_n$ denote the set of all poly-harmonic Green functions of order $n \in \mathbb{N}$.

This set in particular contains the poly-harmonic Green-Almansi $G_n$ and the poly-harmonic iterated Robin functions $R_{n;\alpha,\beta}$, including the poly-harmonic iterated Green $G_n$ and Neumann functions $N_n$, of order $n$ and all convolutions of such kernel functions of proper lower orders. The convolution

$$K_m \hat{K}_n(z, \zeta) = -\frac{1}{\pi} \int_D K_m(z, \tilde{\zeta}) \hat{K}_n(\tilde{\zeta}, \zeta) d\tilde{\xi} d\eta$$

(16)

of two poly-harmonic iterated Green functions $K_m \in PHG_m, \hat{K}_n \in PHG_n$ is a hybrid poly-harmonic Green function of order $m + n$, [52], $K_m \hat{K}_n \in PHG_{m+n}$.

Theorem 8. Let the $m$ boundary conditions and additional side conditions, if needed, for $K_m(z, \zeta)$ for $z \in \partial D, \zeta \in D$ be denoted by $B_m$ such that $K_m(z, \zeta)$ satisfies

- $K_m(\cdot, \zeta)$ is polyharmonic of order $m$ in $D \setminus \{\zeta\}$,
• $K_m(z, \zeta) + \frac{|z - \zeta|^{2(m-1)}}{(m-1)!} \log |z - \zeta|^2$ is polyharmonic of order $m$ in $D$ for any $\zeta \in D$,

• $K_m(z, \zeta)$ satisfies the conditions $B_m$ for $z \in \partial D, \zeta \in D$,

then $K_m \hat{K}_n(z, \zeta)$ satisfies as a function of $z$ for any $\zeta \in D$ the boundary value problem

$$ (\partial_z \partial_{\bar{z}})^m K_m \hat{K}_n(z, \zeta) = \hat{K}_n(z, \zeta) \text{ in } D, \quad (17) $$

$$ B_m(K_m \hat{K}_n(z, \zeta)) = -\frac{1}{\pi} \int_D (B_m(K_m(z, \tilde{z}))) \hat{K}_n(\tilde{z}, \zeta) d\tilde{z} d\tilde{\eta} \text{ on } \partial D, \quad (18) $$

and as a function of $\zeta$ for any fixed $z \in D$

$$ (\partial_\zeta \partial_{\bar{\zeta}})^n (K_m \hat{K}_n(z, \zeta)) = K_m(z, \zeta) \text{ in } D, \quad (19) $$

$$ \hat{B}_n(K_m \hat{K}_n(z, \zeta)) = -\frac{1}{\pi} \int_D K_m(z, \tilde{\zeta}) \hat{B}_n(\tilde{\zeta}, \zeta) \tilde{\zeta} d\tilde{\eta} d\tilde{\zeta} \text{ on } \partial D. \quad (20) $$

In cases where $K_m(z, \zeta)$ and $\hat{K}_n(z, \zeta)$ are both symmetric then $K_m \hat{K}_n(z, \zeta) = \hat{K}_n K_m(\zeta, z)$ follows.

### 2.4 A poly-analytic Pompeiu-Vekua operator.

Not all boundary value problems for a differential operator are well-posed. Well-posed boundary value problems are called natural boundary value problems. While the Schwarz problem is natural for the poly-analytic operator as the Riquier problem for the poly-harmonic operator, the iterated Dirichlet problem is not well-posed for the poly-analytic and the iterated Robin problem in general not for the poly-harmonic operator. In these cases of unnatural boundary conditions solvability conditions are required. For the iterated Dirichlet problems for the poly-analytic operator in [14] is shown:

**Theorem 9.** The iterated Dirichlet problem for the poly-analytic operator for a domain $D$ with harmonic Green function $G_1(z, \zeta) = -\log |z - \zeta|^2 + h_1(z, \zeta),$

$$ \partial_z^{\mu} w(z) = f(z), \quad z \in D; \quad \partial_\zeta^{\mu} w(\zeta) = \gamma_\mu(\zeta), \quad \zeta \in \partial D, \quad 0 \leq \mu \leq n - 1, $$

with data $f \in L_2(D; \mathbb{C})$, $2 < p$, $\gamma_\mu \in C(\partial D; \mathbb{C})$, $0 \leq \mu \leq n - 1$, is solvable if and
only if for $0 \leq \mu \leq n - 1$,

\[
\frac{1}{2\pi i} \int_{\partial D} \gamma_{\mu}(\zeta) \partial_{\zeta} h_1(z, \zeta) d\zeta \\
+ \sum_{\nu=\mu+1}^{n-1} \frac{1}{2\pi i} \int_{\partial D} \gamma_{\nu}(\zeta_{\nu-\mu+1}) \left( \frac{1}{\pi} \int_{D} \right)^{\nu-\mu} \partial_{\zeta_{\nu}} h_1(z, \zeta_{\nu}) \prod_{\lambda=1}^{\nu-\mu} \frac{d\xi_{\lambda} d\eta_{\lambda}}{\zeta_{\lambda} - \zeta_{\lambda+1}} - \zeta_{\nu-\mu+1} (21)
\]

The solution then is given as

\[
w(z) = \sum_{\mu=0}^{n-1} \frac{1}{2\pi i} \int_{\partial D} \frac{(-1)^\mu (\zeta - z)^\mu}{\mu! (\zeta - z)} \gamma_{\mu}(\zeta) d\zeta - \frac{1}{\pi} \int_{D} \frac{(-1)^{n-1} (\zeta - z)^{n-1}}{(n-1)! (\zeta - z)} f(\zeta) d\xi d\eta.
\]

(22)

For investigating the iterated Neumann problem for the poly-analytic operator

\[
\partial_{\zeta}^n w(z) = f(z), \quad z \in D; \quad \partial_{\zeta_{\nu}} \partial_{\zeta}^n w(\zeta) = \gamma_{\mu}(\zeta), \quad \zeta \in \partial D, \quad \partial_{\zeta}^n w(z_0) = c_{\mu}, \quad 0 \leq \mu \leq n - 1,
\]

for proper data the higher order Pompeiu-Vekua operator $T_{0,n}$ is altered. This is achieved on the basis of

**Lemma 5.** (Integration by parts). For $f, g \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$

\[
\frac{1}{\pi} \int_{D} f z g \, dx dy = -\frac{1}{\pi} \int_{D} f g z \, dx dy - \frac{1}{2\pi i} \int_{\partial D} f g d\zeta,
\]

\[
\frac{1}{\pi} \int_{D} f \bar{z} g \, dx dy = -\frac{1}{\pi} \int_{D} f g \bar{z} \, dx dy + \frac{1}{2\pi i} \int_{\partial D} f g dz
\]

are valid.

**Corollary 1.** For any $z_1, z_2 \in \mathbb{C}$

\[
\frac{1}{\pi} \int_{D} \frac{dx dy}{(z - z_1)^2 (z - z_2)} = -\frac{1}{\pi} \int_{D} \frac{dx dy}{(z - z_1)(z - z_2)^2} + \frac{1}{2\pi i} \int_{\partial D} \frac{dz}{(z - z_1)(z - z_2)}
\]

holds.

For a motivation of the construction of a poly-analytic integral operator related to the iterated Neumann boundary condition for a bounded regular domain $D$ and $z, z_0, \zeta \in D$ the kernel function

\[
C_1(z, z_0, \zeta) = \frac{1}{\pi} \int_{D} \log \frac{\zeta_1 - z}{\zeta_1 - z_0} \frac{d\xi_1 d\eta_1}{(\zeta_1 - \zeta)^2} = \frac{1}{\pi} \int_{D} \log \frac{\zeta_1 - z}{\zeta_1 - z_0} \frac{1}{\zeta_1 - \zeta} \frac{d\xi_1 d\eta_1}{\zeta_1 - \zeta_1}
\]
Lemma 5 then implies
\[ C_1(z, z_0, \zeta) = \frac{1}{\pi} \int_D \left[ \frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right] \frac{d\xi_1 d\eta_1}{\zeta_1 - \zeta} + \frac{1}{2\pi i} \int_{\partial D} \log \frac{\zeta_1 - z}{\zeta_1 - z_0} \frac{d\bar{\zeta}_1}{\zeta_1 - \zeta} \]
and differentiating
\[ \partial_z C_1(z, z_0, \zeta) = \frac{1}{\zeta - z} = C_0(z, \zeta). \]
This observation is the first step in the induction proof for a higher order Pompeiu-Vekua operator.

**Theorem 10.** For \( z, z_0 \in D, \zeta \in \mathbb{C}, n \in \mathbb{N} \) the kernel function
\[
C_n(z, z_0, \zeta) = \left( \frac{1}{\pi} \int_D \right)^n \log \frac{\zeta_n - z}{\zeta_n - z_0} \prod_{k=0}^{n-2} \frac{d\xi_{n-k} d\eta_{n-k-1}}{\zeta_n - \zeta_k} \frac{d\xi_1 d\eta_1}{\zeta_1 - \zeta},
\]
as consecutive primitives of the Cauchy kernel \( C_0(z, \zeta) \), satisfies for \( n \in \mathbb{N} \)
\[ \partial^2_z C_n(z, z_0, \zeta) = \frac{1}{\zeta - z}, \quad C_n(z_0, z_0, \zeta) = 0. \]

**Proof.** For induction rewrite on the basis of Corollary 1
\[
C_{n+1}(z, z_0, \zeta) = \left( \frac{1}{\pi} \int_D \right)^{n+1} \log \frac{\zeta_{n+1} - z}{\zeta_{n+1} - z_0} \prod_{k=0}^{n-2} \frac{d\xi_{n+1-k} d\eta_{n+1-k-1}}{\zeta_{n+1} - \zeta_k} \frac{d\xi_2 d\eta_2}{\zeta_2 - \zeta} \frac{d\xi_1 d\eta_1}{\zeta_1 - \zeta} \frac{1}{2\pi i} \int_{\partial D} \left( \frac{1}{\pi} \int_D \right)^n \log \frac{\zeta_n - z}{\zeta_n - z_0} \prod_{k=0}^{n-2} \frac{d\xi_{n-k} d\eta_{n-k-1}}{\zeta_n - \zeta_k} \frac{d\xi_2 d\eta_2}{\zeta_2 - \zeta} \frac{d\bar{\zeta}_1}{\zeta_1 - \zeta} + I_{n+1}(z, z_0, \zeta), \tag{23}
\]
with
\[
I_{n+1}(z, z_0, \zeta) = \frac{1}{2\pi i} \int_{\partial D} \left( \frac{1}{\pi} \int_D \right)^n \log \frac{\zeta_{n+1} - z}{\zeta_{n+1} - z_0} \partial_{\xi_{n+1}} \log \frac{\zeta_n - \zeta_{n+1}}{\zeta_n - z_0} d\xi_{n+1} d\eta_{n+1} \prod_{k=1}^{n-1} \frac{d\xi_{n+1-k} d\eta_{n+1-k}}{\zeta_{n+1-k} - \zeta_k} \frac{d\bar{\zeta}_1}{\zeta_1 - \zeta}.
\]
Lemma 5 then implies
\[
I_{n+1}(z, z_0, \zeta) = -\frac{1}{2\pi i} \int_{\partial D} \left( \frac{1}{\pi} \int_D \right)^n \left( \frac{1}{\zeta_{n+1} - z} - \frac{1}{\zeta_{n+1} - z_0} \right) \log \frac{\zeta_n - \zeta_{n+1}}{\zeta_n - z_0} d\xi_{n+1} d\eta_{n+1} \prod_{k=1}^{n-1} \frac{d\xi_{n+1-k} d\eta_{n+1-k}}{\zeta_{n+1-k} - \zeta_k} \frac{d\bar{\zeta}_1}{\zeta_1 - \zeta}.
\]
\[-\frac{1}{2\pi i} \int_{\partial D} \left( \frac{1}{\pi} \int_D \right)^{n-1} \frac{1}{2\pi i} \int_D \log \frac{\zeta_{n+1} - z}{\zeta_{n+1} - z_0} \log \frac{\zeta_n - \zeta_{n+1}}{\zeta_n - z_0} d\zeta_{n+1} \]
\times \prod_{k=1}^{n-1} \frac{d\xi_{n+1-k} d\eta_{n+1-k}}{\zeta_{n+1-k} - \zeta_{n-k} \frac{d\zeta_1}{\zeta_1 - \zeta}}.

As the second term is an analytic function of \( z \) the Pompeiu operator implies
\[ \partial_z I_{n+1}(z, z_0, \zeta) = I_n(z, z_0, \zeta). \]
Observing
\[ I_1(z, z_0, \zeta) = \frac{1}{2\pi i} \int_{\partial D} \log \frac{\zeta_1 - z}{\zeta_1 - z_0} \frac{d\zeta_1}{\zeta_1 - \zeta}, \quad \partial_z I_1(z, z_0, \zeta) = 0, \]
from (23)
\[ \partial_z^n C_{n+1}(z, z_0, \zeta) = \frac{1}{\pi} \int_D \frac{1}{\zeta_1 - z} \frac{d\xi_1 d\eta_1}{\zeta_1 - \zeta_1 - \zeta} + \partial_z^n I_{n+1}(z, z_0, \zeta) \]
\[ = -\frac{1}{\zeta} \left( T(\zeta) - T(z) \right) + I_1(z, z_0, \zeta), \]
and hence
\[ \partial_z^{n+1} C_{n+1}(z, z_0, \zeta) = \frac{1}{\zeta - z} \]
is seen. \( \square \)

**Definition 1.** The function \( C_n(z, z_0, \zeta) \) is called poly-analytic Cauchy kernel of order \( n \) for the domain \( D \).

2. For a bounded domain \( D \) the integral operator
\[ V_n f(z, z_0) = -\frac{1}{\pi} \int_D f(\zeta) C_n(z, z_0, \zeta) d\xi d\eta, \quad f \in L_1(D; \mathbb{C}), \quad 1 \leq n, \]
\[ V_0 f(z) = -\frac{1}{\pi} \int_D f(\zeta) C_0(z, \zeta) d\xi d\eta, = T f(z) \quad f \in L_1(D; \mathbb{C}), \]
is called Pompeiu-Vekua operator of order \( n \).
For \( z, z_0 \in D, n \in \mathbb{N} \) the operator \( V_n \) satisfies
\[ \partial_z^n V_n f(z, z_0) = V_0 f(z), \quad \partial_z^{n+1} V_n f(z, z_0) = f(z), \quad V_n f(z_0, z_0) = 0, \quad 1 \leq n, \]
where the derivatives are taken in distributional sense.

**Theorem 11.** In a planar domain \( D \subset \mathbb{C} \) with harmonic Green function \( G_1(z, \zeta) = -\log |\zeta - z|^2 + h_1(z, \zeta) \) the homogeneous Neumann-n problem
\[ \partial^\alpha \partial^\mu w = f \quad \text{in} \quad D, \quad f \in C^\alpha(\overline{D}; \mathbb{C}), \quad 0 < \alpha < 1, \]
\[ \partial_{\zeta} \partial^\mu w = 0, \quad \text{on} \quad \partial D, \quad \partial^\mu w(z_0) = 0, \quad 0 \leq \mu \leq n - 1, \]
where $z_0 \in D$ is a fixed point, is uniquely solvable if and only if
\[
\frac{1}{2\pi i} \int_{\partial D} f(\zeta) h_{1\zeta}(z, \zeta) d\zeta + \frac{1}{\pi} \int_D f(\zeta) h_{1\zeta\zeta}(z, \zeta) d\xi d\eta = 0,
\]
\[
\frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{\mu} h_{1\zeta\mu}(z, \zeta) \prod_{k=1}^{\mu} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_k - 1} d\zeta
\]
\[
+ \frac{1}{\pi} \int_D f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{\mu} h_{1\zeta\mu}(z, \zeta) \prod_{k=2}^{\mu} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_k - 1 (\zeta_1 - \zeta)^2} d\xi d\eta = 0 \text{ for } 1 \leq \mu \leq n - 1,
\]
where $\zeta_0 = \zeta$. The solution then is
\[
w(z) = -\frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-1} \log \frac{\zeta_n - z}{\zeta_n - z_0} \prod_{k=1}^{n-1} \frac{d\xi_{n-k} d\eta_{n-k}}{\zeta_n - \zeta_{n-k} - 1} d\zeta + V_{n-1} f(z, z_0).
\]
(25)

**Proof.** 1. For $n = 1$ the Neumann problem
\[
\partial_\nu w = 0 \text{ on } \partial D, w(z_0) = 0, z_0 \in D,
\]
for the Cauchy-Riemann equation
\[
w_\tau = f \text{ in } D, f \in C^\alpha(\overline{D}; \mathbb{C}), 0 < \alpha < 1,
\]
is solvable if and only if
\[
\frac{1}{2\pi i} \int_{\partial D} f(\zeta) h_{1\zeta}(z, \zeta) d\zeta + \frac{1}{\pi} \int_D f(\zeta) h_{1\zeta\zeta}(z, \zeta) d\xi d\eta = 0.
\]
(26)
The solution then is
\[
w(z) = -\frac{1}{2\pi i} \int_{\partial D} f(\zeta) \log \frac{\zeta - z}{\zeta - z_0} d\zeta - \frac{1}{\pi} \int_D f(\zeta) \frac{z - z_0}{(\zeta - z)(\zeta - z_0)} d\xi d\eta.
\]
(27)
Here the area integral is $V_0 f(z) - V_0 f(z_0)$, an unimportant modification of (25) for the case $n = 1$. For convenience the proof from [14] is repeated. The problem is reducible to a Dirichlet problem for analytic functions. But the result can just be verified.

Obviously, the function (27) satisfies as well the side condition $w(z_0) = 0$ as the differential equation $w_\tau = f$ because of the property of the Pompeiu operator
\[
Tf(z) = -\frac{1}{\pi} \int_D f(\zeta) \frac{d\xi d\eta}{\zeta - z},
\]
see [1, 9]. For verifying the boundary condition the solvability condition (26) is needed. The outward normal derivative $\partial_\nu w$ on the boundary $\partial D$ is expressed as
\(i\partial_\nu ds_\zeta = \partial_\zeta d\zeta - \partial_\zeta d\xi\). Applied to the Green function \(G_1(z, \zeta) = -\log |\zeta - z|^2 + h_1(z, \zeta)\) gives, see (5), (6),

\[
2 \left[ \frac{1}{\zeta - z} - h_1(z, \zeta) \right] d\zeta = -i\partial_\nu G_1(z, \zeta) ds_\zeta = 2ip_1(z, \zeta) ds_\zeta
\]

with the Poisson kernel \(p_1(z, \zeta)\) of the domain \(D\), satisfying

\[
\lim_{z \to \zeta_0 \in \partial D} \frac{1}{2\pi} \int_{\partial D} \gamma(\zeta) p_1(z, \zeta) ds_\zeta = \gamma(\zeta_0)
\]

for continuous \(\gamma\).

Denoting \(z' = \partial_\nu z\) and calculating from (27)

\[
w_\nu(z)z' - w_\nu(z)\overline{z'} = \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \frac{d\xi}{\zeta - \zeta'} + \Pi f(z)z' - f(z)\overline{z'}
\]

with the Ahlfors-Beurling operator

\[
\Pi f(z) = -\frac{1}{\pi} \int_D \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta
\]

and taking the solvability condition (26) into account gives

\[
w_\nu(z)z' - w_\nu(z)\overline{z'} = \frac{1}{2\pi i} \int_{\partial D} i\nu f(z) p_1(z, \zeta) \frac{\zeta(\zeta)}{\zeta(\zeta)} ds_\zeta z' - f(z)\overline{z'}
\]

\[
+ \frac{1}{\pi} \int_D f(\zeta) \partial_\zeta^2 G_1(z, \zeta) d\xi d\eta.
\]

Letting \(z\) approach a boundary point \(\zeta_0\) leads to

\[
i\partial_\nu w(\zeta_0) = \lim_{z \to \zeta_0} \left[ w_\nu(z)z' - w_\nu(z)\overline{z'} \right] = 0.
\]

On the other hand assuming the Neumann problem has a solution proper manipulations of the Cauchy-Pompeiu representation formula for solutions to the Dirichlet problem for analytic functions and an integration shows that it has the form (27). Then necessarily the solvability condition (26) holds, see [10].

2. The Neumann-n problem is equivalent to the system

\[
\partial_\nu w = \omega \text{ in } D, \quad \partial_\nu w = 0 \text{ on } \partial D, \quad w(z_0) = 0,
\]

\[
\partial_\nu^{n-1} \omega = f \text{ in } D, \quad \partial_\nu^{n-1} \omega = 0 \text{ on } \partial D, \quad \partial_\nu^{n-1} \omega(z_0) = 0, \quad 0 < \mu < n - 2.
\]

Thus by induction assumption \(w\) is given by (27) if and only if (26) holds, where \(f\) is replaced by \(\omega\), and \(\omega\) is given by the right-hand side of (25) if and only if (24) is valid with \(n - 1\) instead of \(n\). In detail

\[
w(z) = -\frac{1}{2\pi i} \int_{\partial D} \omega(\zeta_{n-1}) \log \frac{\zeta_{n-1} - z}{\zeta_{n-1} - z_0} d\xi_{n-1}
\]

\[
- \frac{1}{\pi} \int_D \omega(\zeta_{n-1}) \left( \frac{1}{\zeta_{n-1} - z} - \frac{1}{\zeta_{n-1} - z_0} \right) d\xi_{n-1} d\eta_{n-1}, \quad (28)
\]
\[ \omega(\zeta_{n-1}) = -\frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=1}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ - \frac{1}{\pi} \int_D f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=1}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} \frac{d\xi_1 d\eta_1}{(\zeta_1 - \bar{\zeta})^2} d\xi d\eta \] (29)

if and only if

\[ \frac{1}{2\pi i} \int_{\partial D} \omega(\zeta_{n-1}) h_{1\zeta_{n-1}}(z, \zeta_{n-1}) d\zeta_{n-1} + \frac{1}{\pi} \int_D \omega(\zeta_{n-1}) h_{1\zeta_{n-1}}(z, \zeta_{n-1}) d\zeta_{n-1} d\eta_{n-1} = 0, \] (30)

\[ \frac{1}{2\pi i} \int_{\partial D} f(\zeta) h_{1\zeta}(z, \zeta) d\zeta + \frac{1}{\pi} \int_D f(\zeta) h_{1\zeta}(z, \zeta) d\xi d\eta = 0, \]

\[ \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{\mu} h_{1\zeta^\mu}(\zeta, \zeta) \prod_{k=1}^{\mu} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ + \frac{1}{\pi} \int_D f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{\mu} h_{1\zeta^\mu}(z, \zeta) \prod_{k=2}^{\mu} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} \frac{d\xi_1 d\eta_1}{(\zeta_1 - \bar{\zeta})^2} d\xi d\eta = 0, \quad 1 \leq \mu \leq n - 2, \] (31)

where in the integral before the last \( \zeta_0 = \zeta \) has to be used.

Inserting (29) into (30) requires the evaluation of

\[ J_1 = \frac{1}{2\pi i} \int_{\partial D} \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=1}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ \times \log \frac{\zeta_{n-1} - z}{\zeta_{n-1} - z_0} d\zeta_{n-1}, \]

and of

\[ J_2 = \frac{1}{2\pi i} \int_{\partial D} \frac{1}{\pi} \int_D f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=2}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} \frac{d\xi_1 d\eta_1}{(\zeta_1 - \bar{\zeta})^2} d\xi d\eta \]

\[ \times \log \frac{\zeta_{n-1} - z}{\zeta_{n-1} - z_0} d\zeta_{n-1}. \]

Because of the Gauss theorem

\[ \frac{1}{2\pi i} \int_{\partial D} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \log \frac{\zeta_{n-1} - z}{\zeta_{n-1} - z_0} d\zeta_{n-1} \]

\[ = -\frac{1}{\pi} \int_D \left[ \frac{1}{\zeta_{n-1} - \zeta_{n-2}} \log \frac{\zeta_{n-1} - z}{\zeta_{n-1} - z_0} \right. \]

\[ + \left( \frac{1}{\zeta_{n-1} - z} - \frac{1}{\zeta_{n-1} - z_0} \right) \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} d\xi_{n-1} d\eta_{n-1}, \]
consequently

\[ J_1 = -\frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-1} \log \frac{\zeta_n - z}{\zeta_n - z_0} \prod_{k=1}^{n-1} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ - \frac{1}{\pi} \int_D \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=1}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ \times \left( \frac{1}{\zeta_{n-1} - z} - \frac{1}{\zeta_{n-1} - z_0} \right) d\xi_{n-1} d\eta_{n-1}, \]

and

\[ J_2 = -\frac{1}{\pi} \int_D f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-1} \log \frac{\zeta_n - z}{\zeta_n - z_0} \prod_{k=2}^{n-1} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ - \frac{1}{\pi} \int_D \frac{1}{\pi} \int_D f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=2}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ \times \left( \frac{1}{\zeta_{n-1} - z} - \frac{1}{\zeta_{n-1} - z_0} \right) d\xi_{n-1} d\eta_{n-1}. \]

From

\[ w(z) = - J_1 - \frac{1}{\pi} \int_D \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=1}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ \times \left( \frac{1}{\zeta_{n-1} - z} - \frac{1}{\zeta_{n-1} - z_0} \right) d\xi_{n-1} d\eta_{n-1} \]

\[ - J_2 - \frac{1}{\pi} \int_D \frac{1}{\pi} \int_D f(\zeta) \left( \frac{1}{\pi} \int_D \right)^{n-2} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} \prod_{k=2}^{n-2} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_{k-1}} d\zeta \]

\[ \times \left( \frac{1}{\zeta_{n-1} - z} - \frac{1}{\zeta_{n-1} - z_0} \right) d\xi_{n-1} d\eta_{n-1}, \]

the representation (25) follows.

For the solvability conditions (29) has to be inserted into (30). Again the Gauss theorem in the form

\[ \frac{1}{2\pi i} \int_{\partial D} \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} h_{1\zeta_{n-1}}(z, \zeta_{n-1}) d\zeta_{n-1} \]

\[ - \frac{1}{\pi} \int_D \left[ \frac{1}{\zeta_{n-1} - \zeta_{n-2}} h_{1\zeta_{n-1}}(z, \zeta_{n-1}) + \log \frac{\zeta_{n-2} - \zeta_{n-1}}{\zeta_{n-2} - z_0} h_{1\zeta_{n-1}\zeta_{n-2}}(z, \zeta_{n-1}) \right] d\xi_{n-1} d\eta_{n-1} \]
The inhomogeneous Neumann-\(n\) problem

Remark 6. shows if and only if for 0

\[ \log \pi \sum_{\rho=0}^{n-1-\mu} \frac{\gamma_{\mu+\rho}(\zeta)}{\mu!} (z - z_0)^{\mu} - \frac{1}{2\pi} \int_{\partial D} \gamma_{\mu}(\zeta) \left( \frac{1}{\pi} \int_{D} \right)^{\mu} \log \frac{\zeta - z}{\zeta - z_0} \prod_{k=1}^{\mu} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_k-1} \]

This on the basis of (30) together with (31) implies (24).

\[ \boxdot \]

Remark 6. The inhomogeneous Neumann-\(n\) problem

\[ \partial_\nu^2 w = f \text{ in } D, f \in C^\alpha(\bar{D}; \mathbb{C}), \ 0 < \alpha < 1, \]

\[ \partial_\nu \partial_\xi^2 w = \gamma_\mu, \text{ on } \partial D, \ \gamma_\mu \in C(\partial D; \mathbb{C}), \ \partial_\nu^2 w(z_0) = c_\mu, \ 0 \leq \mu \leq n-1, \]

where \(z_0 \in D\) is a fixed point, can be investigated. It is likely that the solution is

\[ w(z) = \sum_{\mu=0}^{n-1} \left[ \frac{\gamma_{\mu}(z - z_0)^{\mu}}{\mu!} - \frac{1}{2\pi} \int_{\partial D} \gamma_{\mu}(\zeta) \left( \frac{1}{\pi} \int_{D} \right)^{\mu} \log \frac{\zeta - z}{\zeta - z_0} \prod_{k=1}^{\mu} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_k-1} \right] \]

if and only if for 0 \(\leq \mu \leq n-2\), where \(\zeta_0 = \zeta\) is used,

\[ \sum_{\rho=0}^{n-1-\mu} \frac{1}{2\pi} \int_{\partial D} \gamma_{\mu+\rho}(\zeta) \left( \frac{1}{\pi} \int_{D} \right)^{\rho} h_{1\zeta_\rho}(z, \zeta_\rho) \prod_{k=1}^{\rho} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_k-1} \]

\[ + \frac{1}{2\pi} \int_{\partial D} f(\zeta) \left( \frac{1}{\pi} \int_{D} \right)^{n-1-\mu} h_{1\zeta_\rho}(z, \zeta_{n-1-\mu}) \prod_{k=1}^{n-1-\mu} \frac{d\xi_k d\eta_k}{\zeta_k - \zeta_k-1} = 0, \]

and

\[ \frac{1}{2\pi} \int_{\partial D} \gamma_{n-1}(\zeta) h_{1\zeta}(z, \zeta) d\xi + \frac{1}{2\pi} \int_{\partial D} f(\zeta) h_{1\zeta}(z, \zeta) d\xi = 0. \]
3 Appendix. Explicit examples.

While for theoretical reasons the existence of kernel functions is important for practical problems from mathematical physics and technics knowledge of the explicit form of them are essential. Also passing from special examples to a general concept is for didactic reasons a proper way. For constructing Schwarz kernels and harmonic Green and Neumann functions as a new possibility the parqueting-reflection principle is developed [56] and often applied, see e.g. [57, 58, 59, 60].

3.1 Poly-analytic Schwarz kernels.

The analytic Schwarz kernel $S(z, \zeta)$ for a domain $D$ satisfies for an analytic function $w$ in $D$ and $z \in D$

$$w(z) = \frac{1}{2\pi} \int_{\partial D} \text{Re} w(\zeta) S(z, \zeta) ds_\zeta + ic$$

with arbitrary $c \in \mathbb{R}$. For admissible domains with Green function $G_1(z, \zeta) = -\log |\zeta - z|^2 + h_1(z, \zeta)$ it is

$$S(z, \zeta) = -i \left[ \frac{\zeta}{\zeta - z} + h_1(\zeta, z) \right].$$

- The unit disc $\mathbb{D}$

The analytic Schwarz kernel is $S(z, \zeta) = \frac{\zeta + z}{\zeta - z}$, see e.g. [9]. The modified Cauchy-Schwarz-Pompeiu representation formula states

$$w(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \text{Re} w(\zeta) \frac{\zeta + z}{\zeta - z} d\zeta + i \text{Im} w(0)$$

$$-\frac{1}{2\pi} \int_{\mathbb{D}} \left( \frac{w_\zeta(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{w_\zeta(\zeta) 1 + z\zeta}{\zeta} \right) d\xi d\eta, z \in \mathbb{D}.$$ 

Its poly-analytic generalization is

$$w(z) = \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{2\pi i \nu!} \int_{\partial \mathbb{D}} \text{Re} \partial_\zeta^\nu w(\zeta) \frac{\zeta + z}{\zeta - z} (\zeta - z + \zeta - z)^\nu \frac{d\zeta}{\zeta}$$

$$+ i \sum_{\nu=0}^{n-1} \frac{\text{Im} \partial_\zeta^\nu w(0)}{\nu!} (z + \bar{z})^\nu$$

$$- \frac{(-1)^{n-1}}{2\pi (n-1)!} \int_{\mathbb{D}} \left( \frac{\partial_\zeta^2 w(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\partial_\zeta^2 w(\zeta)}{\zeta} \frac{1 + z\zeta}{\zeta} \right) (\zeta - z + \zeta - z)^n d\xi d\eta, z \in \mathbb{D},$$

[12].
• Circular ring $R_{r,1} = \{0 < r < |z| < 1\}$

$$S(z, \zeta) = \frac{\zeta + z}{\zeta - z} + 2 \sum_{k=1}^{\infty} \left( \frac{r^{2k} \zeta - z}{r^{2k} \zeta - \zeta - r^{2k} z} + \frac{r^{2k} z - \zeta}{r^{2k} z - z - r^{2k} \zeta} \right), \ z \in R_{r,1}, \ \zeta \in \partial R_{r,1},$$

see [61, 62].

• Half plane $H^+ = \{0 < \text{Im} z\}$

$$S(z, t) = \frac{2}{t - z}, \ z \in H^+, \ t \in \mathbb{R},$$

see [26].

• Quarter plane $Q^{++} = \{0 < \text{Re} z, 0 < \text{Im} z\}$

$$S(z, \zeta) = \begin{cases} \frac{4z}{z^{2} - z}, & 0 < \xi, \ \eta = 0, \ z \in Q^{++}, \\ \frac{4iz}{\eta^{2} + 2z}, & 0 < \eta, \ \xi = 0, \ z \in Q^{++}, \end{cases}$$

$$\zeta = \xi + i\eta, \text{ see } [63, 64].$$

• Unbounded sector $S_{\frac{\pi}{n}} = \{0 < \text{arg} z < \frac{\pi}{n}\}$

$$S(z, \zeta) = \sum_{k=0}^{n-1} \left( \frac{1}{\zeta - ze^{-\frac{2k\pi i}{n}}} - \frac{1}{\zeta - ze^{\frac{2k\pi i}{n}}} \right), \ \zeta \in \partial S_{\frac{\pi}{n}}, \ z \in S_{\frac{\pi}{n}},$$

see [65].

• Upper half of $\mathbb{D}, \mathbb{D}^+ = \mathbb{D} \cap H^+$

$$S(z, \zeta) = \begin{cases} \frac{\zeta + z}{\zeta - z} - \frac{\zeta + z}{\zeta - z}, & |\zeta| = 1, \ 0 < \text{Im} \zeta, \ z \in \mathbb{D}^+, \\ 2 \left( \frac{1}{\zeta - z} - \frac{z}{1 - \zeta} \right), & |\text{Re} \zeta| < 1, \ \text{Im} \zeta = 0, \ z \in \mathbb{D}^+, \end{cases}$$

see [66].

• Isosceles orthogonal triangle with the vertices 0, 1, $i$

$$S(z, \zeta; z_0) = \sum_{m,n \in \mathbb{Z}} \left[ g_{m,n}(z, \zeta) - \frac{1}{2} \left[ g_{m,n}(z_0, \zeta) + g_{m,n}(\overline{z_0}, \zeta) \right] \right],$$

$$g_{m,n}(z, \zeta) = \frac{1}{\zeta - z - 2m - 2ni} + \frac{1}{\zeta + zi - (2m + 1) - (2n + 1)i} + \frac{1}{\zeta + z - (2m + 2) - 2ni}, \ m, n \in \mathbb{Z},$$

see [17, 67].
• Upper half of concentric ring $R_{r,1}^{+} = \{0 < r < |z| < 1, 0 < \text{Im}z\}$

$$S(z, \zeta) =$$

$$\left\{ \begin{array}{l}
\left\{ \frac{\zeta + 2}{\zeta - 2} - \frac{\zeta + 2}{\zeta - 2} \right\} + 2 \sum_{n=1}^{\infty} r^{2n} \left[ \frac{\zeta}{r^{2n+1} - \zeta} - \frac{\zeta}{r^{2n+1} - \zeta} - \frac{\zeta}{r^{2n} - \zeta} + \frac{\zeta}{r^{2n} - \zeta} \right], \quad |\zeta| = 1, \ 0 < \text{Im} \zeta, \ z \in R^+,

\left\{ \frac{\zeta + 2}{\zeta - 2} - \frac{\zeta + 2}{\zeta - 2} \right\} + 2 \sum_{n=1}^{\infty} r^{2n} \left[ \frac{\zeta}{r^{2n+1} - \zeta} - \frac{\zeta}{r^{2n+1} - \zeta} - \frac{\zeta}{r^{2n} - \zeta} + \frac{\zeta}{r^{2n} - \zeta} \right], \quad |\zeta| = r, \ 0 < \text{Im} \zeta, \ z \in R^+,

2 \left\{ \frac{1}{1 - \xi} - \frac{1}{1 - \eta} \right\} + \sum_{n=1}^{\infty} r^{2n} \left[ \frac{1}{\xi - r^{2n} \zeta - \xi} - \frac{1}{\xi - r^{2n} \zeta - \xi} + \frac{\xi}{\xi - r^{2n} \zeta - \xi} + \frac{\zeta}{\xi - r^{2n} \zeta - \xi} \right], \quad \zeta \in [-1, r] \cup [r, 1], \ z \in R^+,
\end{array} \right.$$ 

see [66].

• Quarter ring $R_{r,1}^{++} = \{0 < r < |z| < 1, 0 < x = \text{Re}z, 0 < y = \text{Im}z\}$

$$S(z, \zeta) =$$

$$\left\{ \begin{array}{l}
2 \left[ \frac{\xi^2 + \zeta^2}{\xi^2 - \zeta^2} - \frac{\xi^2 + \zeta^2}{\xi^2 - \zeta^2} \right] + 4 \sum_{k=1}^{\infty} r^{4k} \left[ \frac{\xi^2}{r^{4k} \xi^2 - \zeta^2} - \frac{\zeta^2}{r^{4k} \zeta^2 - \zeta^2} + \frac{\xi^2}{r^{4k} \xi^2 - \zeta^2} - \frac{\zeta^2}{r^{4k} \zeta^2 - \zeta^2} \right], \quad |\zeta| = 1, 0 < \xi, 0 < \eta,

-2 \left[ \frac{\xi^2 + \zeta^2}{\xi^2 - \zeta^2} - \frac{\xi^2 + \zeta^2}{\xi^2 - \zeta^2} \right] - 4 \sum_{k=1}^{\infty} r^{4k} \left[ \frac{\xi^2}{r^{4k} \xi^2 - \zeta^2} - \frac{\zeta^2}{r^{4k} \zeta^2 - \zeta^2} + \frac{\xi^2}{r^{4k} \xi^2 - \zeta^2} - \frac{\zeta^2}{r^{4k} \zeta^2 - \zeta^2} \right], \quad |\zeta| = r, 0 < \xi, 0 < \eta,

4 \left[ \frac{\xi}{\xi^2 - \zeta^2} - \frac{\xi^2}{1 - \xi^2} \right] + \sum_{k=1}^{\infty} r^{4k} \left( \frac{\xi}{r^{4k} \xi^2 - \zeta^2} - \frac{\zeta^2}{r^{4k} \zeta^2 - \zeta^2} + \frac{\xi^2}{r^{4k} \xi^2 - \zeta^2} - \frac{\zeta^2}{r^{4k} \zeta^2 - \zeta^2} \right), \quad \eta < \xi < 1, 0 < \eta < 0,

-4i \left[ \frac{\eta^2}{1 + \eta^2} + \frac{\eta^2}{1 + \eta^2} \right] + \sum_{k=1}^{\infty} r^{4k} \left( \frac{\eta}{r^{4k} \eta^2 + \eta^2} - \frac{\eta^2}{r^{4k} \eta^2 + \eta^2} + \frac{\eta^2}{r^{4k} \eta^2 + \eta^2} - \frac{\eta^2}{r^{4k} \eta^2 + \eta^2} \right), \quad \xi = 0, \ 0 < \eta < 1,
\end{array} \right.$$ 

$z \in R_{r,1}^{++}, \zeta = \xi + i\eta$, see [68].

• Half hexagon $P^+$ with the corners $\pm 2, \pm 1 + i\sqrt{3}$

$$S(z, \zeta) = 2 \sum_{m+n \in \mathbb{Z}} [q_{m,n}(z, \zeta) - q_{m,n}(0, \zeta)] \zeta^m +$$

$$\left\{ \begin{array}{l}
- \frac{2(2\xi - 3)}{(2\xi - 3)^2 + 3}, \quad \zeta \in \partial_1 P^+,

\frac{2\xi}{\xi^2 + 3}, \quad \zeta \in \partial_2 P^+,

\frac{2(2\xi + 3)}{(2\xi + 3)^2 + 3}, \quad \zeta \in \partial_3 P^+,

\frac{2}{\xi}, \quad \zeta \in \partial_4 P^+,
\end{array} \right.$$
with the segments \( \partial_k P^+ \) on the respective lines of \( \partial P^+ \) between the points \([2, 1 + i\sqrt{3}], k = 1; [1 + i\sqrt{3}, -1 + i\sqrt{3}], k = 2; [-1 + i\sqrt{3}, -2], k = 3; [-2, 2], k = 4; \)

\[
q_{m,n}(z, \zeta) \in \left\{ \frac{3(\zeta - \omega_{m,n} - 2)^2}{(\zeta - \omega_{m,n} - 2)^3 - (z - 2)^3}, \quad \frac{3(\zeta - \omega_{m,n} + 1 - i\sqrt{3})^2}{(\zeta - \omega_{m,n} + 1 - i\sqrt{3})^3 - (z + 1 - i\sqrt{3})^3}, \quad \frac{3(\zeta - \omega_{m,n} + 2)^2}{(\zeta - \omega_{m,n} + 2)^3 - (z + 2)^3} \right\},
\]

and \( \omega_{m,n} = 3m + i\sqrt{3}n, m + n \in 2\mathbb{Z}, \) see [68].

- Circle sector \( S_{1,\frac{z}{n}} = \{|z| < 1, 0 < \arg z < \frac{\pi}{n}\} \)

\[
S(z, \zeta) = \left\{ \begin{array}{ll}
\sum_{k=0}^{n-1} \left( \frac{\zeta + ze^{\frac{2k\pi i}{n}}}{\zeta - ze^{\frac{2k\pi i}{n}}} - \frac{\overline{\zeta} + ze^{\frac{2k\pi i}{n}}}{\overline{\zeta} - ze^{\frac{2k\pi i}{n}}} \right), & |\zeta| = 1, \ 0 < \arg \zeta < \frac{\pi}{n}, \ z \in S_{1,\frac{z}{n}}, \\
\sum_{k=0}^{n-1} \left( \frac{1}{\zeta - ze^{\frac{2k\pi i}{n}}} - \frac{z}{e^{\frac{2k\pi i}{n}} - z} \right), & 0 < |\zeta| < 1, \ \arg \zeta \in \{0, \frac{\pi}{n}\}, \ z \in S_{1,\frac{z}{n}},
\end{array} \right.
\]

see [69].

- Hyperbolic half plane, lens \( D = \mathbb{D} \cap D_m(r), \) and lunes \( \mathbb{D} \setminus D_m(r), D_m(r) \setminus \mathbb{D} \)

\( D_m(r) = \{|z - m| < r\}, \) \( 1 + r^2 = m^2 \)

\[
S(z, \zeta) = \left\{ \begin{array}{ll}
\left[ \frac{2\zeta}{\zeta - z} - 1 + \frac{2(\zeta - m)^2}{(\zeta - m)^2 + z - m} - 1 \right], & \zeta \in \partial D \cap \partial \mathbb{D}, \ z \in D, \\
\left[ \frac{2(\zeta - m)}{\zeta - z} - 1 + \frac{2(\zeta - m)^2}{(\zeta - m)^2 - zr^2} - 1 \right], & \zeta \in \partial D \cap \partial D_m(r), \ z \in D,
\end{array} \right.
\]

see [70].

### 3.2 Poly-harmonic Green functions.

Examples for Green, Neumann, Robin, and hybrid Green functions

- Unit disc \( \mathbb{D} = \{|z| < 1\} \)

\[
G_1(z, \zeta) = \log \left| \frac{1 - z\overline{\zeta}}{\zeta - z} \right|^2,
\]

\[
N_1(z, \zeta) = -\log |(\zeta - z)(1 - z\overline{\zeta})|^2
\]

with \( \partial_{\nu} N_1(z, \zeta) = -2 \) on \( \partial \mathbb{D}, \)
\[ R_{1, \alpha, \beta}(z, \zeta) = \log \left| \frac{1 - \overline{z} \zeta}{\zeta - z} \right|^2 + \begin{cases} 
\quad 2\beta \sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^k + (\zeta z)^k}{\alpha + k\beta}, & -\frac{\alpha}{\beta} \notin \mathbb{N}, \\
\quad 2\beta \sum_{\substack{k=1, \\ k \neq k_0}}^{\infty} \frac{(z\overline{\zeta})^k + (\zeta z)^k}{\alpha + k\beta} + 2[(z\overline{\zeta})^{k_0} \log(z\overline{\zeta}) + (\zeta z)^{k_0} \log(\zeta z)], \\
\quad k_0 = -\frac{\alpha}{\beta} \in \mathbb{N}, \end{cases} \]

see [71, 56].

Explicit bi-harmonic Green functions for \( D \)

\[ G_2(z, \zeta) = |\zeta - z|^2 G_1(z, \zeta) - (1 - |z|^2)(1 - |\zeta|^2), \]
\[ G_2(z, \zeta) = |\zeta - z|^2 G_1(z, \zeta) + (1 - |z|^2)(1 - |\zeta|^2) \left[ \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \zeta z)}{\zeta z} \right], \]
\[ N_2(z, \zeta) = |\zeta - z|^2 \left[ 4 + N_1(z, \zeta) \right] - 4 \sum_{k=2}^{\infty} \frac{(z\overline{\zeta})^k + (\zeta z)^k}{k^2} \\
\quad - 2(z\overline{\zeta} + \zeta z) \log|1 - z\overline{\zeta}|^2 \\
\quad + (1 + |z|^2)(1 + |\zeta|^2) \left[ \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \zeta z)}{\zeta z} \right], \]

see [50, 69].

Iterated Robin function

\[ R_{2, \alpha, \beta}(z, \zeta) = G_2(z, \zeta) + 2\beta(|z|^2 + |\zeta|^2 - 2) \sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^k + (\zeta z)^k}{(\alpha + k\beta)(k + 1)} \\
\quad - 4\beta^2 \sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^k + (\zeta z)^k}{(\alpha + k\beta)^2(k + 1)}, \quad -\frac{\alpha}{\beta} \notin \mathbb{N}, \]

see [54, 55]. For \( R_{3, \alpha, \beta} \) see [54].

Green-Neumann function

\[ G_1 N_1(z, \zeta) = -|\zeta - z|^2 \log|\zeta - z|^2 \\
\quad - (1 - |z|^2) \left[ 4 + \frac{1 - z\overline{\zeta}}{z\overline{\zeta}} \log(1 - z\overline{\zeta}) + \frac{1 - \zeta z}{\zeta z} \log(1 - \zeta z) \right] \\
\quad - (\zeta - z) \frac{1 - z\overline{\zeta}}{z} \log(1 - z\overline{\zeta}) - \frac{(\zeta - z)(1 - \zeta z)}{z} \log(1 - \zeta z), \]

see [50, 72].
Neumann-Green function

\[ N_1 G_1(z, \zeta) = -|\zeta - z|^2 \log |\zeta - z|^2 \]

\[ - (1 - |\zeta|^2) \left[ 4 + \frac{1 - z\zeta}{z\zeta} \log(1 - z\zeta) + \frac{1 - \zeta\bar{z}}{z\zeta} \log(1 - \zeta\bar{z}) \right] \]

\[ + \frac{(\zeta - z)(1 - z\zeta)}{\zeta} \log(1 - \zeta\bar{z}) + \frac{(\zeta - z)(1 - z\zeta)}{\zeta} \log(1 - \zeta\bar{z}), \]

see [50, 72].

Tri-harmonic Green function for \( \mathbb{D} \)

\[ G_3(z, \zeta) = \frac{|\zeta - z|^4}{4} \log \left| \frac{1 - z\zeta}{\zeta - z} \right|^2 + \frac{1}{4} (1 - |z|^2)(1 - |\zeta|^2)(z\zeta + \zeta\bar{z} - 4) \]

\[ - \frac{1}{4} (1 - |z|^2)(1 - |\zeta|^4) \left[ \frac{\log(1 - z\zeta)}{(z\zeta)^2} + \frac{\log(1 - \zeta\bar{z})}{(z\zeta)^2} + \frac{1}{z\zeta} + \frac{1}{\bar{z}\zeta} \right] \]

\[ - \frac{1}{2} (1 - |z|^2)(1 - |\zeta|^2)(|z|^2 + |\zeta|^2) \left[ \frac{\log(1 - z\zeta)}{z\zeta} + \frac{\log(1 - \zeta\bar{z})}{\bar{z}\zeta} \right] \]

\[ + (1 - |z|^2)(1 - |\zeta|^2) \sum_{k=0}^{\infty} \frac{(z\zeta)^k + (\bar{z}\zeta)^k}{(k + 1)^2}, \]

see [69].

Tri-harmonic Neumann function for \( \mathbb{D} \)

\[ N_3(z, \zeta) = \frac{3}{2} (|z|^4 + |\zeta|^4) + 5(1 + |z|^2)(1 + |\zeta|^2) + 2(|z|^2 + |\zeta|^2 + 6) \]

\[ + \frac{1}{4} (1 - |z|^2)(1 - |\zeta|^2)(z\zeta + \zeta\bar{z}) + \frac{|\zeta - z|^4}{4} \log \left| \frac{1 - z\zeta}{\zeta - z} \right|^2 \]

\[ - [2(2 + |z|^2)(2 + |\zeta|^2) + \frac{1}{2} (|z|^4 + |\zeta|^4)] \log |1 - z\zeta|^2 \]

\[ + \frac{1}{2} (1 + |z|^2)(1 + |\zeta|^2)|z|^2 + 2(2 + |z|^2)(2 + |\zeta|^2) + 4(|z|^2 + |\zeta|^2 + 2)] \left[ \frac{\log(1 - z\zeta)}{z\zeta} + \frac{\log(1 - \zeta\bar{z})}{\bar{z}\zeta} \right] \]

\[ - \frac{1}{4} (1 + |z|^4)(1 + |\zeta|^4) \left[ \frac{\log(1 - z\zeta)}{(z\zeta)^2} + \frac{\log(1 - \zeta\bar{z})}{(z\zeta)^2} + \frac{1}{z\zeta} + \frac{1}{\bar{z}\zeta} \right] \]

\[ + \sum_{k=1}^{\infty} \frac{8}{k^2} \left( \frac{1 + |z|^2)(1 + |\zeta|^2)}{(k + 1)^2} - \frac{4(|z|^2 + |\zeta|^2) + 6}{k^2} \right) [(z\zeta)^2 + (\bar{z}\zeta)^2], \]

see [69].
Poly-harmonic Green-Almansi function for \( \mathbb{D} \):
\[
G_n(z, \zeta) = \frac{|z - \zeta|^{2(n-1)}}{(n-1)!} \log \left| \frac{1 - z \bar{\zeta}}{1 - \zeta \bar{z}} \right|^2
- \sum_{\mu=1}^{n-1} \frac{1}{\mu} |\zeta - z|^{2(n-1-n)} (1 - |z|^2)^{\mu} (1 - |\zeta|^2)^{\mu},
\]
z, \zeta \in \mathbb{D}, z \neq \zeta, n \in \mathbb{N},
\]
see [24, 25, 11].

- Upper (lower) half of \( \mathbb{D} \), \( \mathbb{D}^\pm = \mathbb{D} \cap H^\pm (H^\pm = \{ \mathbb{R} \}) \)
\[
G_1(z, \zeta) = \log \left| \frac{(1 - \zeta \bar{z})(\zeta - \bar{z})}{(z - \zeta)(1 - z \bar{\zeta})} \right|^2, N_1(z, \zeta) = - \log |(\zeta - z)(\zeta - \bar{z})(1 - z \bar{\zeta})(1 - \zeta \bar{z})|^2,
\]
also
\[
\tilde{N}_1(z, \zeta) = 2 \log |z\bar{\zeta}|^2 - \log |(\zeta - z)(\zeta - \bar{z})(1 + z \bar{\zeta})(1 - z \bar{\zeta})|^2,
\]
see [66] and also [73].
\[
G_2(z, \zeta) = |\zeta - \bar{z}|^2 \log \left| \frac{\zeta - \bar{z}}{1 - z \bar{\zeta}} \right|^2 - |\zeta - z|^2 \log \left| \frac{\zeta - z}{1 - \zeta \bar{z}} \right|^2
+ (1 - |z|^2)(1 - |\zeta|^2) \left[ \frac{\log (1 - z \bar{\zeta})}{z \bar{\zeta}} + \frac{\log (1 - \zeta \bar{z})}{\zeta \bar{z}} - \frac{\log (1 - z \bar{\zeta})}{z \bar{\zeta}} - \frac{\log (1 - \zeta \bar{z})}{\zeta \bar{z}} \right],
\]
see [69].
\[
G_3(z, \zeta) = \frac{|\zeta - z|^4}{4} \log \left| \frac{1 - z \bar{\zeta}}{\zeta - z} \right|^2 - \frac{|\zeta - \bar{z}|^4}{4} \log \left| \frac{1 - \zeta \bar{z}}{\zeta - \bar{z}} \right|^2
- \frac{1}{4} (1 - |z|^2)(1 - |\zeta|^2) \left[ (z - \bar{z})(\zeta - \bar{\zeta}) - 4 \sum_{k=1}^\infty \frac{(z \bar{\zeta})^k + (\bar{z} \zeta)^k - (\zeta \bar{z})^k - (z \bar{\zeta})^k}{k+1} \right]
+ \frac{1}{z \bar{\zeta}} + \frac{1}{\zeta \bar{z}} - \frac{1}{z \bar{\zeta}} - \frac{1}{\zeta \bar{z}}
+ \frac{\log (1 - z \bar{\zeta})}{(z \bar{\zeta})^2} + \frac{\log (1 - \zeta \bar{z})}{(\zeta \bar{z})^2} - \frac{\log (1 - z \bar{\zeta})}{(z \bar{\zeta})^2} - \frac{\log (1 - \zeta \bar{z})}{(\zeta \bar{z})^2}
+ \frac{1}{2} (1 - |z|^2)(1 - |\zeta|^2)(|z|^2 + |\zeta|^2) \left[ \frac{\log (1 - z \bar{\zeta})}{z \bar{\zeta}} + \frac{\log (1 - \zeta \bar{z})}{\zeta \bar{z}} - \frac{\log (1 - z \bar{\zeta})}{z \bar{\zeta}} - \frac{\log (1 - \zeta \bar{z})}{\zeta \bar{z}} \right].
\]
see \[69\].

\[ N_2(z, \zeta) = 8(|z|^2 + |\zeta|^2) - |\zeta - z|^2 \log |\zeta - z|^2 - |\zeta - \overline{z}|^2 \log |\zeta - \overline{z}|^2 \\
- |\zeta + z|^2 \log |1 - z\overline{\zeta}|^2 - |\zeta + \overline{z}|^2 \log |1 - z\zeta|^2 \\
- 4 \sum_{k=1}^{\infty} \frac{(z\overline{\zeta})^k + (\overline{z}\zeta)^k + (z\zeta)^k + (\overline{z}\overline{\zeta})^k}{k^2} \\
+ (1 + |z|^2)(1 + |\zeta|^2) \left[ \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \zeta\overline{z})}{z\zeta} \\
+ \frac{\log(1 - z\overline{\zeta})}{\overline{z}\overline{\zeta}} + \frac{\log(1 - \zeta\overline{z})}{\overline{z}\zeta} \right], \]

see \[69\].

\[ N_3(z, \zeta) = 3(|z|^4 + |\zeta|^4) + 10(1 + |z|^2)(1 + |\zeta|^2) + 4(|z|^2 + |\zeta|^2 + 6) \\
- \frac{1}{4} \left[ \frac{1 + |z|^4 + |\zeta|^4}{|z|^2|\zeta|^2} + |z|^2 + |\zeta|^2 - 1 \right] (z + \overline{z})(\zeta + \overline{\zeta}) \\
+ \frac{|\zeta - z|^4}{4} \log \left| \frac{1 - z\overline{\zeta}}{\zeta - z} \right|^2 + \frac{|\zeta - \overline{z}|^4}{4} \log \left| \frac{1 - z\overline{\zeta}}{\zeta - \overline{z}} \right|^2 \\
+ \sum_{k=1}^{\infty} \left[ \frac{8}{k^3} - \frac{(1 + |z|^2)(1 + |\zeta|^2)}{(k + 1)^2} - \frac{4(|z|^2 + |\zeta|^2 + 6)}{k^2} \right] \\
\times [(z\overline{\zeta})^k + (\overline{z}\zeta)^k + (z\zeta)^k + (\overline{z}\overline{\zeta})^k] \\
- [2(2 + |z|^2)(2 + |\zeta|^2) + \frac{1}{2}(|z|^4 + |\zeta|^4)] \log |1 - z\overline{\zeta}|(1 - z\zeta)|^2 \\
+ \left[ \frac{1}{2} (1 + |z|^2)(1 + |\zeta|^2)(|z|^2 + |\zeta|^2) + 4(|z|^2 + |\zeta|^2 + 2) \right] \\
\times \left[ \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \zeta\overline{z})}{z\zeta} + \frac{\log(1 - z\overline{\zeta})}{\overline{z}\overline{\zeta}} + \frac{\log(1 - \zeta\overline{z})}{\overline{z}\zeta} \right] \\
- \frac{1}{4} (1 + |z|^4)(1 + |\zeta|^4) \left[ \frac{\log(1 - z\overline{\zeta})}{(z\overline{\zeta})^2} + \frac{\log(1 - \zeta\overline{z})}{(\overline{z}\overline{\zeta})^2} \\
+ \frac{\log(1 - z\overline{\zeta})}{(z\zeta)^2} + \frac{\log(1 - \zeta\overline{z})}{(\overline{z}\zeta)^2} \right], \]

see \[69\]. The tetra-harmonic Green and Neumann functions \( G_4 \) and \( N_4 \) for disc sectors \( S_{\frac{1}{n}} = \{|z| < 1, 0 < \arg z < \frac{\pi}{n}\} \), \( n \in \mathbb{N} \) in particular for \( n = 1 \) and for \( \mathbb{D} \) are also provided in \[69\].
- Right (left) half of $\mathbb{D}, \mathbb{D}_r = \mathbb{D} \cap \{0 < \text{Re} z\} (\mathbb{D}_l = \mathbb{D} \cap \{\text{Re} z < 0\})$

$$G_1(z, \zeta) = \log \left| \frac{1 - \frac{z\zeta + \bar{\zeta}}{\zeta - z}}{1 + z\zeta} \right|^2, \quad N_1(z, \zeta) = -\log \left| (\zeta - z)(1 - \bar{\zeta})(1 + z\zeta)(\zeta + \bar{\zeta}) \right|^2,$$

see [74].

- Disc sector $S_\alpha = \{z : |z| < 1, 0 < \arg z < \alpha\pi\}, 0 < \alpha < 2$

$$G_1(z, \zeta) = \log \left| \frac{1 - (\bar{z}\zeta)^{\frac{1}{\alpha}} - (z\zeta)^{\frac{1}{\alpha}}}{\zeta - z} \right|^2,$$

$$N_1(z, \zeta) = -\log \left| \left(\frac{1}{\zeta} - \frac{1}{z} \right) \left(\frac{1}{\bar{\zeta}} - \frac{1}{\bar{z}} \right) \left(1 - \bar{z}\zeta\right) \left(1 - z\bar{\zeta}\right) \left(1 - (z\zeta)^{\frac{1}{\alpha}}\right) \left(1 - (\bar{z}\zeta)^{\frac{1}{\alpha}}\right) \right|^2,$$

see [56, 75].

- Almaty apple $D = \{z + \bar{z} < 1, |z - 1| < 1\}$

$$G_1(z, \zeta) = \log \left| \frac{1 - z\zeta - (1 - \bar{z})\bar{\zeta}}{\zeta - z} \right|^2,$$

$$\tilde{N}_1(z, \zeta) = -\log \left| \left(\frac{1}{\zeta} - \frac{1}{z} \right) \left(1 - \bar{z}\zeta\right) \left(1 - z\bar{\zeta}\right) \left(\zeta - (1 - \bar{z})\bar{\zeta}\right) \left(1 - (z\zeta)\bar{\zeta}\right) \left(1 - (1 - z)\zeta\bar{\zeta}\right) \right|^2,$$

$$N_1(z, \zeta) = \tilde{N}_1(z, \zeta) - c, c = \frac{3}{2\pi} \int_{\partial D \cap \{|z - 1| = 1\}} \tilde{N}_1(z, \zeta) ds_z,$$

see [74].

- Domain bounded by two circular arcs $D = \{|z| < 1, 0 < |z|\sin(\alpha - \theta) + (z + \bar{z})\sin \theta - \sin(\alpha + \theta)\}, 0 < \alpha < \pi, \theta = \frac{\pi}{n}, n \in \mathbb{N}$

$$G_1(z, \zeta) = \log |P_1(z, \zeta)|^2, \quad N_1(z, \zeta) = -\log |Q_1(z, \zeta)|^2,$$

$$P_1(z, \zeta) = \frac{1 - \bar{z}\zeta}{\zeta - z} \prod_{k=1}^{n-1} \frac{\bar{z}\zeta \sin(\alpha + k\theta) - (\bar{z} + \zeta) \sin k\theta - \sin(\alpha - k\theta)}{(1 + z\zeta) \sin k\theta + z \sin(\alpha - k\theta) - \zeta \sin(\alpha + k\theta)},$$

$$Q_1(z, \zeta) = (\zeta - z)(1 - \bar{z}\zeta) \prod_{k=1}^{n-1} \left[ (\bar{z}\zeta \sin(\alpha + k\theta) - (\bar{z} + \zeta) \sin k\theta - \sin(\alpha - k\theta)) \right] \times \left[ (1 + z\zeta) \sin k\theta + z \sin(\alpha - k\theta) - \zeta \sin(\alpha + k\theta) \right],$$

see [60].

- Upper half planes $\mathbb{R}^+, \mathbb{R}^r$, quarter plane $\mathbb{Q}^{++}$
\[ \mathbb{H}^+ = \{ z : i(z - \overline{z}) < 0 \} : \quad G_1(z, \zeta) = \log \left| \frac{\zeta - z}{\zeta - \overline{z}} \right|^2, \quad h_1(z, \zeta) = \frac{1}{\zeta - z}, \quad N_1(z, \zeta) = -\log |(\zeta - z)(\zeta - \overline{z})|^2, \]

see [13],

Poly-harmonic Green-Almansi function for \( \mathbb{H}^+ \)

\[ G_n(z, \zeta) = \log \left| \frac{\zeta - z}{\zeta - \overline{z}} \right|^2, \quad h_n(z, \zeta) = \frac{1}{\zeta - z}, \quad N_n(z, \zeta) = -\log \left| \frac{(\zeta - z)(\zeta - \overline{z})}{\zeta - \overline{z}} \right|^2, \]

see [26].

\[ \mathbb{H}^- = \{ z : 0 < (z + \overline{z}) \} : \quad G_1(z, \zeta) = \log \left| \frac{\zeta + \overline{z}}{\zeta - z} \right|^2, \quad h_1(z, \zeta) = \frac{1}{\zeta + z}, \quad N_1(z, \zeta) = -\log |(\zeta + \overline{z})(\zeta - z)|^2, \]

\[ \mathbb{Q}^{++} = \{ z : i(z - \overline{z}) < 0 < (z + \overline{z}) \} : \quad G_1(z, \zeta) = \log \left| \frac{(\zeta - z)(\zeta + \overline{z})}{(\zeta - \overline{z})(\zeta + z)} \right|^2, \quad h_1(z, \zeta) = \frac{1}{\zeta - z} + \frac{1}{\zeta + z} - \frac{1}{\zeta + z}, \quad N_1(z, \zeta) = -\log |(\zeta - \overline{z})(\zeta + \overline{z})(\zeta - z)(\zeta + z)|^2, \]

see [13].

- Strip \( S_1 = \{ z \in \mathbb{C} : z = e^{2ita} + 2i\alpha e^{ita}, 0 < t < 1 \}, 0 < \alpha < \pi, a \in \mathbb{R}^+ \)

\[ G_1(z, \zeta) = \left| \frac{\sin \pi \frac{z - e^{2ita}}{2ia e^{ita}}}{\sin \pi \frac{\zeta - z}{2ia e^{ita}}} \right|^2, \quad N_1(z, \zeta) = -\log \left| \sin \pi \frac{\zeta - e^{2ita}}{2ia e^{ita}} \sin \pi \frac{\zeta - z}{2ia e^{ita}} \right|^2, \]

see [76].

- Hyperbolic half plane, lens \( D = \mathbb{D} \cap D_m(r), \) and lunes \( \mathbb{D} \setminus D_m(r), D_m(r) \setminus \mathbb{D} \)

\[ D_m(r) = \{ |z - m| < r \}, 1 + r^2 = m^2 \]

\[ G_1(z, \zeta) = \log \left| \frac{1 - z\overline{\zeta} m(\overline{\zeta} + z) - (1 + z\overline{\zeta})}{\zeta - z} \right|^2, \quad N_1(z, \zeta) = -\log |(\zeta - z)(1 - z\overline{\zeta})(\zeta + z - m(1 + z\overline{\zeta}))(1 + z\overline{\zeta} - m(\overline{\zeta} + z))|^2, \]

see [56, 70].
Hyperbolic strip $D = \mathbb{D} \setminus \{D_{-m_1(r_1)} \cup D_{m_2(r_2)}\}$

$D_m(r) = \{|z - m| < r\}, 1 + r^2 = m^2, -1 < r_1 - m_1 < 0 < m_2 - r_2 < 1$

$G_1(z, \zeta) = \log |P(z, \zeta)|^2, \quad N_1(z, \zeta) = -\log |Q(z, \zeta)|^2,$

$P(z, \zeta) = \frac{1 - \bar{z}\zeta}{\zeta - z}\frac{\zeta - z_1}{\zeta - z_2}\prod_{k=1}^{\infty} \frac{1 - \bar{z}_{4k-1}\zeta(1 - \bar{z}_{4k}\zeta - z_{4k+1})}{1 - \bar{z}_{4k-1}\zeta - z_{4k}}\frac{\zeta - z_{4k+2}}{1 - \bar{z}_{4k-1}\zeta - z_{4k}},$

$Q(z, \zeta) = (\zeta - z)(1 - \bar{z}\zeta)\prod_{k=1}^{\infty} \frac{\zeta - z_{2k-1}}{\zeta + 1}\frac{1 - \bar{z}_{2k-1}\zeta - z_{2k}}{\bar{z}_{2k-1}(1 + \zeta)}\frac{\zeta - 1}{\bar{z}_{2k}(1 - \zeta)}.$

$\alpha = m_1m_2 + 1, \beta = m_1 + m_2, z_1 = -\frac{m_1(\bar{\zeta} + 1)}{\bar{\zeta} + m_1}, z_2 = \frac{m_2\bar{\zeta} - 1}{\bar{\zeta} - m_2},$

$z_{2k+3} = \frac{\alpha z_{2k-1} - \beta}{\alpha - \beta z_{2k-1}}, \quad z_{2k+4} = \frac{\alpha z_{2k} + \beta}{\alpha + \beta z_{2k}}, \quad k \in \mathbb{N},$

[76, 77, 78].

Degenerated ring $D = \{\frac{1}{2} < |z - \frac{1}{2}|, |z| < 1\} = \{1 < |2z - 1|, |z| < 1\}$

$z_{2k} = \frac{(k - 1)z - k}{k\bar{z} - (k + 1)}$, $z_{2k+1} = \frac{(k + 1)\bar{z} - k}{(k + 2)\bar{z} - (k + 1)}$,

$\hat{z}_{2k} = \frac{(k + 1)\bar{z} - (k + 2)}{k\bar{z} - (k + 1)} = \frac{1}{\hat{z}_{2k+2}}, \quad \hat{z}_{2k+1} = \frac{(k + 3)z - (k + 2)}{(k + 2)z - (k + 1)} = \frac{1}{\hat{z}_{2k+3}}, \quad k \in \mathbb{N},$

$G_1(z, \zeta) = \log |P_1(z, \zeta)|^2, \quad N_1(z, \zeta) = -\log |Q_1(z, \zeta)|^2, z, \zeta \in D, \zeta \neq z,$

$P_1(z, \zeta) = \frac{z\bar{\zeta} - \zeta - \bar{z}}{\bar{\zeta} - z}\frac{\zeta - z_{2k+1}}{\bar{\zeta} - \hat{z}_{2k+1}}\prod_{k=0}^{\infty} \frac{\zeta - z_{2k+1}}{\bar{\zeta} - \hat{z}_{2k+2}}\zeta - \hat{z}_{2k+1},$

$Q_1(z, \zeta) = \frac{z\bar{\zeta} - \zeta - \bar{z}}{\bar{\zeta} - z}\frac{\zeta + 1 - 2z}{\bar{\zeta} - z}\frac{\zeta + 1 - 2z}{\bar{\zeta} - z}\frac{\zeta + 1 - 2z}{\bar{\zeta} - z},$

$\times \prod_{k=0}^{\infty} \left[\frac{\zeta - z_{2k+1}}{\zeta - \hat{z}_{2k+1}}\frac{\zeta - \hat{z}_{2k+1}}{\zeta - \hat{z}_{2k+2}}\right].$

see [79].

Plane circular rectangle $D = \{0 < i(\bar{z} - z), |z + 1| < \sqrt{2}, |z - i\sqrt{3}|\}$

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\[ G_1(z, \zeta) = \log |P(z, \zeta)|^2, \quad P(z, \zeta) = \frac{1}{P_0(z, \zeta)} \prod_{k \in \mathbb{N}_0} P_{2k+1}(z, \zeta) P_{2k+2}(z, \zeta), \]

with

\[ P_k(z, \zeta) = \zeta - z_k^+ 1 + z_k^+ \zeta - z_k^+ 1 + z_k^+ \zeta, \quad k \in \mathbb{N}_0, \]

where

\[ z_0^+ = z, \quad z_1^+ = \frac{m_1 \bar{z} - 1}{z + m_1 i}, \quad m_1 = \sqrt{3}, \]

\[ z_{k+1}^+ = \frac{\alpha_k z_{k-1}^+ + \beta_k i}{\alpha_k - \beta_k i z_k^+}, \quad \alpha_k = m_k m_{k+1} - 1, \quad \beta_k = m_k - m_{k+1}, m_{k+1} = \frac{m_1 m_k + 1}{m_1 + m_k}, \]

\[ z_{k+1}^+ = 1 - \frac{z_k^+}{1 + z_k^+}, \quad k \in \mathbb{N}, \]

see [80].

- Equilateral triangle \( T \) with corner point in \( -1, 1, i\sqrt{3} \)

\[ G_1(z, \zeta) = \begin{cases} 
\log \left| \prod_{m+n \in \mathbb{Z}} \left( \frac{(z - \omega_{m,n} - 1)^3 - (\zeta - 1)^3}{(z - \omega_{m,n} - 1)^3 - (\zeta - 1)^3} \right)^2 \right|, & \zeta = \bar{\zeta} \quad \text{or} \quad \zeta = -\frac{1}{2}(1 + i\sqrt{3}\zeta) + \frac{\sqrt{3}}{2}(\sqrt{3} + i), \\
\log \left| \prod_{m+n \in \mathbb{Z}} \left( \frac{(z - \omega_{m,n+1})^3 - (\zeta + 1)^3}{(z - \omega_{m,n+1})^3 - (\zeta + 1)^3} \right)^2 \right|, & \zeta = \bar{\zeta} \quad \text{or} \quad \zeta = \frac{1}{2}(1 - i\sqrt{3})\zeta - \frac{\sqrt{3}}{2}(\sqrt{3} - i), 
\end{cases} \]

for \( z \in T \) and with \( \omega_{m,n} = 3m + i\sqrt{3}n, \quad m + n \in 2\mathbb{Z}, \) see [71, 81].

- Concentric ring \( R_{r,1} = \{ 0 < r < |z| < 1 \} \)

\[ G_1(z, \zeta) = \frac{\log |z|^2 \log |\zeta|^2}{\log r^2} + \log \left| \frac{1 - z \zeta}{\zeta - z} \right| \prod_{k=1}^{\infty} \frac{z^{2k} - 1 - r^{2k} z^{2k}}{z^{2k} - r^{2k} \zeta^{2k}}, \]

\[ N_1(z, \zeta) = -\log \left| (\zeta - z)(1 - z\zeta) \prod_{k=1}^{\infty} \left( 1 - r^{2k} \zeta \right) \left( 1 - \frac{r^k}{z\zeta} \right) \left( 1 - r^{2k} z \zeta \right) \right|^2, \]
\[ G_2(z, \zeta) = |\zeta - z|^2 G_1(z, \zeta) \]
\[ + 2 \text{Re} \left[ \frac{(1 - |z|^2) \log |\zeta|^2}{\log r^2} + \frac{(1 - |\zeta|^2) \log |z|^2}{\log r^2} \right] \]
\[ - \frac{r^2 (1 - |z|^2)}{1 - r^2} \frac{\zeta}{z} \log |\zeta|^2 - \frac{r^2 (1 - |\zeta|^2)}{1 - r^2} \frac{z}{\zeta} \log |z|^2 \]
\[ + \frac{z \zeta \log |z|^2 \log |\zeta|^2}{\log r^2} - (1 - r^2) \frac{\log |z|^2 \log |\zeta|^2}{(\log r^2)^2} \]
\[ + \frac{r^2}{1 - r^2} \frac{(1 - |z|^2)(1 - |\zeta|^2) \log r^2}{z \zeta} \]
\[ + (1 - |\zeta|^2) \left[ \frac{1 - |z|^2}{z \zeta} \log (1 - z \zeta) \right] \]
\[ - (1 - r^2) \sum_{n=1}^{\infty} \left( \frac{z}{r^{2n} \zeta} \log \left( 1 - r^{2n} \zeta \right) - \frac{1}{r^{2n} z \zeta} \log (1 - r^{2n} z \zeta) \right) \]
\[ - \sum_{n=1}^{\infty} \frac{(r^{2n} |\zeta|^2 - 1)(1 - |z|^2)}{r^{2n} z \zeta} \log (1 - r^{2n} z \zeta) \]
\[ - \sum_{n=1}^{\infty} (r^{2n} - |\zeta|^2) \left[ \frac{|z|^2 - r^2}{z \zeta} \log \left( 1 - r^{2n} z \zeta \right) - (1 - r^2) \zeta \log \left( 1 - r^{2n} z \zeta \right) \right] \]
\[ - \sum_{n=1}^{\infty} (1 - r^{2n}) \left[ \frac{(|z|^2 - r^2) \zeta}{z} \log \left( 1 - r^{2n} z \zeta \right) - (1 - r^2) z \zeta \log \left( 1 - r^{2n} z \zeta \right) \right] \]
\[ - \sum_{n=1}^{\infty} \frac{(1 - r^{2n})}{r^{2n} z} \zeta \log \left( 1 - r^{2n} z \zeta \right) + (1 - r^2) \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{r^{2n} |\zeta|^2 - 1}{r^{2n}} \left[ \frac{z}{r^{2k} \zeta} \log \left( 1 - r^{2(n+k)} z \zeta \right) - \frac{1}{r^{2k} z \zeta} \log (1 - r^{2(n+k)} z \zeta) \right] \right. \]
\[ - (r^{2n} - |\zeta|^2) \left[ \frac{r^{2k} \zeta}{z} \log \left( 1 - r^{2(n+k)} z \zeta \right) - r^{2k} \frac{z \zeta}{\zeta} \log \left( 1 - r^{2(n+k)} z \zeta \right) \right] \]
\[ - (1 - r^{2n}) \left[ \frac{r^{2k} \zeta}{z} \log \left( 1 - r^{2(n+k)} z \zeta \right) - r^{2k} \frac{z \zeta}{\zeta} \log \left( 1 - r^{2(n+k)} z \zeta \right) \right] \]
\[ + \frac{1 - r^{2n}}{r^{2n}} \left[ \frac{z \zeta}{r^{2k} \zeta} \log \left( 1 - r^{2(n+k)} z \zeta \right) - \frac{1}{r^{2k} z \zeta} \log \left( 1 - r^{2(n+k)} z \zeta \right) \right] \],

see [61].

- Upper half ring \( R_{r,1}^+ = \{ 0 < r < |z| < 1, 0 < y = \text{Im} z \} \)
\[ G_1(z, \zeta) = \log \left| \frac{1 - z\zeta}{\zeta - z} \right| \quad \text{see \cite{66}.} \]

- Quarter ring \( R^+_{r,1} = \{0 < r < \|z\| < 1, 0 < x = \text{Re}z, 0 < y = \text{Im}z\} \)

\[ G_1(z, \zeta) = \log \left| \frac{\zeta - z^2}{\zeta^2 - z^2} \right| \quad \text{see \cite{68}.} \]

- Ring sector \( S_{r,1,\alpha} = \{0 < r < \|z\| < 1, 0 < \arg z < \alpha \pi\}, \frac{1}{2} \leq \alpha \)

\[ G_1(z, \zeta) = \log \left| \frac{1 - z^\alpha \zeta^\alpha}{\zeta^\alpha - z^\alpha} \right| \quad \text{see \cite{82}.} \]

- Half hexagon \( P^+ \) with the corners \( \pm 2, \pm 1 + i\sqrt{3} \)
\[ G_1(z, \zeta) = \begin{cases} 
\log \prod_{m+n \in \mathbb{Z}} \frac{(z-\omega_{m,n}-2)^3-(\zeta-2)^3}{(z-\omega_{m,n}-2)^3-(\zeta-2)^3}, & \zeta \in \partial_1 P^+ \cup \partial_4 P^+, z \in P^+, \\
\log \prod_{m+n \in \mathbb{Z}} \frac{(z-\omega_{m,n}+1-i\sqrt{3})^3-(\zeta+1+i\sqrt{3})^3}{(z-\omega_{m,n}+1-i\sqrt{3})^3-(\zeta+1+i\sqrt{3})^3}, & \zeta \in \partial_2 P^+, z \in P^+, \\
\log \prod_{m+n \in \mathbb{Z}} \frac{(z-\omega_{m,n}+1+i\sqrt{3})^3-(\zeta+1-i\sqrt{3})^3}{(z-\omega_{m,n}+1+i\sqrt{3})^3-(\zeta+1-i\sqrt{3})^3}, & \zeta \in \partial_3 P^+ \cup \partial_4 P^+, z \in P^+, 
\end{cases} \]

with the segments \( \partial_k P^+ \) on the respective lines of \( \partial P^+ \) between the points \([2, 1 + i\sqrt{3}], k = 1; [1 + i\sqrt{3}, -1 + i\sqrt{3}], k = 2; [-1 + i\sqrt{3}, -2], k = 3; [-2, 2], k = 4; \) and \( \omega_{m,n} = 3m + i\sqrt{3}n, m + n \in 2\mathbb{Z}, \) see [68].

**Remark 7.** Obviously the iteration process for creating higher order iterated Green, Neumann, Robin, and hybrid Green functions is involved and even for the unit disc only the first few iterated Green functions are explicitly evaluated. But their normal derivatives at the boundary of \( \mathbb{D} \),

\[ g_n(z, \zeta) = -\frac{1}{2} \partial_{\zeta} G_n(z, \zeta), z \in \mathbb{D}, \zeta \in \partial \mathbb{D}, \]

\[ g_n(z, \zeta) = -\frac{1}{\pi} \int_{\mathbb{D}} G_1(z, \zeta) g_{n-1}(\zeta, \zeta) d\zeta d\eta, \quad n \in \mathbb{N}, \]

i.e. the poly-harmonic Poisson kernels of any order \( n \) are calculated, see [83]. The expressions are involved and their deduction is solely based on their properties

- \( \partial_2 \partial_\zeta g_1(z, \zeta) = 0, \partial_2 \partial_\bar{\zeta} g_n(z, \zeta) = g_{n-1}(z, \zeta), \quad 2 \leq n, \)
- \( \lim_{z \to t, |z| < 1, |t| = 1} \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) g_1(z, \zeta) d\frac{\zeta}{\zeta} = \gamma(t), \quad \text{for } \gamma \in C(\partial \mathbb{D}; \mathbb{C}), \)
- \( \lim_{z \to t, |z| < 1, |t| = 1} \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) g_2(z, \zeta) d\frac{\zeta}{\zeta} = 0 \quad \text{for } \gamma \in C(\partial \mathbb{D}; \mathbb{C}), \)
- \( \lim_{z \to t, |z| < 1, |t| = 1} g_n(z, \zeta) = 0 \quad \text{for } 2 < n \text{ and } |\zeta| = 1, \)
- \( g_n(\cdot, \zeta) \in C^{2n}(\mathbb{D}; \mathbb{C}) \) for any \( \zeta \in \partial \mathbb{D}, \)
- \( g_n(z, \zeta), \partial_2 g_n(z, \zeta), \partial_{\bar{\zeta}} g_n(z, \zeta) \in C(\mathbb{D} \times \partial \mathbb{D}; \mathbb{C}), \quad n \in \mathbb{N}. \)

The first ones are for the disc \( \mathbb{D} \)

\[ g_1(z, \zeta) = \frac{1}{1 - z\zeta} + \frac{1}{1 - \bar{z}\zeta} - 1, \]

\[ g_2(z, \zeta) = (1 - |z|^2) \left[ 1 + \frac{\log(1 - z\zeta)}{z\zeta} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta} \right], \]
\[ g_3(z, \zeta) = (1 - |z|^2) \left[ 1 + \sum_{k=2}^{\infty} \frac{(z\zeta)^{k-1} + (\zeta z)^{k-1}}{k^2} \right] \]
\[ - \frac{1 - |z|^4}{2} \left[ \frac{1}{2} + \sum_{k=2}^{\infty} \frac{(z\zeta)^{k-1} + (\zeta z)^{k-1}}{k(k+1)^2} \right], \]

\[ g_4(z, \zeta) = -(1 - |z|^2) \left[ 1 + \sum_{k=2}^{\infty} \frac{(z\zeta)^{k-1} + (\zeta z)^{k-1}}{k^2} \right] \]
\[ + \frac{1 - |z|^4}{2!} \left[ \frac{1}{2!} + \sum_{k=2}^{\infty} \frac{(z\zeta)^{k-1} + (\zeta z)^{k-1}}{k^2(k+1)} \right] \]
\[ + \frac{1 - |z|^2}{2!} \left[ \frac{1}{2!} + \sum_{k=2}^{\infty} \frac{(z\zeta)^{k-1} + (\zeta z)^{k-1}}{k^2(k+1)} \right] \]
\[ - \frac{1 - |z|^6}{3!} \left[ \frac{1}{3!} + \sum_{k=2}^{\infty} \frac{(z\zeta)^{k-1} + (\zeta z)^{k-1}}{k(k+1)(k+2)} \right], \]

see also [84]. In a sequence of papers Z. Du and his collaborators extended these higher order Poisson kernels to the half plane and to respective domains in higher dimensions, see e.g. [85].

References


[48] Ch. Riquier. *Sur quelques problèmes relatifs à l’équation aux dérivées partielles* \((\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^nu = 0\). C.R. 181, 490-491 (1925); J. de Math. (9) 5(1926), 297 - 393 (French).


