ON A LOGARITHMIC HARDY-BLOCH TYPE SPACE II

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Abstract. Suppose $0 < p \leq \infty$ and $-\infty < \alpha < \infty$. Let $BH_{p,\alpha}$ denote the logarithmic Hardy-Bloch type space of those functions $f$ which are analytic in the unit disk $D$ and satisfy

$$\|f\|_{p,\alpha} = \sup_{0 < |z| < 1} (1 - |z|)(\log \frac{e}{1 - |z|})^\alpha M_p(|z|, f') < \infty.$$ 

In this paper, we mainly obtain the relation between the logarithmic Hardy-Bloch type space $BH_{p,\alpha}$ and the Hardy space $H^p$ (or the Dirichlet space $D^p_{p-1}$). We also give the characterization of lacunary series on $BH_{p,\alpha}$ when $1 < p \leq \infty$.

1. Introduction

Let $D$ denote the open unit disk of the complex plane, and let $H(D)$ denote the set of all analytic functions on $D$. If $0 < r < 1$ and $f \in H(D)$, we set

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \max_{0 \leq t \leq 2\pi} |f(re^{it})|.$$ 

For $0 < p \leq \infty$, the Hardy space $H^p$ consists of those functions $f \in H(D)$ for which

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(r, f) < \infty.$$ 

We refer to [1] for the theory of Hardy spaces. The Dirichlet space $D^p_{p-1}$ consists of those functions $f \in H(D)$ for which

$$\int_D (1 - |z|)^{p-1} |f'(z)|^p dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dx dy$ denotes the normalized Lebesgue area measure on $D$. It is easy to obtain that $f \in D^p_{p-1}$ if and only if

$$\int_0^1 (1 - r)^{p-1} M_p^p(r, f') dr < \infty, \quad f \in H(D).$$ 

The space $D^p_{p-1}$ is closely related to $H^p$. A classical result of Littlewood and Paley [12] asserts that

$$H^p \subset D^p_{p-1}, \text{ if } 2 \leq p < \infty.$$ 

On the other hand, see, e.g. [16, 19], we have

$$D^p_{p-1} \subset H^p, \text{ if } 0 < p \leq 2.$$ 

It has been proved that the inclusions are strict when $p \neq 2$. If $0 < p < 2$, Vinogradov proved in [19] that there exists a Blaschke product $B$ which does not belong to $D^p_{p-1}$ for every $p \in (0, 2)$. Girela and Peláez [7] gave a sufficient and

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necessary condition for which the lacunary series \( f \in D_{p-1}^p \). Using this condition we can immediately deduce that \( D_{p-1}^p \neq H^p \) if \( p \neq 2 \). We can also see [10] for the special case \( p = 1 \).

Given \( 0 < p \leq \infty \) and \( -\infty < \alpha < \infty \). Let \( BH_{p,\alpha} \) denote the logarithmic Hardy-Bloch space, that is, the set of those \( f \in H(D) \) such that

\[
\|f\|_{p,\alpha} = \sup_{0 < r < 1} (1 - r)(\log \frac{e}{1 - r})^\alpha M_p(r, f') < \infty.
\]

We remark that the space \( BH_{p,1} \) denoted by the second author and Xiaoming Wu in [20] is a special case (\( \alpha = 1 \)) to which we denote here.

It is easy to prove that the space \( BH_{p,\alpha} \) is complete under the norm

\[
\|f\| = \|f\|_{p,\alpha} + |f(0)|.
\]

Moreover, the space \( BH_{p,\alpha} \) is Banach space when \( p > 1 \) and \( \alpha \in (-\infty, +\infty) \). The relation between the logarithmic Hardy-Bloch spaces can be easily checked that

\[
BH_{p_1,\alpha} \supset BH_{p_2,\alpha}, \quad 0 < p_1 < p_2 \leq \infty,
\]

and

\[
BH_{p,\alpha_1} \supset BH_{p,\alpha_2}, \quad -\infty < \alpha_1 < \alpha_2 < \infty.
\]

The Bloch space \( B \) (cf. [9, 15]) consists of those functions \( f \in H(D) \) for which

\[
M_\infty(r, f') = O\left(\frac{1}{1 - r}\right).
\]

With the terminology just introduced, we immediately obtain that the space \( BH_{\infty,L}^0 \) is the Bloch space \( B \). The following result is known (see [5, 8] for example).

**Theorem A.** Let \( 0 < p < \infty \) and \( f \in BH_{p,L}^0 \). Then

\[
M_p(r, f) = O\left((\log \frac{1}{1 - r})^\beta\right),
\]

where

(i) \( \beta = 1/p \), for \( 0 < p < 2 \).

(ii) \( \beta = 1/2 \), for \( 2 \leq p < \infty \).

The indexes above are optimal.

We shall give the corresponding results for the general \( \alpha \) in section 2.

Throughout this paper, \( C \) will denote a positive constant depending only on the displayed parameters but not necessarily the same from one occurrence to the next. Also, \( A \asymp B \) means that there exist constants \( C_1, C_2 > 0 \) such that \( C_1 A \leq B \leq C_2 A \).

## 2. The Relation Between BH\(_{p,\alpha}\) and \( H^p \)

With some calculations, we get the following lemma, which plays an important role in this paper.

**Lemma 2.1.** If \( a, b > 0, \ p \in (0, \infty) \), and

\[
f(z) = \frac{1}{(1 - z)^a(\log \frac{e}{1 - z})^b}, \quad z \in D,
\]

then

(i) \( f \in H^p \) if and only if \( f \in D_{p-1}^p \) if and only if \( ap < 1 \) and \( b > 0 \), or \( ap = 1 \) and \( bp > 1 \);

(ii) \( f \in BH_{p,\alpha} \) if and only if \( ap < 1 \) and \( b > 0 \), or \( ap = 1 \) and \( b \geq \alpha \).

We firstly give the corresponding results of Theorem A for the general \( \alpha \) as in the following two theorems.
Theorem 2.1. Let $0 < p < 2$ and $-\infty < \alpha < \infty$.

(a) If $1/p < \alpha < \infty$, then $\mathcal{BH}_{p,\alpha} \subseteq \mathcal{H}^p$.
(b) If $\alpha = 1/p$ and $f \in \mathcal{BH}_{p,\alpha}$, then $M_p(r, f) = O((\log \log \frac{r}{M})^{1/p})$.
(c) If $-\infty < \alpha < 1/p$ and $f \in \mathcal{BH}_{p,\alpha}$, then $M_p(r, f) = O((\log \frac{e}{r})^{1/(p-\alpha)})$.

The exponents both in (b) and (c) are sharp.

Theorem 2.2. Let $2 \leq p < \infty$ and $-\infty < \alpha < \infty$.

(a) If $1/2 < \alpha < \infty$, then $\mathcal{BH}_{p,\alpha} \subseteq \mathcal{H}^p$.
(b) If $\alpha = 1/2$ and $f \in \mathcal{BH}_{p,\alpha}$, then $M_p(r, f) = O((\log \log \frac{r}{M})^{1/2})$.
(c) If $-\infty < \alpha < 1/2$ and $f \in \mathcal{BH}_{p,\alpha}$, then $M_p(r, f) = O((\log \frac{e}{r})^{1/(2-\alpha)})$.

The estimates in (b) and (c) when $2 < p < \infty$ and $0 \leq \alpha \leq 1/2$ are sharp.

Remark 2.1. The result in [20] that $\mathcal{BH}_{p,\alpha} \subseteq \mathcal{H}^p$ ($0 < p < \infty$) is also the special case (for $\alpha = 1$) of Theorem 2.1 and 2.2.

In order to give the proof of Theorem 2.1 and 2.2, we need the following three lemmas.

Lemma 2.2. [5] If $0 < p < 2$, then there is a constant $C_p$ depending only on $p$ such that

$$
\|f\|_{\mathcal{H}^p}^p \leq C_p \left( |f(0)|^p + \int_0^1 (1-r)^{p-1} M_p^p(r, f') dr \right), \quad \text{for all } f \in \mathcal{H}^p(D).
$$

Lemma 2.3. [11] If $2 \leq p < \infty$, then there is a constant $C_p$ depending only on $p$ such that

$$
\|f\|_{\mathcal{H}^p} \leq C_p \left( |f(0)| + \left( \int_0^1 (1-r)M_p^2(r, f') dr \right)^{1/2} \right), \quad \text{for all } f \in \mathcal{H}^p(D).
$$

Lemma 2.4. Suppose $0 < p < \infty$ and $-\infty < \alpha < \infty$. Let

$$
f(r, \rho) = \frac{(1-\rho)^p(\log \frac{e}{r})^\alpha}{(1-r\rho)^p(\log \frac{e}{r\rho})^\alpha}, \quad 0 < r, \rho < 1.
$$

Then

$$
f(r, \rho) \leq 1, \quad \text{if } -\infty < \alpha \leq 1,
$$

and

$$
f(r, \rho) \leq M, \quad \text{if } 1 < \alpha < \infty,
$$

where $M = \max\{1, \alpha^\alpha e^{(1-\alpha)p}\}$.

To give the proof of Lemma 2.4, we still need the following lemma.

Lemma 2.5. [20] If $0 < \alpha, \beta < \infty$, $x \in (0, e]$, then $f(x) = x^\alpha (\log x)^\beta$ increases on $(0, e^{1-\beta/\alpha}]$, decreases on $[e^{1-\beta/\alpha}, e]$.

Proof of Lemma 2.4. Suppose $0 < p < \infty$, $-\infty < \alpha < \infty$ and $0 < r, \rho < 1$. We set

$$
f(r, \rho) = \frac{(1-\rho)^p(\log \frac{e}{r})^\alpha}{(1-r\rho)^p(\log \frac{e}{r\rho})^\alpha}.
$$

First consider the case $\alpha > 1$. By Lemma 2.5, we obtain that

$$
f(r, \rho) \leq 1,
$$

if $0 < 1 - r\rho < e^{1-\alpha}$, and

$$
f(r, \rho) \leq \frac{(e^{1-\alpha})^p(\log \frac{e}{r\rho})^\alpha}{1} = \alpha^\alpha e^{(1-\alpha)p},
$$
if $e^{1-\alpha} \leq 1 - r \rho < 1$. For the case $0 < \alpha \leq 1$, since $e^{1-\alpha} \geq 1$, Lemma 2.5 implies
\[ f(r, \rho) \leq 1. \]
For the case $\alpha \leq 0$, notice that for fixed $0 < \rho < 1$, the function $g(r) = 1/(1 - r \rho)^{\alpha}$ is increasing on $(0, 1]$. It follows that
\[ f(r, \rho) = \frac{g(r)}{g(1)} \leq 1. \]
This completes the proof.

**Proof of Theorem 2.1.** Suppose $0 < p < 2$, $-\infty < \alpha < \infty$. Take $f \in \mathcal{BH}_{p,L}^\alpha$ and assume, without loss of generality, that $f(0) = 0$. For $0 < r < 1$, set $f_r(z) = f(rz)$. Applying Lemma 2.2 to $f_r$, we deduce that
\[
M_p^p(r, f) \leq C \int_0^1 \frac{(1 - \rho)^{p-1}}{(1 - r \rho)^{\alpha p}} d\rho.
\]
Then by Lemma 2.4, we trivially obtain that if $1/p < \alpha < \infty$, then
\[
M_p^p(r, f) \leq C \int_0^1 \frac{d\rho}{(1 - \rho)^{\alpha p}} < \infty,
\]
which is equivalent to saying that $\mathcal{BH}_{p,L}^\alpha \subset H^p$. We take the test function
\[
f(z) = \left(1 - z\right)^{1/p} \left(\log e^{1-r}\right)^{\alpha}, \quad z \in \mathbb{D}.
\]
It follows from Lemma 2.1 that $f \in H^p \setminus \mathcal{BH}_{p,\alpha}$. This shows that the inclusion is strict and we finish the proof of (i).

If $-\infty < \alpha \leq 1/p$, we write
\[
M_p^p(r, f) \leq C \int_0^r \frac{d\rho}{(1 - \rho)^{\alpha p}} + C \int_r^1 \frac{d\rho}{(1 - \rho)^{\alpha p}} \int_r^1 (1 - \rho)^{p-1} d\rho.
\]
It follows that
\[
M_p(r, f) = O\left(\left(\log \frac{e}{1-r}\right)^{1/p-\alpha}\right), \quad \text{for } \alpha < 1/p,
\]
and
\[
M_p(r, f) = O\left(\left(\log \frac{e}{1-r}\right)^{1/p}\right), \quad \text{for } \alpha = 1/p.
\]
Now we take
\[
f(z) = \frac{1}{(1 - z)^{1/p} \left(\log \frac{e}{1-r}\right)^\alpha}, \quad z \in \mathbb{D},
\]
which belongs to $\mathcal{BH}_{p,\alpha}$ by Lemma 2.1. A direct computation shows that
\[
M_p(r, f) \asymp \left(\left(\log \frac{e}{1-r}\right)^{1/p}\right)^{\alpha} \quad \text{as } r \to 1, \quad \text{for } \alpha = 1/p,
\]
and
\[
M_p(r, f) \asymp \left(\left(\log \frac{e}{1-r}\right)^{1/p-\alpha}\right)^{\alpha} \quad \text{as } r \to 1, \quad \text{for } \alpha \in (-\infty, 1/p).
\]
This shows the exponents in (2) and (3) cannot be improved, and we finish the proof. \(\square\)
The proof of Theorem 2.2 by using Lemma 2.3 is similar to that of Theorem 2.1, so we omit the details here. Let us simple remark, for $2 \leq p < \infty$ and $1/2 < \alpha < \infty$, the function

$$f(z) = \frac{1}{(1-z)^{1/p}(\log \frac{e}{1-z})^{(1/p+\alpha)/2}}, \quad z \in \mathbb{D}$$

again shows that inclusion in $(a)$ of Theorem 2.2 is sharp. We defer showing the exponents in $(b)$ and $(c)$ of Theorem 2.2 are sharp when $2 < p < \infty$ and $0 \leq \alpha \leq 1/2$.

We see that the $(b)$ and $(c)$ of Theorem 2.1 and 2.2 are equivalent to saying that $BH_{p,\alpha} \nsubseteq H^p$ when $0 < p < 2$ and $\alpha \leq 1/p$ or $2 \leq p < \infty$ and $\alpha \leq 1/2$. In order to obtain an accurate inclusion relationship, we shall consider whether $H^p \subset BH_{p,\alpha}$ or not in those cases. And we obtain most of the results as in the following theorems.

**Theorem 2.3.** Given $0 < p < 2$ and $1/2 < \alpha \leq 1/p$, there exists a function $f \in H^p$ but $f / \notin BH_{p,\alpha}$.

**Theorem 2.4.** Suppose $0 < p < \infty$ and $\alpha \leq 0$. Then $H^p \subset BH_{p,\alpha}$. Moreover, the inclusion is strict when $1 < p < \infty$.

**Theorem 2.5.** Suppose $1 < p < \infty$ and $0 < \alpha < 1/2$, there exists a function $f \in H^p$ but $f / \notin BH_{p,\alpha}$.

To complete the proof of Theorem 2.3, we need the following lemma.

**Lemma 2.6.** [4] Let $\phi$ be a positive increasing function in $(0,1)$ such that

$$\int_0^1 (1-r)\phi^2(r)dr < \infty.$$ 

Let $p \in (0,2)$. Then there exists a function $f \in H^p$ given by a power series with lacunary gaps

$$(2.1) \quad f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \quad \text{with } \inf_k (n_{k+1}/n_k) = \lambda > 1,$$

such that

$$M_p(r, f') \geq \phi(r), \quad 0 < r < 1$$

**Proof of Theorem 2.3.** Given $0 < p < 2$. Let

$$\phi(r) = (1-r)^{-1} \left( \frac{1}{p} + (\log \frac{e}{1-r})^{p/2} \log \log \frac{e}{1-r} \right)^{-1/p}, \quad 0 < r < 1.$$ 

It is easy to check that $\phi$ is a positive increasing function in $(0,1)$ which satisfies

$$(2.2) \quad \int_0^1 (1-r)\phi^2(r)dr < \infty$$

and

$$(2.3) \quad \int_0^1 (1-r)^{p-1}(\log \frac{e}{1-r})^{p/2-1}\phi^p(r)dr = \infty.$$ 

Now Lemma 2.6 shows that there exists a lacunary series $f \in H^p$ such that $M_p(r, f') \geq \phi(r)$. Therefore, (2.3) implies that

$$\infty = \int_0^1 (1-r)^{p-1}(\log \frac{e}{1-r})^{p/2-1}\phi^p(r)dr$$

$$\leq \int_0^1 (1-r)^{p-1}(\log \frac{e}{1-r})^{p/2-1}M_p^p(r, f')dr$$
On the other hand, if $0 < p < 2$, $1/2 < \alpha \leq 1/p$ and for every $g \in \mathcal{BH}_{p,\alpha}$, it follows that
\[
\int_0^1 (1 - r)^{p-1}(\log \frac{e}{1 - r})^{p/2 - 1}M_p^\alpha(r, g')dr \leq \int_0^1 \frac{\|g\|_{p,\alpha}}{(1 - r)(\log \frac{e}{1 - r})^{1 - p/2 + \alpha \beta}}dr < \infty.
\]

This shows that the function $f \in H^p \setminus \mathcal{BH}_{p,\alpha}$. \hfill \Box

Our next objective is to give the characterizations of lacunary series on $\mathcal{BH}_{p,\alpha}$ when $1 < p \leq \infty$, for which we need some results in $L^p$ behavior of power series of analytic functions.

Let $\varphi$ denote a non-negative increasing function defined on $(0, 1]$. Then $\varphi$ is called a normal weight (cf. [18]) if the following conditions are satisfied:

(i) There exists a constant $\alpha > 0$ such that the function $\varphi(x)/x^\alpha(0 < x < 1)$ is almost increasing.

(ii) There exists a constant $\beta > 0$ such that the function $\varphi(x)/x^\beta(0 < x < 1)$ is almost decreasing.

A non-negative real function is almost increasing if there exists a constant $C > 0$ such that $x < y$ implies $\varphi(x) \leq C\varphi(y)$. An almost decreasing function is defined similarly. It is not difficult to check that the function
\[
\varphi(x) = x^\alpha(\log \frac{e}{x})^\beta
\]
is a normal weight when $\alpha > 0$ and $-\infty < \beta < \infty$.

The space $H(p, q, \varphi)(0 < p, q \leq \infty)$, introduced in [18] for $q = \infty$ and [13] for $q < \infty$, consists of those $f \in H(D)$ for which the function $F(r) = \varphi(1 - r)M_p(r, f)$ belongs to the space $L^q(dr/(1 - r))$. The norm in $H(p, q, \varphi)$ is given by
\[
||f||_{H(p, q, \varphi)} = ||F||_{L^q(dr/(1 - r))}.
\]

For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $D$, define the polynomials
\[
\Delta_j f(z) = \sum_{k=2^j}^{2^{j+1} - 1} a_k z^k, \quad \text{for } j \geq 1,
\]
\[
\Delta_0 f(z) = a_0 + a_1 z.
\]

Mateljevic and Plavlovic proved the following two results in [13].

**Theorem B.** Let $\varphi$ be a normal weight, $f \in H(D)$ and $0 < p \leq \infty$, $0 < q \leq \infty$. Then $f' \in H(p, q, \varphi)$ if the sequence $\{2^{-j}\varphi(2^{-j})||\Delta_j f||_{H^p}\}_{j=0}^\infty$ belongs to $l^q$.

**Theorem C.** Let $\varphi$ be a normal weight, $f \in H(D)$ and $1 < p < \infty$, $0 < q < \infty$. Then the sequence $\{2^{-j}\varphi(2^{-j})||\Delta_j f||_{H^p}\}_{j=0}^\infty$ belongs to $l^q$ if $f' \in H(p, q, \varphi)$.

Suppose $-\infty < \alpha < \infty$. Notice that $f \in \mathcal{BH}_{p,\alpha}$ if and only if $f' \in H(p, \infty, \varphi)$, where $\varphi(x) = x(\log \frac{e}{x})^\alpha$. Hence, using Lemma B and C, we trivially have the following result.

**Proposition 2.1.** Suppose $1 < p < \infty$, $-\infty < \alpha < \infty$. Then $f \in \mathcal{BH}_{p,\alpha}$ if and only if
\[
\sup_j j^\alpha||\Delta_1 f||_{H^p} < \infty.
\]

Now we recall a well known result about the characterization of power series with Hadamard gaps which belong to Hardy spaces.

**Lemma 2.7.** [23] If $f \in H(D)$ which satisfies (2.1), then, for every $p \in (0, \infty)$, $f \in H^p$ if and only if $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.  

Theorem 2.6. Suppose $1 < p \leq \infty$, then a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $BH_{p, \alpha}$ if and only if $a_k = O\left(\frac{1}{k^\alpha}\right)$.

Proof. We first consider the case $1 < p < \infty$. Let $f$ be a power series with lacunary gaps, then there exist at most $C_\lambda = \log_2 2 + 1$ of the $n_k$'s in the set

$$I(n) = \{j \in \mathbb{N} : 2^n \leq j < 2^{n+1}\}, \quad n = 1, 2, \ldots.$$  

Thus, by proposition 2.1 and Lemma 2.7, we have

$$f \in BH_{p, \alpha} \iff \sup_{j \geq 0} j^\alpha \|\Delta_j f\|_{L_p} < \infty$$

$$\iff \sup_{j \geq 0} j^\alpha \|\Delta_j f\|_{L^2} < \infty$$

(2.4)

$$\iff \sup_{k \geq 0} k^{2\alpha} |a_k|^2 < \infty$$

$$\iff |a_k| = O\left(\frac{1}{k^\alpha}\right).$$

For the case $p = \infty$. If $f = \sum_{k=0}^{\infty} a_k z^k \in BH_{\infty, \alpha}$, apply Cauchy formula to write

$$a_k = \frac{1}{2\pi i} \int_{|\xi| = \rho} \frac{f'(\xi)}{\xi^k} d\xi, \quad 0 < \rho < 1.$$  

Thus

$$|a_k| \leq \frac{1}{2\pi} \int_{|\xi| = \rho} \frac{|f'(\xi)|}{|\xi|^k} d\xi \leq \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\|f\|_{L_p}^\alpha}{\rho^\alpha(1 - \rho)(\log \frac{e}{1 - \rho})^\alpha} d\rho, \quad 0 < \rho < 1.$$  

Let $\rho = 1 - 1/k$, we obtain that

$$a_k = O\left(\frac{1}{k^\alpha}\right).$$

If $f$ is a lacunary series, we have $a_k = O\left(\frac{1}{k^\alpha}\right)$.

On the other hand, given a lacunary series $f$, Theorem B shows that

$$\sup_{0 < r < 1} (1 - r)(\log \frac{e}{1 - r})^\alpha |f'(z)| \leq C \sup_{j} j^\alpha \|\Delta_j f\|_{L^\infty} \leq C \sup_{j} j^\alpha a_j.$$  

Since $a_k = O\left(\frac{1}{k^\alpha}\right)$, we have trivially $f \in BH_{\infty, \alpha}$. \qed

**Proof of Theorem 2.4.** Suppose $0 < p < \infty$ and $f \in H^p$. It is well known that there exists a constant $C > 0$ such that

$$|f'(z)| \leq \frac{C}{1 - |z|} \left|\frac{f(z)}{z}\right|.$$  

This implies that for fixed $1/2 < r < 1$,

$$M^p_r(f, f') = \frac{1}{2\pi} \int_{0}^{2\pi} \left|f'(re^{it})\right|^p dt \leq \frac{C}{(1 - r)^p} \int_{0}^{2\pi} \left|f(re^{it})\right|^p dt \leq \frac{C}{(1 - r)^p} M^p_r(f, f)$$  

Now by the definition of the $BH_{\infty, \alpha}$, it follows that $H^p \subset BH_{\infty, \alpha}$ when $\alpha \leq 0$.

Suppose $1 < p < \infty$ and $-\infty < \alpha < 1/2$. It follows from Lemma 2.7 and (2.4) that the function

$$g(z) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} z^{n_k}$$

28 Mar 2021 05:33:23 PDT
belongs to $BH_{p,\alpha}$ but not to $H^p$. This finishes the proof.

**Proof of Theorem 2.5.** Suppose $1 < p < \infty$ and $0 < \alpha < 1/2$. By Lemma 2.7 and Theorem 2.6, it suffices to prove that there exists a sequence $\{a_k\}$ which satisfies
\[ \sum_{k=0}^\infty |a_k|^2 < \infty \quad \text{and} \quad \sup_k k^\alpha |a_k| = \infty. \]

We set $a_k = k^\alpha/2$ when $k = j^{(2/\alpha)+1}$, $j = 1, 2, 3, \ldots$ and $a_k = \frac{1}{k}$ the others, where $[x]$ represents the maximum integer not exceeding $x$. Clearly, the sequence $\{a_n\}$ we construct here satisfies
\[ \sup_k k^\alpha |a_k| = \infty. \]

It also follows that
\[ \sum_{k=0}^\infty |a_k|^2 \leq \sum_{k=0}^\infty \frac{1}{k^2} + \sum_{j=0}^\infty \frac{1}{j^{((2/\alpha)+1)\alpha}} < \infty. \]

This finishes the proof.

**Lemma 2.8.** Let $0 \leq \alpha \leq 1/2$, then there exists a constant $C > 0$ such that
\[ \sum_{k=0}^\infty \frac{1}{(k+1)^{2\alpha} r^{2k+1}} \geq C \left( \log \log \frac{e}{1-r} \right), \quad \text{if } \alpha = 1/2, \]
and
\[ \sum_{k=0}^\infty \frac{1}{(k+1)^{2\alpha} r^{2k+1}} \geq C \left( \log \frac{e}{1-r} \right)^{1-2\alpha}, \quad \text{if } \alpha \in [0, 1/2). \]

for $r \in [1/e, 1)$.

**Proof.** This proof uses the method of Lemma 1 in [3]. Suppose $0 \leq \alpha \leq 1/2$, fix $r \in [1/e, 1)$ and set
\[ f(x) = \frac{1}{(x+1)^{2\alpha} r^{2x+1}}. \]

Since $f(x)$ is a decreasing function in $(0, \infty)$, we have
\[ \sum_{k=0}^\infty \frac{1}{(k+1)^{2\alpha} r^{2k+1}} \geq \int_0^\infty \frac{1}{(x+1)^{2\alpha} r^{2x+1}} dx. \]

We set
\[ x = \log_2 y - \log_2 \log \frac{1}{r} - 1. \]

This, together with the inequalities
\[ \log \frac{1}{r} \leq e(1-r^2), \quad \text{for } 1/e < r < 1 \]
and
\[ \log \frac{1}{r} \geq 0, \quad \text{for } 1/e < r < 1 \]

imply that
\[
\int_0^\infty \frac{1}{(x+1)^{2\alpha} r^{2x+1}} dx = \frac{1}{\log 2} \int_{2\log \frac{1}{r}}^\infty \frac{1}{(\log y - \log \log \frac{1}{r})^{2\alpha}} \frac{e^{-y}}{y} dy \\
\geq \frac{1}{\log 2} \int_{2e(1-r^2)}^\infty \frac{1}{(\log y - \log \log \frac{1}{r})^{2\alpha}} \frac{e^{-y}}{y} dy \\
\geq \frac{1}{\log 2e^{4e}} \int_{2e(1-r^2)}^{4e} \frac{1}{y(\log y)^{2\alpha}} dy.
\]
Then we can easily obtain that
\[\sum_{k=0}^{\infty} \frac{1}{(k+1)^{2\alpha}r^{2k+1}} \geq C(\log \log \frac{e}{1-r})^{\alpha} \quad \text{for } \alpha = 1/2,\]
and
\[\sum_{k=0}^{\infty} \frac{1}{(k+1)^{2\alpha}r^{2k+1}} \geq C(\log \frac{e}{1-r})^{1-2\alpha} \quad \text{for } \alpha \in [0, 1/2).\]
This finishes the proof of Lemma 2.7.

Now, we can show that the exponents in Theorem 2.2 when \(2 \leq p < \infty\) and \(\alpha \leq 1/2\) are best possible. In those cases, we set
\[f(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha}z^{2k+1}}, \quad z \in \mathbb{D}.\]
Then (2.4) implies that \(f \in \mathcal{BH}_{p,\alpha}\). Let \(z = re^{it}\), Use Zygmund’s result on gap series (see Theorem 8.20 in [23]) to write
\[M^p_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha}z^{2k+1}} \right|^p dt \geq C \left( \sum_{k=0}^{\infty} \frac{1}{(k+1)^{2\alpha}r^{2k+1}} \right)^{p/2}\]
This and Lemma 2.8 show the exponents in Theorem 2.2 when \(2 \leq p < \infty\) and \(\alpha \leq 1/2\) are sharp.

Our next objective is to give the relation between \(\mathcal{BH}_{\infty,\alpha}\) and \(H^\infty\).

**Theorem 2.7.**
(a) If \(1 < \alpha < \infty\), then \(\mathcal{BH}_{\infty,\alpha} \subsetneq H^\infty\).
(b) If \(1/2 < \alpha \leq \infty\), then there exists a function \(f \in H^\infty \setminus \mathcal{BH}_{\infty,\alpha}\). However, \(\mathcal{BH}_{\infty,\alpha} \subset H^2\), when \(1/2 < \alpha \leq 1\).
(c) If \(-\infty < \alpha \leq 1\), then there exists a function \(f \in \mathcal{BH}_{\infty,\alpha} \setminus H^\infty\).
(d) If \(-\infty < \alpha \leq 0\), then \(H^\infty \subsetneq \mathcal{BH}_{\infty,\alpha}\).

In [22], Zygmund observes that if \(f \in H(\mathbb{D})\), then
\[|f'(pe^{i\theta})|d\rho = o \left( (\log \frac{1}{1-r})^{1/2} \right), \quad \text{as } r \to 1^-,
\]
for almost all \(\theta\) in the set of those \(\theta\) where \(f\) has a nontangential limit at \(e^{i\theta}\).
Zygmund also proved in [22] that the estimate (2.5) is sharp in \(L = \{f : f \in H^2, f(z) = \sum_{n=0}^{\infty} a_n z^{2n}\}\). It follows from in [23] that \(L \in H^p\) for each \(p > 0\), so that (2.5) is sharp in \(\bigcap_{0 < p < \infty} H^p\). And Hallenbeck proved in [10] that (2.5) is sharp in \(H^\infty\).

**Proof of Theorem 2.6.** We first suppose \(1 < \alpha < \infty\) and \(f \in \mathcal{BH}_{\infty,\alpha}\), and without loss of generality, that \(f(0) = 0\), then
\[|f(re^{it})| \leq \int_0^r |f'(pe^{i\theta})|d\rho \leq C \int_0^r \frac{d\rho}{(1-\rho)(\log \frac{e}{1-\rho})^\alpha} \leq C \int_0^1 \frac{d\rho}{(1-\rho)(\log \frac{e}{1-\rho})^\alpha} < \infty.
\]
Which implies
\[\mathcal{BH}_{\infty,\alpha} \subset H^\infty.
\]
For \(1/2 < \alpha < \infty\) and \(f \in \mathcal{BH}_{\infty,\alpha}\), we have
\[\int_0^r |f'(pe^{i\theta})|d\rho = O \left( (\log \frac{1}{1-r})^{1-\alpha} \right).
\]
The fact that (2.5) is sharp in $H^\infty$ implies that there exists a function $f \in H^\infty$ but $f \notin \mathcal{BH}_{\infty,\alpha}$ for $1/2 < \alpha < \infty$.

For the case $1/2 < \alpha \leq 1$, let $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{BH}_{\infty,\alpha}$. Applying Parseval’s formula, we have

$$\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} = \frac{1}{2\pi} \int_{0}^{2\pi} |f'(re^{i\theta})|^2 d\theta = O \left( \frac{1}{(1-r)^2 (\log \frac{1}{1-r})^{2\alpha}} \right),$$

which, integrating twice, gives

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta = O(1).$$

This implies $f \in H^2$. Thus,

$$\mathcal{BH}_{\infty,\alpha} \subset H^2, \quad \text{for } 1/2 < \alpha \leq 1.$$ And we finish the proof of (a) and (b).

It is known that $H^\infty \subset \mathcal{BH}_{\infty,L}^0$ (See for instance [2]). And it is easy to prove that the function

$$f(z) = \log \log \frac{e}{1-z}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{BH}_{\infty,L}^1$, but not to $H^\infty$. Bearing in mind the fact that

$$\mathcal{BH}_{p,L}^{\alpha_1} \supset \mathcal{BH}_{p,L}^{\alpha_2}, \quad \text{if } -\infty < \alpha_1 < \alpha_2 < \infty,$$

we can finish the proof of (c) and (d).

Now we shall estimate the growth of $f \in \mathcal{BH}_{p,\alpha}(0 < p < \infty)$. The following two lemmas are needed.

**Lemma 2.9.** If $0 < p < \infty$, $0 < \alpha < \infty$ and $0 < r < 1$, then there exists a constant $C_{p,\alpha}$ depending only on $p$ and $\alpha$ such that

$$\int_{0}^{r} \frac{1}{(1-s)^{1+1/p}(\log \frac{e}{1-s})^\alpha} ds \leq \frac{C_{p,\alpha}}{(1-r)^{1/p}(\log \frac{e}{1-r})^\alpha}.$$  

**Proof.** For $p \in (0, \infty)$ and $\alpha \in (0, \infty)$, we set $r_0 = 1 - e^{1-2p/\alpha}$. Using Lemma 2.5, we easily obtain that if $r_0 \geq 1$

$$\int_{0}^{r} \frac{1}{(1-s)^{1+1/p}(\log \frac{e}{1-s})^\alpha} ds \leq \frac{1}{(1-r)^{1/p}(\log \frac{e}{1-r})^\alpha} \int_{0}^{r} \frac{1}{(1-s)^{1+1/2p}} ds \leq \frac{2p}{(1-r)^{1/p}(\log \frac{e}{1-r})^\alpha}.$$  

On the other hand, if $r_0 < 1$, then for $r \leq r_0$, we have

$$\int_{0}^{r} \frac{1}{(1-s)^{1+1/p}(\log \frac{e}{1-s})^\alpha} ds \leq \int_{0}^{r} \frac{1}{(1-s)^{1+1/2p}} ds \leq 2pe.$$  

For $r > r_0$, we write

$$\int_{0}^{r} \frac{1}{(1-s)^{1+1/p}(\log \frac{e}{1-s})^\alpha} ds \leq \int_{0}^{r_0} \frac{1}{(1-s)^{1+1/p}(\log \frac{e}{1-s})^\alpha} ds + \int_{r_0}^{r} \frac{1}{(1-s)^{1+1/p}(\log \frac{e}{1-s})^\alpha} ds \leq 2pe + \frac{1}{(1-r)^{1/2p}(\log \frac{e}{1-r})^\alpha} \int_{0}^{r} \frac{1}{(1-s)^{1+1/2p}} ds \leq \frac{C_{p,\alpha}}{(1-r)^{1/p}(\log \frac{e}{1-r})^\alpha}.$$  

This finishes the proof. \[\square\]
We firstly fix \( \alpha \in (-\infty, \infty) \). When \( 1 \leq p < \infty \), let \( q \) be the conjugate index, then, by Cauchy formula and Hölder’s inequality, we obtain that

\[
|f'(z)| \leq \frac{1}{2\pi} \int_{|\xi| = \rho} \frac{|f'(|\xi|)|}{|\xi - z|} \, d\xi
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{|re^{i\theta} - z|} \, dt
\]

\[
\leq M_p(\rho, f') \left\{ \int_0^{2\pi} \frac{dt}{|re^{i\theta} - z|^q} \right\}^{1/q},
\]

where \( |z| = r < \rho < 1 \). Let \( \rho = (1 + r)/2 \) and simple calculate, we deduce

\[
|f'(z)| \leq \frac{C\|f\|_{p,\alpha}}{(1 - r)^{1+1/p}(\log \frac{e}{1-r})^\alpha}.
\]

When \( p < 1 \), under the preliminary assumption that \( f(z) \neq 0 \) in \( |z| < 1 \), the function \( F(z) = |f'(z)|^p \) is analytic and

\[
M_1(r, F) = \{M_p(r, f')\}^p \leq \frac{C^p\|f\|_{p,\alpha}^p}{(1 - r)^{p+1}(\log \frac{e}{1-r})^{\alpha p}}.
\]

Then we have

\[
|f'(z)|^p = |F(z)| \leq \frac{C^p\|f\|_{p,\alpha}^p}{(1 - r)^{p+1}(\log \frac{e}{1-r})^{\alpha p}}.
\]

Thus

\[
|f'(z)| \leq \frac{C\|f\|_{p,\alpha}}{(1 - r)^{1+1/p}(\log \frac{e}{1-r})^\alpha}.
\]

If \( f'(z) \) has zeros, we fix \( \rho < 1 \) and use Lemma 2.10 to write

\[
f'(\rho z) = f_1(z) + f_2(z),
\]

where \( f_1 \) and \( f_2 \) do not vanish and

\[
\|f_n\|_p \leq 2M_p(\rho, f') \leq \frac{2\|f\|_{p,\alpha}}{(1 - \rho)(\log \frac{e}{1-r})^\alpha}.
\]

Since \( f_n(z) \neq 0 \), it follows that

\[
|f'(\rho e^{i\theta})| \leq |f_1(\rho e^{i\theta})| + |f_2(\rho e^{i\theta})| \leq \frac{C\|f\|_{p,\alpha}}{(1 - r)^{1+1/p}(\log \frac{e}{1-r})^\alpha}.
\]

Assume, without loss of generality, that \( f(0) = 0 \) and apply the relation

\[
f(\rho e^{i\theta}) \leq |f(0)| + \int_0^\rho |f'(s e^{i\theta})| \, ds.
\]

We obtain

\[
|f(\rho e^{i\theta})| \leq C\|f\|_{p,\alpha} \int_0^\rho \frac{1}{(1 - s)^{1+1/p}(\log \frac{e}{1-r})^\alpha} \, ds.
\]
If $\alpha \leq 0$, we have
\[
|f(re^{it})| \leq C\|f\|_{p,\alpha} \int_0^r \frac{1}{(1-s)^{1+1/p}(\log \frac{r}{1-s})^\alpha} ds
\]
\[
\leq C\|f\|_{p,\alpha} \int_0^r \frac{1}{(\log \frac{r}{1-s})^\alpha} \frac{1}{(1-s)^{1+1/p}} ds
\]
\[
\leq \frac{C\|f\|_{p,\alpha}}{(1-r)^{1/p}(\log \frac{r}{1-s})^\alpha}.
\]

If $\alpha > 0$, we apply Lemma 2.9 to write
\[
|f(re^{it})| \leq C\|f\|_{p,\alpha} \int_0^r \frac{1}{(1-s)^{1+1/p}(\log \frac{r}{1-s})^\alpha} ds
\]
\[
\leq \frac{C\|f\|_{p,\alpha}}{(1-r)^{1/p}(\log \frac{r}{1-s})^\alpha}.
\]
This finishes the proof of Theorem 2.6. \hfill\Box

A complex-value function defined in $\mathbb{D}$ is said to be \textit{univalent} if it is analytic and one-to-one there. We let $\mathcal{U}$ be the class of all univalent functions in $\mathbb{D}$.

**Theorem D.** [14] Suppose that $0 < p < \infty$. If $f \in \mathcal{U}$ and $\int_0^1 \mathcal{M}_p(z,f) dr < \infty$, then $f \in H^p$.

For $0 < p < \infty$ and $-\infty < \alpha < \infty$. We set
\[
H_{p,\alpha} = \{ f \in H(\mathbb{D}), \sup_{z \in \mathbb{D}} (1-|z|)^{1/p} (\log \frac{1}{1-|z|})^\alpha |f(z)| < \infty \}.
\]

It is easy to check that $\mathcal{B}H_{p,\alpha}$ and the classic Bloch space $\mathcal{B}$ are included in $H_{p,\alpha}$. For every $0 < p < \infty$ and $-\infty < \alpha < \infty$, the function
\[
f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad z \in \mathbb{D}
\]
is in $\mathcal{B}$ but not in $H^p$. This implies that there exists a function $f$ such that $f \in H_{p,\alpha}$ but $f \notin H^p$. On the other hand, if $0 < p < \infty$ and $1/p < \alpha < \infty$, we set
\[
g(z) = \frac{1}{(1-z)^{1/p}(\log \frac{r}{1-z})^{(1/p+\alpha)/2}}, \quad z \in \mathbb{D}.
\]
Then $g \in \mathcal{U}$ and $g \in H^p \setminus H_{p,\alpha}$. The arguments above show that for $0 < p < \infty$, the space $H^p$ and the spaces $H_{p,\alpha} (1/p < \alpha < \infty)$ do not include each other. However, using Theorem D, we can easily obtain the following theorem.

**Theorem 2.9.** Suppose $0 < p < \infty$ and $1/p < \alpha < \infty$. Then $(\mathcal{U} \cap H_{p,\alpha}) \subsetneq (\mathcal{U} \cap H^p)$.

3. The relation between $\mathcal{B}H_{p,\alpha}$ and $\mathcal{D}_{p-1}^p$

In this section, we obtain the relation between $\mathcal{B}H_{p,\alpha}$ and $\mathcal{D}_{p-1}^p$ all the cases except $0 < p \leq 1$ and $0 < \alpha < 1/p$. We also obtain sharp estimate on the growth of the integral means of $\mathcal{D}_{p-1}^p$-functions.

**Theorem 3.1.**

(i) If $0 < p < \infty$ and $\alpha > 1/p$, then $\mathcal{B}H_{p,\alpha} \subset \mathcal{D}_{p-1}^p$.

(ii) If $0 < p < \infty$ and $\alpha \leq 0$, then $\mathcal{D}_{p-1}^p \subset \mathcal{B}H_{p,\alpha}$.

(iii) If $1 < p < \infty$ and $0 < \alpha < 1/p$, then $\mathcal{D}_{p-1}^p$ and $\mathcal{B}H_{p,\alpha}$ do not include each other.
Proof. (i) Suppose $0 < p < \infty$ and $\alpha > 1/p$. Take $f \in \mathcal{D}_{p-1}^p$. Then
\[
M_p(r, f') < \frac{\|f\|_{p, \alpha}}{(1 - r)(\log \frac{1}{1 - r})^{\alpha}}.
\]
A direct computation shows that
\[
\int_0^1 (1 - r)^{p-1} M_p^p(r, f') \, dr < \infty.
\]
This implies that $f \in \mathcal{D}_{p-1}^p$. Hence, $\mathcal{B}H_{p, \alpha} \subset \mathcal{D}_{p-1}^p$.

(ii) Suppose now that $0 < p < \infty$ and $\alpha \leq 0$. If $f \in \mathcal{D}_{p-1}^p$, then
\[
\lim_{r \to 1^-} \int_r^1 (1 - \rho)^{p-1} M_p^p(\rho, f') \, d\rho = 0.
\]
Since the integral means $M_p(\rho, f')$ increase with $\rho$, it follows that
\[
\int_1^r (1 - \rho)^{p-1} M_p^p(\rho, f') \, d\rho \geq M_p^p(\rho, f') \int_r^1 (1 - \rho)^{p-1} \, d\rho \geq C_p M_p^p(r, f')(1 - r)^p.
\]
Letting $r$ tends to 1, we obtain that
\[
M_p(r, f') = o \left( (1 - r)^{-1} \right), \quad \text{as } r \to 1^-.
\]
This and the definition of $\mathcal{B}H_{p, \alpha}$ show that $\mathcal{D}_{p-1}^p \subset \mathcal{B}H_{p, \alpha}$ when $\alpha \geq 0$.

(iii) For the case $1 < p < \infty$ and $0 < \alpha < 1/p$. Let us recall a well known result in [7]: If $f$ satisfies (2.1), then $f \in \mathcal{D}_{p-1}^p$ if and only if
\[
\sum |a_k|^p < \infty.
\]
Using this and Theorem 2.5, we immediately obtain that the function
\[
g(z) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} z^{\alpha}
\]
belongs to $\mathcal{B}H_{p, \alpha}$ but not to $\mathcal{D}_{p-1}^p$.

To complete the proof, we still need to find a function $f$ such that $f \in \mathcal{D}_{p-1}^p$ but $f \notin \mathcal{B}H_{p, \alpha}$ in this case. By Theorem 2.6 and (3.1), it suffices to prove that there exists a sequence $\{a_k\}$ which satisfies
\[
\sum |a_k|^p < \infty \quad \text{and} \quad \sup_k k^\alpha |a_k| < \infty.
\]
For $1 < p < \infty$ and $0 < \alpha < 1/p$. We set $a_k = k^{\alpha/2}$ when $k = j^{[2/\alpha] + 1}$, $j = 1, 2, 3, \ldots$ and $a_k = \frac{1}{k^{\alpha/2}}$ the others, where $[x]$ represents the maximum integer not exceeding $x$. Clearly, the sequence $\{a_n\}$ we construct here satisfies $\sup_k k^\alpha |a_k| = \infty$.

It also follows that
\[
\sum_{k=0}^{\infty} |a_k|^p \leq \sum_{k=0}^{\infty} \frac{1}{k^\alpha} + \sum_{j=0}^{\infty} \frac{1}{j^{[2/\alpha]+1}p\alpha/2} < \infty.
\]
This finishes the proof. \[\square\]

The following result is an improvement about the integral means of Dirichlet spaces in [7].

**Theorem 3.2.** If $2 < p < \infty$ and $f \in \mathcal{D}_{p-1}^p$, then
\[
M_p(r, f) = o \left( (\log \frac{1}{1 - r})^{1/2 - 1/p} \right), \quad \text{as } r \to 1.
\]
The exponent above is the best possible.

In fact, this Theorem is a byproduct of the following two lemmas.
Lemma 3.1. [8] If $2 < p < \infty$ and $f \in \mathcal{D}_p^{p-1}$, then

$$
\int_0^1 \frac{1}{1-\rho} \left( \frac{e}{1-\rho} \right)^{-p/2} M_p(r, f) \, dr < \infty.
$$

Lemma 3.2. [7] If $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$, then there exists a function $f \in \mathcal{D}_p^{p-1}$ such that

$$\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{it})| \, dt \right) \neq o \left( \left( \frac{1}{1-r} \right)^\beta \right), \quad \text{as } r \to 1^-.$$

**Proof of Theorem 3.2.** Suppose $2 < p < \infty$ and $f \in \mathcal{D}_p^{p-1}$. Applying Lemma 3.1, we obtain that

$$\lim_{r \to 1^-} \int_r^1 \frac{1}{1-\rho} \left( \frac{e}{1-\rho} \right)^{-p/2} M_p(r, f) \, d\rho = 0. \quad (3.2)$$

It follows that

$$\int_r^1 \frac{1}{1-\rho} \left( \frac{e}{1-\rho} \right)^{-p/2} M_p(r, f) \, d\rho \geq M_p(r, f) \int_r^1 \frac{1}{1-\rho} \left( \frac{e}{1-\rho} \right)^{-p/2} \, d\rho \geq C_p M_p(r, f) \left( \frac{1}{1-r} \right)^{1-p/2}.$$

This together with (3.2) imply that

$$M_p(r, f) = o \left( \left( \frac{1}{1-r} \right)^{1/2-1/p} \right), \quad \text{as } r \to 1.$$

Since

$$\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{it})| \, dt \right) \leq M_2(r, f)$$

and the fact that the integral means $M_p(r, f')$ increase with $p$, Lemma 3.2 shows that the exponent $1/2 - 1/p$ is the best possible. This finishes the proof.

**References**


ON A LOGARITHMIC HARDY-BLOCH TYPE SPACE II


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